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FACULTY OF ARTS AND PHILOSOPHY

From epistemic possibilities to constructive reasoning with open assumptions

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Outline

Verificationist Principle of Truth

- Intuitionistically, the semantics of truth values for proposition is crucially substituted by the explanation of proof conditions;

Definition (Verification Principle of Truth)

The notion of truth is defined as existence of a proof.

- With the explanation of the notion of judgment (act of proving vs. proof-object), the analysis of proof conditions for A turns into that of assertion conditions for 'A is true'.

Arithmetical interpretation of proofs

- The standard interpretation for intuitionistic truth is given arithmetically: $\vdash_{Int} F$ means that F is a theorem of Peano Arithmetic;
- The strongest formulation is given by the modal reading of intuitionistic provability (Gödel (1933)):

$$\vdash_{Int} F \Rightarrow \quad \vdash_{S4} P(F) \mid \forall A \subseteq F, \vdash_{S4} \Box A.$$

- provability $P(x, y)$ is interpreted as 'x is a code of a proof of a formula having a code y' for a theory containing Peano Arithmetic (PA).

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Conditions for Hypothetical Reasoning

"There is a special case, where the combination of syllogism has a different nature, that appears to resemble the usual logical figures, and which really seems to presuppose the hypothetical judgement from logic. This occurs when a construction is defined through some relation in a construction, without being directly evident how to provide it. It seems one assumes here that the sought was constructed, and a chain of hypothetical judgements derives from the assumptions." (Brouwer (1907), pp.124-125)

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Conditions for Hypothetical Reasoning (2)

Kreisel (1962): The implication $p \to q$ can be asserted, if and only if we possess a construction r, which, joined to any construction proving p (supposing the latter be effected), would automatically effect a construction proving q;

Conditions for Hypothetical Reasoning (2)

Kreisel (1962): The implication $p \rightarrow q$ can be asserted, if and only if we possess a construction r, which, joined to any construction proving p (supposing the latter be effected), would automatically effect a construction proving q;

van Dalen (1979): A proof p of $A \rightarrow B$ is a construction which assigns to each proof q of A a proof $p(q)$ [p, provided that q] of B, plus a verification that p indeed satisfies these conditions.

Two different interpretations!

Proof Conditions-interpretation: A proof p of ‘$A \rightarrow B$ true’ is given as the pair of proof-objects $< a, b >$, such that one obtains a formal object of a function type $f =< a, b >$, which is the construction for the implicational relation $f : (A \rightarrow B)$.

Two different interpretations!

Proof Conditions-interpretation: A proof p of 'A $\rightarrow$ B true' is given as the pair of proof-objects $< a, b >$, such that one obtains a formal object of a function type $f =< a, b >$, which is the construction for the implicational relation $f:(A \rightarrow B)$.

Assertion Conditions-interpretation: In order to establish 'A true $\Rightarrow$ B true,' one requires that the satisfaction of the conditions that make the proposition A true, can be transformed constructively into the satisfaction of the conditions that make the proposition B true (all functions with domain A and range B):

$$\frac{x:A \vdash b:B}{\lambda((x)b):A \rightarrow B}$$

Premises vs. Assumptions

Martin-Löf (1996) - analysis of the notion of hypothetical judgement (based on Gentzen's sequent calculus):

$$< prop : type >$$
$$(x_1/a_1 : A_1)A_1 \ true$$
$$(A_1 \ true)x_2/a_2 : A_2$$
$$\vdots$$
$$(x_1/a_1 : A_1, \ldots, x_{n-i}/a_{n-i} : A_{n-1})A_n \ true$$
$$A \ true$$

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Remarks

- Whenever appropriate proof constructions for A_1 *true*, $\ldots$, A_n *true* are given, a construction for A *true* is also provided;
- The assertion conditions interpretation is reduced to the proof conditions interpretation without circularity (essential under the arithmetical interpretation);
- Formally, the introduction rule for assumptions is justified as an elimination rule on constructions:

$$\dfrac{\dfrac{a_i : A_i}{A_i \ \mathit{true}}\ \text{Construction Elimination Rule/ Truth definition}}{x_i : A_i}\ \text{Assumption Introduction Rule}$$

