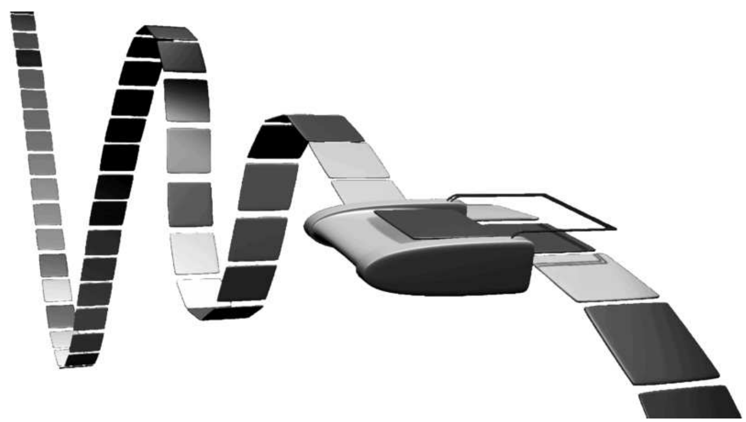
"you can't hide behind a definition" Church's and Post's practices of symbolic logic



Liesbeth De Mol
Centre for Logic and Philosophy of Science, Belgium
elizabeth.demol@ugent.be

Introduction

Specific Question: How did Church, Post (and Turing) arrive at their respective theses, i.e., the formalization of certain intuitive notions (1936)? What was the meaning of what is now known as the Church-Turing thesis *originally*?

General Question: What is the role of "practices" of symbolic logic for the "discovery" of a result like the Church-Turing thesis?

Background Motivation: "*The lesson seems to be this: we cannot fully understand our own conceptual scheme without plumbing its historical roots, but in order to appreciate those roots, we may well have to filter them back through our own ideas*" – Judson C. Webb, 1980

– Foundations of (theoretical) computer science

– The computer then and now: the physical realization of a universal Turing machine?

– The physical Church-Turing thesis and hypercomputability

– The limits of (human) computing

Introduction (Continued)

- The Church-Turing thesis: contrasting the now and then
- Post's practice
- Church's practice
- "you can't hide behind a definition": the Church-Turing thesis as a natural law (Post) or a definition (Church)?
- Discussion

1. The Church-Turing thesis now and then

The "Church-Turing" thesis then....

- **Development of Mathematical Logic** "professional philosophers have taken very little interest in it, presumably because they found it too mathematical. On the other hand, most mathematicians, have taken very little interest in it, because they found it too philosophical" (Skolem, 1928)
- **Formalizing the whole of mathematics** *Principia Mathematica*
- **Decision problems** "Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: to devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers" (Hilbert, 1901)
- **Significance** "[T]he contemporary practice of mathematics, using as it does heuristic methods, only makes sense because of this undecidability. When the undecidability fails then mathematics, as we now understand it, will cease to exist; in its place there will be a mechanical prescription for deciding whether a given sentence is provable or not" (Von Neumann, 1927)

The Church-Turing thesis then...

Church's thesis "We now define the notion [...] of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ-definable function of positive integers.)"

Turing's thesis "The expression 'there is a general process for determining...' has been used throughout this section as equivalent to 'there is a machine which will determine...'. This usage can be justified if and only if we can justify our definition of 'computable' [...] According to my definition, a number is computable if its decimal expansion can be written down by a machine"

$\Rightarrow$ If true, then there are problems that cannot be decided in finite time (e.g. the halting problem)

...and now

⇒ What can be considered as a computation in the physical world?

CCT Every effectively calculable function (or anything that is "equivalent" to it) is Turing computable (or any other notion of computability equivalent to this)

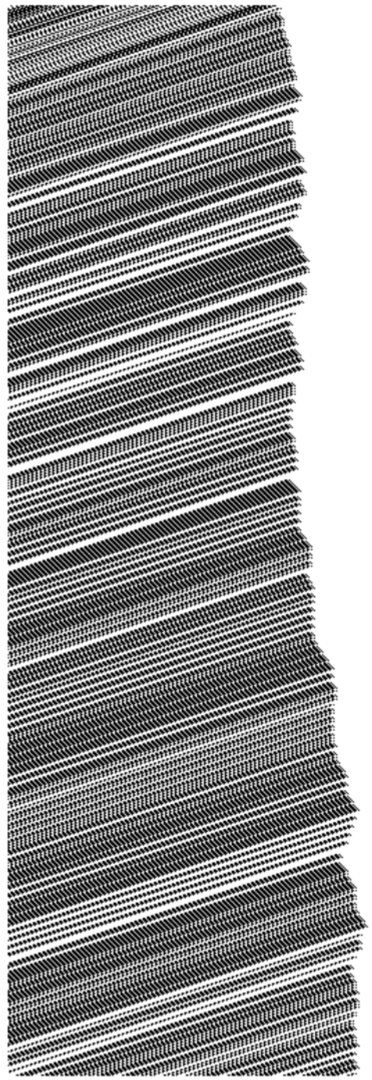
CCT Every effectively calculable function is a computable function (wikipedia)

