A verificationist modal language for contextual computations

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A Preamble

- $S4$ relates to IL via necessity as provability;
- There are both Kripke and categorical semantics from K to Constructive $S4$, see e.g. [Bierman and de Paiva, 1996], [Bellin et al., 2001], [Alechina et al., 2001];
- Less work has been done on the *contextual* notion of derivability, see e.g. CK in [Mendler and de Paiva, 2005], type-theoretical style in [Pfenning and Davies, 2001] or the AI-related notion e.g. [McCarthy, 1993];
- *What interpretation of possibility?*

The Interpretation

We interpret modalities w.r.t. global and local computations:

1. *"A is true" is necessary if and only if a computation for A is globally valid, i.e. valid under no specific context*:

$$\Box(A\ true) \Leftrightarrow ((\emptyset)A)$$

2. *"A is true" is possible if and only if a computation for A is locally valid, i.e. valid under some specific context*:

$$\Diamond(A\ true) \Leftrightarrow ((\Gamma)A)$$

$\mathcal{L}^{ver}$

A standard intuitionistic semantics for non-modal formulas:

$$\mathcal{L}^{ver} := \phi \mid \top \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \supset B.$$

$C1^{ver}$ $K_i \nvDash \bot$ and $K_i \vDash \top$;
$C2^{ver}$ for all ϕ, $K_i \vDash \phi$ iff $(\phi, v(K_i))$;
$C3^{ver}$ $K_i \vDash A \vee B$ iff $K_i \vDash A$ or $K_i \vDash B$;
$C4^{ver}$ $K_i \vDash A \wedge B$ iff $K_i \vDash A$ and $K_i \vDash B$;
$C5^{ver}$ $K_i \vDash A \supset B$ iff $K_i \vDash A$ implies $K_i \vDash B$;
$C6^{ver}$ $K_i \vDash \neg A$ iff $\forall K_j \geq K_i$, it holds $K_j \vDash A \supset \bot$.

$\mathcal{L}^{ctx}$

- Formulas valid under *uncertain or unverified information* by a modal language that contains both empty and non-empty contexts;
- a context is given as a set of non-verified formulas, admissible in view of missing refutation:

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Definition

For any $K_i \in \mathbb{K}$, an informational context $\Gamma : \mathcal{V} \mapsto \mathcal{W}$ for K_i consists of a finite set of injective functions $\gamma_1, \gamma_2, \ldots, \gamma_n$ such that $\gamma_i := x_i \mapsto A_i$. We then say that the truth of A_i is admissible in K_i if $K_h \not\models \neg A_i$ for all $K_h \leq K_i$.

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$$\mathcal{L}^{ctx} := \phi \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A$$

The notion of a model for $\mathcal{L}^{ctx}$ is formulated by modifying the definition of a model for $\mathcal{L}^{ver}$ with the newly defined order among knowledge states:

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Definition

A model for $\mathcal{L}^{ctx}$ is a tuple $M^{ctx} = \{\mathbb{K}, \leq^{\gamma}, v\}$, where $\mathbb{K}$ is a nonempty set ranging over $\{(\Gamma)K_i, (\Gamma')K_j, \dots\}$; $\leq^{\gamma}$ is a reflexive ordering relation over members of $\mathbb{K}$ such that if $(\Gamma)K_i$ and $\Gamma \subseteq_{\gamma'} \Gamma'$, then $(\Gamma)K_i \leq^{\gamma'} (\Gamma')K_j$; v is a verification function $v : \mathbb{K} \mapsto 2^{\mathcal{W}}$.

$\mathcal{L}^{ctx}$ valuation

A is a consequence of knowledge state K_i in the context of unverified information Γ:

$C1^{ctx}$ $K_i \vDash^{\Gamma} \phi$ iff $(\phi, v((\Gamma)K_i))$;
$C2^{ctx}$ $K_i \vDash^{\Gamma} \bot$ iff $(A, v((\Gamma)K_i))$ and $K_h \vDash^{\emptyset} \neg A$, for some $K_h \leq^{\gamma} K_i$;
$C3^{ctx}$ $K_i \vDash^{\Gamma} A \vee B$ iff $K_i \vDash^{\Gamma} A$ or $K_i \vDash^{\Gamma} B$;
$C4^{ctx}$ $K_i \vDash^{\Gamma} A \wedge B$ iff $K_i \vDash^{\Gamma} A$ and $K_i \vDash^{\Gamma} B$;
$C5^{ctx}$ $K_i \vDash^{\Gamma} A \supset B$ iff $K_i \vDash^{\Gamma} A$ implies $K_i \vDash^{\Gamma} B$;
$C6^{ctx}$ $K_i \vDash^{\Gamma} \Box A$ iff for all $(\Gamma')K_j \geq^{\gamma} (\Gamma)K_i$, it holds $K_j \vDash^{\Gamma \cup \gamma} A$;
$C7^{ctx}$ $K_i \vDash^{\Gamma} \Diamond A$ iff there is a $(\Gamma')K_j \geq^{\gamma} (\Gamma)K_i$ such that $K_j \vDash^{\Gamma \cup \gamma} A$.

