Centre for Logic and Philosophy of Science
Centrum voor Logica en Wetenschapsfilosofie

UNIVERSITEIT GENT

FACULTY OF ARTS AND PHILOSOPHY

Paraconsistent Intuitionistic Logic

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Outline

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Intuitionist Negation: Intuitive Interpretation

“$\neg p$ can be asserted if and only if we possess a construction which from the supposition that a construction p were carried out, leads to a contradiction.” (Heyting, 1956, p. 98)

Introduction

Standard Intuitionistic Negation

Intuitionist Negation: Intuitive Interpretation

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Intuitionist Negation: Proof Theoretic Interpretation

In intuitionistic logic **INT**, negation is characterized by the axiom schemas **RED** (reductio) and **EFQ** (ex falso quodlibet).

RED $(A \supset \neg A) \supset \neg A$
EFQ $A \supset (\neg A \supset B)$

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EFQ $A \supset (\neg A \supset B)$

Clash of the Interpretations!

In view of the intuitive interpretation of intuitionisitic negation, it is hard to see why the logic **INT** validates the inference rule **EFQ**.

FOR The construction of a contradiction doesn't guarantee the construction of any formula whatsoever.

Introduction

Standard Intuitionistic Negation

Heyting's Answer

"Now suppose that $\vdash \neg p$, that is, we have deduced a contradiction from the supposition that p were carried out. Then, in a sense, this can be considered as a construction, which joined to a proof of p (which cannot exist) leads to a proof of q." (Heyting, 1956, p. 102)

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IMPLIES There are no constructions for contradictions!

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IMPLIES There are no constructions for contradictions!

HOWEVER This has been refuted time and again by the history of scientific practice.

FOR People seem to find it quite difficult to come up with theories that do not contain contradictions, which is only possible if these theories contain constructions for those contradictions.

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Standard Intuitionistic Negation

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The Normative Answer

A theory should not contain constructions for contradictions!

BUT — I agree!

HOWEVER — The present inconsistent theories have to be put to use as long as no consistent replacement theories have been constructed.

$\Rightarrow$ It is necessary to cope efficiently with the theories at hand!

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Aim of this talk

A Twofold Aim

To present a version of intuitionistic logic

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- that can efficiently cope with inconsistent theories, and
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$\Rightarrow$ I will do so by relying on the *adaptive logics approach* — based on Batens (2001,2007,201x) and Lycke (201x).

Outline

First Proposal

The overall rejection of the axiom schema **EFQ**.

Paraconsistent Intuitionistic Logic

Main Idea

First Proposal

The overall rejection of the axiom schema **EFQ**.

$\Rightarrow$ The logic **INTuN** (intuitionistic logic with gluts for negation).

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Paraconsistent Intuitionistic Logic

The Logic **INTuN**: Language Schema

Preliminary Remark

For reasons of simplicity, I here limit myself to the propositional case!

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The Language Schema(s) of **INTuN**

Language	*Letters*	*Logical Symbols*	*Well–Formed Formulas*
$\mathcal{L}$	$\mathcal{S}$	$\sim, \wedge, \vee, \supset$	$\mathcal{W}$
$\mathcal{L}^{\perp}$	$\mathcal{S}, \perp$	$\sim, \wedge, \vee, \supset$	$\mathcal{W}^{\perp}$

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The Negation Set $\mathcal{N}$

$\mathcal{N} = \{\sim A \mid A \in \mathcal{W}\}$.

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Paraconsistent Intuitionistic Logic

Axioms and Rules

The axiom system of **INTuN** is obtained by adding the axiom schema **RED** (reductio) to the axiom system of positive intuitionistic logic **INT**.

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A⊃1	$A \supset (B \supset A)$
A⊃2	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A∧1	$(A \wedge B) \supset A$
A∧2	$(A \wedge B) \supset B$
A∧3	$A \supset (B \supset (A \wedge B))$
A∨1	$A \supset (A \vee B)$
A∨2	$B \supset (A \vee B)$
A∨3	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
RED	$(A \supset \sim A) \supset \sim A$
MP	$A \supset B, A \Rightarrow B$

Absurdity

The axiom schema **A**$\perp$ may also be added to the axiom system of **INTuN**.

$$\mathrm{A}\perp \qquad \perp \supset A$$

Paraconsistent Intuitionistic Logic

The Logic **INTuN**: Proof Theory

Absurdity

The axiom schema **A**$\bot$ may also be added to the axiom system of **INTuN**.

A$\bot$ $\quad \bot \supset A$

REMARK $\sim A$ and $A \supset \bot$ are not interdefinable!

$\Rightarrow$ Negation is NOT interpreted in terms of absurdity.

Absurdity

The axiom schema **A**$\perp$ may also be added to the axiom system of **INTuN**.