(P)CCT The Church-Turing thesis (formerly commonly known simply as Church's thesis) says that any real-world computation (!) can be translated into an equivalent computation involving a Turing machine (Mathworld)

Physical CTT A Turing machine can do what any physically realizable system can do

Strong CCT "A probabilistic Turing machine can efficiently simulate any reasonable model of computation" (Kaye et al, 2007)

2. Post's practices



Post's practices: Two (hypo)theses

1921 (!): Post's thesis I (P1) Every generated set of sequences on a given set of letters $a_1, a_2, ..., a_\mu$ is a subset of the set of assertions of a system in normal form with primitive letters $a_1, a_2, ..., a_\mu, a'_1, a'_2, ..., a'_\nu$, i.e., the subset consisting of those assertions of the normal system involving the letters $a_1, a_2, ..., a_\mu$

1936: Post's thesis II (P2) A decision problem is considered intuitively solvable iff. the problem is 1-given and one can set-up a finite 1-process which is a 1-solution to the problem.

$\Rightarrow$ *Where do these two logically equivalent formulations come from?*

The starting point: Post's PhD (1920)

Introduction to a general theory of elementary propositions, 1921

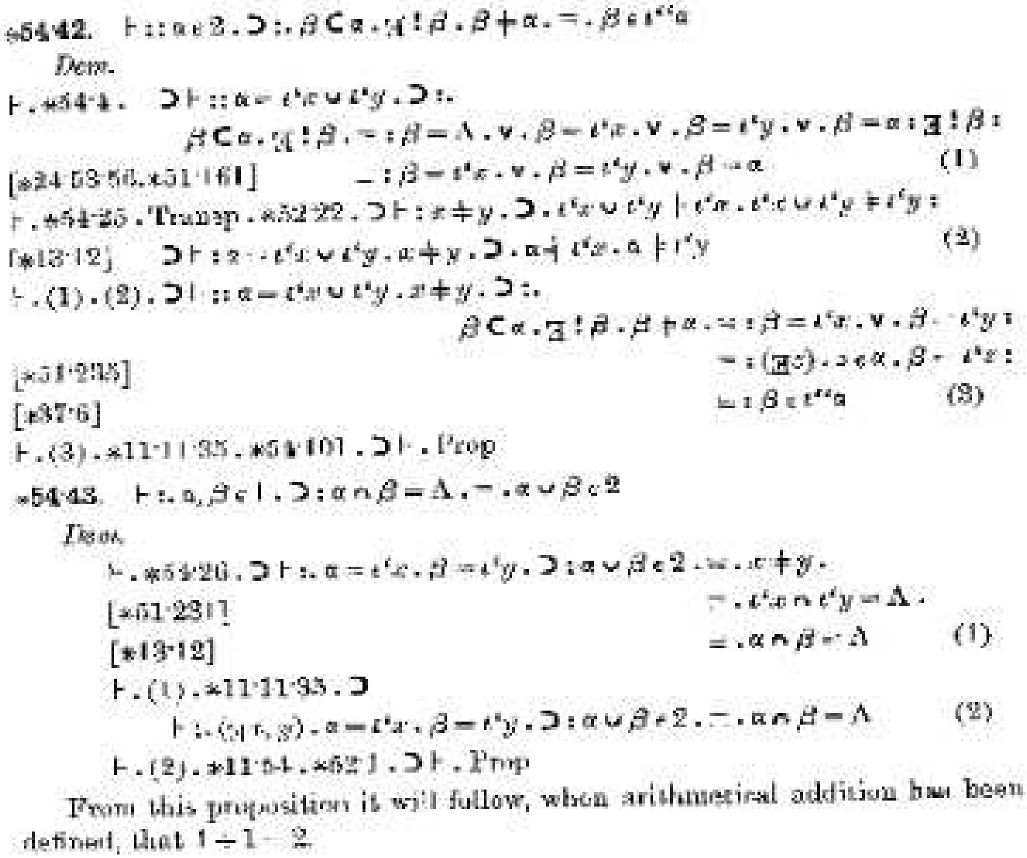
⇒ **Principia Mathematica** $(\sim, \vee)$

- Reducing the whole of mathematics to symbolic logic
- BUT (!) "[...] we might take cognizance of the fact that the system of 'Principia' is but one particular development of the theory [...] and so [one] might construct a general theory of such developments." (Post, 1921)

⇒ **Survey of Symbolic Logic** (Clarence I. Lewis, 1918)

- "A mathematical system is any set of strings of recognizable marks in which some of the strings are taken initially and the remainder derived from these by operations performed according to rules which are *independent of any meaning assigned to the marks* [...] Whatever the mathematician *has in his mind* when he develops a system, what he *does* is to set down certain marks and proceed to manipulate them" (Lewis, 1918) ⇒ "Mathematics without Meaning"
- "We have regarded the system of 'Principia' and the generalizations thereof as purely *formal developments* [in Lewis' sense] (Post, 1921)
- "This meaning of + and - is convenient to bear in mind as a guide to thought, but in the actual development they are to be considered merely as symbols which we manipulate in a certain way"

Post's PhD



*54·42. ⊢ :: α ∈ 2 . ⊃ :. β ⊂ α . ∃! β . β ≠ α . ≡ . β ∈ ι"α

Dem.

