Adaptive Logics using the Minimal Abnormality strategy are Π^1_1-complex *

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Abstract. In this paper complexity results for adaptive logics using the Minimal Abnormality strategy are presented. It is proven here that the consequence set of some premise sets is Π^1_1-complete. So, the complexity results in (Horsten and Welch, 2007) are mistaken for the adaptive logics using Minimal Abnormality strategy.

Keywords: Adaptive logics, Minimal Abnormality, complexity, dynamic proofs, Π^1_1-complete sets.

1. Introduction

In their (2007) Horsten en Welch prove that the $\mathbf{CLuN^r}$-consequence set of a premise set is maximally Σ^0_3-complete[1]. They give a concrete premise set with a Σ^0_3-complete consequence set. These results are very useful, especially because they are easily generalizable to all adaptive logics in standard format. However, their results for $\mathbf{CLuN^m}$ are mistaken. A simple analysis of the definition leads them to deciding that Π^1_1 is an upper bound to the complexity. In the following lemmas they reduce this upper bound to Σ^0_3, concluding that the same complexity results hold for both related logics. The proof for this reduction is very hard to follow; their proposition 13 is dubious and some definitions are sloppy. In this paper I will show that the reduction is indeed mistaken. I will construct a premise set with a Π^1_1-complete consequence set for an arbitrary adaptive logic (that has a classical disjunction). This makes Π^1_1 the lowest upper bound for the complexity of $\mathbf{CLuN^m}$.

I will proceed by showing that the $\mathbf{CLuN^m}$-consequence set of a particular infinite but recursive premise set Γ_R is (at least) as complex as a set that is the solution of a graph theoretic problem that is known to be Π^1_1-hard. The graph problem is whether some statement holds for all paths in a recursive but infinite graph. There may be non-enumerably many paths in such a graph. In the adaptive logic problem,

* Research for this paper was supported by subventions from Ghent University and from the Fund for Scientific Research – Flanders. I am indebted to Diderik Batens for comments on a former draft. I am also indebted to Dexter Kozen for his useful help on Π^1_1-hard problems.

[1] For an introduction to these complexity classes, see (Rogers, 1967).

every path will correspond to a set of minimally abnormal models of the premise set. The property that has to hold for all paths in the graph will correspond to a property that has to hold for all models in the sets of minimally abnormal models corresponding to these paths. Hence, every such set of models has to be checked, in order to know which conclusions can be drawn from the premise set. So, to express the consequence set mathematically, one needs a statement with a universal quantifier that ranges over a non-enumerable amount of objects, which is to say that it is at least Π^1_1-complete.

First the graph problem is presented. Next, adaptive logics are semantically and proof theoretically defined. In section 4 the actual proof of the crucial theorem is given. Section 5 describes the adaptive proofs for the relevant premise set. Section 6 discusses the possibility to define, in a first order language, a finite premise set with similar properties. Finally, some philosophical conclusions are formulated.

2. A Π^1_1-complete problem expressed in graph theory

Let $I\!N$ denote the natural numbers without 0 and let $I\!N_0 = I\!N \cup \{0\}$. Let $G = (I\!N, R)$ be a recursive directed graph with nodes $I\!N$ and edges $R \in \mathcal{P}(I\!N^2)$. In this paper only graphs with set of nodes $I\!N$ will be considered and for this reason the relation R will fully define the graph.

The paths of the graph defined by a relation R are the elements of the set $\mathit{Paths}(R)$, which is defined as follows:

DEFINITION 1. *The function* $\mathit{Paths}\colon \mathcal{P}(I\!N^2) \to \mathcal{P}(I\!N \to I\!N_0)$ *expressing the paths of a graph, is defined by:*
$\pi \in \mathit{Paths}(R)$ *iff (there is some* $m \in I\!N$ *such that for all* $n \leq m$, $(\pi(n), \pi(n+1)) \in R$ *and for all* $m > n$, $\pi(m) = 0$*) or (for all* $n \in I\!N$, $(\pi(n), \pi(n+1)) \in R$*).*
The function $\mathit{Paths}\colon \mathcal{P}(I\!N^2) \times I\!N \to \mathcal{P}(I\!N \to I\!N_0)$ *expressing the paths of a graph starting in a node* i*, is defined by:*
$\pi \in \mathit{Paths}(R, i)$ *iff* $\pi \in \mathit{Paths}(R)$ *and* $\pi(1) = i$*.*

