Constructive contextual modal judgments for reasoning from open assumptions

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Outline

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Reasoning by Open Assumptions

- Task: a constructive reading of the formula

$$A\ true[x_1 : A_1, \ldots, x_n : A_n]$$

  based on open assumptions
  (= not abstracted from given constructions);

- Objectives:
  - Natural reasoning, assumptions without strict justification;
  - Computational processes with partial information (e.g. partial evaluation: a function considers part of its input code as given).

- Logical Take: express epistemic states in terms of modalities.

References

- Approaches for the connection between modalities and provability:
  - Sequential approach – [Sambin, Valentini (1982)]
  - Curry-Howard correspondance – [Bellin et al. (2001)]
  - Intuitionistic Semantics for Modal Logic – [Williamson (1992)], [Simpson (1994)], [Bierman, de Paiva (2000)], [Alechina et al. (2001)]
  - Provability and Logic of Proofs – [Artemov (2001)], [Kramer (2008)].
  - Type theories and Constructive Modalities: [Pfenning, Davies (2001)], [Davies, Pfenning (2001)], [Nanevski et al. (2008)].

Type Theory with judgmental modalities

- We present a type theory including modal operators with judgmental scope;

- Additional judgments of the theory:
  - "it is necessary that proposition A is true" – $\Box(A\ true)$;
  - "it is possible that proposition A is true" – $\Diamond(A\ true)$.

- Using modalities to express epistemic states (in particular to model the contextual basis of propositional contents);

- Separated treatment of constructions and assumptions:
  - categorical fragment;
  - assumption-based fragment.

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Modalities and Conditions

- '*A* is true' is necessary:
  - '*A* is true' is known;
  - Proof-conditions for *A* are actually satisfied;
  - The context of assumptions for *A true* has been emptied;
  - *A true* holds under any context extension $\emptyset, \Delta$.

- '*A* is true' is possible:
  - '*A* is true' can be known;
  - Proof-conditions for *A* are satisfiable;
  - *A* holds up to refutation of its conditions;
  - cf. Kolmogorov's 'pseudo-truths' and Pfenning's 'proof irrelevance';
  - There is a non-empty context of assumptions for *A true*;
  - *A true* holds under some context extension Γ, Δ.

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Structure of the language

- polymorphic language:
  - $\mathcal{K} : \{type, type_{inf}\}$;
  - constructive truth (*true*);
  - weaker truth up to refutation ($true^*$);

- *type*-constructors composed by listing, application, abstraction and pairing for $\wedge, \vee, \rightarrow, \forall, \exists$;

- $\rightarrow$ is material implication: a λ-term presented *together with* one of its α-terms;

- $type_{inf}$: admissible A from $\neg(A \rightarrow \bot)$ to $x : A$;

- $\supset$ is functional implication: abstraction on the admissible construction for the antecedent.

Categorical Fragment

Definition (Type and Truth Formation)

Standard type introduction rule and constructive definition of truth with Reflexivity, Symmetry and Transitivity on types (omitted for brevity):

$$\frac{a : A}{A : type} \quad \text{Type formation}$$

$$\frac{a : A}{A \; true} \quad \text{Truth Definition}$$

Categorical Fragment (2)

Definition (Typing Rules)

$$\frac{a : A \qquad b : B}{(a, b) : A \wedge B \; true} \; I\wedge$$

$$\frac{a : A}{l(a) : A \vee B \; true} \; \text{Left } I\vee \qquad \frac{b : B}{r(b) : A \vee B \; true} \; \text{Right } I\vee$$

$$\frac{a : A \qquad A \; true \vdash b : B}{a(b) : A \rightarrow B \; true} \; I \rightarrow \text{(Implication)}$$

$$\frac{a_1 : A_1, \ldots, a_n : A_n \qquad [A_i \; true] \vdash b : B \qquad \lambda((a_i(b))A, B)}{(\forall a_i : A_i)B \; type} \; I\forall$$

$$\frac{a_1 : A_1, \ldots, a_n : A_n \qquad [a_i : A_i] \vdash b : B \qquad (< a_i, b >, A, B)}{(\exists a_i : A_i)B \; type} \; I\exists$$

$$\frac{a : A}{\neg A \rightarrow \bot} \; I\bot$$

Interpreting Assumptions

Definition (Informational Type and Weak Truth Formation)