⊢ . *54·4 . ⊃ ⊢ :: α = ι'x ∪ ι'y . ⊃ :.
β ⊂ α . ∃! β . ≡ : β = Λ . v . β = ι'x . v . β = ι'y . v . β = α : ∃! β :
[*24·53·56 . *51·161] ≡ : β = ι'x . v . β = ι'y . v . β = α (1)
⊢ . *54·25 . Transp . *52·22 . ⊃ ⊢ : x ≠ y . ⊃ . ι'x ∪ ι'y ≠ ι'x . ι'x ∪ ι'y ≠ ι'y :
[*13·12] ⊃ ⊢ : α = ι'x ∪ ι'y . x ≠ y . ⊃ . α ≠ ι'x . α ≠ ι'y (2)
⊢ . (1) . (2) . ⊃ ⊢ :: α = ι'x ∪ ι'y . x ≠ y . ⊃ :.
β ⊂ α . ∃! β . β ≠ α . ≡ : β = ι'x . v . β = ι'y :
[*51·235] ≡ : (∃z) . z ∈ α . β = ι'z :
[*37·6] ≡ : β ∈ ι"α (3)
⊢ . (3) . *11·11·35 . *54·101 . ⊃ ⊢ . Prop

*54·43. ⊢ :. α, β ∈ 1 . ⊃ : α ∩ β = Λ . ≡ . α ∪ β ∈ 2

Dem.

⊢ . *54·26 . ⊃ ⊢ :. α = ι'x . β = ι'y . ⊃ : α ∪ β ∈ 2 . ≡ . x ≠ y .
[*51·231] ≡ . ι'x ∩ ι'y = Λ .
[*13·12] ≡ . α ∩ β = Λ (1)
⊢ . (1) . *11·11·35 . ⊃
⊢ :. (∃x, y) . α = ι'x . β = ι'y . ⊃ : α ∪ β ∈ 2 . ≡ . α ∩ β = Λ (2)
⊢ . (2) . *11·54 . *52·1 . ⊃ ⊢ . Prop

From this proposition it will follow, when arithmetical addition has been defined, that 1 + 1 = 2.

?

→ Post's thesis 1

Post's "programme" (1918–1920)

⇒ Two directions of generalization to study systems of symbolic logic

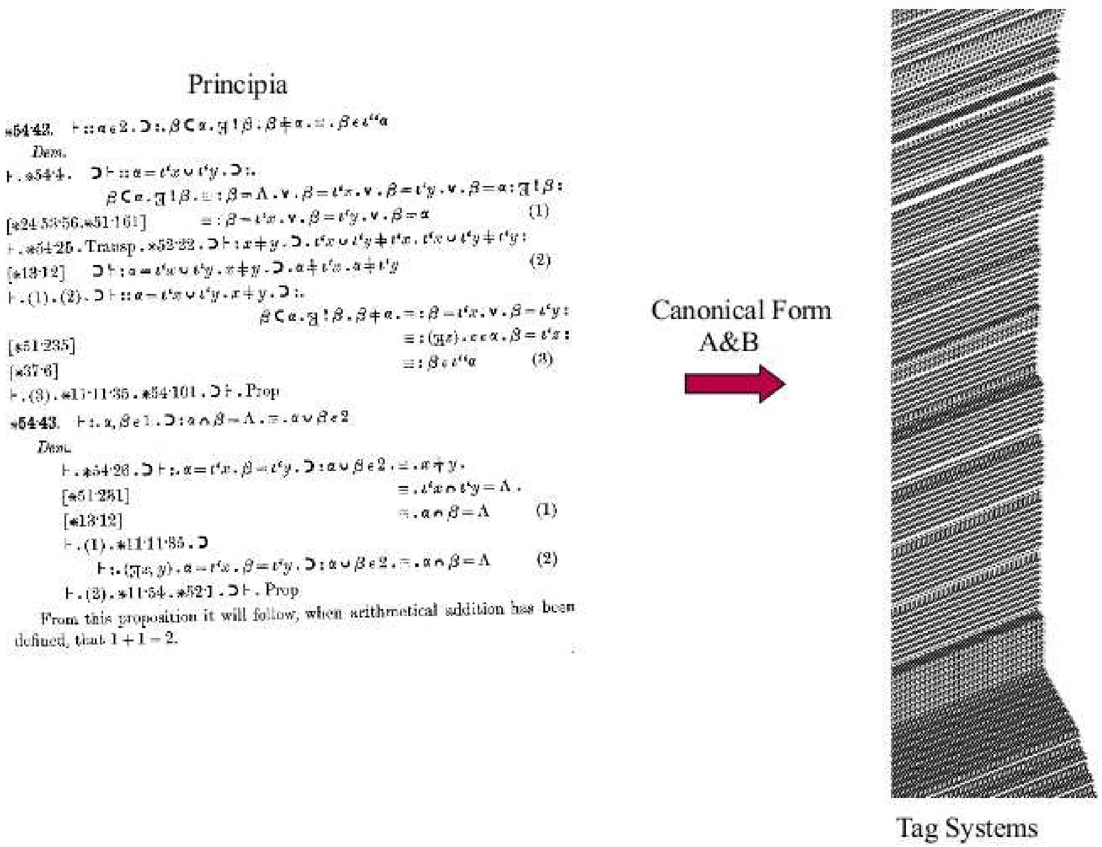
1. Development of a *general* theory of systems of symbolic logic (in his PhD):
   - (PhD) A general framework for systems of logic regarded as systems for inferences via finitary symbol manipulation: *Generalization by Postulation, systems in canonical form A*
   - Development of many-valued logics
2. Generalization of the main results (Completeness, decidability and consistency of propositional logic (information about *all* assertions) to other parts of *Principia* and ultimately mathemaics.
   - "we believe that, inasmuch as the theory of elementary propositions is at the base of the complete system of *Principia*, this broadened outlook upon the theory will serve to prepare us for a similar analysis of that complete system, and so ultimately of mathematics." (Post,1921)