DEFINITION 2. *The function* $\mathit{WF}\colon \mathcal{P}(I\!N^2) \to \mathcal{P}I\!N$ *expressing the set of nodes from which only finite paths start, is defined by:*
$n \in \mathit{WF}(R)$ *iff for all* $\pi \in \mathit{Paths}(R, n)$*, there is a* $m \in I\!N$*, such that* $\pi(m) = 0$*.*

I shall use $\langle i_1, i_2, i_3, \ldots\rangle$ as a shorthand for the path π with $\pi(1) = i_1$, $\pi(2) = i_2$, $\pi(3) = i_3$, etc. A path is infinite iff for all $i \in I\!N$, $\pi(i) \neq 0$. The function $\mathit{lth}(\pi)$ denotes the length of a path π ($\mathit{lth}(\pi) = \omega$ iff π is infinite and $\mathit{lth}(\pi) = \max\{i | i \in I\!N; \pi(i) \neq 0\}$ otherwise) and $\mathit{lst}(\pi)$

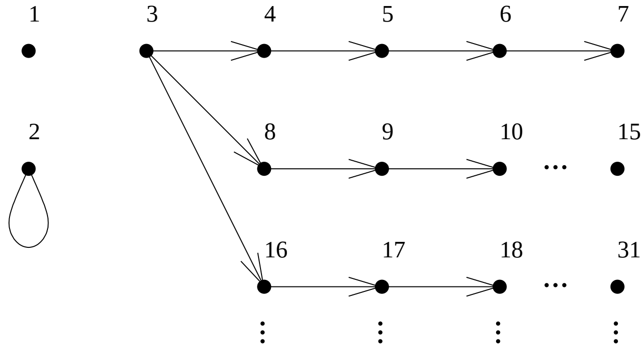

Figure 1. The example graph $(\mathbb{N}, R_1)$

denotes the last item of a path π ($lst(\pi) = \pi(lth(\pi))$), which is obviously undefined for infinite paths.

It is a well known fact that the set $WF(R)$ is Π^1_1-complete (see (Kozen, 2002) and (Rogers, 1967)). The following example graph $(\mathbb{N}, R_1)$ is clarifying (see also figure 1). R_1 is defined by $(i, j) \in R_1$ iff $(i, j) = (2, 2)$ or ($j = i + 1$ and $i > 3$ and $j \neq 2^k$ for some $k \in \mathbb{N}$) or ($i = 3$ and $j = 2^k$ for some $k > 1$). Only in node 2 infinite paths start (the path $\langle 2, 2, 2, \ldots \rangle$). In node 1 only the path $\langle 1, 0, 0, \ldots \rangle$ starts. The paths that start in node 3 are arbitrary long but finite (the paths $\langle 3, 2^k, 2^k + 1, \ldots, 2^{k+1} - 1, 0, 0, \ldots \rangle$, where $k \geq 2$). Hence, $WF(R_1) = \mathbb{N} - \{2\}$.

3. Adaptive logics

3.1. The standard format of **AL**

In this section adaptive logics are very briefly presented (see (Batens, 2007) for an overview and (Batens, 2004) for the philosophical basis). An adaptive logic in standard format is defined as a triple consisting of:

- a **LLL**: a monotonic, reflexive, transitive and compact extension of classical logic (**CL**) which has a characteristic semantics,

- a set of abnormalities: a set of **LLL**-contingent formulas Ω, characterized by a (possibly restricted) logical form, and

- a strategy (the most important strategies in **AL** are ‘Reliability’ and ‘Minimal Abnormality’).

The standard format demands that the **LLL**-language, next to its own standard logical symbols, also encompasses formulas with the standard logical symbols of **CL**. They must behave classically, i.e. they should function in a **CL**-standard manner (e.g. $M \models \check{\neg} A$ iff $M \not\models A$). In this paper, we will denote the **CL**-symbols by means of $\check{\neg}$ (negation) and $\check{\vee}$ (disjunction).

3.2. The proof theory of **AL**

The proof theory of an **AL** consists of a set of inference rules (determined by the **LLL** and Ω) and a marking definition (determined by Ω and the chosen strategy). A line of an annotated **AL**-proof consists of five elements: (1) a line number i, (2) a formula A, (3) the name of a rule and the line numbers of the rule premises, (4) a condition consisting of a set of abnormalities $\Theta \subset \Omega$. A stage s of a proof is the subproof that is completed up to line number s. The inference rules govern the addition of lines. There are 3 types of rules.