An *information type* $type_{inf}$ is constructed by running a test over the finite set of given derivations to check that no construction for $A \to \bot$ is given:

$$\frac{\neg(A \to \bot)}{A\ type_{inf}} \quad \text{Informational Type formation}$$

$$\frac{A\ type_{inf} \qquad x : A}{A\ true^{*}} \quad \text{Hypothetical Truth Definition}$$

Interpreting Assumptions (2)

Definition (Typing Rules)

- B is true up to a refutation of A *true*:

$$\frac{A\ type_{inf} \qquad x : A \vdash b : B}{x : A \vdash B\ true^{*}} \quad \text{Function Formation}$$

- standard dependent functional construction (abstraction):

$$\frac{A\ type_{inf} \qquad x : A \vdash B\ true^{*}}{((x)b) : A \supset B\ true} \quad \text{Functional Abstraction}$$

- translation to standard dependent type formation (application):

$$\frac{A\ type_{inf} \qquad x : A \vdash B\ true^{*} \qquad a : A}{(x(b))(a) = b[a/x] : B\ type[a/x]} \quad \beta\text{-conversion}$$

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Modal Extension

Definition (Start Rules)

$$\frac{}{\Gamma, a : A, \Delta \vdash A \; true} \quad \text{Premise Rule}$$

$$\frac{}{\Gamma, x : A, \Delta \vdash A \; true^{*}} \quad \text{Hypothesis Rule}$$

$$\frac{a : A}{\Box(A \; true)} \quad \Box - \textit{Formation}$$

$$\frac{x : A}{\Diamond(A \; true)} \quad \Diamond - \textit{Formation}$$

Generalized Contextual Format

Definition (Necessitation Context)

For any context Γ, the global context $\Box\Gamma$ is given by
$\bigcup\{\Box(A_1\ true), \ldots, \Box(A_n\ true)\}$.

Definition (Normal Context)

For any context Γ, the local context $\Diamond\Gamma$ is given by
$\bigcup\{\circ(A_1\ true), \ldots, \circ(A_n\ true) \mid \circ = \{\Box, \Diamond\}\}$ and for at least one A_i it holds $\circ = \Diamond$.

Modal Rules

Definition (Introduction and Elimination for $\Box$)

$$\frac{\Gamma \vdash A\ true}{\Box\Gamma \vdash \Box(A\ true)}\ I\Box$$

$$\frac{\Box\Gamma \vdash \Box(A\ true) \qquad \Delta, a{:}A \vdash b{:}B}{\Gamma, \Delta \vdash B\ true}\ E\Box$$

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Modal Rules

Definition (Introduction and Elimination for $\Box$)

$$\frac{\Gamma \vdash A\ \mathit{true}}{\Box\Gamma \vdash \Box(A\ \mathit{true})}\ I\Box$$

$$\frac{\Box\Gamma \vdash \Box(A\ \mathit{true}) \qquad \Delta, a{:}A \vdash b{:}B}{\Gamma, \Delta \vdash B\ \mathit{true}}\ E\Box$$

Definition (Introduction and Elimination for $\Diamond$)

$$\frac{\Gamma, x{:}A \vdash B\ \mathit{true}^{*}}{\Box\Gamma, \Diamond(A\ \mathit{true}) \vdash \Diamond(B\ \mathit{true})}\ I\Diamond$$

$$\frac{\Gamma, \Delta \vdash A\ \mathit{true}^{*} \qquad \Box\Gamma, \Diamond(A\ \mathit{true}) \vdash \Diamond(B\ \mathit{true})}{\Gamma, \Delta \vdash B\ \mathit{true}^{*}}\ E\Diamond$$

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Substitution on Terms and Truth

Theorem (Substitution on truth predicates)

1. *If* $\Gamma, x : A, \Delta \vdash B$ *true** *and* $\Gamma, \Delta \vdash a : A$, *then* $\Gamma, \Delta \vdash [x/a]B$ *true.*
2. *If* $\Box\Gamma, \Diamond(A\ true), \Box\Delta \vdash \Diamond(B\ true)$ *and* $\Box\Gamma, \Box\Delta \vdash \Box(A\ true)$, *then* $\Box\Gamma, \Box\Delta \vdash \Box(B\ true)$.

where $[x/A]B$ is the substitution of occurrences of x in B by a (proven by induction and the Premise Rule) and the modal part is induced from the Modal Introduction Rules.