Post's account of an anticipation: towards the reversal of Post's programme

- **Ambitions of a Procter fellow**: deciding the "finiteness problem" (Entscheidungsproblem) for first-order logic and ultimately the whole of *Principia*: "Since *Principia* was intended to formalize all of existing mathematics, Post was proposing no less than to find a single algorithm for all of mathematics." (Davis, 1994)
- **Methodology (influence Lewis): Simplification through generalization**:
  "Perhaps the chief difference in method between the present development and its more complete successors is its preoccupation with the outward forms of symbolic expressions, and possible operations thereon, rather than with logical concepts as clothed in, or reflected by, correspondingly particularized symbolic expressions, and operations thereon. While this in part is perhaps responsible for the fragmentary nature of our development, it also allows greater freedom of method and technique."

Account of an anticipation: towards the reversal of Post's programme

Method (influence Lewis): Simplification through generalization:



Account of an anticipation: the problem of "tag"

- A proof of the solvability of the finiteness problem was not that straightforward, and Post began to focus on a problem closely connected with the finiteness problem, the unification problem.

⇒ Unification problem and the problem of "Tag"

- To determine for two expressions what substitutions would make those expressions identical.

- The general problem proving intractable, successive simplifications thereof were considered, one of the last being this problem of "tag". Again, after the finiteness problem for systems in canonical form A involving primitive functions of only one argument was solved, an attempt to solve the problem for systems going, it seemed, but a little beyond this one argument case, led once more essentially to the selfsame problem of "tag". The solution of this problem thus appeared as a vital stepping stone in any further progress to be made.

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000$

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~~001~~$\underbrace{10111011101000000}_{A_0}$ $\Rightarrow$ Periodicity!

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000$
~~101~~110111010000001101
~~110~~11101000000 11011101
~~111~~01000000110111011101
~~010~~0000011011101110100
~~000~~001101110111010000
~~001~~$\underbrace{10111011101000000}_{A_0} \Rightarrow$ Periodicity!

$\Rightarrow$ Two decision problems (finiteness problems) for tag systems: the halting and reachability problem

The frustrating problem of "Tag".

⇒ **Exploring tag systems: pencil-and-paper computations and "observations" to find a solution to the problem.**

⇒ **Results**

⇒ Observation of three classes of behavior: periodicity, production of ϵ (halt) and unbounded growth.

- Infinite class with $v = 1$ or $\mu = 1$ is decidable ("trivial") (Wang, 1963)
- Infinite class with $\mu = v = 2$ is decidable (involves "considerable labor") (De Mol,2010)
- Infinite class with $\mu = 2, v = 3$ was called "intractable" (See also Minsky, 1967; De Mol, 2009)
- Infinite class with $\mu > 2, v = 2$ Post identifies as being of "bewildering complexity"

⇒ *Principia* vs. Lewis-like Abstract form ("mathematics without meaning") → shift to an analysis of the behavior → limitations of Lewis' ideal mathematics

The frustrating problem of "Tag".

"While considerable effort was expanded on the case $\mu = 2, v > 2$, but little progress resulted, such a simple basis as $0 \rightarrow 00, 1 \rightarrow 1101, v = 3$, proving intractable. For a while the case $v = 2$, $\mu > 2$, seemed to be more promising, since it seemed to offer a greater chance of a finely graded series of problems. But when this possibility was explored in the early summer of 1921, it rather led to an overwhelming confusion of classes of cases, with the solution of the corresponding problem depending more and more on problems in ordinary number theory. Since it had been our hope that the known difficulties of number theory would, as it were, be dissolved in the particularities of this more primitive form of mathematics, the solution of the general problem of "tag" appeared hopeless, and with it our entire program of the solution of finiteness problems. This *frustration* [my emphasis], however, was largely based on the assumption that "tag" was but a minor, if essential, stepping stone in this wider program." (Post,1965)

$\Rightarrow$ *Trigger reversal of Post's programme + inspiration for normal form* (De Mol, 2006)

From canonical form C....