PREM If $A \in \Gamma$

$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \vee Dab(\Theta)$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

The classical disjunction of the members of a finite $\Delta \subset \Omega$, $Dab(\Delta)$, is called a *Dab-formula*. $Dab(\Delta)$ is a *minimal Dab*-formula of stage s iff $Dab(\Delta)$ is derived at stage s on the condition $\emptyset$ and no $Dab(\Delta')$ with $\Delta' \subset \Delta$ is derived on the condition $\emptyset$. The most important strategies are Reliability and Minimal Abnormality.

DEFINITION 3. *Marking definition for Reliability.*
Where $Dab(\Delta_1)$, ..., $Dab(\Delta_n)$ are the minimal Dab-formulas derived on the condition $\emptyset$ at stage s, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$, and Δ is the condition of line i, line i is marked at stage s iff $\Delta \cap U_s(\Gamma) \neq \emptyset$.

DEFINITION 4. *Marking definition for Minimal Abnormality.*
Where $Dab(\Delta_1)$, ..., $Dab(\Delta_n)$ are the minimal Dab-formulas derived on the condition $\emptyset$ at stage s, $\Phi^\circ_s(\Gamma)$ is the set of all sets that contain

one member of each Δ_i, $\Phi_s(\Gamma)$ are the $\varphi \in \Phi^\circ_s(\Gamma)$ that are not proper supersets of a $\varphi' \in \Phi^\circ_s(\Gamma)$, A is the formula and Δ is the condition of line i, line i is marked at stage s iff

(i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or

(ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line on which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.

Two types of derivability are defined. A formula A is *derived at a stage* iff A is derived on an unmarked line at the stage. A formula A is *finally derived* at stage s iff A is derived on an unmarked line i at stage s and line i will not be marked in any extension of the stage. The finally derivable consequences of a premise set are independent of the stage and constitute the consequence sets for $\mathbf{AL^r}$ and $\mathbf{AL^m}$: $Cn_{\mathbf{AL^r}}(\Gamma)$, respectively $Cn_{\mathbf{AL^m}}(\Gamma)$, and their consequence relations: $\vdash_{\mathbf{AL^r}}$, respectively $\vdash_{\mathbf{AL^m}}$.

3.3. The semantics of **AL**

$Dab(\Delta)$ is a *minimal Dab-consequence* of Γ iff $\Gamma \vDash_{\mathbf{LLL}} Dab(\Delta)$ and, for all $\Delta' \subset \Delta$, $\Gamma \nvDash_{\mathbf{LLL}} Dab(\Delta')$. Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal Dab-consequences of Γ, let $U(\Gamma) =_{df} \Delta_1 \cup \Delta_2 \cup \ldots$. Finally, where M is a **LLL**-model, $Ab(M) =_{df} \{A \in \Omega \mid M \models A\}$.

DEFINITION 5. *Reliable model and the corresponding semantical consequence relation $\vDash_{\mathbf{AL^r}}$.*
*A **LLL**-model M of Γ is reliable iff $Ab(M) \subseteq U(\Gamma)$. $\Gamma \vDash_{\mathbf{AL^r}} A$ iff all reliable models of Γ verify A.*

DEFINITION 6. *Minimally abnormal model and the corresponding semantical consequence relation $\vDash_{\mathbf{AL^m}}$.*
*A **LLL**-model M of Γ is minimally abnormal iff there is no **LLL**-model M' of Γ for which $Ab(M') \subset Ab(M)$. $\Gamma \vDash_{\mathbf{AL^m}} A$ iff all minimally abnormal models of Γ verify A.*

3.4. $\mathbf{CLuN^m}$ and $\mathbf{CLuN^r}$

Let us consider the inconsistency-adaptive logics $\mathbf{CLuN^m}$ and $\mathbf{CLuN^r}$. The lower limit logic is the paraconsistent logic **CLuN**. **CLuN** is the full positive fragment of **CL** with simple gluts for the negation connective. For any formula A, both A and $\sim A$ may be true in **CLuN**. The set of abnormalities is $\Omega = \{\exists(A \& \sim A) \mid A \in \mathcal{F}\}$, with $\mathcal{F}$ the set of open or closed formulas and $\exists$ the existential closure. The strategies are respectively Minimal Abnormality (**CLuN$^{\mathbf{m}}$**) and Reliability (**CLuN$^{\mathbf{r}}$**). Where the name of the logic is not mentioned, **CLuN** is meant. For example, the expression "models of Γ" will refer to the **CLuN**-models of Γ.