Remarks (cont'd)

- Is there any constructive reading of the formula $[x_1 : A_1, \ldots, x_n : A_n]A$ *true* which does not require the substitution procedure $x_i/a_i : A_i$?
- Why should one want to do so?
  - to provide the meaning of possibility
  - to formalize natural reasoning, where assumptions may be not strictly justified by constructions

Necessity: the meaning of satisfied conditions

Sundholm (2003) - analysis of the necessity operator occurring in a judgment:

1. Necessarily A is true

   reading of standard modal logic; universally quantified function over a set;

2. A is necessarily true

   equivalent to 1 under equi-assertibility conditions;

3. 'A is true' is necessary

   the proper judgmental form.

Necessity: the meaning of satisfied conditions

- '$\Box$(A is true)' $\Rightarrow$ 'A is true' is known";
- Categorical judgment A *true*: proof-conditions for A are satisfied;
- Dependent judgment A *true*: 'A is true' is known, provided proof-conditions for $(A_1, \ldots, A_n)$ are satisfied.

Possibility: the meaning of satisfiable conditions

What about possibility?

1. Possibly *A* is true

   reading of standard modal logic; existentially quantified function over a set;

2. *A* is possibly true

   reduced to 1;

3. '*A* is true' is possible:
   - A solution is to use interdefinability of modalities:

     'A is true' is possible $\equiv$ *'A is false is not known';*

   - Problem: It makes no sense under the proof-conditions interpretation.

Possibility: the meaning of satisfiable conditions

- '$\Diamond$(A is true)' $\Rightarrow$ 'A is true' can be known;
- Categorical judgment A *true*: it simply reduces to the proof-condition interpretation;
- Dependent judgment A *true*: possibility is conditional provability;
- $\Diamond$(A *true*) $\Leftrightarrow$ there is some minimal world in which the conditions for A *true* are satisfiable;

Possibility: the meaning of satisfiable conditions (cont'd)

- The explanation of $\Diamond(\Gamma \vdash A)$ *true* should not be based on a proof-object $< g, a >$, such $g : \bigwedge \Gamma$ and $a : A$;
- Reasoning is kept at th elevel of the assertion-conditions interpretation, rather than at the level of proof-objects;
- Such an interpretation is possible by introducing judgmental modalities in a calculus for a type-theoretical language with an 'up to refutation' condition.

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1. Conceptual Background: conditions for truth

2. Epistemic Modalities

3. Modal contextual type theory
   - The propositional approach
   - The judgmental approach

4. Conclusions

Outline

Type Theory with propositional modalities (1)

- Modal versions of type theory (Pfenning, Davies 2001 and Nanevski et al. 2008) use modalities to speak about dependent truth by internalizing the modalities as *propositional operators*;
- The additional judgments of the theory are
  - "proposition 'A is necessary' is true" ($\Box A$ *true*)
  - "proposition 'A is possible' is true" ($\Diamond A$ *true*)

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Type Theory with propositional modalities (2)

- ($\Box A$ *true*) means that A stays true under further assumptions being formulated;
- A *valid* is inferred from A *true* and can be used hypothetically;

$$\frac{\vdash A\ \mathit{true}}{A\ \mathit{valid}} \qquad \frac{A\ \mathit{valid}}{\Gamma \vdash A\ \mathit{true}}$$

$$\frac{\Delta; - \vdash A\ \mathit{true}}{\Delta; \Gamma \vdash \Box A\ \mathit{true}}\ I\Box \qquad \frac{\Delta; \Gamma \vdash \Box A\ \mathit{true} \qquad \Delta, A\ \mathit{valid}, \Gamma \vdash C\ \mathit{true}}{\Delta; \Gamma \vdash C\ \mathit{true}}\ E\Box$$

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Type Theory with propositional modalities (4)