Post Productions Systems

Defined by Σ, a finite set of initial words $\in \Sigma^*$ and a finite set of production rules of the form:

$$g_{11}P_{i_1^1}g_{12}P_{i_2^1}\dots g_{1m_1}P_{i_{m_1}^1}g_{1_{(m_1+1)}}$$

$$g_{21}P_{i_1^2}g_{22}P_{i_2^2}\dots g_{2m_2}P_{i_{m_2}^2}g_{2_{(m_2+1)}}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$g_{k1}P_{i_1^k}g_{k2}P_{i_2^k}\dots g_{km_k}P_{i_{m_k}^k}g_{k_{(m_k+1)}}$$

produce

$$g_1P_{i_1}g_2P_{i_2}\dots g_mP_{i_m}g_{(m+1)}$$

with each $g_{i,j}, P_{i,j} \in \Sigma^*$

To systems in normal form....

Defined by Σ, one initial word $\in \Sigma^*$ and a finite set of production rules of the form:

$$
\begin{array}{ccc}
g_i P_i & 1101 P_i & \text{~~1101~~}11011101000000 \\
 & \textit{produces} & \\
P_i g_{i'} & P_i 001 & 11011101000000001
\end{array}
$$

with each $g_i, g_{i'}, P_i \in \Sigma^*$.[a]

[a] Note! Tag systems are a subclass of normal systems.

To Post's normal form theorem, "the most beautiful theorem in mathematics" (Minsky, 1961)

Given a system in canonical form C with primitive letters $a_1, a_2, ..., a_\mu$, a system in normal form with primitive letters $a_1, a_2, ..., a_\mu, a'_1, a'_2, ..., a'_{\mu'}$ can be set up such that the assertions of the system in canonical form are exactly those assertions of the system in normal form which involve no other letters than $a_1, a_2, ..., a_\mu$.

"May I suggest that the tricks employed in my paper [...] were forced on me by ~~b~~ the ever more restricted formal means left me by the required ever simpler forms of basis. But the canonical form is powerful, and reduction to it should be natural instead of tricky." (Post in a letter to Church, dated July 29, 1943) $\Rightarrow$ Closing the circle....

The anticipation....

"In view of the generality of the system of *Principia Mathematica*, and its seeming inability to lead to any other generated set of sequences on a given set of letters than those given by our normal systems, we are led to the following generalization", i.e., Post's thesis I (Davis,1982):

> **Post's Thesis I.** *Every **generated** set of sequences on a given set of letters* $a_1, a_2, ..., a_\mu$ *is a subset of the set of assertions of a system in **normal form** with primitive letters* $a_1, a_2, ..., a_\mu, a'_1, a'_2, ..., a'_\mu$*, i.e., the subset consisting of those assertions of the normal system involving the letters* $a_1, a_2, ..., a_\mu$.

Given thesis I + idea reversal programme:

> "[...] the finiteness problem for the class of all normal systems is unsolvable"

> "A complete logic is impossible"

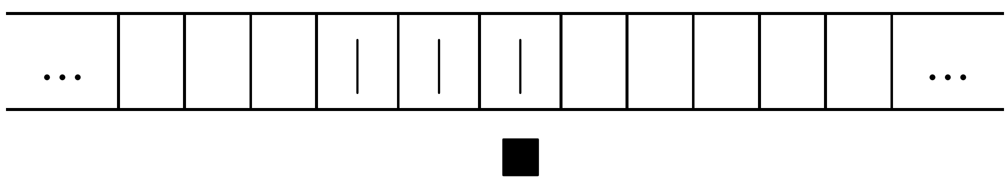
From normal form (1921) to Post's machines (1936)

Post's Thesis I $\stackrel{?}{\Rightarrow}$ *Post's thesis II*

~~1101~~11011101000000

produces $\stackrel{?}{\Rightarrow}$

11011101000000001



Generated sets $\stackrel{?}{\Rightarrow}$ *Solvability*

Post's thesis II. Motivations.

"[for the thesis to obtain its full generality] an analysis should be made of all the possible ways the human mind can set up finite processes to generate sequences."

"While the formal reductions of Part I [the reduction from canonical form A to B to C to normal form] should make it a relatively simple matter to supply the details of the development outlined [i.e. the (theorems)] that development owes its significance entirely to the universal character of our characterization of an arbitrary generated set of sequences [...] Establishing this universality is not a matter for mathematical proof, but of *psychological analysis of the mental processes involved in combinatory mathematical processes* [m.i.]. [...] Actually, we can present but fragments of the *proposed analysis of finite processes* [m.i.]. [...] This theme [the idea that there exist problems we humans cannot solve] will protrude itself ever so often in *our immediate task of obtaining an analysis of finite processes* [m.i.]."

"The real question at issue is: What are the possible processes that can be carried out in computing a number?" (Turing, 1936)

⇒ Post made an analysis similar to Turing's!

Post's thesis II

Post's thesis II *A decision problem is considered intuitively solvable iff. the problem is 1-given and one can set-up a finite 1-process which is a 1-solution to the problem*

3. Church's practices

$$(\lambda[m]\cdot(\lambda[n]\cdot(\lambda[f]\cdot(\lambda[x]\cdot m[f][n[f][x]]))))[5][2] == \lambda[f]\cdot(\lambda[x]\cdot f[f[f[f[f[f[f[x]]]]]]])$$

Church's practices: One definition

1936: Church's thesis "We now define the notion [...] of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ-definable function of positive integers.)"