4. Expressing the graph problem in propositional CLuN$^{\mathbf{m}}$

Let $!A$ abbreviate $A \wedge \neg A$ and let R be an arbitrary relation in $\mathcal{P}(I\!N^2)$. The premise set Γ_R that results in a Π^1_1-complete consequence set is defined as follows:

$$\Gamma_R = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \cup \Delta_5$$

$$\Delta_1 = \{!p_i^n \check{\vee} !p_j^n | i, j, n \in I\!N; i \neq j\}$$

$$\Delta_2 = \{!p_i^n \check{\vee} !p_j^{n+1} | i, j, n \in I\!N; (i, j) \notin R\}$$

$$\Delta_3 = \{!r_n \check{\vee} !p_i^n | i, n \in I\!N\}$$

$$\Delta_4 = \{q \check{\vee} !r_n | n \in I\!N\}$$

$$\Delta_5 = \{(q \check{\vee} !p_i^1) \supset u_i | i \in I\!N\}$$

Let Δ abbreviate $\Delta_1 \cup \Delta_2 \cup \{\check{\neg} !p_i^1\}$. Let a *series of normalities* be a set of formulas $\{\check{\neg} !p_{g(n)}^n | n \in I\} \cup \{!p_l^n | l \in I\!N; n \in I; l \neq g(n)$ or $n = \max(I) + 1\}$ with I an interval within the natural numbers and g a function in $I \rightarrow I\!N$. Infinite series of normalities have infinite intervals and $\max([i, \omega[)$ is obviously not defined for any i and the part of the clause in the definition of the set after 'or' has to be omitted for infinite series. We will refer to a specific series of normalities with interval I and function g as $SerNor$(I,g).

In what follows, I will prove that each minimally abnormal model for Γ_R corresponds to a path of the graph and that all paths have their minimally abnormal models. In these minimally abnormal models all of the $\{!r_n | n \in I\!N\}$ will come out true if and only if the model corresponds to an infinite path. The sentential letter q is derivable from the premisses if and only if at least one of the $\{!r_n | n \in I\!N\}$ is false. If all paths of the graph that start in a certain node are finite, then every minimally abnormal model for Γ_R that corresponds to these paths will falsify some $!r_n$ and therefore they all verify q. If a path is infinite than q may be true or false in the corresponding model, and therefore q will not be a **CLuN$^{\mathbf{m}}$**-consequence in that particular case.

Table I. A (part of a) model for the premises in Γ_R, where the graph $G = (\mathbb{N}, R)$ has a path π, with $\pi(1) = i'$, $\pi(n) = i$, $\pi(n') = i''$, and $lth(\pi) = n'$. The truthvalues for the $!p^l_k$ are given in the table. Remark that lemma 1 holds for this model.

$_k\diagdown^l$	1		n		n'	$n'+1$	$n'+2$
	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$
$i-1$	1	$\ldots$	1	$\ldots$	1	1	1
i	1		0		1	1	1
$i+1$	1		1		1	1	1
	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
$i'-1$	1		1		1	1	1
i'	0	$\ldots$	1	$\ldots$	1	1	0
$i'+1$	1		1		1	1	1
	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
$i''-1$	1		1		1	1	1
i''	1		1		0	1	1
$i''+1$	1	$\ldots$	1	$\ldots$	1	1	1
	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$

DEFINITION 7. *The function $Mp : \mathcal{P}(\mathcal{W}) \to \mathcal{P}(\mathcal{M})$, expressing the set of models of a set of formulas that verify the same series of normalities with an interval that contains the number 1, is defined by:*
$\Upsilon \in Mp(\Gamma)$ iff there is an interval $I \subseteq \mathbb{N}$ and a function $g\colon I \to \mathbb{N}$, such that $1 \in I$ and $\Upsilon = \{M | M \models \Gamma \cup SerNor(I, g)\}$.

LEMMA 1. *For any $n \in \mathbb{N}$, all* **LLL***-models for Δ_1 verify at least all but one of the abnormalities in $\{!p^n_k | k \in \mathbb{N}\}$. Every model that verifies at least all but one of the abnormalities in $\{!p^n_k | k \in \mathbb{N}\}$ for any $n \in \mathbb{N}$ is a model for Δ_1.*

In the next lemmas we will state that there is a bijection between the set of sets of models for Δ that verify the same series of normalities and the set of all paths π of the graph that start in node i.

