“That will do”: logics of deontic necessity and sufficiency*

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Abstract

We study a logic for deontic necessity and sufficiency (often interpreted as obligation, resp. strong permission), as originally proposed in [28]. Building on earlier work in modal logic, we provide a sound and complete axiomatization for it, consider some standard extensions, and study other important properties. After that, we compare this logic to the logic of “obligation as weakest permission” from [3].

1 Intro: Deontic Necessity and Sufficiency

Deontic Necessity and Sufficiency Following Anderson [1], let us use the term *deontic logic* for any normal modal logic in which the truth axiom, $\Box p \supset p$, is invalid. The most well-known example of such a logic is Standard Deontic Logic (henceforth **SDL**).[1] **SDL** is often presented as “the logic of ought, permitted, and forbidden”, which are themselves taken to be interdefinable.

In this paper, we will start from a more abstract, and perhaps less ambitious interpretation of (modal) deontic logics, viz. as logics of *deontic necessity* – logics of the necessary conditions for the satisfaction of the obligations that hold in a given discursive context (defined in terms of a speaker, addressee, normative system, ...). According to this interpretation, where φ is a formula, $\mathsf{N}\varphi$ is read as: “necessarily, if all your obligations are satisfied, then φ is the case”. This view is far from new: it can be seen as the basic intuition behind the well-known Andersonian-Kangerian reduction of **SDL** to **K**, which has been generalized to a broad class of (normal) modal deontic logics.[2]

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[1] **SDL** is the extension of the minimal modal logic **K** with the axiom (D): $\Box\varphi \supset \neg\Box\neg\varphi$.

[2] See [4] for a general introduction to this topic; see also [19] for a more recent discussion and formal results.

This immediately gives rise to the question: what about sufficient conditions for the satisfaction of our all-things-considered obligations – what about *deontic sufficiency*? Can we also define an operator for this notion in the object language, and what should be its formal properties? In more prosaic terms: what is the logic of expressions such as "φ will do" or "it suffices that φ", when interpreted as assertions about the norms that apply?

The general notion of modal sufficiency – as a counterpart to the necessity that is modeled by normal modal logics – has been studied in various works.[3] The more specific concept of deontic sufficiency under scrutiny here displays strong links to what is often called "strong permission" or "free choice permission" in the deontic logic literature.[4] In fact, deontic or normative sufficiency is sometimes thought to be the very core of this concept, which in turn explains the logical properties that are often attributed to it.[5] However, just as is the case with deontic necessity and obligation, interpreting deontic sufficiency as a kind of permission gives rise to some well-known paradoxes.[6] This is, in our opinion, not a reason to deny every link between deontic sufficiency and (strong) permission entirely – just as we would not deny that there is an important link between obligation and deontic necessity. Rather, one has to distinguish various senses of obligation and permission, and hence to pinpoint exactly under what interpretation these concepts have certain properties.[7]

This paper The starting point of the present paper is a minimal bi-modal logic for deontic necessity and what we call deliberative (or practical) necessity, which we extend with an operator for deontic sufficiency. This logic was proposed by van Benthem in the late 1970s [28].[8] Our main contribution consists in studying its formal properties, drawing on earlier work concerning modal logics of (necessity and) sufficiency. In addition, we provide a detailed comparison of this logic and the logic of "obligation as weakest permission" from [3].

There are various motivations for this work. First of all, it is interesting from the perspective of formal semantics of natural language expressions: given that normative claims about sufficiency are pervasive in everyday discourse and in

[3] See Section 3.1 and the references provided there.

[4] See [2, footnote 1] for a brief history of these concepts; a key reference is [18]. In more recent work, Asher and Bonevac argue that free choice permission should be modeled in terms of a (weak) default conditional [5], in order to avoid certain paradoxes. We return to the notion of strong permission in Section 2. However, our aim here is not to propose some (let alone *the right*) logic of strong permission, but rather to shed new light on this and similar notions from the viewpoint of modal logics for sufficiency.

[5] See e.g. [25, 3].

[6] See, in particular, [5] where a number of such paradoxes are listed for strong permission. See also [2] for a more recent discussion of these paradoxes.

[7] Thus, our overall approach to deontic logic is pluralistic and pragmatic, following Hansson's recent work [15]: deontic logic is a formal tool that allows us to disambiguate between various interpretations of natural language expressions that concern the normative realm – it's aim is *not* to develop *the* logic of "ought", "permitted", and "forbidden".

[8] Even though it received quite a few citations, the exact history of this brief paper has turned out to be elusive. Most likely, this is the extended abstract of a presentation at the DLMPS conference held in Hannover (1979) (van Benthem, personal communication).

ethical theories, it is worthwhile developing exact and unified formal accounts of them. Second, if we enrich the language of **SDL** with an operator for deontic sufficiency, we can express and distinguish various well-known concepts of obligation and permission at the object level. This in turn gives us a unifying framework within which one can study the interaction between those notions. Third, if we want to understand an intelligent agent's ability to check whether all the norms that apply to a given situation have been satisfied, it is essential that this agent can handle claims concerning deontic sufficiency.[9] And finally, at a more technical level, the logics to be introduced illustrate several interesting problems that concern the properties and axiomatic characterization of modal logics, and thereby show both their virtues and limitations.

The paper is structured as follows. In the next section, we introduce the language of the logics and their informal interpretation, which will allow us to explain the motivation for this paper in more detail. Sections 3 and 4 contain the technical core of the paper: here we provide a formal semantics and axiomatization of the base logic and some of its extensions and study their properties. After that, we compare **DNS** to the logic of "obligation as weakest permission" from [3] (Section 5). We end the paper with some concluding remarks and prospects for future research.

All proofs of theorems can be found in the appendix.

2 Informal interpretation of the language

Formal Language The formal language $\mathcal{L}$ is built up from a set $\mathcal{S}$ of sentential variables $p, q, \ldots$, (classical) connectives $\top, \bot, \neg, \vee, \wedge, \supset, \equiv$ and three modal operators: N, S, and $\Box$. We treat $\bot, \neg, \vee$ as primitive; the other connectives are defined from them in the usual way. Let $\mathcal{W}$ be the set of formulas in $\mathcal{L}$. We use $\varphi, \psi, \ldots$ as metavariables for members of $\mathcal{W}$ and $\Delta, \Gamma, \ldots$ for subsets of $\mathcal{W}$. The set of formulas that do not contain the operator S is denoted by $\mathcal{W}_{\overline{\mathsf{S}}}$.

Informal Interpretation of the Operators As explained in the introduction, we shall read N and S as operators for deontic necessity, resp. sufficiency. Hence, $\mathsf{N}\varphi$ means that φ is a necessary condition for the satisfaction of all obligations, and $\mathsf{S}\varphi$ means that φ is a sufficient condition for the satisfaction of all obligations. Moreover, we interpret N and S as operators of *relative* or *practical* deontic necessity and sufficiency, respectively. This requires some clarification.

We assume that whenever we are in a certain context, there is a range $\mathcal{A}$ of deliberative[10] alternatives – thought of as possible worlds – which are relevant to that context.[11] The deontic notions we are interested in here are always

[9] See e.g. [2, pp. 29-30], where it is argued that strong permission allows one to give a positive test for the legality of an action token.

[10] This term is borrowed from Thomason [27], who speaks of a deliberative ought in the context of deontic temporal logic.

[11] The (important, since more realistic) extension of our framework to cases where the deliberative alternatives consist of arbitrary *sets* of worlds is investigated in [31, 30].

relative to this $\mathcal{A}$. In other words, we want to speak about the alternatives in $\mathcal{A}$ that are acceptable. Let us use $\mathcal{O}$ to refer to the subset of $\mathcal{A}$ which consists exactly of those deontically acceptable alternatives. N is then used to express necessary conditions for membership in $\mathcal{O}$, and S to express sufficient conditions for membership in $\mathcal{O}$ *relative to* $\mathcal{A}$. In other words, $\mathsf{S}\varphi$ means that whenever something is an alternative, and whenever it makes φ true, then it is deontically acceptable.[12]

The $\Box$-operator is used to speak about properties of *all* the deliberative alternatives – in other words, to quantify over $\mathcal{A}$. This has four important consequences. First, we can distinguish between "trivial" necessary and sufficient conditions for the obligations to be satisfied – expressed by $\mathsf{N}\varphi \wedge \Box\varphi$ and $\mathsf{S}\varphi \wedge \Box\neg\varphi$ respectively – and more "significant" conditions – expressed by $\mathsf{N}\varphi \wedge \Diamond\neg\varphi$ and $\mathsf{S}\varphi \wedge \Diamond\varphi$.

Second, $\Box$ allows us to model the interaction between deontic necessity and sufficiency, and deliberative – or, as Hansson [15] calls it, "practical" – necessity. For instance, the validity of $\mathsf{N}\varphi \supset (\Box(\varphi \supset \psi) \supset \mathsf{N}\psi)$ indicates that whenever φ is deontically necessary, and one cannot obtain φ without also making ψ true, then also ψ is deontically necessary.[13] Likewise, if φ is sufficient ($\mathsf{S}\varphi$), and if for all deliberative alternatives, ψ implies that φ ($\Box(\psi \supset \varphi)$), then also ψ is sufficient ($\mathsf{S}\psi$).[14]

Third, the operator $\Box$ allows us to express the link between (deontic) sufficiency and deontic necessity in the object language of the logic by a single principle:

$$(\star) \qquad (\mathsf{N}\varphi \wedge \mathsf{S}\psi) \supset \Box(\psi \supset \varphi)$$

This axiom implies e.g. that if φ is necessary ($\mathsf{N}\varphi$) and we know that there is a deliberative alternative in which ψ holds but φ fails ($\Diamond(\psi \wedge \neg\varphi)$), then ψ is *not* sufficient ($\neg\mathsf{S}\psi$).

Last but not least, the $\Box$-operator allows us to encode properties of the deliberative alternatives themselves. This requires some clarification. In the **SDL**-semantics, the set of deliberative alternatives is in a sense implicit, i.e. it consists of the entire domain of the model. Making this explicit would imply that $\Box$ is a universal modality [14] and yield all the **S5**-properties for it. We discuss this option in some detail in Section 4.2

One may also take the set of deliberative alternatives to be the set of those worlds that are physically possible from the viewpoint of the present world, which could justify the adoption of **S4** for $\Box$. As shown in Section 4.1, the addition of a suitable collection of axioms to our basic logic allows one to characterize such cases. However, we will start from the assumption that $\Box$ is just a **K**-modality.[15]

[12] We leave open the possibility that $\mathcal{O}$ is empty, so that in our base logic, N is weaker than the traditional O-operator of **SDL**. We return to this point in Section 4.1.

[13] See [15, p. 15]: "The derived norms should include not only that which follows logically from the basic (or explicit) norms but also that which follows by practical necessity."

[14] Note that we omit the adverb "deontically"; we will only do so when it is clear from the context that deontic necessity or sufficiency is at stake.

[15] One interesting application for such a weak reading of $\Box$ is the following: consider an

Example Let us now illustrate the use of the three operators by means of an informal example. Suppose it's Friday morning and John and Roy are contemplating what they will do over the weekend. John notes that the fridge is nearly empty, and hence that they must go shopping at some point. Also, Roy reminds John of their long-standing promise to visit their parents. Now, given certain practical considerations, they cannot visit both John's parents and Roy's parents in one weekend, but they should at least try to visit either Roy's parents or John's parents. Finally, John also remarks that the lane has to be mown this weekend. After giving it some more thought, they both settle on the following plan: Roy will mow the lane while John does the shopping on Saturday morning; in the afternoon they visit John's parents. They agree that if they follow this plan, they can still do whatever they like on Sunday and rest assured that all their duties have been fulfilled.

Of course, not all the information in this short story can be captured by means of the abstract language we have just introduced. For instance, we cannot speak about temporal aspects, involved agents and actions, possible preferences of those agents, nor do we have the machinery to deal with the dynamics that is involved in the reasoning of John and Roy. What we want to focus on here is those parts of the story that *can* be modeled in $\mathcal{L}$, and the gain in expressive power this shows in comparison to logics like **SDL**.

The set of alternatives $\mathcal{A}$ in the above story consists of a (possibly infinite) set of ways in which Roy and John could organize their weekend. The deontic necessities can be represented as follows: "shopping must be done" ($\mathsf{N}s$), "either John's parents or Roy's parents must be visited" $\mathsf{N}(v_j \vee v_r)$, and "the lane must be mown" ($\mathsf{N}m$). The final sentence of the story indicates that Roy and John agree that it is sufficient that shopping is done, John's parents are visited, and the lane is mown: $\mathsf{S}(s \wedge v_j \wedge m)$. In fact, it indicates that any state of affairs that makes these three propositions true, is "deontically ok".

Of course, implicitly, much more information about deontic necessity and sufficiency is available in this case. For instance, it is usually assumed that one can mow one's lane without visiting one's parents: $\Diamond(m \wedge \neg(v_j \vee v_r))$, and that one can do both without shopping: $\Diamond(m \wedge (v_j \vee v_r) \wedge \neg s)$. And hence, given these background assumptions, our rational agents know that just mowing the lane, or just mowing the lane and visiting John's parents will not do: $\neg\mathsf{S}m$, $\neg\mathsf{S}(m \wedge v_j)$.

In **SDL**, we can only express deontic necessities, and hence we are never able to pinpoint exactly what has to be done in order to make sure everything is (deontically) "allright".[16] In the present case, this means that we cannot express the conditions under which a given plan for the weekend is a good plan. All we can do is point out that a given sketch of a plan does not guarantee the satisfaction of all obligations; e.g. we can say that $\mathsf{N}s$ and $\Diamond(m \wedge \neg s)$. But there

indeterministic, discrete temporal order on the set of worlds, and let the set of deliberative alternatives of some state consist of all its possible "next" states. The operators N and S then allow us to speak about those next states that are deontically acceptable.

[16] As we will show below, the logic **DNS** displays a proper gain in expressive power to **SDL**, also at a more technical level.

is no single formula which allows us to express that $m \wedge (v_j \vee v_r) \wedge s$ will do. This informal claim is made exact by Observation 3 in Section 4.4.

Different notions of obligation and permission Once we extend the language of **SDL** with an operator for deontic sufficiency, we can distinguish between different concepts of obligation and permission as they occur in natural language. Let us start with the notion of obligation. On the one hand, this term may refer to one of many deontic necessities that an agent faces in a given situation: e.g. she may have to ensure that she gets to the office in time, but also that the children have their sandwiches before they go to school. Whatever the logic of such obligations may be, it seems rather unnatural to assume that in the end, all of them are just equivalent expressions of one and the same thing.

On the other hand, in some specific cases, the term "obligation" can refer to a single thing. For instance, in [3], Singer's example of a kid drowning in a pond is considered. The authors write that "What the agent ought to do here is be moral, i.e. save the child in a way that complies with all other requirements of morality." [3, p. 7]. Only that which falls under this (admittedly, fairly abstract) description would then count as *the* obligation of the agent.[17]

Whereas **SDL** explicates one notion of obligation in plural, other systems from the deontic logic literature – in particular the one from [3] – are rather based on the second, singular concept of obligation.[18] In $\mathcal{L}$, we can express both concepts at the object-level: obligations in plural can be represented by N, whereas a unique obligation can be expressed by the combination of N and S.

The same applies to the concept of permission: at least two notions of permission can be captured within $\mathcal{L}$. On the one hand, there is the notion of permission from **SDL** (often called "weak permission"), which is just the dual of deontic necessity. That is, something is permitted if and only if there is at least one acceptable alternative which makes it true.

On the other hand, there is the notion of "strong permission" which is captured by our S-operator. That is, when φ is strongly permitted, this means that any means of making φ true is "deontically allright". But this is exactly how we interpret deontic sufficiency, as explained above.

3 DNS: formal aspects

In this section, we define an exact formal semantics for $\Box$, N and S. This gives us the base logic **DNS**. As mentioned in the introduction, this logic was originally proposed by van Benthem in [28]. Here we study the main technical properties of

[17]Note that this distinction is orthogonal to the usual distinction in terms of the sources or grounds of an obligation: one may still have several obligations that derive from one basic principle – such as "you have to be on time for the meeting, and be on time for dinner" and given various normative sources, one may still claim that there is a unique obligation (call it the "overriding obligation") that has to be fulfilled.

[18]We return to the work of Anglberger, Roy and Gratzl in Section 5, showing how this logic relates to our approach.

this logic: axiomatizations, finite model property, expressive power, Anderson-Kangerian reductions, and interpolation.