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In the remaining, $\perp$ is taken to be included in the characterization of the logic **INTuN**.

HOWEVER $\perp$ is not allowed in the premises nor in the conclusion of an **INTuN**–proof!

($\perp$ has mainly been introduced for technical reasons)

Defining Proofs

An **INTuN**–proof is a finite sequence of well–formed formulas (wffs) each of which is a premise, an axiom or follows from wffs earlier in the list by means of a rule of inference.

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Derivability

$\Gamma \vdash_{\textbf{INTuN}} A$ iff there is an **IntuN**–proof of the formula A from $B_1, ..., B_n \in \Gamma$.

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INTuN–Models

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  - $v : \mathcal{S} \cup \mathcal{N} \times W \mapsto \{0, 1\}$ is an assignment function.

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  - $v : \mathcal{S} \cup \mathcal{N} \times W \mapsto \{0, 1\}$ is an assignment function.
- The following hereditariness condition is introduced:
  - For $A \in \mathcal{S} \cup \mathcal{N}$ and $w, w' \in W$, if Rww' and $v(A, w) = 1$ then $v(A, w') = 1$.

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Paraconsistent Intuitionistic Logic

The Logic **INTuN**: Semantics

INTuN–Valuations

- The assignment function v of M is extended to a valuation function v_M in the following way:
  - For $A \in \mathcal{S}$, $v_M(A, w) = 1$ iff $v(A, w) = 1$.
  - $v_M({\sim}A, w) = 1$ iff, for all $w' \in W$, if Rww' then $v_M(A, w) = 0$ or $v({\sim}A, w) = 1$.
  - $v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = 1$ and $v_M(B, w) = 1$.
  - $v_M(A \vee B, w) = 1$ iff$v_M(A, w) = 1$ or $v_M(B, w) = 1$.
  - $v_M(A \supset B, w) = 1$ iff, for all $w' \in W$, if Rww' then $v_M(A, w') = 0$ or $v_M(B, w') = 1$.

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  - $v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = 1$ and $v_M(B, w) = 1$.
  - $v_M(A \vee B, w) = 1$ iff$v_M(A, w) = 1$ or $v_M(B, w) = 1$.
  - $v_M(A \supset B, w) = 1$ iff, for all $w' \in W$, if Rww' then $v_M(A, w') = 0$ or $v_M(B, w') = 1$.
- Validity is defined as truth at the actual world w_0 in all models.
- Semantic consequence is defined as truth preservation at the actual world w_0.

Outline

Paraconsistent Intuitionistic Logic

The Logic **INTuN**: Advantages and Disadvantages

Advantage

Because of the overall rejection of the axiom schema **EFQ**, the logic **INTuN** doesn't explode in the face of inconsistencies.

$\Rightarrow$ The logic **INTuN** can cope with inconsistent theories.

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Disadvantage

Most applications of the axiom schema **RAA** (reductio ad absurdum) aren't valid either!

$$\text{RAA} \quad (A \supset B) \supset ((A \supset {\sim}B) \supset {\sim}A)$$

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RAA $(A \supset B) \supset ((A \supset {\sim}B) \supset {\sim}A)$

HOWEVER **RAA** captures the intuitive meaning of intuitionistic negation. Hence, most applications of **RAA** should be valid.

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Inconsistency–Adaptive Intuitionistic Logic

Main Idea

Adaptive Logics?

Adaptive Logics are formal logics that were developed to explicate dynamic (reasoning) processes (both monotonic and non–monotonic ones).

e.g. Handling inconsistency, induction, abduction, default reasoning,...

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The logic **INTuN**$^{\mathbf{m}}$ adds all *unproblematic instantiations* of **RAA** to the logic **INTuN**.

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$\Rightarrow$ The logic $\mathbf{INTuN^m}$ captures the intuitive meaning of intuitionistic negation.

HOW? By interpreting a premise set as consistent as possible.

= Proceed as a classical intuitionist logician would, i.e. suppose that no constructions for inconsistencies can be obtained, except for those inconsistencies of which you can prove otherwise.

Outline

General Characterization

General Characterization

1. Lower Limit Logic (**LLL**)
2. Set of Abnormalities Ω
3. Adaptive Strategy

Inconsistency–Adaptive Intuitionistic Logic

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Adaptive Consequences

$$\frac{\Gamma \vdash_{\mathbf{INTuN}} (\bigvee(\Delta) \supset \bot) \supset B \qquad (\Delta \text{ a finite subset of } \Omega)}{\Gamma \vdash_{\mathbf{INTuN^m}} B \qquad (\textit{unless } Dab(\Delta) \supset \bot \text{ cannot be true})}$$

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$(\Delta = \emptyset)$ B is a final **INTuN$^{\mathbf{m}}$**–consequence of Γ.