LEMMA 2. *For every* **LLL***-model M for Δ there is some $\pi \in Paths(R, i)$, such that $M \models SerNor([1, lth(\pi)], \pi)$[2].*

Proof. Suppose there is a model M of Δ that does not verify $SerNor(I', g)$ for any function g and any interval $I' = [1, k]$ or $I' = [1, \omega[$. This is impossible because of lemma 1 and $M \models \check{\neg}!p^1_i$. Now suppose that there is a model M for Δ that verifies $SerNor(I, f)$ for some function f and some interval I, with $\pi \notin Bra(R, i)$ and π defined from f by the

[2] Where $a = \omega$, the interval notation $[1, a]$ refers to $[1, \omega[$.

following: $\pi(j) = f(j)$ for all $j \in I$ and $\pi(j) = 0$ for all $j \notin I$. This would mean that for some $j \in I\!N$, $M \models \check{\neg}!p_k^j$ and $M \models \check{\neg}!p_l^{j+1}$ whereas $(k,l) \notin R$. This is in contradiction with $M \models !p_k^j \check{\vee} !p_l^{j+1}$ ($M \models \Delta_2$).

LEMMA 3. *For every path $\pi \in Paths(R,i)$ there is a* **LLL***-model for Δ, such that $M \models SerNor([1, lth(\pi)], \pi)$.*

Proof. Let π be an arbitrary path in $Paths(R,i)$. If π is a finite path, define M_π as the model that verifies the combination of series of normalities $\bigcup\{SerNor(I_n, f_n) | f_n(n(lth(\pi)+1)+j) = \pi(j);\ I_n = [n(lth(\pi)+1)+1, n(lth(\pi)+1)+lth(\pi)];\ j \in I\!N;\ n \in I\!N_0\}$. If π is an infinite path, simply let M_π be the model that verifies the series of normalities $SerNor([1,\omega[,\pi)$. The constructed model M_π is a model for Δ_1 because of the fact that for any $n \in I\!N$, it verifies at least all but one of the abnormalities in $\{!p_k^n | k \in I\!N\}$ and because of lemma 1. It is a model for Δ_2 as well since from the construction of M_π follows that for no $j \in I\!N$, $M_\pi \models \check{\neg}!p_k^j$ and $M_\pi \models \check{\neg}!p_l^{j+1}$ with $(k,l) \notin R$. Therefore non of the members of Δ_2 will be falsified by M_π. Trivially, it is also a model for $\check{\neg}!\pi_i^1$, and therefore M_π is a model for Δ.

LEMMA 4. *The function $f\colon Paths(R,i) \to Mp(\Delta)$ where $f(\pi) = \{M | M \models \Delta \cup SerNor([1, lth(\pi)], \pi)\}$ and its inverse are both total functions. In other words, f defines a bijection between $Paths(R,i)$ and $Mp(\Delta)$.*

Proof. This is a consequence of lemmas 2 and 3 and the definition of the functions f and $SerNor$.

LEMMA 5. *"For all models M of Δ there is an $n \in I\!N$, such that for all $k \in I\!N$, $M \models !p_k^n$" (1) iff "for all minimally abnormal models M of $\Delta \cup \Delta_3$, there is an $m \in I\!N$ such that $M \not\models !r_m$" (2).*

Proof. I will first prove the left to right direction of the lemma and then the right to left direction.

($\Rightarrow$) Suppose (1) is true, but (2) is not. If (2) is false then there is a minimally abnormal model M for $\Delta \cup \Delta_3$, such that for all $m \in I\!N$, $M \models !r_m$. Because M is minimally abnormal, there is no $m \in I\!N$, such that there is a model M' of $\Delta \cup \Delta_3$ that verifies the same abnormalities as M in $\{!p_k^l | k,l \in I\!N\} \cup \{!r_k | k \in I\!N - \{m\}\}$ but $M' \not\models !r_m$. But there is such a natural number m. (1) warrants that for every model of Δ, and hence also every model of $\Delta \cup \Delta_3$, there is an $n \in I\!N$, such that for all $k \in I\!N$, $M \models !p_k^n$. Take this n to be the m we are looking for. Therefore, the only relevant premises $\{!p_k^m \check{\vee} !r_m | m \in I\!N\}$, can be made true without $!r_m$ having to be true. Since $!r_m$ does not occur in any other formula in $\Delta \cup \Delta_3$, changing only the thruth value for $!r_m$ in M, results in a model M' of $\Delta \cup \Delta_3$. We have derived a contradiction.