$\Rightarrow$ *Where does this formulation which is logically equivalent to Post's theses come from?*

The starting point: Church's PhD

- "Lewis and Langford's Symbolic Logic was around. No, that may have been later, but certainly the book by C.I. Lewis was available. But there was nothing about the sort of thing I wanted to teach, logic directed towards math rather than the philosophical aspects of logic. Well, I am not sure; there may have been a book of that sort. Of course [David] Hilbert and Wilhelm Ackermann's Grundzuege der theoretischen Logik was in existence at that time, but it was in German. While the grad students were supposed to learn German, as a practical matter I could not have used it as a textbook. So I used written notes of my own and things like that." (Interview with Aspray, 1984)
- Subject matter: the axiom of choice:

  For any set A, all of whose member are non-empty sets, there exists a set B which contains exactly one element from each of the sets belonging to A

"To deny what seems intuitively natural"....

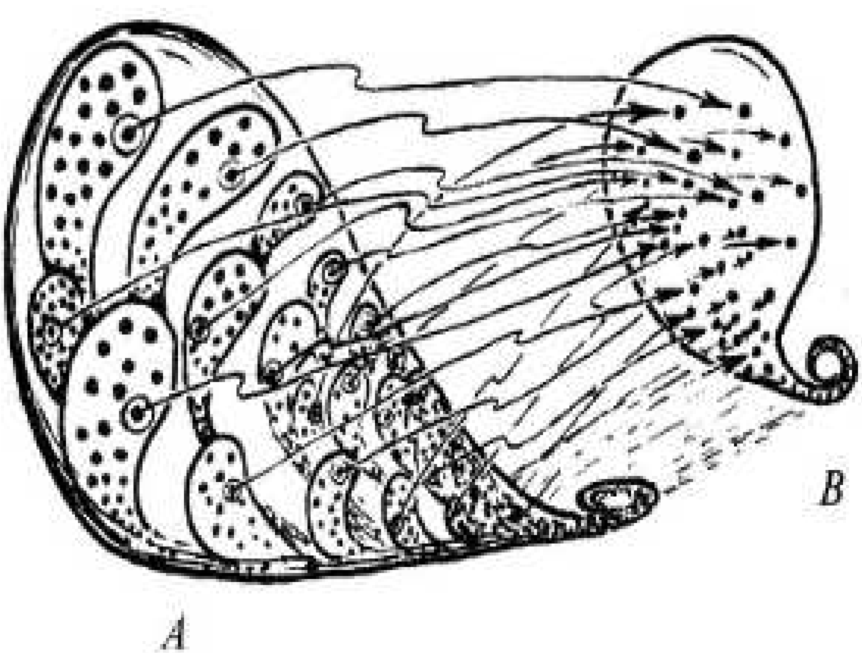
Goal? Prove the independence of the axiom of choice: "The object of this paper is to consider the possibility of setting up a logic in which the axiom of choice is false."

Church's "experimental" approach?:

"If any one of these involve a contradiction it is reasonable to expect that a systematic examination of its properties will ultimately reveal this contradiction. But if a considerable body of theory can be developed on the basis of one of these postulates without obtaining inconsistent results, then this body of theory, when developed, could be used as presumptive evidence that no contradiction exists." (Church, 1927)

"We shall examine briefly the consequences of each of the postulates just stated when taken in conjunction with Postulates 1-5 and C, taking the same experimental attitude as that which we took in the case of Postulates A, B and C" (Church, 1927)

Church's PhD



→ Church's Thesis?

Towards variant systems of symbolic logic: Motivations

Yet another formalization of mathematics *after* (!) Gödel "In this paper we present a set if postulates for the foundation of formal logic" (Church, 1932)

Going beyond Gödel "It is worth observing, however, that there may be a possibility of proving that there is no formula **A** such that both **A** and ∼**A** are consequences of our postulates [...] This is conceivable on account of the entirely formal character of the system which makes it possible to abstract from the meaning of the symbols and to regard the proving of theorems (of formal logic) as a game played with marks on paper according to a certain arbitrary set of rules" (Church 1933) " I was working on a project for a radically different formulation of logic which would (as I saw it at the time) escape some of the unfortunate restrictiveness of type theory. In a way I was seeking to do the very thing that Gödel proved impossible" (Church in a letter to Dawson, July 25, 1983) ∼ Post

Introduction of the λ-operator to denote functions: **Ex.** "$x^4 + x$ is smaller than 1000" vs. "x^4+x is a primitive recursive function" → $\lambda x.x^4+x$

Towards variant systems of symbolic logic: From the criterion of consistency....

> "We do not attach any character of uniqueness or absolute truth to any particular system of logic. The entities of formal logic are abstractions, invented because of their use in describing and systematizing facts of experience or observation, and their properties, determined in rough outline by this intended use, depend for their exact character on the arbitrary choice of the inventor. [T]here exist, undoubtedly, more than one formal system whose use as a logic is feasible, and of these systems one may be more pleasing or more convenient than another, but it cannot be said that one is right and the other wrong." (Church, 1932)

⇒ Criterion of consistency

... to the "experimental" method (revisited)?