$\Rightarrow \quad Cn_{\mathbf{INTuN}}(\Gamma) \subseteq Cn_{\mathbf{INTuN^m}}(\Gamma)$

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$(\Delta = \emptyset)$ B is a final $\mathbf{INTuN^m}$–consequence of Γ.

$\Rightarrow$ $Cn_{\mathbf{INTuN}}(\Gamma) \subseteq Cn_{\mathbf{INTuN^m}}(\Gamma)$

$(\Delta \neq \emptyset)$ B is a conditional $\mathbf{INTuN^m}$–consequence of Γ.

$\Rightarrow$ B is a final $\mathbf{INTuN^m}$–consequence of Γ as well if $\bigvee(\Delta)$ can safely be interpreted as false (at all *reachable worlds*).

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⇒ The $\mathbf{INTuN^m}$–consequences of a premise set are defined by reference to the set of *minimally abnormal* **INTuN**–models of that premise set.

i.e. $\Gamma \vDash_{\mathbf{INTuN^m}} A$ iff, for all minimally abnormal **INTuN**–models of Γ, $v_M(A, w_0) = 1$.

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Minimally Abnormal Models of a Premise Set

- The set of *reachable worlds* $Reach(M)$ of an **INTuN**–model M.
  - $Reach(M) = \{w \in W \mid Rw_0w \text{ is the case in } M\}$.

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- The *abnormal part* $Ab(M)$ of an **INTuN**–model M.
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  - $Ab(M) = \{A \in \Omega \mid \text{for some } w \in Reach(M), v_M(A, w) = 1\}$.
- An **INTuN**–model M of Γ is a *minimally abnormal* model of Γ iff there is no **INTuN**–model M' of Γ such that $Ab(M') \subset Ab(M)$.

Outline

Inconsistency–Adaptive Intuitionistic Logic

The Adaptive Logic **INTuN**$^{\mathbf{m}}$: Proof Theory (1)

General Features

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- An **INTuN^m**–proof is a succession of stages, each consisting of a sequence of lines.
  - Adding a line = to move on to a next stage
- Each line consists of 4 elements:
  - Line number
  - Formula
  - Justification
  - Adaptive condition = set of abnormalities

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- Deduction Rules
- Marking Criterium
    - Dynamic proofs

Deduction Rules

PREM — If $A \in \Gamma$:

$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU — If $A_1, \ldots, A_n \vdash_{\mathbf{INTuN}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC — If $A_1, \ldots, A_n \vdash_{\mathbf{INTuN}} (Dab(\Theta) \supset \bot) \supset B$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

Definition

$Dab(\Delta) = \bigvee(\Delta)$ for Δ a finite subset of Ω.

Marking Criterium: Minimal Abnormality Strategy

- Minimal Reachable *Dab*–formulas

  $Dab(\Delta)$ is a minimal reachable *Dab*–formula of Γ at stage s of the proof iff $(Dab(\Delta) \supset \bot) \supset \bot$ is derived at stage s on the condition $\emptyset$ and there is no $\Delta' \subset \Delta$ such that $(Dab(\Delta') \supset \bot) \supset \bot$ is derived at stage s on the condition $\emptyset$.

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- Minimal Choice Sets

  - A choice set of $\Sigma = \{\Delta_1, \Delta_2, ...\}$ is a set that contains an element out of each member of Σ.
  - A minimal choice set of Σ is a choice set of Σ of which no proper set is a choice set of Σ as well.

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- The Set $\Phi_s(\Gamma)$

  The set $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, ..., \Delta_n\}$, where $Dab(\Delta_1), ..., Dab(\Delta_n)$ are the minimal reachable *Dab*–formulas of Γ at stage s of the proof.

Marking Criterium: Minimal Abnormality Strategy

- Marking Definition

Line i is marked at stage s of the proof iff, where A is derived on condition Δ at line i,

(i) there is no $\Delta' \in \Phi_s(\Gamma)$ such that $\Delta' \cap \Delta = \emptyset$, or
(ii) for some $\Delta' \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\Delta' \cap \Theta = \emptyset$.

Derivability

A is derived from Γ at stage s of a proof iff A is the second element of an unmarked line at stage s.

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REMARK Derivability is stage–dependent

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Final Derivability

- A is finally derived from Γ on a line i of a proof at stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.
- $\Gamma \vdash_{\mathbf{INTuN^m}} A$ iff A is finally derived on a line of a proof from Γ.