Structural Rules (1)

Theorem (Weakening)

1. *If* $\Gamma \vdash B$ *true, then* $\Gamma, a : A \vdash B$ *true.*
2. *If* $\Gamma \vdash B$ *true***, then* $\Gamma, x : A \vdash B$ *true***.*
3. *If* $\Box\Gamma \vdash \Box(B$ *true*$)$*, then* $\Box\Gamma, \Box(A$ *true*$) \vdash \Box(B$ *true*$)$*.*
4. *If* $\Diamond\Gamma \vdash \Diamond(B$ *true*$)$*, then* $\Diamond\Gamma, \Diamond(A$ *true*$) \vdash \Diamond(B$ *true*$)$*.*

Theorem (Contraction)

1. *If* $\Gamma, a_1 : A, a_2 : A \vdash B$ *true, then* $\Gamma, a : A \vdash [a_1 \approx a_2 / a]B$ *true.*
2. *If* $\Gamma, x_1 : A, x_2 : A \vdash B$ *true***, then* $\Gamma, x : A \vdash [x_1 \approx x_2 / x]B$ *true***.*
3. *If* $\Box\Gamma, a_1 : A, a_2 : A \vdash \Box(B$ *true*$)$*, then* $\Box\Gamma, \Box(A$ *true*$) \vdash \Box(B$ *true*$)$*.*
4. *If* $\Box\Gamma, x_1 : A, x_2 : A \vdash \Diamond(B$ *true*$)$*, then* $\Box\Gamma, \Diamond(A$ *true*$) \vdash \Diamond(B$ *true*$)$*.*

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Structural Rules (2)

Theorem (Exchange)

1. *If* $\Gamma, a_1 : A, a_2 : A \vdash B$ *true, then* $\Gamma, a_2 : A, a_1 : A \vdash B$ *true.*
2. *If* $\Gamma, x_1 : A, x_2 : A \vdash B$ *true***, then* $\Gamma, x_2 : A, x_1 : A \vdash B$ *true***.*
3. *If* $\Box\Gamma, a_1 : A, a_2 : A \vdash \Box(B\ true)$*, then* $\Box\Gamma, a_2 : A, a_1 : A \vdash \Box(B\ true)$*.*
4. *If* $\Box\Gamma, x_1 : A, x_2 : A \vdash \Diamond(B\ true)$*, then* $\Box\Gamma, x_2 : A, x_1 : A \vdash \Diamond(B\ true)$*.*

Local Soundness and Completeness

- Soundness by local reduction and expansion on $\Box(A\ true)$ in terms of terms substitution;
- Completeness by local expansion on $\Box(A\ true)$ with a side condition on multiple simultaneous substitutions on contexts;
- Soundness by local reduction on $\Diamond(A\ true)$ in terms of the use of the Hypothesis Rule;
- Completeness by local expansion on $\Diamond(A\ true)$.

Models

- Weakening of the truth-values model
  - the poset $\{1,0\}$ that satisfies inhabitness and intensional identity;

- types as pairs $A = [a, \rightarrow]$, with a the verification term and $\rightarrow$ the evaluation function:
  - $A = [a, \rightarrow] = \{1\}$ if $x \rightarrow a = 1$ and $A : type = 1$
  - $A = [a, \rightarrow] = \emptyset$ if $x \rightarrow a = undefined$ and $A : type_{inf} = 1$
  - $A = [a, \rightarrow] = \{0\}$ if $x \rightarrow a = 0$ and $A : type = 0$

- $type_{inf}$ admits undefinability:
  - preserving only symmetricity;
  - inhabitness is not guaranteed ('super-modest types');

- Semantics of $cKT_{\Box,\Diamond}$ obtained by a composed set of (non-standard) Kripke models $\mathcal{M}^{(\mathcal{L}^{ver} \cup \mathcal{L}^{inf})}$.

Remarks and Open Issues

- Modal type theory for refutable contents
  - allows constructive systems in knowledge representation;
  - applications in non-monotonic knowledge processes by data retraction;
  - automatic reasoning for systems including misinformation;
- Multi-staged information processes:
  - obtained by adding a multi-modal format and a signature system;
  - implement security and reliability relations;
    - Trusted Communications ([Primiero, Taddeo (2010)])
    - Data Accessibility in Networks for Distributed Computing ([Primiero (2010)]).

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