($\Leftarrow$) Suppose (2) is true, but (1) isn't. If (1) is false then there is a model of Δ, such that for all $n \in \mathbb{N}$, there is a $k \in \mathbb{N}$, $M \models \check{\neg}!p^n_k$. Therefore, this model M is also a model for $\Delta \cup \Delta_3$ when $M \models !r_n$ for all $n \in \mathbb{N}$ (this makes all the formulas of Δ_3 true). If (2) is true, then M is not minimally abnormal. Hence there is a model that makes a proper subset of the abnormalities of M true. This model makes a proper subset of the abnormalities of M in $\{!p^l_k | k, l \in \mathbb{N}\}$ true or at least one of the $!r_n$ false. The first is impossible because this would mean that for some n, more than one abnormality is falsified (c.f. lemma 1). The second part is impossible because, knowing that per $n \in \mathbb{N}$ one $!p^n_{f(n)}$ is false in M, if $!r_m$ is false in M for some $m \in \mathbb{N}$, then $!p^m_{f(m)} \check{\vee} !r_m$ could never be verified by M.

LEMMA 6. *For any adaptive logic in standard format one can prove that:*

(a) if B does not not have any subformulas in common with a set of formulas $\Gamma \cup \{A\}$, then: $\Gamma \cup \{A \check{\supset} B\} \models_{\mathbf{AL^m}} B$ iff $\Gamma \models_{\mathbf{AL^m}} A$, and

(b) where B is an abnormality, $\Gamma \models_{\mathbf{AL^m}} A \check{\vee} B$ iff $\Gamma \cup \{\check{\neg} B\} \models_{\mathbf{AL^m}} A$.

Let a model $M \in \mathcal{M}_\Gamma$ iff $M \models_{\mathbf{CLuN}} \Gamma$ and $M \in \mathcal{M}^m_\Gamma$ iff $M \models_{\mathbf{CLuN^m}} \Gamma$.

COROLLARY 1. $i \in WF(R)$ *iff* $\Gamma_R \models_{\mathbf{CLuN^m}} u_i$.

Proof.

$$i \in WF(R)$$

iff

$$\text{for all } \pi \in Paths(R, i), \text{ there is a } j \in \mathbb{N} \text{ such that } \pi(j) = 0$$

iff (in view of lemma 4)

$$\text{for all } \Upsilon \in Mp(\Delta), \text{ there is a } j \in \mathbb{N} \text{ such that } (f^{-1}(\Upsilon))(j) = 0 \tag{1}$$

iff

$$\text{for all } M \in \mathcal{M}_\Delta, \text{ there is a } j \in \mathbb{N} \text{ such that for all } n \; M \models !p^j_n \tag{2}$$

iff (with lemma 5)

$$\text{for all } M \in \mathcal{M}^m_{\Delta\cup\Delta_3},\ \text{there is a } j \in I\!N \text{ such that} \\ M \not\models !r_j \tag{3}$$

iff

$$\text{for all } M \in \mathcal{M}^m_{\Delta\cup\Delta_3\cup\Delta_4}\text{: } M \models q$$

iff (in view of the definition of semantic consequence for $\mathbf{CLuN^m}$)

$$\Delta \cup \Delta_3 \cup \Delta_4 \models_{\mathbf{CLuN^m}} q$$

iff (obtained by application of lemma 6b)

$$\Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4 \models_{\mathbf{CLuN^m}} !p_i^1 \vee q$$

iff (using lemma 6a)

$$\Gamma_R \models_{\mathbf{CLuN^m}} u_i$$

THEOREM 1. *The consequence set of propositional adaptive logics with infinite premise sets may be* Π^1_1*-complete.*

5. Proof theoretical approach

Of course the same results can be obtained by looking for syntactical consequences of the above defined Γ_R. I will not prove the crucial lemma, but illustrate what the object-proofs would look like for $\Gamma_R \vdash u_i$ when $i \in WF(R)$.

LEMMA 7.

$$\begin{aligned}
\phi \in \Phi(\Gamma_R) \textit{ iff } \phi = \ & \{!p_i^1 | i \in I\!N - \{\pi_1(1)\}\} \cup \{!r_1\} \cup \\
& \vdots \\
& \{!p_i^{lth(\pi_1)} | i \in I\!N - \{\pi_1(lth(\pi_1))\}\} \cup \{!r_{lth(\pi_1)}\} \cup \\
& \{!p_i^{lth(\pi_1)+1} | i \in I\!N\} \cup \\
& \{!p_i^{n_1+1} | i \in I\!N - \{\pi_2(1)\}\} \cup \{!r_{n_1+1}\} \cup \\
& \vdots \\
& \{p_i^{n_1+lth(\pi_2)} | i \in I\!N - \{\pi_2(lth(\pi_2))\}\} \cup \{!r_{n_1+lth(\pi_2)}\} \cup \\
& \{!p_i^{n_1+lth(\pi_2)+1} | i \in I\!N\} \cup \\
& \{!p_i^{n_2+1} | i \in I\!N - \{\pi_3(1)\}\} \cup \{!r_{n_2+1}\} \cup \\
& \vdots \\
& \{!p_i^{n_2+lth(\pi_3)} | i \in I\!N - \{\pi_3(lth(\pi_3))\}\} \cup \{!r_{n_2+lth(\pi_3)}\} \cup \\
& \{!p_i^{n_2+lth(\pi_3)+1} | i \in I\!N\} \cup \\
& \vdots,
\end{aligned}$$