3.1 Semantics

We presuppose that a context coincides with a possible world w, and that at every such w, the set of deliberative alternatives to w is a set of possible worlds. This gives us the following definition:

Definition 1. *A* **DNS**-frame *is a triple* $F = \langle W, R_\Box, R_\mathsf{N}, v\rangle$*, where:*

(i) W *is a non-empty set, the* domain *of* F
(ii) $R_\Box \subseteq W \times W$
(iii) $R_\mathsf{N} \subseteq R_\Box$

A **DNS**-model *is a quadruple* $M = \langle W, R_\Box, R_\mathsf{N}, v\rangle$*, where* $\langle W, R_\Box, R_\mathsf{N}\rangle$ *is a* **DNS***-frame and* $v : \mathcal{S} \to \wp(W)$ *is a valuation function.*

Here, $R_\Box(w)$ represents the set of deliberative alternatives at w, and $R_\mathsf{N}(w)$ consists exactly of those alternatives that are deontically acceptable.

Definition 2. *Where* $M = \langle W, R_\Box, R_\mathsf{N}, v\rangle$ *is a* **DNS***-model and* $w \in W$*,*

$M, w \not\models \bot$
$M, w \models p$ *iff* $w \in v(p)$
$M, w \models \neg\varphi$ *iff* $M, w \not\models \varphi$
$M, w \models \varphi \vee \psi$ *iff* $M, w \models \varphi$ *or* $M, w \models \psi$
$M, w \models \Box\varphi$ *iff* $M, w' \models \varphi$ *for all* $w' \in R_\Box(w)$
$M, w \models \mathsf{N}\varphi$ *iff* $M, w' \models \varphi$ *for all* $w' \in R_\mathsf{N}(w)$
$M, w \models \mathsf{S}\varphi$ *iff for all* $w' \in R_\Box(w)$ *such that* $M, w' \models \varphi$*,* $w' \in R_\mathsf{N}(w)$

Local/global validity in a model, local/global frame validity, and semantic consequence (denoted by $\Vdash_{\mathbf{DNS}}$) are defined as usual.

Note that in view of the above semantics, N and $\Box$ are normal modalities. This means that e.g. N is not suitable when we want to express conflicting obligations in a non-trivial way — even if **DNS** does not trivialize them, they will lead to deontic explosion, i.e. the property that everything is obligatory. In order to accommodate such conflicts in a sensible way, one may use non-standard semantics (e.g. multi-relational or neighbourhood-semantics) for N, or one may weaken the underlying logic of the connectives – see e.g. [11] for a recent overview of such strategies. However, in the present paper we focus on the traditional normal modal logic case, leaving such generalizations for future work.

One can rephrase the present semantics in terms of a more regular Kripke-semantics as well.[19] Consider a **DNS**-model $M = \langle W, R_\Box, R_\mathsf{N}, v\rangle$. Define the

[19] This is not too surprising: it is well-known that the logic of modal sufficiency *in the absolute sense* coincides with a normal bi-modal logic – see e.g. [16, 9, 24]. Here, we just show how this correspondence can be generalized to the case of sufficiency relative to a (variable) set of alternatives, as given by $R_\Box(w)$.

accessibility relation $R_{\overline{\mathsf{N}}} = R_{\Box} \setminus R_{\mathsf{N}}$. We have: $M, w \models \mathsf{S}\varphi$ iff for all $w' \in R_{\Box}(w)$ such that $M, w' \models \varphi$, $w' \in R_{\mathsf{N}}(\varphi)$ iff for all $w' \in R_{\overline{\mathsf{N}}}(w)$, $M, w' \models \neg\varphi$. Hence, instead of using a non-standard semantic clause for S, we can just use the standard clause:

$$M, w \models \overline{\mathsf{N}}\varphi \text{ iff for all } w' \in R_{\overline{\mathsf{N}}}(w),\ M, w' \models \varphi$$

and define $\mathsf{S}\varphi$ by $\overline{\mathsf{N}}\neg\varphi$.

As this alternative semantics highlights, N and $\overline{\mathsf{N}}$ ($\mathsf{S}\neg$) are perfectly symmetric in **DNS**. Whereas the former allows us to quantify over all the states in $R_{\mathsf{N}}(w)$, the latter allows us to speak about the states in $R_{\Box}(w) \setminus R_{\mathsf{N}}(w)$. As a result, we obtain the following:

Observation 1. *Suppose that $\Gamma \Vdash_{\mathbf{DNS}} \varphi$. Let Γ', φ' be the result of replacing in Γ, φ each occurrence of N with $\mathsf{S}\neg$ and each occurrence of S with $\mathsf{N}\neg$. Then $\Gamma' \Vdash_{\mathbf{DNS}} \varphi'$.*

Also, this alternative semantics at once makes it clear that one may treat $\Box$ as a defined operator in **DNS**: simply take R_{N} and $R_{\overline{\mathsf{N}}}$ as primitive, assume that $R_{\mathsf{N}} \cap R_{\overline{\mathsf{N}}} = \emptyset$, and put $\Box\varphi =_{\mathsf{df}} \mathsf{N}\varphi \wedge \overline{\mathsf{N}}\varphi$. Our main reason for not doing this is that it puts the cart before the horse, conceptually speaking: only once we specify what the deliberative alternatives are, can we speak about deontic sufficiency (or equivalently, about necessity relative to $R_{\overline{\mathsf{N}}}$). Still, it is important to keep this alternative formulation in mind in what follows.

3.2 Axiomatization

The fragment of **DNS** without the S-operator is just a normal bi-modal version of the minimal modal logic **K**, with the additional bridging axiom $\Box\varphi \supset \mathsf{N}\varphi$. The interesting part is of course the S-operator. To see why this operator is not normal, it suffices to note that $\mathsf{S}\top$ is invalid in any interesting **DNS**-model (i.e. in any **DNS**-model for which $R_{\mathsf{N}} \subset R_{\Box}$). Nevertheless, one can give a sound and complete, Hilbert-style axiomatization of **DNS**.

Definition 3. *The set of* **DNS***-axioms is obtained by closing the set of all instances of the following set of axiom schemas under modus ponens (MP) and necessitation for $\Box$ (NEC):*

(CL)	*a set of axioms schemas that are complete for Classical Logic (***CL***)*
($K_{\Box}$)	$\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi)$
(K_{N})	$\mathsf{N}(\varphi \supset \psi) \supset (\mathsf{N}\varphi \supset \mathsf{N}\psi)$
(B)	$\Box\varphi \supset \mathsf{N}\varphi$
(NS)	$\mathsf{N}\varphi \supset (\mathsf{S}\psi \supset \Box(\psi \supset \varphi))$
(UC)	$(\mathsf{S}\psi \wedge \mathsf{S}\varphi) \supset \mathsf{S}(\psi \vee \varphi)$
(OR)	$\mathsf{S}(\varphi \vee \psi) \supset (\mathsf{S}\varphi \wedge \mathsf{S}\psi)$
(EQ_{S})	$\Box(\varphi \equiv \psi) \supset (\mathsf{S}\varphi \equiv \mathsf{S}\psi)$
($Triv_{\mathsf{S}}$)	$\mathsf{S}\bot$

$\Gamma \vdash_{\mathbf{DNS}} \varphi$ *iff there are* $\psi_1, \ldots, \psi_n \in \Gamma$ *such that* $(\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi$ *is a* **DNS**-*axiom.*

We briefly comment on each of the axioms that are characteristic for the S-operator. (NS) was already introduced in Section 2; it just states that whatever is deontically sufficient always (practically) entails whatever is deontically necessary. (UC) states that if two propositions are both sufficient, then so is their disjunction – we will pay closer attention to this axiom in Section 5. Its converse, (OR) is usually known as the principle of "free choice permission". In the presence of replacement of equivalents for S – which is here axiom (EQ$_\mathsf{S}$) –, (OR) is equivalent to the axiom $(\mathsf{S}\varphi \wedge \Box(\psi \supset \varphi)) \supset \mathsf{S}\psi$, stating that whenever something is sufficient, then anything that (practically) entails it is also sufficient. Finally, (Triv$_\mathsf{S}$) states that any trivial proposition is always sufficient: since there are no $\bot$-states, every $\bot$-state is (vacuously) acceptable.

Let us list a number of properties of $\vdash_{\mathbf{DNS}}$ to illustrate the power of these axioms:

Observation 2. *For all* φ, ψ*:*[20]

1. $\mathsf{N}\varphi, \Diamond(\psi \wedge \neg\varphi) \vdash_{\mathbf{DNS}} \neg\mathsf{S}\psi$
2. $\mathsf{S}\varphi \vdash_{\mathbf{DNS}} \mathsf{S}(\varphi \wedge \psi)$
3. $\Box\varphi \vdash_{\mathbf{DNS}} \mathsf{S}\neg\varphi$
4. $\mathsf{S}\varphi \wedge \mathsf{N}\varphi, \mathsf{S}\psi \wedge \mathsf{N}\psi \vdash_{\mathbf{DNS}} \Box(\varphi \equiv \psi)$
5. $\mathsf{S}\varphi \wedge \Diamond\varphi \vdash_{\mathbf{DNS}} \neg\mathsf{N}\neg\varphi$
6. $\mathsf{S}\varphi \wedge \Box(\psi \supset \varphi) \vdash \mathsf{S}\psi$

Item 1 follows from (NS) by classical logic and the definition of $\Diamond$. Item 2 follows from (OR) and (EQ$_\mathsf{S}$), in view of the theorem $\Box(\varphi \equiv (\varphi \vee (\varphi \wedge \psi)))$. Item 3 follows from (Triv$_\mathsf{S}$), (EQ$_\mathsf{S}$), and **S5**-properties as follows: suppose $\Box\varphi$. Hence, $\Box(\neg\varphi \equiv \bot)$. By (Triv$_\mathsf{S}$), and (EQ$_\mathsf{S}$), this yields $\mathsf{S}\neg\varphi$.

Items 4 and 5 follow from **K**-properties of $\Box$ and (NS). For item 4, suppose the antecedent holds. By $\mathsf{S}\varphi$ and $\mathsf{N}\psi$, using (NS), we can derive $\Box(\psi \supset \varphi)$. Likewise, from $\mathsf{S}\psi$ and $\mathsf{S}\varphi$, we can derive $\Box(\varphi \supset \psi)$ and hence we get $\Box(\varphi \equiv \psi)$. For item 5, suppose $\mathsf{S}\varphi$ and $\mathsf{N}\neg\varphi$. Then by (NS), we get $\Box(\varphi \supset \neg\varphi)$ and hence $\Box\neg\varphi$. Finally, item 6 follows from (OR) and (EQ$_\mathsf{S}$).

The following is proved in the appendix:

Theorem 1 (Soundness and Strong Completeness)**.** $\Gamma \vdash_{\mathbf{DNS}} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{DNS}} \varphi$.

The proof of soundness is a matter of routine; it suffices to check that all the **DNS**-axioms are (globally) valid in every **DNS**-model. For the proof of (strong) completeness, we need to adapt the standard technique of canonical models. We make two disjoint copies of the usual canonical model M^c, and then "merge" these copies to form a new model M^+ for which we can prove the Truth Lemma. This can moreover be done in different ways, yielding completeness results for

[20]We skip set brackets around the premises to simplify notation.

certain extensions of **DNS** in turn (see Section 4). See Appendix A for the details.[21]

One specific variant of this copy-and-merge method already occurred in earlier work by Passy and co-workers, where they call it the "important construction" [24, 10, 9]. The main difference with the present paper is that their $\Box$ is a universal modality, whereas here it is just a **K**-modality. In addition, we consider other variants of the same technique and establish a generic truth lemma for all such constructions (see Theorem 9 in Appendix A).

3.3 Van Benthem's original characterization of DNS

Van Benthem [28] also proposes an axiomatic characterization of **DNS**, or, as he calls it, $\mathbf{K_{D,m}}$. However, he builds up his axiomatization in a more stepwise fashion, ending up with slightly different axioms. In this section, we briefly show that the result is equivalent to our characterization in Definition 3, and hence sound and complete w.r.t. the **DNS**-semantics.[22]

Van Benthem's axiomatization goes as follows:

(i) all theorems of **CL**
(ii) **K** for $\Box$ and N
(iii) our axiom schemas (B), (NS), (UC), and (Triv$_{\mathsf{S}}$)
(iv) the axiom schema $\Box\neg\varphi \supset \mathsf{S}\varphi$
(v) the inference rule: if $\vdash \varphi \supset \psi$, then $\vdash \mathsf{S}\psi \supset \mathsf{S}\varphi$

We show first that (iv) and (v) are derivable in **DNS**, and hence sound. (iv) follows since $\Box\neg\varphi \vdash \Box(\varphi \equiv \bot)$ (which holds by **K**-properties of $\Box$) and by axioms (Triv$_{\mathsf{S}}$) and (EQ$_{\mathsf{S}}$). (v) follows by from (OR), (EQ$_{\mathsf{S}}$), and (NEC) for $\Box$.[23] Note that the axiom (Triv$_{\mathsf{S}}$) is redundant in view of van Benthem's axiom (iv) and (NEC) for $\Box$.

To prove completeness of van Benthem's axiomatization, we need to show that it yields both (OR) and (EQ$_{\mathsf{S}}$). Let us start with the former: suppose $\mathsf{S}(\varphi \vee \psi)$. By **CL**-properties, $\vdash \varphi \supset (\varphi \vee \psi)$ and $\vdash \psi \supset (\varphi \vee \psi)$. Hence, we can use (v) to derive both $\mathsf{S}\varphi$ and $\mathsf{S}\psi$, and hence also $\mathsf{S}\varphi \wedge \mathsf{S}\psi$. To prove (EQ$_{\mathsf{S}}$), suppose $\Box(\varphi \equiv \psi)$ and $S\varphi$. By **K**-properties, we can derive $\Box\neg(\psi \wedge \neg\varphi)$. Hence by (iv), $\mathsf{S}(\psi \wedge \neg\varphi)$. By (UC), we get $\mathsf{S}(\varphi \vee (\psi \wedge \neg\varphi))$. By **CL**-properties, $\vdash \psi \supset (\varphi \vee (\psi \wedge \neg\varphi))$, and hence by rule (v), we can derive $\mathsf{S}\psi$. By a symmetric argument in φ and ψ, we can derive $\mathsf{S}\varphi$ from $\Box(\varphi \equiv \psi)$ and $\mathsf{S}\psi$.

Thus, van Benthem's logic is essentially the same as ours. Curiously, this paper has gone largely unnoticed in the deontic logic literature. Still, the link between modal sufficiency and permission has received some attention in other

[21] Roughly speaking, M^c can be seen as a bounded morphic image of M^+, in the sense of [6]. However, spelling out this link in its entirety is a rather tedious task and does not alter the main ideas behind the technique.

[22] Van Benthem does not prove completeness in his own paper. Earlier on he claims about the S-fragment of his logic that one can obtain a completeness proof "using the Henkin method of Lemmon and Scott as a heuristic device".

[23] That is, suppose $\vdash \varphi \supset \psi$. Hence, $\vdash (\varphi \vee \psi) \equiv \psi$. By (NEC), $\vdash \Box((\varphi \vee \psi) \equiv \psi)$. Hence, by (EQ$_{\mathsf{S}}$), $\vdash \mathsf{S}\psi \supset \mathsf{S}(\psi \vee \varphi)$. Applying **CL**-properties and (OR), we have $\vdash \mathsf{S}\psi \supset \mathsf{S}\varphi$.

papers. For instance, Humberstone [16] remarks that his so-called "inaccessibility operator" – which corresponds to an absolute version of our operator $\overline{\mathsf{N}}$ above – has applications in the context of deontic logic, where it can be used to model strong permission. He refers (as van Benthem does, independently) to the validity of (OR) for strong permission as a motivation for this idea. In more recent work, Roy et al. propose a logic of "obligation as weakest permission" [3]. The relation between their logic and **DNS** is delicate: we will discuss it at length in Section 5.

3.4 Further meta-results for DNS

Beside soundness and strong completeness, a number of standard results of normal (mono)modal logics can be generalized to **DNS**. We list them here; their proof is given in Section 4.

(i) **DNS** has the finite model property and is decidable. See Section 4.3.

(ii) S cannot be defined in the S-less fragment of **DNS** – see Section 4.4.

(iii) **DNS** can be reduced to $\mathbf{K_d}$, i.e. the minimal modal logic over a language extended with a propositional constant d – see Section 4.5.

(iv) **DNS** satisfies interpolation – see Section 4.6.

4 Extensions of DNS

We now turn to various extensions of **DNS**. We first show how a number of these can be axiomatized and discuss the problem of axiomatization of **DNS**-extensions more generally (Section 4.1). Next, we briefly discuss these extensions from the viewpoint of their intended application, viz. as *deontic* logics (Section 4.2). Finally, we return to the meta-properties mentioned at the end of the previous section and study these for **DNS**-extensions in general (Sections 4.3-4.6).