- ($\Diamond A$ *true*) means that there is no further assumption that can be done in the context that makes A true;
- in such a world we can still assume that A is true, but any further inference induces only possible contents;

$$\frac{\Gamma \vdash A\ true}{\Gamma \vdash A\ poss} \qquad \frac{\Gamma \vdash A\ poss \quad A\ true \vdash C\ poss}{\Gamma \vdash C\ poss}$$

$$\frac{\Delta; \Gamma \vdash A\ poss}{\Delta; \Gamma \vdash \Diamond A\ true}\ I\Diamond \qquad \frac{\Delta; \Gamma \vdash \Diamond A\ true \quad \Delta, A\ true \vdash C\ poss}{\Delta; \Gamma \vdash C\ poss}\ E\Diamond$$

Type Theory with propositional modalities (4)

- One separates predications of truth from predications of validity and possibiliyt;
- Modalities make explicit the representation of the syntactical machinery already given by CTT;
- Morever, it needs additional judgments such as *A valid* and *A poss* (in the semantics) and *A verif*, *A hyp* (in the corresponding sequent calculus);
- The formulation in Nanevski et al. (2008) is more detailed by a more analytic presentation of contextual validity;

Outline

Type Theory with judgmental modalities (1)

- A judgmental theory of modalities will have modal expressions whose operators are extendend to the judgmental scope;
- The additional judgments of the theory will be respectively of the form:
  - "it is necessary that proposition A is true" – $\Box(A\ true)$;
  - "it is possible that proposition A is true" – $\Diamond(A\ true)$.
- Aim is to give separated treatment of constructions and assumptions: a categorical and an assumption-based fragment of the language are defined;

Categorical Fragment

- Standard type introduction rules and definition of truth are used for categorically justified propositions (identity rules that define Reflexivity, Simmetry and Transitivity on types are omitted for brevity):

$$\frac{< A : type >}{a : A \; type} \quad \text{Type formation}$$

$$\frac{a : A}{A \; true} \quad \text{Truth Definition}$$

Categorical Fragment (2)

$$\frac{a : A \qquad b : B}{(a, b) : A \wedge B \ true} \ I\wedge$$

$$\frac{a : A}{l(a) : A \vee B \ true} \ \text{Left } I\vee \qquad \frac{b : B}{r(b) : A \vee B \ true} \ \text{Right } I\vee$$

$$\frac{a : A \qquad A \ true \vdash b : B}{a(b) : A \rightarrow B \ true} \ I \rightarrow \text{(Implication)}$$

$$\frac{a_1 : A_i, \ldots, a_n : A_i \qquad [A_i \ true] \vdash b : B \qquad \lambda((a_i(b))A, B)}{(\forall a_i : A_i)B \ type} \ I\forall$$

$$\frac{a_1 : A_i, \ldots, a_n : A_i \qquad [a_i : A_i] \vdash b : B \qquad (< a_i, b >, A, B)}{(\exists a_i : A_i)B \ type} \ I\exists$$

$$\frac{a : A}{\neg A \rightarrow \bot} \ I\bot$$

Interpreting Assumptions

- A new type format, called *type*$_{inf}$ for *information type* is introduced;
- For the construction of a judgment $A\ type_{inf}$ one runs a test over the finite set of given derivations to check that no construction for $A \rightarrow \bot$ is given;

$$\frac{\neg(A \rightarrow \bot)}{A\ type_{inf}} \quad \text{Informational Type formation}$$

$$\frac{A\ type_{inf} \qquad x : A}{A\ true^{*}} \quad \text{Hypothetical Truth Definition}$$

Interpreting Assumptions (2)

- On this interpretation one defines functional expressions of $type_{inf}$, saying that B is true up to a refutation of A *true*:

$$\frac{A\ type_{inf} \quad x:A \vdash b:B}{x:A \vdash B\ true^*}$$

- the weak truth predicate induces the standard dependent functional construction by abstraction

$$\frac{A\ type_{inf} \qquad x:A \vdash B\ true^*}{((x)b) : A \supset B\ true} \quad \text{Functional Abstraction}$$