where $\langle \pi_1, \pi_2, \ldots \rangle$ is some infinite enumeration of possibly empty finite paths of the graph or $\langle \pi_1, \ldots, \pi_m \rangle$ is some finite enumeration of possibly empty finite paths in the graph, except for π_m, which is an infinite path in the graph, and where $\langle n_1, n_2, n_3, \ldots \rangle = \langle lth(\pi_1)+1, lth(\pi_1)+lth(\pi_2)+2, lth(\pi_1)+lth(\pi_2)+lth(\pi_3)+3, \ldots \rangle$.

From this lemma follows that every path in the graph corresponds to one or more elements of $\Phi(\Gamma_R)$ (the $\phi \in \Phi(\Gamma_R)$ that have same π_1 – see lemma above). In other words, the paths of the graph define a partition on $\Phi(\Gamma_R)$.

The conclusions are only finally derived after an infinite proof (all premises have to be inserted). Suppose all premises in $\Delta_1 \cup \Delta_2 \cup \Delta_3$ are inserted in the proof on lines 1.1 to 1.ω. The order nor the position of these premises matters. They are alternated with the following lines 2.$i.j$.1 for any $i, j \in I\!N$ (from 2.1.1.1 to 2.$\omega.\omega$.9):

2.i.j.1	$q \vee !r_j$	Prem	$\emptyset$	
2.i.j.2	q	2.$i.j$.1; RC	$\{!r_j\}$	
2.i.j.3	$q \vee !p_i^1$	2.$i.j$.2; RU	$\{!r_j\}$	
2.i.j.4	$!p_i^1 \vee !p_j^1$	Prem	$\emptyset$	IF $i \neq j$
2.i.j.5	$!p_i^1$	2.$i.j$.4; RC	$\{!p_j^1\}$	IF $i \neq j$
2.i.j.6	$q \vee !p_i^1$	2.$i.j$.5; RU	$\{!p_j^1\}$	IF $i \neq j$
2.i.j.7	$(q \vee !p_i^1) \supset u_i$	Prem	$\emptyset$	
2.i.j.8	u_i	2.$i.j$.3, 2.$i.j$.7; RU	$\{!r_j\}$	
2.i.j.9	u_i	2.$i.j$.6, 2.$i.j$.7; RU	$\{!p_j^1\}$	IF $i \neq j$

Also for these lines the order of occurrence is of no importance as long as the order within the finite blocks 2.$i.j$.1 to 2.$i.j$.9 is respected. As a consequence, all the lines can easily be written in such a way that they all occur in an infinite list of lines. Call this list stage s of the proof. Using lemma 7, it is demonstrable that u_i is finally derived at this stage s of the proof iff $i \in WF(R)$.

6. Expressing the graph problem in predicative CLuN$^{\mathbf{m}}$

In a predicative language a finite premise set is sufficient to express the problem. Let Q be the weak system for first order arithmetic defined in (Boolos et al., 2002) and $T_R(x, y)$ a formula that represents the recursive relation R (which defines the edges of the aforementioned graphs) in Q (there is such a formula in view of theorem 16.16a in (Boolos et al., 2002)). The language has a sentential letter q, binary predicates P, S

and U, variables x, y and z, a constant 0, and a successor function $'$. It is provable that the problem $\Gamma'_R \vdash_{\mathbf{CLuN^m}} Ux$, where x is an arbitrary natural number, is Π^1_1-hard.