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Example

Set of Unreliable Formulas

Example

1	$p \supset q$	–;PREM	$\emptyset$
2	$\sim q$	–;PREM	$\emptyset$
3	$\sim(r \wedge \sim r) \supset q$	–;PREM	$\emptyset$
4	$\sim r$	–;PREM	$\emptyset$
5	r	–;PREM	$\emptyset$

Set of Unreliable Formulas

$\Phi_5(\Gamma) = \emptyset$

Inconsistency–Adaptive Intuitionistic Logic

The Adaptive Logic $\mathbf{INTuN^m}$: Example

Example

1	$p \supset q$	–;PREM	$\emptyset$
2	$\sim q$	–;PREM	$\emptyset$
3	$\sim(r \wedge \sim r) \supset q$	–;PREM	$\emptyset$
4	$\sim r$	–;PREM	$\emptyset$
5	r	–;PREM	$\emptyset$
6	$\sim p$	1, 2;RC	$\{q \wedge \sim q\}$

Set of Unreliable Formulas

$\Phi_6(\Gamma) = \emptyset$

Inconsistency–Adaptive Intuitionistic Logic

The Adaptive Logic **INTuN$^{\mathbf{m}}$**: Example

Example

1	$p \supset q$	–;PREM	$\emptyset$
2	$\sim q$	–;PREM	$\emptyset$
3	$\sim(r \wedge \sim r) \supset q$	–;PREM	$\emptyset$
4	$\sim r$	–;PREM	$\emptyset$
5	r	–;PREM	$\emptyset$
6	$\sim p$	1, 2;RC	$\{q \wedge \sim q\}$
7	$(((r \wedge \sim r) \vee (q \wedge \sim q)) \supset \bot) \supset \bot$	2,3;RU	$\emptyset$

Set of Unreliable Formulas

$\Phi_7(\Gamma) = \{\ \{r \wedge \sim r\}, \{q \wedge \sim q\}\ \}$

Example

1	$p \supset q$	–;PREM	$\emptyset$	
2	$\sim q$	–;PREM	$\emptyset$	
3	$\sim(r \wedge \sim r) \supset q$	–;PREM	$\emptyset$	
4	$\sim r$	–;PREM	$\emptyset$	
5	r	–;PREM	$\emptyset$	
6	$\sim p$	1, 2;RC	$\{q \wedge \sim q\}$	✓
7	$(((r \wedge \sim r) \vee (q \wedge \sim q)) \supset \bot) \supset \bot$	2,3;RU	$\emptyset$	

Set of Unreliable Formulas

$\Phi_7(\Gamma) = \{\ \{r \wedge \sim r\}, \{q \wedge \sim q\}\ \}$

Inconsistency–Adaptive Intuitionistic Logic

The Adaptive Logic $\mathbf{INTuN^m}$: Example

Example

1	$p \supset q$	–;PREM	$\emptyset$	
2	$\sim q$	–;PREM	$\emptyset$	
3	$\sim(r \wedge \sim r) \supset q$	–;PREM	$\emptyset$	
4	$\sim r$	–;PREM	$\emptyset$	
5	r	–;PREM	$\emptyset$	
6	$\sim p$	1, 2;RC	$\{q \wedge \sim q\}$	✓
7	$(((r \wedge \sim r) \vee (q \wedge \sim q)) \supset \bot) \supset \bot$	2,3;RU	$\emptyset$	
8	$((r \wedge \sim r) \supset \bot) \supset \bot$	4,5;RU	$\emptyset$	

Set of Unreliable Formulas

$\Phi_8(\Gamma) = \{ \{r \wedge \sim r\} \}$

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Inconsistency–Adaptive Intuitionistic Logic

The Adaptive Logic $\mathbf{INTuN^m}$: Example

Example

1	$p \supset q$	–;PREM	$\emptyset$
2	$\sim q$	–;PREM	$\emptyset$
3	$\sim(r \wedge \sim r) \supset q$	–;PREM	$\emptyset$
4	$\sim r$	–;PREM	$\emptyset$
5	r	–;PREM	$\emptyset$
6	$\sim p$	1, 2;RC	$\{q \wedge \sim q\}$
7	$(((r \wedge \sim r) \vee (q \wedge \sim q)) \supset \bot) \supset \bot$	2,3;RU	$\emptyset$
8	$((r \wedge \sim r) \supset \bot) \supset \bot$	4,5;RU	$\emptyset$

Set of Unreliable Formulas

$\Phi_8(\Gamma) = \{ \{r \wedge \sim r\} \}$

Conclusion

To Conclude

The logic **INTuN$^{\mathbf{m}}$** can cope efficiently with inconsistent theories, and captures the intuitive meaning of intuitionistic negation as well.

More Results

- There is a semantic as well as a proof theoretic characterization of full predicative **IntuN** and **IntuN$^{\mathbf{m}}$**.
- Soundness and completeness for both **IntuN** and **IntuN$^{\mathbf{m}}$** have been proven.

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