> "Whether the system of logic which results from our postulates is adequate for the development of mathematics, and whether it is wholly free from contradiction, are questions which we cannot answer except by conjecture. Our proposal is to seek at least an empirical answer to these questions by carrying out in some detail a derivation of the consequences of our postulates, and it is hoped either that the system will turn out to satisfy the conditions of adequacy and freedom from contradiction or that it can be made to do so by modifications or additions." (Church, 1932)

> "Our present project is to develop the consequences of the foregoing set of postulates, until a contradiction is obtained from them, or until the development has been carried so far consistently as to make it empirically probably that no contradiction can be obtained from them. And in this connection it is to be remembered that just such empirical evidence, although admittedly inconclusive, is the only existing evidence of the freedom from contradiction of any system of mathematical logic which has a claim to adequacy." (Church, 1933)

⇒ **Confronted with the problems of this approach?:** Church's set of postulates proven inconsistent by his PhD students (Kleene and Rosser, 1935)

λ – The ultimate operator

- Symbols: $\lambda, (,), x, y, z, \ldots$
- λ-formulas:
  - the variables
  - If P is a λ-formula containing x as a free variable then $\lambda x[P]$ $(\lambda x.P)$ is also a λ-formula.
  - If M and N are λ-formulas then so is $\{M\}(N)$
- Rules of conversion:
  1. *Reduction.* To replace any part $((\lambda x\ M)\ N)$ of a formula by $S^x_N M|$ provided that the bound variables of M are distinct both from x and from the free variables of N. For example to change $\{\lambda x[x^2]\}(2)$ reduces to 2^2
  2. *Expansion* To replace any part $S^x_N M|$ of a formula by $((\lambda x\ M)\ N)$ provided that $((\lambda x\ M)\ N)$ is well-formed and the bound variables of M are distinct both from x and from the free variables in N. For example, 2^2 can be expanded to $\{\lambda x[x^2]\}(2)$
  3. *Change of bound variable* To replace any part M of a formula by $S^x_y M|$ provided that x is not a free variable of M and y does not occur in M. For example changing $\{\lambda x[x^2]\}$ to $\{\lambda y[y^2]\}$

λ – The ultimate operator: an example

- Defining the natural numbers:

$$
\begin{aligned}
&1 \rightarrow \lambda yx.yx, \\
&2 \rightarrow \lambda yx.y(yx), \\
&3 \rightarrow \lambda yx.y(y(yx)), \\
&\ldots
\end{aligned}
$$

- The successor function S:

$$S \rightarrow \lambda abc.b(abc)$$

$$
\begin{aligned}
& \Big(\lambda abc.b(abc)\Big)\Big(\lambda yx.y(yx)\Big) = S(2) \\
\rightarrow \quad & \lambda bc.b\Big(\Big(\lambda yx.y(yx)\Big)bc\Big) \\
\rightarrow \quad & \lambda bc.b\Big(\Big(\lambda x.b(bx)\Big)c\Big) \\
\rightarrow \quad & \lambda bc.b(b(bc)) = 3
\end{aligned}
$$

Surprised by λ....

The heuristic method revisited (again)?

> "We [Church and Kleene] kept thinking of specific such functions, and of specific operations for proceeding from such functions to others. I kept establishing the functions to be λ-definable and the operations to preserve λ-definability." (Kleene, 1981)

> "Our object is to prove empirically (!) that the system is adequate for the theory of positive integers, by exhibiting a construction of a significant portion of the theory within the system" (Kleene, 1935)

> "The results of Kleene are so general and the possibilities of extending them apparently so unlimited that one is led to the conjecture that a formula can be found to represent any particular constructively defined function of positive integers whatever." (Church, January 1935)

$\Rightarrow$ Every effectively calculable function is λ-definable

Convinced by the formalism "Turing's definition of computability was intrinsically plausible, whereas with the other two, a person became convinced only after he investigated and found, much by surprise, how much could be done with the definition." (Kleene in an interview with Aspray, 1985)

...and being "careful" about λ

- First informal formulation Church's thesis I in February 1934; public announcement: April, 1935!
- In need of more support: argument by confluence (+ step-by-recursive step argument)

⇒ **1936: Church's thesis** "We now define the notion [...] of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ-definable function of positive integers.)"

4. “you can’t hide behind a definition”

A definition or a hypothesis?