$$\Gamma'_R = Q \cup \Delta'_1 \cup \Delta'_2 \cup \Delta'_3 \cup \Delta'_4 \cup \Delta'_5$$

$$\Delta'_1 = \forall x \forall y \forall z (x = y \check{\vee} !Pxz \check{\vee} !Pyz)$$

$$\Delta'_2 = \forall x \forall y \forall z (T_R(x,y) \check{\vee} !Pxz \check{\vee} !Pyz')$$

$$\Delta'_3 = \forall x \forall y (!Sy \check{\vee} !Pxy)$$

$$\Delta'_4 = \forall x (q \check{\vee} !Sx)$$

$$\Delta'_5 = \forall x ((q \check{\vee} !Px0') \supset Ux)$$

7. Conclusion and philosophical comments

I have proved that the set $WF(R)$ of nodes in which only finite paths start is reducible to the $\mathbf{CLuN^m}$-consequence set of a propositional infinite but recursive premise set as well as to the $\mathbf{CLuN^m}$-consequence set of a predicative finite set. So, Horsten and Welch's 'proof' that propositional $\mathbf{CLuN^m}$ with infinite premise sets is maximally Σ^0_3-complex is mistaken.

The results in this paper can immediately be generalized to all adaptive logics that have an object language in which classical disjunction is present or definable and in which the logic's abnormalities can be expressed[3]. In the metatheoretic proofs of this paper, I only refer to general properties of adaptive logics.

Adaptive logics that use the Minimal Abnormality strategy turn out to be even more complex than Horsten and Welch thought. Although they would probably see this result as an even bigger problem for adaptive logics, I insist that a very complex consequence set is not necessarily problematic for the logic. In (Batens et al., 2007) Horsten and Welch's negative reflections on their complexity results are rebutted. However, there is more. The great complexity of the set of finally derivable consequences can be seen as a positive property, as long as the proofs themselves are not complex.

The type of logics under consideration are not candidates for the standard of deduction, but determine the formally correct reasoning steps within a certain problem solving context. Given that perspective,

[3] There is also a $\supset$ in Δ_5. But since it is clear from section 5 that $Cn_{\mathbf{CLuN^m}}(\Gamma_R) \cap \{u_i | i \in \mathbb{N}\} = Cn_{\mathbf{CLuN^m}}(\Gamma''_R) \cap \{u_i | i \in \mathbb{N}\}$, where $\Gamma''_R = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \{u_i \check{\vee} !p^1_j | i, j \in \mathbb{N}; i \neq j\} \cup \{u_i \check{\vee} !r_n | i, n \in \mathbb{N}\}$, Δ_4 and Δ_5 can be replaced by sets of $\supset$-free formulas

a logic can cope with a problem if there exists a function that translates the relation between problem and solution into the logic's actual stable consequence relation between the premises, which correspond to the problem, and a conclusion, which corresponds to a solution. If the complexity of such a logic is defined as the maximal complexity of its stable consequence set (as Horsten and Welch do), it is clear that a less complex logic must fail to handle some complex problems. In that sense, one cannot blame the designer of the logic if his logic is as complex as the problems for which he wants to explicate the reasoning processes.

Apparently, dynamic reasoning forms enable reasoning towards the solutions of far more complex problems than usual logics. Logics with a consequence set that is less than Π^1_1-complete, cannot explicate the reasoning processes for problems for which the solution is only expressible using a universal quantifier over a non-enumerable amount of objects. The example I gave copes with a problem that refers to the possibly non-enumerable amount of paths of a recursive graph. It is a technical example, but one can easily devise other game theoretic and graph related problems, that can be handled using adaptive logic. And of course, the task of interpreting the premises as normally as possible itself is sensible and has the same complexity. These problems are not expressible in terms of first order classical logic nor are they expressible in terms of the vast majority of other first order logics (they all are less than Π^1_1-complex).

Although one cannot develop logics that are less complex than the reasoning processes one wants to explicate, it is important that the actual proofs provide the user with insights in the problem that he tries to solve. Therefore, they should not be hard to construct or to verify. In adaptive logics these proofs are constructed and verified using the definition of derivability at a stage, which is essentially not more complex then the monotonic lower limit logic. Final derivability defines the stable goal of the reasoning, even if, generally, this goal is not reachable in finite time. One does not need this goal to obtain the next line in the proof or to check whether a concrete proof is correct. Therefore, if one wants to attack adaptive logics on their complexity, one should really focus on the complexity of derivability at a stage.

I have explained that the complexity of the stable derivability notion cannot be used as an argument against adaptive logics. Furthermore, it is proven that exactly its Π^1_1-complexity allows the adaptive logic to express problems that are, if one uses classical logic, only expressible in second order languages. Yes, adaptive logics are complex, they are even more complex than Horsten and Welch think, but it is not hard to see that this is an advantage rather than a disadvantage.

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