4.1 Axioms for extensions of DNS

Here, we give an overview of some well-known axioms that can be added to **DNS**, and the corresponding conditions on **DNS**-models. Our positive results are summarized by Theorems 2 and 3 below.

Let $(A_1), \ldots, (A_n)$ be axiom schemas from Tables 1 and 2, and let $(C_1), \ldots, (C_n)$ be the associated frame conditions. We use $\vdash_{\mathbf{DNS}.A_1 \ldots A_n}$ to denote the syntactic consequence relation obtained by adding all instances of $(A_1), \ldots, (A_n)$ to **DNS** (see Definition 3), and we let $\Vdash_{\mathbf{DNS.C_1 \ldots C_n}}$ be the semantic consequence relation obtained by imposing the conditions $(C_1), \ldots, (C_n)$ on **DNS**-models.

Theorem 2. *Where $(A_1), \ldots, (A_n)$ are axiom schemas from Table 1 and $(C_1), \ldots, (C_n)$ are the associated frame conditions:*

$$\Gamma \vdash_{\mathbf{DNS.A_1 \ldots A_n}} \varphi \textit{ iff } \Gamma \Vdash_{\mathbf{DNS.C_1 \ldots C_n}} \varphi$$

Theorem 3. *Where $(A_1), \ldots, (A_n)$ are axiom schemas from Table 2 and $(C_1), \ldots, (C_n)$ are the associated frame conditions:*

$$\Gamma \vdash_{\mathbf{DNS.A_1 \ldots A_n}} \varphi \textit{ iff } \Gamma \Vdash_{\mathbf{DNS.C_1 \ldots C_n}} \varphi$$

(D_{N})	$\mathsf{N}\varphi \supset \neg\mathsf{N}\neg\varphi$	(CD_{N})	R_{N} is serial
(M_{N})	$\mathsf{N}(\mathsf{N}\varphi \supset \varphi)$	(CM_{N})	R_{N} is shift reflexive
(T_{N})	$\mathsf{N}\varphi \supset \varphi$	(CT_{N})	R_{N} is reflexive
(4_{N})	$\mathsf{N}\varphi \supset \mathsf{N}\mathsf{N}\varphi$	($C4_{\mathsf{N}}$)	R_{N} is transitive
(B_{N})	$\varphi \supset \mathsf{N}\neg\mathsf{N}\neg\varphi$	(CB_{N})	R_{N} is symmetric
($C4_{\mathsf{N}}$)	$\mathsf{N}\mathsf{N}\varphi \supset \mathsf{N}\varphi$	($CC4_{\mathsf{N}}$)	R_{N} is dense
(5_{N})	$\neg\mathsf{N}\neg\varphi \supset \mathsf{N}\neg\mathsf{N}\neg\varphi$	($C5_{\mathsf{N}}$)	R_{N} is euclidian
(C_{N})	$\neg\mathsf{N}\neg\mathsf{N}\varphi \supset \mathsf{N}\neg\mathsf{N}\neg\varphi$	(CC_{N})	R_{N} is convergent
($D_{\Box}$)	$\Box\varphi \supset \neg\Box\neg\varphi$	($CD_{\Box}$)	$R_{\Box}$ is serial
($T_{\Box}$)	$\Box\varphi \supset \varphi$	($CT_{\Box}$)	$R_{\Box}$ is reflexive
($M_{\Box}$)	$\Box(\Box\varphi \supset \varphi)$	($CM_{\Box}$)	$R_{\Box}$ is shift reflexive

Table 1: Axiom schemas and frame conditions for **DNS** (part 1).

($D_{\Box}$)	$\Box\varphi \supset \neg\Box\neg\varphi$	($CD_{\Box}$)	$R_{\Box}$ is serial
($M_{\Box}$)	$\Box(\Box\varphi \supset \varphi)$	($CM_{\Box}$)	$R_{\Box}$ is shift reflexive
($T_{\Box}$)	$\Box\varphi \supset \varphi$	($CT_{\Box}$)	$R_{\Box}$ is reflexive
($4_{\Box}$)	$\Box\varphi \supset \Box\Box\varphi$	($C4_{\Box}$)	$R_{\Box}$ is transitive
($B_{\Box}$)	$\varphi \supset \Box\neg\Box\neg\varphi$	($CB_{\Box}$)	$R_{\Box}$ is symmetric
($C4_{\Box}$)	$\Box\Box\varphi \supset \Box\varphi$	($CC4_{\Box}$)	$R_{\Box}$ is dense
($5_{\Box}$)	$\neg\Box\neg\varphi \supset \Box\neg\Box\neg\varphi$	($C5_{\Box}$)	$R_{\Box}$ is euclidian
($C_{\Box}$)	$\neg\Box\neg\Box\varphi \supset \Box\neg\Box\neg\varphi$	($CC_{\Box}$)	$R_{\Box}$ is convergent
(D_{N})	$\mathsf{N}\varphi \supset \neg\mathsf{N}\neg\varphi$	(CD_{N})	R_{N} is serial
(T_{N})	$\mathsf{N}\varphi \supset \varphi$	(CT_{N})	R_{N} is reflexive
(M_{N})	$\mathsf{N}(\mathsf{N}\varphi \supset \varphi)$	(CM_{N})	R_{N} is shift reflexive

Table 2: Axiom schemas and frame conditions for **DNS** (part 2).

The proofs of completeness for both theorems proceed by two distinct variants of the copy-and-merge method that was mentioned in Section 3.2. We refer to Appendix B for the details.

As an example of a case covered by Theorem 2, consider the models in which R_{N} is serial, shift reflexive and transitive, and $R_{\Box}$ is reflexive. The associated logic is completely axiomatized by adding (D_{N}), (SR_{N}), (4_{N}), and $(T_{\Box})$ to **DNS**. A case covered by Theorem 3 is the logic $\mathbf{DNS.D_{\mathsf{N}}.T_{\Box}.5_{\Box}}$ of frames where $R_{\Box}$ is an equivalence relation and R_{N} is serial. We return to the latter system in

Section 5 where it is linked to the logic of "obligation as weakest permission" from [3].

What then about the other combinations of conditions on R_{N} and $R_{\Box}$ – what if we mix conditions from both tables? Here, the results are less pleasing. Indeed, as we will now show, for at least some such combinations, the associated logics (obtained by adding the standard axioms to **DNS**) will be incomplete. Complete logics can be obtained, but there seems to be no general method for doing so.

Consider the logic of frames for which $R_{\Box}$ is euclidian and R_{N} is transitive.[24] This logic is not completely axiomatized by adding to **DNS** all instances of $(5_{\Box})$ and (4_{N}). To obtain a complete logic, on needs to add the following axiom schema – which is not derivable in $\mathbf{DNS}.\mathbf{T}_{\Box}.\mathbf{5}_{\Box}.\mathbf{4}_{\mathsf{N}}$:

(Trans$_{\mathsf{S}}$) $\neg\mathsf{N}\neg\mathsf{S}\varphi \supset \mathsf{S}\varphi$

The soundness proof is relatively straightforward. For completeness, we can use a variant of the copy-and-merge technique; the additional axiom turns out crucial for establishing the transitivity of R_{N}. See Appendix B for a proof outline.

A similar example, though less significant from the viewpoint of deontic logic, is the one where both $R_{\Box}$ and R_{N} are symmetric. If we impose these conditions, all instances of the schema $\varphi \supset \mathsf{SS}\varphi$ become globally valid. Indeed, suppose that φ holds in the present world w and $\mathsf{S}\varphi$ holds at some $w' \in R_{\Box}(w)$. By symmetry of $R_{\Box}$, $w \in R_{\Box}(w')$ and hence $w \in R_{\mathsf{N}}(w')$. But then by the symmetry of R_{N}, $w' \in R_{\mathsf{N}}(w)$. So in order to obtain a complete logic, we need to add these axioms to **DNS**. Again, once we have this additional axiom, we can run the usual completeness proof by means of copy-and-merge.

All in all, this shows that regarding extensions of **DNS**, more general results will not be easy to arrive at. This negative conclusion was already drawn in [16, p. 351] and [13, p. 322]: in the context of modal sufficiency, there seems to be no general, modular method to obtain complete axiomatizations for certain conditions.

4.2 *Deontic* necessity and sufficiency?

Philosophically speaking, not all of the aforementioned extensions of **DNS** can count as genuine deontic logics. Some of the frame conditions seem generally plausible; others seem to make little sense in the context of deontic reasoning. For still others, whether they should be required depends on the specific interpretation of the logic and hence on its application. Let us discuss each of these categories one by one.

First, the condition of seriality for R_{N} is usually conceived as the distinctive feature of deontic logics, as it is valid for Standard Deontic Logic. It is strongly connected to the Kantian principle that "ought implies can". Note that in view

[24] This example is inspired on [13]. In view of Goranko's proof, (Trans$_{\mathsf{S}}$) is not even valid in the stronger logic obtained by adding $(T_{\Box})$, $(5_{\Box})$, and (4_{N}) to the axioms of **DNS**.

of standard modal logic properties and our axiom (B), (D) is equivalent to the following:

(OIC) $\mathsf{N}\varphi \supset \Diamond\varphi$

In the deontic logic literature, (D) and (OIC) have been attacked since, together with other principles of Standard Deontic Logic (or any normal modal logic for that matter), they trivialize deontic conflicts. However, as we explained in Section 3.1, even **DNS** (which invalidates (D)) cannot be said to deal with deontic conflicts in a proper way. In sum, all logics defined in the present paper should be applied to situations in which deontic conflicts are absent; if one wants to take deontic conflicts seriously, weaker variants of the current logics should be devised.[25] Consequently, it seems reasonable to assume the validity of (D) and hence seriality of R_{N}.

Second, reflexivity and symmetry for R_{N} are highly counterintuitive for deontic logics in general. Regarding reflexivity, it is commonly seen as typical of deontic logics that they *in*validate the truth schema $\mathsf{N}\varphi \supset \varphi$, or in semantic terms, that it is not necessarily the case that the current state is deontically acceptable. For similar reasons, one should reject symmetry of R_{N}: that w' is deontically acceptable from the viewpoint of the current world w by no means entails that from the viewpoint of w', w is deontically acceptable.

Third and last, some of the mentioned conditions on R_{N} are controversial, but not totally absurd in the context of deontic logic: shift reflexivity, transitivity, density, and the requirement that R_{N} is euclidean. We will not discuss each of these properties here but refer to [7, Section 6.3] for some arguments for and against them.[26] Toward the end of this section, we will consider one condition that entails each of these four conditions, viz. "uniformity".

Regarding properties of $\Box$, it seems that the choice of logic depends even more on the intended application. We already mentioned a few possible interpretations in Section 2. As we explained there, reflexivity and transitivity of $R_\Box$ can be motivated independently. Seriality of $R_\Box$ is equivalent to the requirement that there must be at least one deliberative alternative that is relevant to the given context. For instance, where we think of $R_\Box(w)$ as the set of potential states of the world at the next time instant, seriality means that there is no last point in time.

Alternatively, one can interpret $\Box$ as a universal or global modality in the sense of [14]. If we leave the definition of models and semantic clauses unaltered, this means we impose the condition that $R_\Box = W \times W$. Let us call the resulting logic $\mathbf{DNS}^{\mathsf{u}}$; we call models (frames) in which $R_\Box$ is the total relation, $\mathbf{DNS}^{\mathsf{u}}$-*models* ($\mathbf{DNS}^{\mathsf{u}}$-frames). Mind that from a syntactic viewpoint, $\mathbf{DNS}^{\mathsf{u}}$ is identical to $\mathbf{DNS.T_\Box.5_\Box}$.[27] Adding a global modality to a given modal logic

[25] See [12] for an excellent survey of conflict-tolerant deontic logics.

[26] See also [23, footnote 20] for a critical discussion of one of Chellas' ideas in the cited passage.

[27] Indeed, every **DNS**-model M in which $R_\Box$ is an equivalence relation, can be seen as a disjoint union of $\mathbf{DNS}^{\mathsf{u}}$-models M_i, where the domain of M_i corresponds to one equivalence class (relative to $R_\Box$) in the domain of M.

is quite common – see e.g. [29] where such an operator $\Box$ is included in a deontic logic based on a logic of preference, and see [14] for a thorough investigation of such constructions in general.

In [13], various extensions of $\mathbf{DNS^u}$ are studied, including the ones where R_{N} is serial and reflexive. This means that, leaving some notational conventions and the choice of primitive operators aside, the completeness results from Section 4.1 that concern $\mathbf{DNS^u}$ and its extensions are not new. However, the general case where $R_{\Box}$ is arbitrary (or merely satisfies a number of weaker conditions) seems to have received little or no attention in the literature, notwithstanding the fact that van Benthem's 1979 paper is often cited in the literature on the "window modality" (which coincides with our S) and inaccessible worlds.

In $\mathbf{DNS^u}$, we can express another common frame condition. Say a model M (a frame F) is *uniform* iff for all $w, w' \in W$, $R_{\mathsf{N}}(w) = R_{\mathsf{N}}(w)$. The logic obtained by adding this condition is axiomatized by adding the following axiom to $\mathbf{DNS^u}$ (see Appendix B.3):

(U) $(\mathsf{N}\varphi \equiv \Box\mathsf{N}\varphi) \wedge (\mathsf{S}\varphi \equiv \Box\mathsf{S}\varphi)$

Theorem 4. *The logic obtained by adding (U) to* $\mathbf{DNS^u}$ *is sound and strongly complete w.r.t. the set of all uniform* $\mathbf{DNS^u}$*-models, i.e.,*

$$\Gamma \vdash_{\mathbf{DNS^u.U}} \varphi \text{ iff } \Gamma \models_{\mathbf{DNS^u.U}} \varphi$$

Adding axiom (D) to $\mathbf{DNS^u.U}$ yields a sound and strongly complete logic w.r.t. the set of all $\mathbf{DNS^u.U}$ models in which R_{N} is serial (see again Appendix B.3). Note that in all $\mathbf{DNS^u.U}$-models, R_{N} is shift reflexive and transitive.[28] Note also that $\mathbf{DNS^u.U}$-models in which R_{N} is reflexive are simply $\mathbf{DNS^u.U}$-models in which $R_{\mathsf{N}} = R_{\Box} = W \times W$.

The requirement of uniformity (or equivalently, the requirement that R_{N} be euclidean) is made in several applications of deontic logic. For instance, in [20], a multi-agent version of such systems is interpreted in terms of "green" (acceptable) states and "red" (unacceptable, violation) states of a distributed computing system, where the green and red states jointly exhaust W. In view of the above results, it is plausible that these systems can be enriched with a deontic sufficiency-operator.

4.3 Finite model property and decidability

Theorem 5. *All* **DNS***-extensions mentioned in Theorems 2 and 3 have the finite model property.*

Since our (finitary) axiomatizations of these logics are sound and complete, we obtain:

Corollary 1. *All* **DNS***-extensions that satisfy the conditions from Theorems 2 and 3 are decidable.*

[28] For shift reflexivity, suppose that $(w, w') \in R_{\mathsf{N}}$. By uniformity, also $(w', w') \in R_{\mathsf{N}}$. For transitivity, suppose that $(w, w'), (w', w'') \in R_{\mathsf{N}}$. By uniformity, also $(w, w'') \in R_{\mathsf{N}}$.

This result is obtained by a combination of the copy-and-merge constructions used to prove completeness of the logics with the standard filtration method. Using another variant of this technique, one can also establish the finite model property and decidability for $\mathbf{DNS^u.U}$ and $\mathbf{DNS^u.U.D_N}$ from the previous section. We refer to Appendix C for the meta-proofs for each of these claims.

As regards other **DNS**-extensions, we encounter the same problems as in Section 4.1. Hence, for these logics, the finite model property remains an open issue.

4.4 Expressive power of DNS and its extensions

In this section, we will show that the S-operator adds expressive power to **DNS** and many of its extensions. To spell out the exact result in full generality, we first define two notions that concern conditions (C) on binary relations R.

Definition 4. (C) *is* preserved by copy *iff for all R that satisfy (C): if $R' = \{(w^i, v^i) \mid i \in \{1,2\}, (w,v) \in R\}$, then R' satisfies (C).*

Most of the standard first-order conditions on frames are preserved by copy: seriality, reflexivity, transitivity, symmetry, etc. One exception is the property that R is the total relation, i.e. $R_\Box = W \times W$.

Definition 5. (C) *is* preserved by copy-merge *iff for all R that satisfy (C): if $R' = \{(w^i, v^j) \mid i, j \in \{1,2\}, (w,v) \in R\}$, then R' satisfies (C).*

Most properties that are preserved by copy are also preserved by copy-merge. In particular, all the conditions that occur in Tables 1 and 2 are both preserved by copy and by copy-merge. One notable exception is functionality, which states that every point has exactly one successor. Examples of conditions that are preserved by copy-merge but not preserved by copy, are the condition that $R_\Box$ is total and the condition of uniformity (see Section 4.2).