Post's position: "Its purpose [...] is not only to present a system of a certain logical potency but also, [...] of psychological fidelity [...] We offer this conclusion at the present moment as a *working hypothesis*. [...] The success of the above program would, for us, change this hypothesis not so much to a definition or to an axiom but to a *natural law*. (Post, 1936)

Church's position: "[The purpose of this paper is] to propose a definition of effective calculability which is thought to correspond satisfactorily to the somewhat vague intuitive notion in terms of which problems of this class are often stated, and to show that not every problem of this class is solvable [...] This [proposed] definition is thought to be justified by the considerations which follow, so far as positive justification can ever be obtained for the selection of a formal definition to correspond to an intuitive notion"

The disagreement between Church and Post: Post’s reaction

- “But to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of the mathematicizing power of *Homo Sapiens* has been made and blinds us to the need of its continual verification.” (Post, 1936)
- “For if symbolic logic has failed to give wings to mathematicians this study of symbolic logic opens up a new field concerned with the fundamental limitations of mathematics, more precisely the mathematics of Homo Sapiens.” (Post to Church, March 24, 1936)
- “Again you argue that you can always as it were withdraw into your shell and say that that is your definition of effective calculability. But you have a result to the effect that the entscheidungsproblem for the engere funktionenkalkül is unsolveable. Recursively of course makes it O.K. But with your criterion recursively would be omitted. But then you should be able to bar any ambitions young man from attempting its solution by any means just as you would rightfully bar him from attempting the trisection of the general angle by straight edge and compass. But now you see you can’t hide behind merely a definition.” (Post to Church, July 10, 1936)
- “Strange as it may seem this Pepis possibility seems almost to reverse our

positions, for you seem to have a lurking suspicion that he may be right while I – to put it crudely but forcibly – am willing to bet a 1000$ that he isn’t, and it would take me five years to save up that sum. [...] P.S. But I am not willing to stake my “immortal” soul in it – which I should were I to adopt your original position. ” (Post to Church, June 9, 1937)

The disagreement between Church and Post: Church's reaction

- "[...] effectiveness in the ordinary sense has not been given an exact definition, and hence the working hypothesis in question has not an exact meaning. To define effectiveness as computability by an arbitrary machine, subject to restrictions of finiteness, would seem to be an adequate representation of the ordinary notion, and if this is done the need for a working hypothesis disappears." (Church, review of Post's 1936 paper)
- "As regards your objections to the identification of recursiveness, or of lambda-definability, with the intuitive notion expressed by "effective", "effective calculable", or some synonym, you seem to be inclined to put the burden of proof on me, which I do not think justified. The fact is that the intuitive notion is vague and inexact, and that what I proposed was to render it exact by giving a formal definition. No proof can be expected, simply because the intuitive notion is inexact; if they maintain that there is something in their notion which makes it more general than recursiveness or lambda-definability, they can be legitimately asked to produce an example, and, failing to do so, stand convicted of making an utterly vague and even pointless assertion. I am thus content to let the matter stand as a challenge." (Church to Post, September 18, 1936)

- "To your position as a whole my reply would be, I think, about as follows. I have been maintaining, all effectively calculable functions are general recursive." In doing so, however, I have not been trying to establish an empirical proposition, or a mathematical proposition. Instead, I have been proposing an exact definition of a phrase ("effectively calculable") which has hitherto had only a vague meaning. Under such circumstances proofs are not to be expected, but only considerations of convenience and naturalness and facts of historical usage and generally accepted connotations." (Church to Post, July 10, 1937)

Why this disgreement?

The Case Church: what remains of the empirical attitude of Church? The safety of a definition (Re: "I have not been trying to establish an empirical proposition, or a mathematical proposition")

The Case Post:

The human limitations and the frustrating problem of "tag": "This problem [the problem of "tag"] itself in its entscheidungsproblem form is a special case of my unsolvable problem (which I hope to get to at least before he end of this letter if not of your patience) and should it too prove unsolvable I will be supplied with the perfect alibie [sic] for a year of frustration." (Post in a latter to Church, May 30, 1936)
"my wife is much worried. So I told her for the first time, the exact history of my mental ups and downs and worse from its first inception in trying to solve the probably unsolvable tag-problem in Princeton and how at 50 experience and lesson of personal importance of failure or success with at best 70-50 $<$ 20 years to go – but I see my 50 years of experience may still not be enough – God help me" (Post in a letter to Church, March 3, 1947)

Reversal Lewis as a reversal of Post's programme: "Perhaps a wider use of logistic would help to free science from a considerable body of "hypotheses" whose value lies not in their logical implications but in their psychological 'suggestiveness' " vs. "Its purpose [...] is not only to present a system of a certain logical potency but also, [...] of psychological fidelity [...] We offer this conclusion at the present moment as a *working hypothesis* (Post, 1936)

5. Discussion

Discussion

- An epistemic irony? ““[T]he contemporary practice of mathematics, using as it does heuristic methods, only makes sense because of this undecidability.” (Von Neumann, 1927)
- Explanation of (paricularities) logical results and philosophical points of view throughout practices of symbolic logic (sources, methods, formalisms, goals, (psychological states?),...)
- Man-Logic Interactions; the creative role of formalisms ∼ man-computer interactions; computer experimentation?

  [...] the creativeness of human mathematics has a counterpart inescapable limitation thereof – witness the absolutely unsolvable (combinatory) problems. Indeed, with the bubble of symbolic logic as universal logical machine finally burst, a new future dawns for it as the indispensable means for revealing and developing those limitations. For [...] Symbolic Logic may be said to be Mathematics become self-conscious.

  Emil L. Post, 1920–21.