$\mathcal{L}^{ctx}$: modal context

We extend modalities to context:

- a global context is such that all of its information remains valid in all extensions of the given context:

Definition

$\Box\Gamma$ is called a *global context* for K_i iff for all $\gamma_i := x_i \mapsto A_i$ in Γ and all $\Gamma' \supseteq \Gamma$ it holds $\vDash^{\Gamma'} A_i$.

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- a local context is such that at some of its information remains valid in one extension of the given context:

Definition

$\Diamond\Gamma$ is called a *local context* for K_i iff for some $\gamma_i := x_i \mapsto A_i$ in Γ, there is a $\Gamma' \supseteq \Gamma$ such that $\models^{\Gamma'} A_i$.

$\mathcal{L}^{glob}$

Let us consider our language $\mathcal{L}^{ctx}$ *restricted* to the set of formulas $\mathcal{L}^{glob} : \{A \mid \models^{\Box\Gamma} A\}$; $\models_{\mathcal{L}^{glob}}$ will be therefore the consequence relation construed by the satisfaction clauses of $\mathcal{L}^{ctx}$ with only global contexts;

- $C1^{glob}$ $K_i \models^{\Box\Gamma} \phi$ iff for every γ, it holds $K_i \models^{\Gamma,\gamma} \phi$;
- $C2^{glob}$ $K_i \models^{\Box\Gamma} \top$;
- $C3^{glob}$ $K_i \models^{\Box\Gamma} A \vee B$ iff $K_i \models^{\Box\Gamma} A$ or $K_i \models^{\Box\Gamma} B$;
- $C4^{glob}$ $K_i \models^{\Box\Gamma} A \wedge B$ iff $K_i \models^{\Box\Gamma} A$ and $K_i \models^{\Box\Gamma} B$;
- $C5^{glob}$ $K_i \models^{\Box\Gamma} A \supset B$ iff $K_i \models^{\Box\Gamma} A$ implies $K_i \models^{\Box\Gamma} B$.

$\mathcal{L}^{glob}$ and local consequence

Theorem

For every $A \in \mathcal{W}$, $\vDash_{\mathcal{L}^{glob}} A$ iff $\vDash_{S4} A$

$\mathcal{L}^{glob}$ and local consequence

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Definition

For every $A \in \mathcal{W}$, $K_i \vDash^{\Diamond\Gamma} A$ iff for some γ it holds $K_i \vDash^{\Gamma\cup\gamma} A$. We denote by $\vDash^{\Diamond\Gamma} A$ a semantic consequence of every K_i with local context $\Diamond\Gamma$.

Calculus $cKT_{\Box\Diamond}$

AXIOMS	
Axioms of IPL	
$K_{\Box}$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
$K_{\Diamond}$	$\Diamond(A \supset B) \supset (\Diamond A \supset \Diamond B)$
$T_{\Box}$	$\Box A \supset A$
$T_{\Diamond}$	$A \supset \Diamond A$
RULES	
Modus Ponens	
Uniform Substitution	
Nec^{glob}	$\vdash^{\Box\Gamma} A \Rightarrow \Box\Gamma \vdash \Box A$
$Weak^{glob}$	$\Box\Gamma \vdash A \Rightarrow \Box\Gamma, \Gamma' \vdash A$

Characterization and Decidability

Theorem

For every set of formulae Γ *and formula* A*, it holds* $\Gamma \vdash_{cKT_{\Box,\Diamond}} A$ *iff either* $\models^{\emptyset} \bigwedge \Gamma \supset A$*, or* $\models^{\Box\Gamma} A$*, or* $\models^{\Diamond\Gamma} A$*.*

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Theorem (Decidability)

$\mathcal{L}^{glob}$ *is the language of the decidable fragment of the theory* $cKT_{\Box,\Diamond}$ *whose class of models is reflexive and transitive.*

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