Let in the remainder $\mathbf{DNS}^+$ be a metavariable for any logic $\mathbf{DNS.C_1^N \ldots C_n^N.C_1^\Box \ldots C_m^\Box}$, where $(C_1^\mathsf{N}), \ldots, (C_n^\mathsf{N})$ are conditions on R_N that are preserved by copy, and where $(C_1^\Box), \ldots, (C_m^\Box)$ are conditions on $R_\Box$ that are preserved by copy-merge. Then the following holds:[29]

Theorem 6. *Where $\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \subseteq \mathcal{W}_{\overline{\mathsf{S}}}$:*

$$\Gamma \Vdash_{\mathbf{DNS}^+} \mathsf{S}\varphi_1 \vee \ldots \vee \mathsf{S}\varphi_n \textit{ iff } \Gamma \Vdash_{\mathbf{DNS}^+} \Box\neg\varphi_1 \vee \ldots \vee \Box\neg\varphi_n$$

In fact, this theorem is obtained in two steps: we first prove it for **DNS** simpliciter, using a specific operation on **DNS**-models (see Appendix D for the details). Next, by its very definition, this operation preserves all conditions on R_N that are safe for copy and all conditions on $R_\Box$ that are safe for copy-merge. In view of our earlier remarks in this section, it follows that Theorem

[29] The restriction that also $\varphi_1, \ldots, \varphi_n$ do not contain the operator S has a reason. Consider e.g. the case where both $R_\Box$ and R_N are required to be symmetric. As explained above, this gives the theorem $p \supset \mathsf{SS}p$ and hence $p \Vdash \mathsf{SS}p$.

6 applies to all the logics mentioned in Theorems 2 and 3. Using a slightly different construction, we can also prove this property for the logics $\mathbf{DNS^u.U}$ and $\mathbf{DNS^u.U.D_N}$ from Section 4.2.[30]

As Theorem 6 shows, there is a gap between deontic necessity and deontic sufficiency in all **DNS**-extensions we considered in this paper. From the viewpoint of the interpretation in terms of obligation and (strong) permission, this is perhaps problematic: after all, we often do and need to infer claims about strong permission on the basis of mere claims about obligation, as our toy example from Section 2 illustrated. However, from a semantic viewpoint it makes perfect sense: when we are at w, we can never exclude that there is some world $w' \in R_\Box(w) \setminus R_\mathsf{N}(w)$ which satisfies every φ that is deontically necessary.

On the basis of the above theorems, we can easily prove that the fragment of each logic $\mathbf{DNS}^+$ without the S-operator is (strictly) less expressive than full **DNS**:

Observation 3. *There is no Δ in the language of $\mathbf{DNS}^+$ without S such that $\Delta \Vdash_{\mathbf{DNS}^+} \mathsf{S}p$ and $\mathsf{S}p \Vdash_{\mathbf{DNS}^+} \varphi$ for all $\varphi \in \Delta$.*

To verify this claim, suppose that such a Δ exists. By Theorem 6, $\Delta \Vdash_{\mathbf{DNS}^+} \Box\neg p$. However, we can easily see that $Sp \nVdash_{\mathbf{DNS}^+} \Box\neg p$, contradicting our supposition.

4.5 The Anderson-Kangerian reduction of DNS

The Anderson-Kangerian reduction of **SDL** to **K** belongs to one of the folklore results within deontic logic. Here, we shall briefly consider the possibility of similar reductions of **DNS** and its extensions.

For the base logic, things are pretty straightforward: we can simply reduce it to **K** in the expected way.[31] More precisely, let $\mathbf{K_d}$ denote the minimal modal logic with $\Box$ as its (only) necessity-operator and a propositional constant d. Let the translation function t from the language of **DNS** to the language of $\mathbf{K_d}$ be defined recursively as follows:

$$t(p) = p \text{ for all } p \in \mathcal{S}$$
$$t(\neg\varphi) = \neg t(\varphi)$$
$$t(\varphi \vee \psi) = t(\varphi) \vee t(\psi)$$
$$t(\mathsf{N}\varphi) = \Box(d \supset t(\varphi))$$
$$t(\mathsf{S}\varphi) = \Box(t(\varphi) \supset d)$$
$$t(\Box\varphi) = \Box t(\varphi)$$
$$t(\Gamma) = \{t(\varphi) \mid \varphi \in \Gamma\}$$

We have:

[30] We explain this at the end of Appendix D. Note that the theorem cannot be generalized to e.g. the logic $\mathbf{DNS^u.U.T_N}$. Indeed, as we explained above, this is the logic of frames for which $R_\mathsf{N} = R_\Box$. In that logic, we can simply define $\mathsf{S}\varphi$ by $\top$.

[31] Also this result is foreshadowed in [28, Section 4], where van Benthem refers to existing modal reductions as proposed by "several deontic logicians".

Theorem 7 (Reduction of **DNS** to $\mathbf{K_d}$)**.** $\Gamma \Vdash_{\mathbf{DNS}} \varphi$ *iff* $t(\Gamma) \Vdash_{\mathbf{K_d}} t(\varphi)$

Due to the presence of the $\Box$-operator in the language of **DNS**, the proof for Theorem 7 is slightly more complicated than the usual meta-proofs for the Anderson-Kangerian reductions of deontic logics, as found e.g. in [4]. This is explained in Appendix E.

What about stronger logics? Here the results are mixed. On the one hand, the **SDL**-variant of **DNS**, which is just **DNS** plus the axiom (D_{N}), can be reduced to $\mathbf{K_d}$ plus the axiom $\Diamond d$, just as in the standard Andersonian reduction. On the other hand, for stronger logics, the analogy breaks down. For instance, it is shown in [4] that the logic $\mathbf{SDL}.\mathbf{SR}_{\mathsf{N}}$ (which is adequate w.r.t. the class of models with an accessibility relation that is shift reflexive) can be characterized by adding the (T)-axiom and $\Diamond d$ to $\mathbf{K_d}$. However, if we take **T** as the logic for $\Box$ and apply the above translation, we get the following validities:

(SR_{N}) $\mathsf{N}(\mathsf{N}\varphi \supset \varphi)$
(ISR_{S}) $\mathsf{S}(\mathsf{S}\varphi \wedge \varphi)$

Whereas (SR_{N}) is just as expected, one can easily construct models that are shift reflexive but for which (ISR_{S}) fails. In fact, this axiom rather seems to correspond to a converse of shift reflexivity, i.e. if $w' \in R_{\Box}(w)$ and $w' \in R_{\mathsf{N}}(w')$, then $w' \in R_{\mathsf{N}}(w)$. Hence, the reduction cannot go through in this simple way. This result suggests that most of the extensions studied in the previous section cannot be reduced to a normal (uni)modal logic.

One notable exception is the logic obtained by adding (UNIF) to $\mathbf{DNS}^{\mathsf{u}}$ (see our discussion at the end of Section 4.2). This logic can be easily reduced to $\mathbf{S5_d}$ (i.e., **S5** with a propositional constant d) using the above translation. The reason is that in this particular case, the conditions imposed on $R_{\Box}$ are "backed up" by the condition of uniformity that is imposed on R_{N}. As a result of this condition, one can identify the truth set of d in a model with the set of worlds that are acceptable from the viewpoint of an arbitrary world w (and hence also from the viewpoint of any other world w'). Likewise, $\mathbf{DNS}^{\mathsf{u}}.\mathbf{U}.\mathbf{D}_{\mathsf{N}}$ can be reduced to the logic obtained by adding the $\Diamond\mathsf{d}$ to $\mathbf{S5_d}$. We explain both points at the end of Appendix E.

4.6 Interpolation

Interpolation (for an arbitrary consequence relation $\vdash$, over a propositional language) is the property that, whenever $\varphi \vdash \psi$, then there is some τ such that (1) $\varphi \vdash \tau$, (2) $\tau \vdash \psi$, and (3) all sentential variables that occur in τ occur both in φ and in ψ. If these conditions hold, we call τ an *interpolant* for $\langle \varphi, \psi \rangle$. It is well-known that this property holds for many normal modal logics, though not for all (see e.g. [21] for an overview of some results in this area).

The proof of interpolation for **K** from [8, Chapter 3, Section 8] can be easily adapted in order to prove interpolation for **DNS**. Recall that **DNS** can be rewritten as a simple bi-modal version of **K**, with no interaction principles (see Section 3.1). So all we need to do is construct a "disjoint copy" of a sequent calculus for **K** (see e.g. [8]), and prove interpolation in the standard way by an induction on the length of the derivation of a sequent.

Interestingly, interpolation fails as soon as we move to relatively weak extensions of **DNS**. Here is an example:[32]

$$\mathsf{S}p \wedge p \vdash_{\mathbf{DNS.T}_\Box} \mathsf{N}q \supset q$$

That is, suppose that $\mathsf{S}p \wedge p$. By simplification, we have $\mathsf{S}p$. By (NS), we can derive $\mathsf{N}q \supset \Box(p \supset q)$. By ($\mathrm{T}_\Box$), $\mathsf{N}q \supset (p \supset q)$. And hence, since we also have p by simplification, we get $\mathsf{N}q \supset q$. However, as we argue in Appendix F, there is no interpolant for $\langle \mathsf{S}p \wedge p, \mathsf{N}q \supset q\rangle$ in $\mathcal{L}$. Note that such an interpolant would allow us to express exactly that a world w is a member of $R_\mathsf{N}(w)$.[33]

The fact that $\mathbf{DNS.T}_\Box$ does not have the interpolation property can be seen to follow from a more general claim in [22]. This claim concerns so-called *union logics*, which can be described as follows. Consider a multi-modal logic with a finite number of normal modalities $\Box_1, \ldots, \Box_n$. Let $\Box\varphi = \bigwedge_{1\leq i\leq n} \Box_i\varphi$ (or equivalently, $\Diamond\varphi = \bigvee_{1\leq i\leq n} \Diamond_i\varphi$). Then we obtain a union logic from this logic if we add one of the following axioms:

($\mathrm{T}_\Box$) $\Box\varphi \supset \varphi$
($4_\Box$) $\Box\varphi \supset \Box\Box\varphi$
($\mathrm{B}_\Box$) $\varphi \supset \Box\Diamond\varphi$
($5_\Box$) $\Diamond\varphi \supset \Box\Diamond\varphi$

Marx and Areces show (by means of a semantic argument) that every union logic obtained in this way violates interpolation [22, Theorem 4.10]. It can easily be checked that also $\mathbf{DNS.T}_\Box$ corresponds to a union logic. First, letting $\overline{\mathsf{N}}\varphi = \mathsf{S}\neg\varphi$, we can easily show that $\overline{\mathsf{N}}$ is a normal modality (cf. our remarks in Section 3.1). Second, as explained in that same section, $\Box\varphi$ is equivalent to $\mathsf{N}\varphi \wedge \overline{\mathsf{N}}\varphi$. Third and last, since ($\mathrm{T}_\Box$) is valid in $\mathbf{DNS.T}_\Box$, this logic is an union logic.

Two further comments are in place. On the one hand, Marx and Areces only use these axioms to illustrate their deepest results, which are essentially semantic. Indeed, it can be shown that weaker axioms such as e.g. the axiom ($\mathrm{SR}_\Box$) already cause interpolation to fail; there is e.g. no interpolant for $\langle \Diamond(\mathsf{S}p \wedge p), \Diamond(\mathsf{N}q \supset q)\rangle$.[34] On the other hand, it is not clear whether these results generalize to *all* **DNS**-extensions defined in the present paper – e.g. seriality of $R_\Box$ does not seem to cause failure of interpolation. We leave a full investigation of this matter for future work.

[32] This example is based on one by Humberstone [17]. In Humberstone's paper, there is only an operator for "necessary and sufficient".

[33] See Lemma 5 in Appendix F.

[34] We sketch a proof of this claim at the end of Appendix F.

5 Obligation as weakest permission versus DNS

In [26], [25], and [3] a logic is developed for "obligation as weakest permission". This logic is meant to capture the deontic aspects of reasoning in strategic games, where we speak about properties of the best actions available to a given agent. Whereas usually in formal models of such games, actions and/or agents are modeled explicitly at the object level, the present logic only speaks about action types (which have, formally, the same status as propositions). Let us explain this briefly – we refer to the cited works for a more elaborate discussion.

Consider a situation in which an agent can choose from a number of distinct actions, where at least some of these are optimal. Whereas the agent is permitted to perform one of those optimal actions, its sole obligation (if there is one at all – mind this important caveat) is to just perform one of them. This means that the concept of obligation that is being used here is the one we referred to as "obligation in singular", and the concept of permission is one of "strong permission" (see Section 2). More specifically, the deontic operators introduced by Anglberger et al. can be read as follows:

$O\varphi$: "φ is the (only) action type that is obligatory"
$P\varphi$: "if an action is of type φ, then it is (strongly) permitted"

As Anglberger et al. explain, $P\varphi$ means that being of type φ is a sufficient condition for any action to be optimal[35], whereas $O\varphi$ means that to be of type φ is a necessary *and* sufficient condition to be optimal.[36]

Anglberger et al. moreover introduce an alethic modality $\Box$, which they interpret as a universal modality. That is, $\Box$ allows us to quantify over all available actions of the agent. $\Box\varphi$ thus means that all of those actions are of type φ. They then propose what they call a "minimal logic" called **5HD** for these three operators. In the remainder, we will focus on the axiomatization of this logic as it is found in [3] and show that, when strengthened in an intuitively plausible way, it can be reconstructed as a fragment of one of the **DNS**-extensions defined in Section 4.1.

Definition 6. *The logic* **5HD** *is axiomatized by all propositional tautologies together with the following axioms and rules:*[37]

(S5$_\Box$)	*All of* **S5** *for* $\Box$
(EQ$_O$)	$\Box(\varphi \equiv \psi) \supset (O\varphi \equiv O\psi)$
(EQ$_P$)	$\Box(\varphi \equiv \psi) \supset (P\varphi \equiv P\psi)$
(FCP)	$P(\psi \vee \varphi) \supset (P\psi \wedge P\varphi)$
(Ought-Perm)	$O\varphi \supset P\varphi$

[35] See e.g. [3, p. 2]:"Under the open reading permission statements identify the sufficient conditions for an action type to be licensed by a given normative system."

[36] See [3, p. 6]: "Assuming that obligation implies permission thus means, in the present setting, that the obligatory action types, i.e. necessary conditions for legality, are also sufficient ones".

[37] This axiomatization is not equivalent to the one given in [25]. We return to this point below.

(Ought-Can)	$O\varphi \supset \Diamond\varphi$
(Weakest-Perm)	$O\varphi \supset (P\psi \supset \Box(\psi \supset \varphi))$
(MP)	*Modus Ponens for* $\supset$
(NEC)	*Necessitation for* $\Box$

Consider now the language $\mathcal{L}$ of **DNS**, and define $P\varphi =_{\mathsf{df}} \mathsf{S}\varphi$ and $O\varphi =_{\mathsf{df}} \mathsf{N}\varphi \wedge \mathsf{S}\varphi$. Hence, strong permission just means deontic sufficiency, and obligation means deontic necessity and sufficiency. Given the informal reading of P and O that was described above, this seems to be a very natural move. However, the resulting operators do not behave in **DNS** the way they do in **5HD**. In fact, **5HD** is in general incomparable to the $O/P/\Box$-fragment of **DNS**. This requires some explanation.

Two obvious differences with **DNS** are the strength of the $\Box$-operator and the (Ought-Can) axiom. Recall that in **DNS**, N and $\Box$ are only assumed to have the **K**-properties in addition to the bridging principle (B). So in order to have a comparable system at all, we will start from $\mathbf{DNS}^{\mathsf{u}}.\mathbf{D}$, which is the extension of **DNS** with the **T**-axiom and **5**-axiom for $\Box$ and the **D**-axiom for N.

It can be easily shown that, given the definitions from the preceding paragraph, all the **5HD**-axioms above are derivable in $\mathbf{DNS}^{\mathsf{u}}.\mathbf{D}$. However, $\mathbf{DNS}^{\mathsf{u}}.\mathbf{D}$ yields more validities than **5HD**. In particular, the following principles are invalid in **5HD**:[38]

(UC_P)	$(P\varphi \wedge P\psi) \supset P(\varphi \vee \psi)$
(Triv_P)	$P\bot$
(Taut-Perm)	$P\top \supset O\top$

Note that, given the above translation, (UC_P) is just our axiom (UC) and (Triv_P) is our axiom ($\mathrm{Triv}_{\mathsf{S}}$). (Taut-Perm) follows (trivially) from the theorem $\mathsf{N}\top$ and our definition of O.