- β-conversion provides the appropriate translation to standard dependent type formation by application:

$$\frac{A\ type_{inf} \qquad x:A \vdash B\ true^* \qquad a:A}{(x(b))(a) = b[a/x]:B\ type[a/x]} \quad \beta\text{-conversion}$$

Introduction Rules for Modal Judgments

- Necessity is validity against any possible state that contains refutable data for the construction of A:

$$A\ true \Leftrightarrow \emptyset \vdash A\ true \Leftrightarrow \Box(A\ true).$$

- Possibility is validity in *some* context in which the conditions for A are not refuted:

$$A\ true^{*} \Leftrightarrow \Gamma \vdash A\ true \Leftrightarrow \Diamond(A\ true)$$

Language (1)

Propositions $:= A; A \wedge B; A \vee B; A \rightarrow B; A \supset B; \neg A \rightarrow \bot;$
Proof terms $:= a : A; (a, b); a(b); \lambda(a(b)); <a, b>;$
Proof variables $:= x : A; (x(b)); (x(b))(a);$
Contexts $:= \Gamma, x : A; \Gamma, a : A; \Box\Gamma; \Diamond\Gamma;$
Judgments $:= A\ true; A\ true^{*}; \Gamma \vdash A\ true; \Diamond(A\ true); \Box(A\ true).$

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Language (2)

$$\frac{}{\Gamma, a : A, \Delta \vdash A\ true}$$ Premise Rule

$$\frac{}{\Gamma, x : A, \Delta \vdash A\ true^*}$$ Hypothesis Rule

$$\frac{a : A}{\Box(A\ true)}$$ $\Box$ – *Formation*

$$\frac{x : A}{\Diamond(A\ true)}$$ $\Diamond$ – *Formation*

Generalized Contextual Format

Definition (Necessitation Context)

For any context Γ, the global context $\Box\Gamma$ is given by $\bigcup\{\Box A_1, \ldots, \Box A_n\}$.

Definition (Normal Context)

For any context Γ, the local context $\Diamond\Gamma$ is given by $\bigcup\{\circ A_1, \ldots, \circ A_n \mid \circ = \{\Box, \Diamond\}\}$ and for at least one A_i it holds $\circ = \Diamond$.

Introduction/Elimination for $\Box$ and $\Diamond$

$$\frac{\Gamma \vdash A\ true}{\Box\Gamma \vdash \Box(A\ true)}\ I\Box \qquad \frac{\Box\Gamma \vdash \Box(A\ true) \qquad \Delta, a:A \vdash b:B}{\Gamma, \Delta \vdash B\ true}\ E\Box$$

$$\frac{\Gamma, x:A \vdash B\ true^{*}}{\Box\Gamma, \Diamond(A\ true) \vdash \Diamond(B\ true)}\ I\Diamond$$

$$\frac{\Gamma, \Delta \vdash A\ true^{*} \qquad \Box\Gamma, \Diamond(A\ true) \vdash \Diamond(B\ true)}{\Gamma, \Delta \vdash B\ true^{*}}\ E\Diamond$$

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Soundness (by local reduction on $\Box(A\ true)$)

$$\dfrac{\dfrac{\begin{array}{c}D_1\\ \Gamma \vdash A\ true\end{array}}{\Box\Gamma \vdash \Box(A\ true)}\,\Box I \qquad \Delta, a{:}A \vdash b{:}B}{\Gamma, \Delta \vdash B\ true}\,\Box E \qquad \Rightarrow_{Redex}$$

$$\begin{array}{c}D_2\\ \Gamma, \Delta \vdash B\ true\end{array}$$

where derivation D_2 is justified in terms of the Premise Rule.

Completeness (by local expansion on $\Box(A\ true)$)

$$\begin{array}{c} \mathcal{D}_1 \\ \Box\Gamma \vdash \Box(A\ true) \end{array} \Rightarrow_{Exp}$$

$$\dfrac{\begin{array}{c} \mathcal{D}_2 \\ \Box\Gamma, a : A \vdash \Box(A\ true) \end{array} \qquad \Box\Gamma, a : A \vdash \Box(A\ true)}{\Gamma \vdash A\ true}\ \Box I$$

with a side condition on multiple simultaneous substitutions on Γ.