How to evaluate these differences? Let us consider them one by one. As regards (UC_P), it should be noted that the following, weaker axiom is **5HD**-valid:[39]

(CUC_P)	$O\tau \supset ((P\varphi \wedge P\psi) \supset P(\varphi \vee \psi))$

Still, it seems somewhat counterintuitive to let the implication from $P\varphi \wedge P\psi$ to $P(\varphi \vee \psi)$ depend on whether there is some obligation at all. Given the informal reading of P, we think that it should be unconditionally valid. Indeed, if every

[38] The failure of these principles in **5HD** can be easily shown in view of the neighbourhood-semantics from [3]. To falsify (UC_P), we can use a two-state model $M = \langle \{w_1, w_2\}, Alt, n_P, n_O, V\rangle$, where we put $n_P(w_1) = n_P(w_2) = \{\{w_1\}, \{w_2\}, \emptyset\}$, $n_O(w_1) = n_O(w_2) = \emptyset$, and $V(p) = \{w_1\}$ for all $\varphi \in \mathcal{S}$. In this model, Pp and $P\neg p$ is true at w_1 but $P(p \vee \neg p)$ is false at w_1. For (Triv_P), construct a model with a single state w such that $n_P(w) = \emptyset$. To falsify (Taut-Perm), construct a model with a single state w such that $n_P(w) = \{w, \emptyset\}$ and $n_O(w) = \emptyset$.

[39] Indeed, suppose $O\tau, P\varphi, P\psi$. By (Weakest-Perm), we have $\Box(\varphi \supset \tau)$ and $\Box(\psi \supset \tau)$. By standard modal logic properties, we have $\Box(((\psi \vee \varphi) \vee \tau) \equiv \tau)$. By (Ought-Perm), we have $P\tau$. By (EQ_P), we can derive $P((\psi \vee \varphi) \vee \tau)$, and hence by (FCP) we have $P(\psi \vee \varphi)$. We are indebted to Olivier Roy for pointing this out (personal communication).

action of type φ is optimal and every action of type ψ is optimal, then it seems unavoidable that also every action of type $\varphi \vee \psi$ is optimal – assuming that an action is of type $\varphi \vee \psi$ iff it is either of type φ or of type ψ or both.

In the earlier versions of the logic spelled out in [26, 25], (UC_P) is valid whereas (FCP) is invalid. In a still more recent paper [2] (which is co-authored by two of the three authors from [25, 3]), the authors reconsider (FCP) and provide some arguments *against* this principle. However, none of these seem to count as arguments against (UC_P).

Also for (Triv_P), there is a weakening that is valid in **5HD**:

(CTriv_P) $\quad P\varphi \supset P\bot$

Where Anglberger et al. consider (CTriv_P), they argue that, although seemingly too strong, it is a harmless principle:

> [...] Obviously, it is consistent for an impossible action $\psi \wedge \neg\psi$ to be permitted. Given the open reading, this is even very plausible: Since there are no tokens of an impossible action type, every token of that very type is OK. [...]

This argument may just as well be used to argue for the plausibility of the stronger axiom (Triv_P): after all, what interests us usually are strong permissions that are non-trivial. And again, in the presence of the operator $\Diamond$, we can easily distinguish those permissions from the trivial ones.

Third and last, there is (Taut-Perm). Quite surprisingly, this axiom is valid in the logic **5HD**$^+$ from [3, Section 4.1], which is the extension of **5HD** with the following infinitary rule:

(R-Conv) $\quad$ From $\{\vdash Pp \supset \Box(p \supset \varphi) \mid p \in \mathcal{S}\}$, to infer $\vdash P\varphi \supset O\varphi$

Indeed, since (trivially) we have $Pp \supset \Box(p \supset \top)$ for any p, we can derive (Taut-Perm) using (R-Conv).[40] According to [3], the rule (R-Conv) encodes exactly the idea that if there is a weakest permission, then this is our obligation. In other words, without this rule, the option is left open that there is no obligation whatsoever, even if every possible action is (strongly) permitted.

Regarding the validity of (Taut-Perm), there are two different positions one may take, neither of which are compatible with the logic **5HD**. On the one hand, one may insist that obligation is just the combination of deontic necessity and sufficiency, and hence that it should be valid. On the other hand, looking at ordinary uses of "it is obligatory that φ", one may insist that in cases where all the alternatives are ok, we will usually deny that one has any obligation whatsoever. But in that case, it seems that φ's not being deliberatively necessary (hence, $\Diamond\neg\varphi$) is a necessary condition for φ's being obligatory. In this case, one should either add an axiom like $O\varphi \supset \Diamond\neg\varphi$, or one should distinguish between "vacuous obligations" and "informative" ones, as we did in Section 2.

[40] In fact, it suffices to add (Taut-Perm) to the system **5HD** in order to obtain a complete characterization of the semantic consequence relation for **5HD**$^+$ from [3, Section 4.1]. This implies that this consequence relation is compact, in contrast to what is claimed in [3]. See [32].

Adding the above three principles to **5HD** suffices to characterize the $O/P/\Box$-fragment of $\mathbf{DNS^u.D}$. Actually, a more general result can be obtained: we can characterize the $O/P/\Box$-fragment of several of the logics defined in Section 4.1 by an appropriate selection from the above axioms. This requires some preparation. Let $\mathcal{W}_O$ be the closure of $\mathcal{S} \cup \{\top, \bot\}$ under the classical connectives and the operators O, P and $\Box$ as defined above. We define the base logic $\mathbf{DNS_O} : \wp(\mathcal{W}_O) \to \wp(\mathcal{W}_O)$ as follows:

Definition 7. *The logic* $\mathbf{DNS_O}$ *is axiomatized by all propositional tautologies together with* ($K_\Box$), ($NEC_\Box$), *modus ponens, and the following axiom schemas:*

(EQ_O)	$\Box(\varphi \equiv \psi) \supset (O\varphi \equiv O\psi)$
(EQ_P)	$\Box(\varphi \equiv \psi) \supset (P\varphi \equiv P\psi)$
(FCP)	$P(\psi \vee \varphi) \supset (P\psi \wedge P\varphi)$
(UC_P)	$(P\varphi \wedge P\psi) \supset P(\varphi \vee \psi)$
($Triv_P$)	$P\bot$
(Taut-Perm)	$P\top \supset O\top$
(Ought-Perm)	$O\varphi \supset P\varphi$
(Weakest-Perm)	$O\varphi \supset (P\psi \supset \Box(\psi \supset \varphi))$

We use $\vdash_{\mathbf{DNS_O.A_1...A_n}}$ to denote the consequence relation obtained by adding the axiom schemas (A_1), ..., (A_n) from Table 3 to $\mathbf{DNS_O}$.

($\mathrm{D}_\Box$)	$\Box\varphi \supset \neg\Box\neg\varphi$	($\mathrm{CD}_\Box$)	$R_\Box$ is serial
($\mathrm{T}_\Box$)	$\Box\varphi \supset \varphi$	($\mathrm{CT}_\Box$)	$R_\Box$ is reflexive
($4_\Box$)	$\Box\varphi \supset \Box\Box\varphi$	($\mathrm{C4}_\Box$)	$R_\Box$ is transitive
($\mathrm{B}_\Box$)	$\varphi \supset \Box\neg\Box\neg\varphi$	($\mathrm{CB}_\Box$)	$R_\Box$ is symmetric
($\mathrm{C4}_\Box$)	$\Box\Box\varphi \supset \Box\varphi$	($\mathrm{CC4}_\Box$)	$R_\Box$ is dense
($5_\Box$)	$\neg\Box\neg\varphi \supset \Box\neg\Box\neg\varphi$	($\mathrm{C5}_\Box$)	$R_\Box$ is euclidian
($\mathrm{C}_\Box$)	$\neg\Box\neg\Box\varphi \supset \Box\neg\Box\neg\varphi$	($\mathrm{CC}_\Box$)	$R_\Box$ is convergent
(D_N)	$O\varphi \supset \Diamond\varphi$	(CD_N)	R_N is serial

Table 3: Axiom schemas and frame conditions for $\mathbf{DNS_O}$.

Theorem 8. *Where* $(A_1), \ldots (A_n)$ *are axioms from Table 3,* $(C_1), \ldots (C_n)$ *are the associated conditions on* **DNS**-*models, and where* $\Gamma \cup \{\varphi\} \subseteq \mathcal{W}_O$*:*

$$\Gamma \vdash_{\mathbf{DNS_O.X_1...X_n}} \varphi \textit{ iff } \Gamma \vdash_{\mathbf{DNS.X_1...X_n}} \varphi$$

What about shift reflexivity? It does not seem easy to come up with an axiom that characterizes it.[41] More generally, many standard properties of R_N

[41]Note that adding the axiom $O(O\varphi \supset \varphi)$ (where $O\psi = \mathsf{N}\psi \wedge \mathsf{S}\psi$) would trivialize the S-operator, in view of the following derivation:

1	$O(O\top \supset \top)$	(by the additional axiom)
2	$\mathsf{S}((\mathsf{N}\top \wedge \mathsf{S}\top) \supset \top)$	(from 1, by the definition of O and $\wedge$-elimination)

(transitivity, density, ...) which could be easily axiomatized in the full language of **DNS** seem hard to characterize in the restricted language of $\mathbf{DNS_O}$. So even if one assumes that the correct notion of obligation should be one of deontic necessity *and* sufficiency, having the notion of deontic necessity in the object language seems to allow for a much greater expressive power. This is another argument in favor of an approach which encompasses both deontic necessity and sufficiency.

6 Summary and Future Work

We re-interpreted van Benthem's minimal deontic logic $\mathbf{K_{D,m}}$ as a logic of deontic necessity and sufficiency **DNS**, and studied some significant formal aspects of it. We argued that this logic allows for a rich explication of various concepts of obligation, permission, and practical necessity in a unifying framework. Finally, we studied the relationship between **DNS** and the logic of obligation as weakest permission from [3].

We mentioned some prospects for future research along our way: e.g. the generalization of the present framework in order to cover weaker (non-aggregative and/or non-monotonous, conflict-tolerant) notions of obligation; the development of a richer semantics in which alternatives are *sets* of states, rather than single states; giving sound and complete axiomatizations for certain first order conditions on **DNS**-frames. Another important task is to develop conditional variants of **DNS**, which allow us to deal with expressions such as "given that φ is the case, ψ will do". Such work can merit from existing work on dyadic deontic logic, even if it will also inherit its major problems.

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3	$\Box(((\mathsf{N}\top \wedge \mathsf{S}\top) \supset \top) \equiv \top)$	(by propositional logic and $\mathrm{NEC}_{\Box}$)
4	$\mathsf{S}\top$	(from 2,3 by EQ_P)
5	$\mathsf{S}\varphi$	(from 4, applying FCP)

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A Completeness for DNS: copy-and-merge

A.1 The need for copies

Completeness boils down to the following claim: for every consistent set $\Delta \subseteq \mathcal{W}$, there is a **DNS**-model M and a world w such that $M, w \models \varphi$ for all $\varphi \in \Delta$. Below, we prove a slightly stronger claim: there is a **DNS**-model M such that for all consistent $\Delta \subseteq \mathcal{W}$, there is a w such that $M, w \models \varphi$ for all $\varphi \in \Delta$. This model M will be called M^+.

We first define the canonical model $M^c = \langle W^c, R^c_\Box, R^c_{\mathsf{N}}, v^c \rangle$ in the standard way, viz. as follows:

(i) W^c is the set of all subsets of $\mathcal{W}$ that are maximally consistent w.r.t. **DNS**
(ii) $R^c_\Box = \{(\Delta, \Theta) \mid \{\psi \mid \Box\psi \in \Delta\} \subseteq \Theta\}$
(iii) $R^c_{\mathsf{N}} = \{(\Delta, \Theta) \mid \{\psi \mid \mathsf{N}\psi \in \Delta\} \subseteq \Theta\}$
(iv) $v^c(\psi) = \{\Delta \mid \psi \in \Delta\}$ for all $\psi \in \mathcal{S}$

It can now be established by standard means that M^c is an **DNS**-model, and for all formulas $\varphi \in \mathcal{W}_{\overline{\mathsf{S}}}$ and all $\Delta \in W^c$, $M^c, \Delta \models \varphi$ iff $\varphi \in \Delta$. This is usually called the truth lemma.[42] However, this lemma does not hold for the entire language. Consider a model M that consists of only two worlds, w_0 and w_1, such that all $\psi \in \mathcal{S}$ are true in both worlds, R_{N} is the identity relation, and $R_\Box$ is the total relation. Let $\Theta_\star = \{\varphi \mid M, w_0 \models \varphi\}$. Note that, since $M, w_1 \models p$ and $w_1 \in R_\Box(w_0) - R_{\mathsf{N}}(w_0)$, $M, w_0 \not\models Sp$ and hence $Sp \notin \Theta_\star$.

By standard means we can show that $\Theta_\star \in W^c$. Moreover, $R^c_\Box(\Theta_\star) = \{\Theta_\star\}$. That is, $\Box\varphi \in \Theta_\star$ iff $M, w_0 \models \Box\varphi$ iff [by the construction and symmetry of M] $M, w_0 \models \varphi$ iff $\varphi \in \Theta_\star$.[43] By the same reasoning, $R^c_{\mathsf{N}}(\Theta_\star) = \{\Theta_\star\}$. But then $M^c, \Theta_\star \models Sp$, whereas $Sp \notin \Theta_\star$.

In other words, if we simply construct the canonical model in the standard way, then we lack a "witness" for the formula $\neg Sp$, i.e. a deliberative alternative in which p holds, but which is not deontically acceptable from the viewpoint of $\Theta_\star$.

To solve this problem, we make two disjoint copies M^1 and M^2 of M^c, take their union, and make some "smart" connections between the points in both. The idea is that whenever some world in M^1 needs a witness for a formula of the form $\neg\mathsf{S}\psi$, we take it from M^2 (and vice versa). This can however be done in many ways. We will first consider arbitrary ways to copy-and-merge M^c and

[42] See e.g. [6, Section 4.2] for an introduction to the notion of canonical models. We will assume familiarity with this technique throughout this appendix.

[43] It is perhaps easier to see that $\Theta_\star \in R^c_\Box(\Theta)$, than that $\Theta_\star$ is the *only* member of $R^c_\Box(\Theta_\star)$. Suppose however that $\Delta \in W^c, \Delta \neq \Theta_\star$. Hence, there is a ψ such that $\psi \in \Delta$, $\neg\psi \in \Theta_\star$. But then $\Box\neg\psi \in \Theta_\star$ and hence $\Delta \notin R^c_\Box(\Theta_\star)$.

establish a sufficient condition for the truth lemma to hold in such constructions. After that, we will consider three specific, concrete ways to copy-and-merge M^c. Whereas the first is aimed at preserving as many properties of R_{N} as possible, the second and third are aimed at preserving as many properties of $R_{\Box}$ as possible.

First some more notation. In the remainder of this appendix, we use i, j, k to range over $\{1, 2\}$. Let $W^i = \{\langle \Delta, i\rangle \mid \Delta \in W^c\}$. So the members of W^1 and W^2 are not sets of formulas, but *indexed* sets of formulas. In order to enhance readibility, we will write Δ^i to refer to $\langle \Delta, i\rangle$. Also, let $W^+ = W^1 \cup W^2$. Note that $W^+ = \{\Delta^1, \Delta^2 \mid \Delta \in W^c\}$.

We define four accessibility relations on W^+ and a valuation over W^+:

$$R^i_{\Box} = \{(\Delta^i, \Theta^i) \mid (\Delta, \Theta) \in R^c_{\Box}\}$$
$$R^{\cup}_{\Box} = R^1_{\Box} \cup R^2_{\Box}$$
$$R^i_{\mathsf{N}} = \{(\Delta^i, \Theta^i) \mid (\Delta, \Theta) \in R^c_{\mathsf{N}}\}$$
$$R^{\cup}_{\mathsf{N}} = R^1_{\mathsf{N}} \cup R^2_{\mathsf{N}}$$
$$v^i(\varphi) = \{\Delta^i \mid \Delta \in v^c(\varphi)\} \text{ for all } \varphi \in \mathcal{S}$$

Where $\mathcal{X} \subseteq W^+ \times W^+$, let $\bar{i}(\mathcal{X}) = \{(\Delta, \Theta) \mid (\Delta^i, \Theta^j) \in \mathcal{X}\}$.

A.2 A Sufficient Condition for the Truth Lemma

Definition 8. *A* **DNS**-*model* $M^+ = \langle W^+, R^+_{\Box}, R^+_{\mathsf{N}}, v^+\rangle$ *is a* smart copy-merge *of* M^c *iff each of the following hold:*

1. *for all* $\varphi \in \mathcal{S}$, $v(\varphi) = v^1(\varphi) \cup v^2(\varphi)$
2. $\bar{i}(R^+_{\Box}) = R^c_{\Box}$
3. $\bar{i}(R^+_{\mathsf{N}}) = R^c_{\mathsf{N}}$
4. $\bar{i}(R^+_{\Box} \setminus R^+_{\mathsf{N}}) = \{(\Delta, \Theta) \in R^c_{\Box} \mid \{\neg\sigma \mid \mathsf{S}\sigma \in \Delta\} \subseteq \Theta\}$

Theorem 9. *If* M^+ *is a smart copy-merge of* M^c*, then for all* $i \in \{1, 2\}$*,* $\Delta \in W^c$*, and* $\psi \in \mathcal{W}$*:* $M^+, \Delta^i \models \psi$ *iff* $\psi \in \Delta$*.*

Proof. Suppose that the antecedent holds. We prove the consequent by an induction on the complexity of ψ. The base case ($\psi \in \mathcal{S}$) is immediate in view of Definition 8.1. For the induction step, the connectives are routine and hence safely left to the reader. This leaves us with the three modal operators:
Case 1: $\psi = \Box\tau$. We have: $M^+, \Delta^i \models \Box\tau$ iff for all $\Theta^j \in R^+_{\Box}(\Delta^i)$, $M^+, \Theta^j \models \tau$ iff [by the induction hypothesis] for all $\Theta^j \in R^+_{\Box}(\Delta_i)$, $\tau \in \Theta$ iff [by Definition 8.2] (†) for all $\Theta \in R^c_{\Box}(\Delta)$, $\tau \in \Theta$. Suppose now that (†) holds. Hence by item (ii) of the construction of M^c, every maximally consistent set $\Theta \supseteq \{\varphi \mid \Box\varphi \in \Delta\}$ contains τ. By a standard proof (relying on the compactness of **DNS** and the axioms and rules for $\Box$), we can derive that $\Box\tau \in \Delta$.

For the other direction, suppose that $\Box\tau \in \Delta$. Hence, every maximal consistent set $\Theta \supseteq \{\varphi \mid \Box\varphi \in \Delta\}$ contains τ. By item (ii) of the construction of M^c, (†) holds.

Case 2: $\psi = \mathsf{N}\tau$. Analogous to the preceding case: replace $\Box$ with N, (ii) with (iii) and Definition 8.2 with Definition 8.3.
Case 3: $\psi = \mathsf{S}\tau$. ($\Rightarrow$) Suppose that $M^+, \Delta^i \models \mathsf{S}\tau$. Hence, for all $\Theta^j \in R^+_\Box(\Delta^i)$ such that $M^+, \Theta^j \models \tau$, $\Theta^j \in R^+_\mathsf{N}(\Delta^i)$. By Definition 8.4 there is no $\Theta \in R_\Box(\Delta)$ such that $\tau \in \Theta$ and $\{\neg\sigma \mid \mathsf{S}\sigma \in \Delta\} \subseteq \Theta$. Hence, there is no maximal consistent extension of $\{\xi \mid \Box\xi \in \Delta\} \cup \{\tau\} \cup \{\neg\sigma \mid \mathsf{S}\sigma \in \Delta\}$. By the compactness of **DNS** and **CL**-properties, this means that there are $\xi_1, \ldots, \xi_n$ and $\sigma_1, \ldots, \sigma_n$ such that each of the following hold:

(a) $\Box\xi_1, \ldots, \Box\xi_n \in \Delta$
(b) $\{\xi_1, \ldots, \xi_n\} \cup \{\tau\} \vdash_{\mathbf{DNS}} \sigma_1 \vee \ldots \vee \sigma_n$
(c) $\mathsf{S}\sigma_1, \ldots, \mathsf{S}\sigma_n \in \Delta$

By Observation 2.7 and (c),

$$\mathsf{S}(\sigma_1 \vee \ldots \vee \sigma_n) \in \Delta \tag{1}$$

By (Nec) and (a),

$$\Box((\xi_1 \wedge \ldots \wedge \xi_n \wedge \tau) \supset (\sigma_1 \vee \ldots \vee \sigma_n)) \in \Delta \tag{2}$$

By (a) and (2),

$$\Box(\tau \supset (\sigma_1 \vee \ldots \vee \sigma_n)) \in \Delta \tag{3}$$

By axiom (OR), (1) and (3), $\mathsf{S}\tau \in \Delta$.

($\Leftarrow$) Suppose that $\mathsf{S}\tau \in \Delta$. Let $\Theta^j \in R^+_\Box(\Delta^i)$ be arbitrary such that $M^+, \Theta^j \models \tau$ — we need to prove that $\Theta^j \in R^+_\mathsf{N}(\Delta^i)$. By the induction hypothesis, $\tau \in \Theta$. Hence, $\neg\tau \notin \Theta$. It follows that $\{\neg\sigma \mid \mathsf{S}\sigma \in \Delta\} \not\subseteq \Theta$. By Definition 8.4, $(\Delta, \Theta) \notin \bar{\imath}(R^+_\Box \setminus R^+_\mathsf{N})$, and hence $(\Delta^i, \Theta^j) \notin R^+_\Box \setminus R^+_\mathsf{N}$. It follows that $(\Delta^i, \Theta^j) \in R^+_\mathsf{N}$ and hence $\Theta^j \in R^+_\mathsf{N}(\Delta^i)$. $\Box$

A.3 Copy-and-merge version 1

As promised, we will now define three concrete ways to copy-and-merge M^c in a smart way. For the first, put

(a) $R^+_\Box = R^\cup_\Box \cup \{(\Delta^i, \Theta^j) \mid (\Delta, \Theta) \in R^c_\Box \text{ and } \{\neg\sigma \mid \mathsf{S}\sigma \in \Delta\} \subseteq \Theta\}$
(b) $R^+_\mathsf{N} = R^\cup_\mathsf{N}$

To see that M^+ is an **DNS**-model, it suffices to check that $R^+_\mathsf{N} \subseteq R^+_\Box$. So suppose that $(\Delta^i, \Theta^j) \in R^+_\mathsf{N}$. By (b), $(\Delta, \Theta) \in R^c_\mathsf{N}$ and $i = j$. Let τ be arbitrary such that $\Box\tau \in \Delta$. By axiom (B) and (MP), also $\mathsf{N}\tau \in \Delta$. Hence, by (iii), $\tau \in \Theta$. So we have shown that $\{\tau \mid \Box\tau \in \Delta\} \subseteq \Theta$. Hence, by (ii), $(\Delta, \Theta) \in R^c_\Box$. Finally, by (b) and since $i = j$, $(\Delta^i, \Theta^j) \in R^+_\Box$.

So it remains to check that M^+ is a smart copy-merge of M^c. But this is obvious in view of the way we defined $R^+_\Box$ and R^+_N. So by Theorem 9 we get the truth lemma for M^+, and hence we obtain a model for every **DNS**-consistent set $\Gamma \subseteq \mathcal{W}$.

A.4 Copy-and-merge version 2

For the second version of copy-and-merge, we put

(a') $R^+_\Box = \{(\Delta^i, \Theta^j) \mid (\Delta, \Theta) \in R^c_\Box\}$
(b') $R^+_{\mathsf{N}} = R^\cup_{\mathsf{N}} \cup \{(\Delta^i, \Theta^j) \in R^+_\Box \mid \{\neg\sigma \mid \mathsf{S}\sigma \in \Delta\} \not\subseteq \Theta\}$

The main difference with the previous construction is that here, we connect *every* Δ^i with Θ^j whenever $(\Delta, \Theta) \in R^+_\Box$. This allows us to preserve a great number of properties of $R^c_\Box$. However, to make sure that formulas of the form $\mathsf{S}\psi$ are respected, we need to enlarge the relation R_{N} in such a way that whenever some τ is sufficient at Δ^i, then every $R_\Box$-accessible world at which τ holds is also R_{N}-accessible.

Checking that M^+ is a **DNS**-model is again easy in view of the (B)-axiom and the construction. Also, in view of (a') and (b') we can easily verify that M^+ is a smart copy-merge of M^c. So we can again infer by Theorem 9 that the truth lemma holds for M^+, which finishes the proof of completeness.

A.5 Copy-and-merge version 3

The third and last copy-merge operation is defined as follows:

(a") $R^+_\Box = \{(\Delta^i, \Theta^j) \mid (\Delta, \Theta) \in R^c_\Box\}$
(b") $R^+_{\mathsf{N}} = \{(\Delta^i, \Theta^j) \mid (\Delta, \Theta) \in R^c_{\mathsf{N}}$ and $j = 1$ or $\{\neg\sigma \mid \mathsf{S}\sigma \in \Delta\} \not\subseteq \Theta\}$

We leave it to the reader to check that once more, every M^+ obtained in this way is a smart copy-merge of M^c and is a **DNS**-model. Note that, in contrast to the two previous constructions, this one is asymmetric w.r.t. the indices 1 and 2. This will turn out instrumental in proving that transitivity and uniformity of R_{N} transfers to R^+_{N} (see Section B.2 and B.3 respectively).

B Extensions of DNS

B.1 Proof of Theorems 2 and 3

To prove the left-right direction of Theorems 2 and 3, it suffices to check that the validity of axioms is preserved in the richer setting of **DNS**-models – we safely leave this to the reader. For the other direction, we need to apply the two techniques of copy-merge from Section A to the stronger logics obtained by adding the axioms. In the remainder, we first illustrate how this works for some cases covered by Theorem 2; the reasoning is completely analogous for the other cases and Theorem 3.

Let **DNS**.$\mathbf{4}_{\mathsf{N}}$ be the logic obtained by adding the axiom (4_{N}) to **DNS**. We need to prove that this is the logic of **DNS**-models in which R_{N} is transitive. The proof proceeds in two steps: first, we define the model M^c as in Section A. We prove that, due to this definition and the axiom (4_{N}), R^c_{N} is transitive. This is done in the standard way as for the logic **K4**: suppose that $(\Delta, \Theta), (\Theta, \Lambda) \in R^c_{\mathsf{N}}$.

By the construction of M^c, $\{\varphi \mid \mathsf{N}\varphi \in \Delta\} \subseteq \Theta$ and $\{\psi \mid \mathsf{N}\psi \in \Theta \subseteq \Lambda$. Let $\mathsf{N}\varphi \in \Delta$. Hence by ($4_{\mathsf{N}}$), $\mathsf{NN}\varphi \in \Delta$. It follows that $\mathsf{N}\varphi \in \Theta$ and hence $\varphi \in \Lambda$. So we have that $\{\varphi \mid \mathsf{N}\varphi \in \Delta\} \subseteq \Lambda$ and hence by the construction of M^c, $(\Delta, \Lambda) \in R^c_{\mathsf{N}}$.

For the second step, let M^+ be defined according to the first method of copy-merge – see Section A.3. Then we have: $R^+_{\mathsf{N}} = R^1_{\mathsf{N}} \cup R^2_{\mathsf{N}}$. It follows immediately that also R^+_{N} is transitive. Hence, we obtain a canonical model for **DNS.4** in which R^+_{N} is transitive.

The same reasoning can be applied to the other conditions on R_{N}: we first prove that the construction of M^c and the axioms ensure that R^c_{N} satisfy them, and next we observe that these conditions are preserved in R^+_{N} in view of its definition.

We now consider the conditions on $R_\Box$ that are covered by Theorem 2. Again, we prove these in the same basic steps: first prove that they hold for $R^c_\Box$, and second show that this transfers to $R^+_\Box$. Since the reasoning in the second step is slightly more intricate here, we summarize the main points here:

($\mathrm{CD}_\Box$) Suppose that $R^c_\Box$ is serial. Let $\Delta^i \in W^+$ be arbitrary. Hence, $\Delta \in W^c$ and hence by seriality, there is a Θ such that $(\Delta, \Theta) \in R^c_\Box$. But then by the construction, $(\Delta^i, \Theta^i) \in R^+_\Box$. Hence, $R^+_\Box$ is serial.

($\mathrm{CT}_\Box$) Suppose that $R^c_\Box$ is reflexive. Let $\Delta^i \in W^+$. Hence, $\Delta \in W^c$ and hence by reflexivity, $(\Delta, \Delta) \in R^c_\Box$. By the construction, $(\Delta^i, \Delta^i) \in R^+_\Box$. Hence, $R^+_\Box$ is reflexive.

($\mathrm{CM}_\Box$) Suppose that $R^c_\Box$ is shift reflexive and that $(\Delta^i, \Theta^j) \in R^+_\Box$. By the construction, $(\Delta, \Theta) \in R^c_\Box$. Hence, by shift reflexivity, $(\Theta, \Theta) \in R^c_\Box$. But then, by the construction, $(\Theta^j, \Theta^j) \in R^+_\Box$ and we are done.

To prove Theorem 3, we use the second construction of copy-and-merge – see Section A.4. Otherwise, the reasoning is the same: first prove that R^c_{N} ($R^c_\Box$) satisfy the respective conditions, and next show how those conditions are transferred to R^+_{N} ($R^+_\Box$).

B.2 Transitivity of R_{N}

We now prove that the logic of frames with $R_\Box$ euclidian and R_{N} transitive is completely axiomatized by **DNS** plus the following axioms:

($5_\Box$) $\Diamond\varphi \supset \Box\Diamond\varphi$
(4_{N}) $\mathsf{N}\varphi \supset \mathsf{NN}\varphi$
($\mathrm{Trans}_{\mathsf{S}}$) $\neg\mathsf{N}\neg\mathsf{S}\varphi \supset \mathsf{S}\varphi$

The proof of soundness is again safely left to the reader. For completeness, we use the third type of copy-and-merge (see Appendix A.5). This gives us the **DNS**-model M^+ which is a smart copy-merge of M^c for the logic **DNS.5**$_\Box$**.4**$_{\mathsf{N}}$**.Trans**$_{\mathsf{S}}$. To show that M^+ satisfies the right conditions, we again proceed by two steps: first, show that $R^c_\Box$ is euclidian and that R^c_{N} is transitive. This is done in the standard way, relying on (respectively) the axioms ($5_\Box$) and (4_{N}). Second, show that (1) $R^+_\Box$ is euclidian and (2) R^+_{N} is transitive.

Proving (1) is straightforward in view of (a") and (b") in the definition of M^+. For the proof that R^+_{N} is transitive, suppose that $(\Delta^i, \Theta^j), (\Theta^j, \Lambda^k) \in R^+_{\mathsf{N}}$. Hence $(\Delta, \Theta), (\Theta, \Lambda) \in R^c_{\mathsf{N}}$ and hence, since R^c_{N} is transitive, (†) $(\Delta, \Lambda) \in R^c_{\mathsf{N}}$. We distinguish the following cases:

$k = 1$. By (b") and (†), we know at once that $(\Delta^i, \Lambda^k) \in R^+_{\mathsf{N}}$.
$k = 2$. By (b"), there is a ψ such that $\mathsf{S}\psi \in \Theta$ and $\psi \in \Lambda$. Hence by the construction of M^c, $\neg\mathsf{N}\neg\mathsf{S}\psi \in \Delta$ and hence, by (Trans$_{\mathsf{S}}$), $\mathsf{S}\psi \in \Delta$. But then also $(\Delta^i, \Lambda^k) \in R^+_{\mathsf{N}}$.

B.3 Completeness for $\mathbf{DNS^u.U}$ and $\mathbf{DNS^u.U.D_{\mathsf{N}}}$

To prove strong completeness for $\mathbf{DNS^u.U}$, we again use the third copy-and-merge technique (see Section A.5). Starting from a canonical model M^c for $\mathbf{DNS^u.U}$. This gives us a new model M^+. It can be easily checked that $R^+_{\Box}$ is an equivalence relation.

Note that by a standard argument (relying on the axiom (U)), we can show that ($\star$) if $(\Delta, \Theta) \in R^c_{\mathsf{N}}$, then for all $\Lambda \in R^c_{\Box}(\Delta)$, $(\Lambda, \Theta) \in R^c_{\mathsf{N}}$.

Suppose now that $(\Delta^i, \Theta^j) \in R^+_{\mathsf{N}}$ and let $\Lambda^k \in R^+_{\Box}(\Delta^i)$ be arbitrary. It follows that $(\Delta, \Theta) \in R^c_{\mathsf{N}}$ and $(\Delta, \Lambda) \in R^c_{\Box}$. By ($\star$), $(\Lambda, \Theta) \in R^c_{\mathsf{N}}$. If $j = 1$, then $(\Lambda^k, \Theta^j) \in R^+_{\mathsf{N}}$ by the construction. If $j \neq 1$, this means that there is a σ such that $\mathsf{S}\sigma \in \Delta$ and $\sigma \in \Theta$. Hence in view of the axiom (U), $\Box\mathsf{S}\sigma \in \Delta$ and hence $\mathsf{S}\sigma \in \Lambda$. But then by the construction, $(\Lambda^k, \Theta^j) \in R^+_{\mathsf{N}}$. So we have shown that

($\star\star$) if $(\Delta^i, \Theta^j) \in R^+_{\mathsf{N}}$, then for all $\Lambda^k \in R^+_{\Box}(\Delta^i)$, $(\Lambda^k, \Theta^j) \in R^+_{\mathsf{N}}$.