Soundness (by local reduction on $\Diamond(A\ true)$)

$$\dfrac{\dfrac{\begin{array}{c}D_1\\ \Gamma, x:A \vdash B\ true^*\end{array}}{\Box\Gamma, \Diamond(A\ true) \vdash \Diamond(B\ true)}\Diamond I \qquad \Gamma, \Delta \vdash A\ true^*}{\Gamma, \Delta \vdash B\ true^*}\Diamond E \qquad \Rightarrow_{Redex}$$

$$\begin{array}{c}D_2\\ \Gamma, \Delta \vdash B\ true^*\end{array}$$

where derivation D_2 is justified in terms of the Hypothesis Rule.

Completeness (by local expansion on $\Diamond(A\ true)$)

$$\cfrac{\begin{array}{c} D_1 \\ \Diamond\Gamma \vdash \Diamond(A\ true) \Rightarrow_{Exp} \end{array}}{}$$

$$\cfrac{\begin{array}{c} D_2 \\ \Gamma, x : A \vdash A\ true^* \end{array} \qquad \Box\Gamma, \Diamond(A\ true) \vdash \Diamond(A\ true)}{\Gamma \vdash A\ true^*}\ \Diamond I$$

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Substitution on Terms and Truth

Theorem (Substitution on terms)

If $\Gamma, x : A, \Delta \vdash B$ *true** *and* $\Gamma, \Delta \vdash a : A$, *then* $\Gamma, \Delta \vdash [x/a]B$ *true.*

where $[x/A]B$ is the substitution of occurrences of x in B by a. This is easily proven by induction and the Premise Rule.

Substitution on Terms and Truth (2)

The formulation of substitution on the different truth predicates and modal judgments defines exchange, weakening and contraction:

Theorem (Substitution on truth predicates)

The inference systems satisfies:

1. *If* $\Gamma, x : A, \Delta \vdash B\ true^*$ *and* $\Gamma, \Delta \vdash A\ true^*$*, then* $\Gamma \vdash B\ true^*$*;*
2. *If* $\Gamma, x : A, \Delta \vdash \Diamond(B\ true)$ *and* $\Gamma, \Delta \vdash A\ true^*$*, then* $\Gamma, \Delta \vdash \Diamond(B\ true)$*;*
3. *If* $\Gamma, x : A, \Delta \vdash \Diamond(B\ true)$ *and* $\Gamma, \Delta \vdash \Diamond(A\ true)$*, then* $\Gamma, \Delta \vdash \Diamond(B\ true)$*;*
4. *If* $\Gamma, a : A, \Delta \vdash B\ true^*$ *and* $\Delta \vdash A\ true^*$*, then* $\Gamma, \Delta \vdash B\ true^*$*;*
5. *If* $\Gamma, a : A, \Delta \vdash \Diamond(B\ true)$ *and* $\Delta \vdash A\ true$*, then* $\Gamma, \Delta \vdash \Diamond(B\ true)$*;*
6. *If* $\Box\Gamma \vdash \Box(A\ true)$ *and* $\Box\Gamma, x : A \vdash \Diamond(B\ true)$*, then* $\Box\Gamma \vdash \Box(A\ true, B\ true)$*.*

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Remarks and Open Issues

- The design of a modal type theory for refutable contents is crucial for using constructive system in knowledge representation;
- Its basic aim is the design of systems for multi-staged information processes (cf modal type theories for staged computation);
- a multi-modal format and a signature system are the next required elements for implementing security and reliability relations;
- There is a composed set of (non-standard) Kripke models $\mathcal{M}^{(\mathcal{L}^{ver} \cup \mathcal{L}^{inf})}$ with respect to which a contextual *KT* with $\Box$ and $\Diamond$ can be proven equivalent (the latter would be the modal system of the syntactic language here introduced).