Next, consider an arbitrary maximal $\mathbf{DNS^u.U}$-consistent set Ξ. We know that $\Xi \in W$ and hence $\Xi^1 \in W^+$. Consider now the generated submodel M^Ξ of M^+, which consists of the following four elements:

(i) $W^\Xi = \{\Delta^i \mid (\Xi^1, \Delta^i) \in R^+_{\Box}\}$
(ii) $R^\Xi_{\Box} = R^+_{\Box} \cap (W^\Xi \times W^\Xi) = W^\Xi \times W^\Xi$
(iii) $R^\Xi_{\mathsf{N}} = R^+_{\mathsf{N}} \cap (W^\Xi \times W^\Xi)$
(iv) $v^\Xi(\psi) = v^+(\psi) \cap W^\Xi$ for all $\psi \in \mathcal{S}$

It can be easily observed that M^+ and M^Ξ are pointwise equivalent for all $\Delta^i \in W^\Xi$, and that $R^\Xi_{\Box} = W^\Xi \times W^\Xi$. Moreover, in view of ($\star\star$), we can derive that M^Ξ is uniform. Hence, we have obtained a $\mathbf{DNS^u.U}$-model which verifies all the members of Ξ in at least one world.

The completeness proof for $\mathbf{DNS^u.U.D_{\mathsf{N}}}$ is entirely analogous; one just needs to observe that R^c_{N} is serial and hence so is R^+_{N} and R^Ξ_{N}.

C Finite Model Property

For the proof of the finite model property, we combine the standard technique of filtration with the copy-and-merge variants that we used for strong completeness of **DNS** and its extensions. Below, we give the outline of the proof for all three types of construction. We start from the supposition that $\nVdash_{\mathbf{DNS}} \varphi$. Hence, there is an **DNS**-model $M = \langle W, R_\Box, R_{\mathsf{N}}, v\rangle$ and $t \in W$ such that $M, t \not\models \varphi$. We then construct a finite model M^f from M, such that also M^f falsifies φ at some state.

The three different variants defined below can be used to establish the finite model property for three different groups of extensions of **DNS**, which in turn correspond to three groups of frame conditions. The three groups of frame conditions are:

1. all frame conditions in Table 1
2. all frame conditions in Table 2
3. uniformity and seriality for R_{N}.

It then suffices to show that whenever M satisfies one or more conditions within one of these groups, so does M^f, when constructed according to the corresponding variant of the copy-and-merge technique. We safely leave this part of the proof to the reader.

Some notation: let Σ be the set of all subformulas of φ.[44] For all $w \in W$, let $|w| = \{v \in W \mid \text{ for all } \psi \in \Sigma : M, w \models \psi \text{ iff } M, v \models \psi\}$.

C.1 Filtration plus copy-and-merge version 1

Let $M^f = \langle W^f, R^f_\Box, R^f_{\mathsf{N}}, v^f\rangle$, where

(i) $W^f = \{|w|^1, |w|^2 \mid w \in W\}$
(ii) $R^f_\Box = \{(|w|^i, |v|^i) \mid (w, v) \in R_\Box\} \cup \{(|w|^i, |v|^j) \mid (w, v) \in R_\Box$ and there is no $\psi \in \Sigma : M, w \models \mathsf{S}\psi, M, v \models \psi\}$
(iii) $R^f_{\mathsf{N}} = \{(|w|^i, |v|^i) \mid (w, v) \in R_{\mathsf{N}}\}$
(iv) For all $\psi \in \Sigma$, $v^f(\psi) = \{|w|^i, |w|^j \mid M, w \models \psi\}$
(v) For all $\psi \in \mathcal{S} - \Sigma$, $v^f(\psi) = W^f$.

Since Σ is finite, W^f is also finite (it contains at most $2 \times 2^{|\Sigma|}$ nodes). To see that M^f is a **DNS**-model, suppose that $(|w|^i, |v|^j) \in R^f_{\mathsf{N}}$. By (ii), $i = j$ and there are $w' \in |w|, v' \in |v|$ with $(w', v') \in R_{\mathsf{N}}$. Hence, since M is a **DNS**-model, $(w', v') \in R^f_\Box$. So by (iii) and since $i = j$, $(|w|^i, |v|^j) \in R^f_\Box$. We now prove the following crucial lemma:

Lemma 1. *Where $i \in \{1, 2\}$, $\psi \in \Sigma$ and $w \in W$: $M, w \models \psi$ iff $M^f, |w|^i \models \psi$.*

Proof. We proceed by an induction on the complexity of ψ. The base case (ψ is a propositional variable) and the induction step for the classical connectives are safely left to the reader. It remains to prove the induction step for the three modal operators:

[44] Hence, $\varphi \in \Sigma$, and each of the following hold: if $\neg\psi \in \Sigma$, then $\psi \in \Sigma$; if $\psi \vee \tau \in \Sigma$, then $\psi, \tau \in \Sigma$; if $\dagger\psi \in \Sigma$ then $\psi \in \Sigma$ for $\dagger \in \{\mathsf{N}, \Box, \mathsf{S}\}$.

Case 1 $\psi = \Box\tau$. ($\Rightarrow$) Suppose that $M, w \not\models \Box\tau$. So there is a $v \in R_\Box(w)$ such that $M, v \not\models \tau$. By the definition of $R^f_\Box$, $|v|^i \in R_\Box(|w|^i)$ and by the induction hypothesis, $M^f, |v|^i \not\models \tau$. It follows that $M^f, |w|^i \not\models \Box\tau$.

($\Leftarrow$) Suppose that $M^f, |w|^i \not\models \Box\tau$. Hence, there is a $|v|^j \in R^f_\Box(|w|^i)$ such that $M^f, |v|^j \not\models \tau$. By the symmetry of the construction, $M^f, |v|^i \not\models \tau$. By the induction hypothesis, $M, v \not\models \tau$. By the definition of $R^f_\Box$, there are $v' \in |v|$ and $w' \in |w|$ such that $v' \in R_\Box(w')$. Since $\tau \in \Sigma$, $M, v' \not\models \tau$ and hence $M, w' \not\models \Box\tau$. Since $\Box\tau \in \Sigma$, $M, w \not\models \Box\tau$.

Case 2 $\psi = \mathsf{N}\tau$. Analogous to case 1; just replace every occurrence of $\Box$ (also in subscripts) with N.

Case 3 $\psi = \mathsf{S}\tau$. ($\Rightarrow$) Suppose that $M, w \models \mathsf{S}\tau$. Let $|v|^j \in R^f_\Box(|w|^i)$ be arbitrary such that $M^f, |v|^j \models \tau$ — we need to prove that $|v|^j \in R^f_\mathsf{N}(|w|^i)$. Note that by the induction hypothesis, $M, v \models \tau$ and hence by the construction $i = j$. By the definition of $R^f_\Box$, there is a $v' \in |v|$, $w' \in |w|$ such that $v' \in R_\Box(w')$. Since $w' \in |w|$, $M, w' \models \mathsf{S}\tau$. It follows that $v' \in R_\mathsf{N}(w')$ and hence $(|w'|^i, |v'|^i) = (|w|^i, |v|^i) = (|w|^i, |v|^j) \in R^f_\mathsf{N}$.

($\Leftarrow$) Suppose that $M, w \not\models \mathsf{S}\tau$. Hence, there is a $v \in R_\Box(w) \setminus R_\mathsf{N}(w)$ such that $M, v \models \tau$. It follows that, for no $\tau' \in \mathcal{W}$, $M, w \models \mathsf{S}\tau'$ and $M, v \models \tau'$. Let $i \neq j$. By the induction hypothesis and the construction of $R^f_\Box$ and R^f_N, $M^f, |v|^j \models \tau$ and $|v|^j \in R^f_\Box(|w|^i) \setminus R^f_\mathsf{N}(|w|^i)$. Hence, $M^f, |w|^i \not\models \mathsf{S}\tau$. $\Box$

C.2 Filtration plus copy-and-merge version 2

Let $M^f = \langle W^f, R^f_\Box, R^f_\mathsf{N}, v^f \rangle$, where

(i) $W^f = \{|w|^1, |w|^2 \mid w \in W\}$
(ii) $R^f_\Box = \{(|w|^i, |v|^j) \mid (w, v) \in R_\Box\}$
(iii) $R^f_\mathsf{N} = \{(|w|^i, |v|^j) \mid (w, v) \in R_\mathsf{N}$ and $i = j$ or there is a $\psi \in \Sigma : M, w \models \mathsf{S}\psi, M, v \models \psi\}$
(iv) For all $\psi \in \Sigma$, $v^f(\psi) = \{|w|^i, |w|^j \mid M, w \models \psi\}$
(v) For all $\psi \in \mathcal{S} - \Sigma$, $v^f(\psi) = W^f$.

Again, it is easily verified that M^f is a **DNS**-model. So we are left proving:

Lemma 2. *Where $i \in \{1, 2\}$, $\psi \in \Sigma$ and $w \in W$: $M, w \models \psi$ iff $M^f, |w|^i \models \psi$.*

Proof. Analogous to the proof for Lemma 1, except for the case where $\psi = \mathsf{S}\tau$:

($\Rightarrow$) Suppose that $M, w \models \mathsf{S}\tau$. Let $|v|^j \in R^f_\Box(|w|^i)$ be arbitrary such that $M^f, |v|^j \models \tau$ — we need to prove that $|v|^j \in R^f_\mathsf{N}(|w|^i)$. Note that by the induction hypothesis, $M, u \models \tau$ for all $u \in |v|$. Also, by the definition of $R^f_\Box$, there is a $v' \in |v|$, $w' \in |w|$ such that $v' \in R_\Box(w')$. Since $w' \in |w|$, $M, w' \models \mathsf{S}\tau$. It follows that (†) $v' \in R_\mathsf{N}(w')$. We now distinguish two cases:

3.1 $j = i$. By (†) and the definition of R^f_N, $|v|^j \in R_\mathsf{N}(|w|^i)$.
3.2 $j \neq i$. By (†) it suffices to note that, for all $v' \in |v|$ and all $w' \in |w|$, $M, v' \models \tau$ and $M, w' \models \mathsf{S}\tau$.

($\Leftarrow$) Suppose that $M, w \not\models \mathsf{S}\tau$. Hence, there is a $v \in R_\Box(w) \setminus R_\mathsf{N}(w)$ such that $M, v \models \tau$. By the induction hypothesis and the construction of $R^f_\Box$ and R^f_N, $M^f, |v|^j \models \tau$ and $|v|^j \in R^f_\Box(|w|^i) \setminus R^f_\mathsf{N}(|w|^i)$. Hence, $M^f, |w|^i \not\models \mathsf{S}\tau$. $\Box$

C.3 Filtration plus copy-and-merge version 3

Let $M^f = \langle W^f, R^f_\Box, R^f_\mathsf{N}, v^f\rangle$, where

(i) $W^f = \{|w|^1, |w|^2 \mid w \in W\}$
(ii) $R^f_\Box = \{(|w|^i, |v|^j) \mid (w, v) \in R_\Box\}$
(iii) $R^f_\mathsf{N} = \{(|w|^i, |v|^j) \mid (w, v) \in R_\mathsf{N}$ and $j = 1$ or there is a $\psi \in \Sigma : M, w \models \mathsf{S}\psi, M, v \models \psi\}$
(iv) For all $\psi \in \Sigma$, $v^f(\psi) = \{|w|^i, |w|^j \mid M, w \models \psi\}$
(v) For all $\psi \in \mathcal{S} - \Sigma$, $v^f(\psi) = W^f$.

We leave the proof of the following to the reader (it proceeds wholly analogously to the proof of Lemma 2):

Lemma 3. *Where $i \in \{1, 2\}$, $\psi \in \Sigma$ and $w \in W$: $M, w \models \psi$ iff $M^f, |w|^i \models \psi$.*

D Proof of Theorem 6

We will first prove the theorem for the base case where $\mathbf{DNS}^+ = \mathbf{DNS}$. So suppose that S does not occur in Γ or in $\varphi_1, \ldots, \varphi_n$. We need to prove:

$$\Gamma \Vdash_{\mathbf{DNS}} \mathsf{S}\varphi_1 \vee \ldots \vee \mathsf{S}\varphi_n \text{ iff } \Gamma \Vdash_{\mathbf{DNS}} \Box\neg\varphi_1 \vee \ldots \vee \Box\neg\varphi_n$$

The right to left direction is easy, in view of the fact that $\Box\neg\varphi \supset \mathsf{S}\varphi$ is a theorem in $\mathbf{DNS}$. For the other direction, we will need some more work.

Suppose that $\Gamma \nVdash_{\mathbf{DNS}} \Box\neg\varphi_1 \vee \ldots \vee \Box\neg\varphi_n$. Let $M = \langle W, R_\Box, R_\mathsf{N}, v\rangle$ and $w_0 \in W$ be such that $M, w_0 \models \psi$ for all $\psi \in \Gamma$ and $M, w_0 \models \neg\Box\neg\varphi_1, \ldots, \neg\Box\neg\varphi_n$. We construct $M' = \langle W', R'_\Box, R'_\mathsf{N}, v'\rangle$ as follows:

(i) $W' = \{w^1, w^2 \mid w \in W\}$
(ii) $R'_\Box = \{(w^i, v^j) \mid i, j \in \{1, 2\}, (w, v) \in R_\Box\}$
(iii) $R'_\mathsf{N} = \{(w^i, v^i) \mid (w, v) \in R_\mathsf{N}\}$
(iv) $v'(\psi) = \{w^1, w^2 \mid w \in v(\psi)\}$ for all $\psi \in \mathcal{S}$

We need to show that (a) for all ψ that do not contain S and all $w \in W$, $M, w \models \psi$ iff $M', w^i \models \psi$ and (b) for all $\psi \in \mathcal{W}_{\overline{\mathsf{S}}}$ and $w^i \in W'$ such that $M', w^i \models \neg\Box\neg\psi$, $M', w^i \not\models \mathsf{S}\psi$. By (a) and the supposition, $M', w^1_0 \models \psi$ for all $\psi \in \Gamma$. By (b), $M', w^1_0 \not\models \mathsf{S}\varphi_1 \vee \ldots \vee \mathsf{S}\varphi_n$. Hence, $\Gamma \nVdash_{\mathbf{DNS}} \mathsf{S}\varphi_1 \vee \ldots \vee \mathsf{S}\varphi_n$.
Ad (a) This is shown by an induction on the complexity of ψ. The base case ($\psi \in \mathcal{S}$) is trivial in view of (iv). So is the inductive step for the classical connectives. For $\psi = \Box\tau$, it suffices to observe that $v \in R_\Box(w)$ iff $v^1, v^2 \in R'_\Box(w^i)$, and hence, by the induction hypothesis, $M, w \models \Box\tau$ iff $M', w^i \models \Box\tau$.

For $\psi = \mathsf{N}\tau$, it suffices to observe that $v \in R_{\mathsf{N}}(w)$ iff $v^i \in R_{\mathsf{N}}(w^i)$ and hence, by the induction hypothesis, $M, w \models \mathsf{N}\tau$ iff $M', w^i \models \mathsf{N}\tau$.
Ad (b) Suppose that $M', w^i \models \neg\Box\neg\psi$ where ψ contains no occurrences of S. Hence, there is a $v^j \in R'_{\Box}(w^i)$ such that $M', v^j \models \psi$. Let $k \neq i$. By the construction, $v^k \in R'_{\Box}(w^i) \setminus R'_{\mathsf{N}}(w^i)$. By (a), $M, v \models \psi$ and hence $M', v^k \models \psi$. Hence, $M, w^i \not\models \mathsf{S}\psi$.

The generalization of this proof to obtain a proof for Theorem 6 is straightforward: we just need to note that whenever a condition on R_{N} ($R_{\Box}$) is preserved by copy (preserved by copy-merge), and whenever it holds for the model M of $\Gamma \cup \{\neg\Box\neg\varphi_1, \ldots, \neg\Box\neg\varphi_n\}$, then it will also hold for the model M' we constructed. But this is just what Definitions 4 and 5 tell us.

Note that in the above construction, R_{N} is not uniform. Hence, proving the same theorem for $\mathbf{DNS}^{\mathsf{u}}.\mathbf{U}$ and $\mathbf{DNS}^{\mathsf{u}}.\mathbf{U}.\mathbf{D}_{\mathsf{N}}$ requires a slightly different construction. Here we define $M' = \langle W', R'_{\Box}, R'_{\mathsf{N}}, v' \rangle$ as follows:

(i) $W' = \{w^1, w^2 \mid w \in W\}$
(ii) $R'_{\Box} = \{(w^i, v^j) \mid i, j \in \{1, 2\}, (w, v) \in R_{\Box}\}$ $(= W' \times W')$
(iii) $R'_{\mathsf{N}} = \{(w^i, v^1) \mid (w, v) \in R_{\mathsf{N}}\}$
(iv) $v'(\psi) = \{w^1, w^2 \mid w \in v(\psi)\}$ for all $\psi \in \mathcal{S}$

Note that this construction is not symmetric. We can now prove each of (a) and (b) as before, with two minor differences. For (a), the case $\psi = \mathsf{N}\tau$, one needs to observe that $v \in R_{\mathsf{N}}(w)$ iff $v^1 \in R_{\mathsf{N}}(w^i)$. For (b), we need to take $k = 2$ instead of $k \neq i$.

E Reducing DNS to $\mathbf{K_d}$

The left-right direction of the reduction theorem is straightforward: it suffices to check that for every **DNS**-axiom φ, $t(\varphi)$ is a **K**-axiom. We safely leave this to the reader.

For the other direction, a more elaborate argument is needed. Suppose that $\Gamma \not\Vdash_{\mathbf{DNS}} \varphi$. Hence, there is an **DNS**-model $M = \langle W, R_{\Box}, R_{\mathsf{N}}, v \rangle$ and $w_0 \in W$ such that $M, w_0 \models \Gamma$ and $M, w_0 \not\models \varphi$. The proof now proceeds in two steps.

First, we unravel the model M around the node w_0, obtaining a model $M' = \langle W', R'_{\Box}, R'_{\mathsf{N}}, v' \rangle$, where

(i) $W' = \{\langle w_0, \ldots, w_n \rangle \mid (w_0, w_1), \ldots, (w_{n-1}, w_n) \in R_{\Box}\}$
(ii) $R'_{\Box} = \{(\langle w_0, \ldots, w_i \rangle, \langle w_0, \ldots, w_i, w_{i+1} \rangle) \mid (w_i, w_{i+1}) \in R_{\Box}\}$
(iii) $R'_{\mathsf{N}} = \{(\langle w_0, \ldots, w_i \rangle, \langle w_0, \ldots, w_i, w_{i+1} \rangle) \mid (w_i, w_{i+1}) \in R_{\mathsf{N}}\}$
(iv) $v'(\psi) = \{\langle w_0, \ldots, w_n \rangle \mid w_n \in v(\psi)$

By standard means, we can prove that (a.1) for all $w_n \in W$ and all ψ, $M', \langle w_0, \ldots, w_n \rangle \models \psi$ iff $M, w_n \models \psi$; and (a.2) for all $x \in W'$, there is at most one $y \in W'$ with $x \in R_{\Box}(y)$.[45]

[45] See [6] for a general discussion of the technique of unraveling, including the means to derive (a.1) and (a.2).

In the second step of the proof, we transform M' into a $\mathbf{K_d}$-model $M'' = \langle W'', R''_{\Box}, v\rangle$, as follows: $W'' = W'$, $R''_{\Box} = R'_{\Box}$, for all $\psi \in \mathcal{S}$, $v''(\psi) = v'(\psi)$, and $v(d) = \{x \in W' \mid \text{ there is a } y \in W' : x \in R'_{\mathsf{N}}(y)\}$. By induction on the complexity of ψ, we can show that for all ψ and all $x \in W'$, $M', x \models \psi$ iff $M'', x \models t(\psi)$. In view of (a.1), $M'', \langle w_0\rangle$ verifies all members of $t(\Gamma \cup \{\neg\varphi\})$ whence we are done.

The unraveling is a necessary step in this proof – even if for the standard proof of the Andersonian reduction as spelled out e.g. in [4], this is not needed. The reason for this complication is that the present theorem concerns the entire language of **DNS**, which includes the operator $\Box$. As a result, we cannot just define the accessibility relation of our $\mathbf{K_d}$-model *ad libitum*: we have to take it over from the original **DNS**-model. This in turn makes it impossible to simply make d true at all worlds w that are deontically acceptable from *some* world. Instead, we first make sure by the unraveling that whenever a world w is deontically acceptable for a world, then it is deontically acceptable for exactly *one* such world. Only then do we apply the usual trick, viz. making d true in those worlds that are acceptable for their (unique) predecessor.

As promised, we also briefly outline the proof for the reduction of $\mathbf{DNS^{u}.U}$ to $\mathbf{S5_d}$. The translation proceeds in the same way as before. We leave it to the reader to check that, under this translation, all $\mathbf{DNS^{u}.U}$-axioms are valid in $\mathbf{S5_d}$.

For the other direction, suppose that $\Gamma \not\models_{\mathbf{DNS^{u}.U}} \varphi$. Let $M = \langle W, R_{\Box}, R_{\mathsf{N}}, v\rangle$ be an $\mathbf{DNS^{u}.U}$-model and $w \in W$, such that $M, w \models \psi$ for all $\psi \in \Gamma$ and $M, w \not\models \varphi$. Construct the $\mathbf{S5_d}$-model $M' = \langle W, R_{\Box}, v'\rangle$, putting $v'(\tau) = v(\tau)$ for all $\tau \in \mathcal{S}$ and $v'(\mathsf{d}) = \{w' \in W \mid R_{\mathsf{N}}(w, w')\}$. Then prove by induction that, for every τ and every $u \in W$, $M, u \models \tau$ iff $M', u \models t(\tau)$. The only difficult cases are $\tau = \mathsf{N}\psi$ and $\tau = \mathsf{S}\psi$; for these, we rely on the fact that $R_{\mathsf{N}}(u) = R_{\mathsf{N}}(w)$ for all $u \in W$.

For the reduction of $\mathbf{DNS^{u}.U.D_{\mathsf{N}}}$ to $\mathbf{S5_d} + \{\Diamond\mathsf{d}\}$, we can run the same argument. Observe that, since R_{N} is serial in M, $v'(\mathsf{d}) \neq \emptyset$.

F No Interpolation for $\mathbf{DNS.T_{\Box}}$

As we argued in the main text, $\mathsf{S}p \wedge p \Vdash_{\mathbf{DNS.T_{\Box}}} \mathsf{N}q \supset q$. Assume now that there is an interpolant for $\langle \mathsf{S}p \wedge p, \mathsf{N}q \supset q\rangle$ — let us call it φ. Note that φ contains no propositional variables. We prove two lemmas about every such φ:

Lemma 4. *For every* **DNS***-frame* $F = \langle W, R_{\Box}, R_{\mathsf{N}}\rangle$*, for all* $w \in W$*, and for all valuations* $v, v' : \mathcal{S} \to \wp(W)$*:* $\langle F, v\rangle, w \models \varphi$ *iff* $\langle F, v'\rangle, w \models \varphi$.

Proof. By an induction on the complexity of φ. Note that the base case is $\varphi = \bot$. For the induction step, we simply rely on the induction hypothesis and the semantic clauses for the various connectives and modal operators. $\Box$

Lemma 5. *Where* $F = \langle W, R_{\Box}, R_{\mathsf{N}}\rangle$*,* $w \in W$*, and* $v : \mathcal{S} \to \wp(W)$*:* $\langle F, v\rangle, w \models \varphi$ *iff* $w \in R_{\mathsf{N}}(w)$.

Proof. (⇒) Suppose that $w \notin R_{\mathsf{N}}(w)$. Let $v' : \mathcal{S} \to \wp(W)$ be such that $v'(q) = R_{\mathsf{N}}(w)$. It follows that $\langle F, v' \rangle, w \models \mathsf{N}q \wedge \neg q$ and hence $\langle F, v' \rangle, w \models \neg(\mathsf{N}q \supset q)$. Since φ is an interpolant for $\langle \mathsf{S}p \wedge p, \mathsf{N}q \supset q \rangle$, this means that $\langle F, v' \rangle, w \not\models \varphi$. Hence by Lemma 4, $\langle F, v \rangle, w \not\models \varphi$.

(⇐) Suppose that $w \in R_{\mathsf{N}}(w)$. Let $v' : \mathcal{S} \to \wp(W)$ be such that $v'(p) = w$. It follows that $\langle F, v' \rangle, w \models \mathsf{S}p \wedge p$. Since φ is an interpolant for $\langle \mathsf{S}p \wedge p, \mathsf{N}q \supset q \rangle$, this means that $\langle F, v' \rangle, w \models \varphi$. By Lemma 4, $\langle F, v \rangle, w \models \varphi$. □

Consider now two models $M^1 = \langle W^1, R^1_\Box, R^1_{\mathsf{N}}, v^1 \rangle$ and $M^2 = \langle W^2, R^2_\Box, R^2_{\mathsf{N}}, v^2 \rangle$, which are defined as follows:

1. where $i \in \{1, 2\}$: $W^i = \{w, w'\}$ and $R^i_\Box = W^i \times W^i$
2. where $i \in \{1, 2\}$ and $\varphi \in \mathcal{S}$: $v^i(\varphi) = W^i$
3. $R^1_{\mathsf{N}} = \{(w, w), (w', w')\}$
4. $R^2_{\mathsf{N}} = \{(w, w'), (w', w)\}$

Note that these models only differ in one respect, viz. the relation R_{N}. In M^1 this relation is reflexive; in M^2 it is not. We can now prove the following (by induction on the complexity of ψ):

Lemma 6. *For all ψ and $u \in \{w, w'\}$: $M^1, u \models \psi$ iff $M^2, u \models \psi$.*

But this means that, in particular, $M^1, w \models \varphi$ iff $M^2, w \models \varphi$. Since R^1_{N} is reflexive, we can derive by Lemma 5 that $M^1, w \models \varphi$. But then also $M^2, w \models \varphi$, which contradicts the fact that R^2_{N} is not reflexive and Lemma 5.

The proof for the failure of interpolation for **DNS.SR**$_\Box$ proceeds in a similar way; the only real difference concerns Lemma 5. The idea here is that φ expresses exactly that there is a $w' \in R_\Box(w)$ such that $w' \in R_{\mathsf{N}}(w')$. In the proof of the lemma, we define $v'(q) = \bigcup_{w' \in R_\Box(w)} R_{\mathsf{N}}(w')$ for the left to right direction, and $v'(p) = \{w'\}$ for the right to left direction. The construction of M^1 and M^2 can just as well be used for this case, since R^1_{N} is shift reflexive and R^2_{N} is not.

G Proof of Theorem 8

We will first give the outline of our proof for the base logic $\mathbf{DNS_O}$, which proceeds along the same lines as the soundness and completeness proof for **DNS**. One direction (soundness of $\mathbf{DNS_O}$ with respect to **DNS**) is easy and has been discussed in the main text. For the other direction (completeness w.r.t. the $O/P/\Box$-fragment of **DNS**), we rely on the soundness and completeness of **DNS**, so that it suffices to show that every $\mathbf{DNS_O}$-consistent set Γ is satisfiable at a point in a **DNS**-model. In order to arrive there, we use once more the copy-and-merge technique, adapting it to present needs.

Let $M = \langle W, R_\Box, R_{\mathsf{N}}, v \rangle$, where

(i) W is the set of all maximal consistent (w.r.t. $\mathbf{DNS_O}$) subsets of $\mathcal{W}_O$.
(ii) $(\Delta, \Theta) \in R_\Box$ iff $\{\varphi \mid \Box\varphi \in \Delta\} \subseteq \Theta$

(iii) $(\Delta, \Theta) \in R_{\mathsf{N}}$ iff $(\Delta, \Theta) \in R_{\Box}$ and there is a φ such that $P\varphi \in \Delta$, $\varphi \in \Theta$
(iv) $v(p) = \{\Delta \in W \mid p \in \Delta\}$ for all $p \in \mathcal{S}$

Let $M = \langle W^+, R^+_{\Box}, R^+_{\mathsf{N}}, v^+\rangle$ where

(i') $W^+ = \{\Delta^1, \Delta^2 \mid \Delta \in W\}$
(ii') $R^+_{\Box} = \{(\Delta^i, \Theta^j) \mid (\Delta, \Theta) \in R_{\Box}, i, j \in \{1, 2\}\}$
(iii') $R^+_{\mathsf{N}} = \{(\Delta^i, \Theta^j) \mid (\Delta, \Theta) \in R_{\mathsf{N}} \mid i = j \text{ or } \{\varphi \mid O\varphi \in \Delta\} \subseteq \Theta\}$
[iv') $v^+(p) = \{\Delta^1, \Delta^2 \mid \Delta \in v(p)\}$ for all $p \in \mathcal{S}$

The first step in the proof is to check that M^+ is a **DNS**-model. It suffices to check that $R^+_{\mathsf{N}} \subseteq R^+_{\Box}$, which is immediate in view of the construction.

Next, we need to establish the following version of the truth lemma:

Lemma 7. *For all $\varphi \in \mathcal{W}_O$ and where $i \in \{1, 2\}$: $M^+, \Delta^i \models \varphi$ iff $\varphi \in \Delta^i$.*

Proof. The base case and the induction step for the connectives and $\Box$ are a matter of routine – we safely leave this to the reader. So we are left with two cases:

Case 1: $\varphi = P\psi = \mathsf{S}\psi$. ($\Rightarrow$) Here, we can follow exactly the same reasoning as in the proof of Theorem 9, left-right direction of Case 3. ($\Leftarrow$) Suppose that $\mathsf{S}\varphi \in \Delta$. Let $\Theta^j \in R^+_{\Box}(\Delta^i)$ be such that $M^+, \Theta^j \models \varphi$. By the induction hypothesis, $\varphi \in \Theta$. Hence by items (iii) and (iii') of the construction, $\Theta^j \in R^+_{\mathsf{N}}(\Delta^i)$. Hence, $M^+, \Delta^i \models \mathsf{S}\varphi$.

Case 2: $\varphi = O\psi = \mathsf{N}\psi \wedge \mathsf{S}\psi$. ($\Rightarrow$) Suppose that $O\psi \notin \Delta$. If $P\psi \notin \Delta$, then we can infer at once (relying on case 1 of the present proof) that $M^+, \Delta^i \not\models P\psi$ and hence $M^+, \Delta^i \not\models O\psi$. So suppose moreover that $P\psi \in \Delta$. It follows that $\Box\psi \notin \Delta$ — otherwise, we can use (EQ$_P$), (Taut-Perm) and (EQ$_O$) to derive that $O\psi \in \Delta$, contradicting our initial supposition.[46] Hence, $\Diamond\neg\psi \in \Delta$.

We now distinguish two cases:

(a) there is no τ such that $O\tau \in \Delta$. Let $\Theta \in R_{\Box}(\Delta)$ be such that $\psi \notin \Theta$.[47] By item (iii') of the construction and $\Theta^i \in R^+_{\mathsf{N}}(\Delta^i)$. Hence, $M^+, \Delta^i \not\models \mathsf{N}\psi$ and hence $M^+, \Delta^i \not\models O\psi$.
(b) there is a τ such that $O\tau \in \Delta$. By axiom (Weakest-Perm), $\Box(\psi \supset \tau) \in \Delta$, but by (EQ$_O$) and since $O\psi \notin \Delta$, $\Box(\psi \equiv \tau) \notin \Delta$. Hence, $\Diamond(\tau \wedge \neg\psi) \in \Delta$. We can infer that there is a $\Theta \in R_{\Box}(\Delta)$ such that $\tau \in \Theta$, $\psi \notin \Theta$. Note that for all τ' such that $O\tau' \in \Delta$, also $\Box(\tau' \equiv \tau) \in \Delta$, and hence $\tau' \in \Theta$. In view of item (iii') of the construction, $\Theta^i \in R^+_{\mathsf{N}}(\Delta^i)$. Hence, $M^+, \Delta^i \not\models \mathsf{N}\psi$ and hence $M^+, \Delta^i \not\models O\psi$.

($\Leftarrow$) Suppose that $O\psi \in \Delta$. Hence, $\mathsf{N}\psi, \mathsf{S}\psi \in \Delta$ and hence by case 1 of the present proof, $M^+, \Delta^i \models \mathsf{S}\psi$. Also, by axiom (Weakest-Perm), for all τ, $S\tau \supset \Box(\tau \supset \varphi) \in \Delta$. Assume now that $M^+ \not\models \mathsf{N}\psi$. Hence by the induction hypothesis, there is a $\Theta^j \in R^+_{\mathsf{N}}(\Delta^i)$ such that $\psi \notin \Theta$. In view of item (iii') of the construction, this can only mean two things:

[46] This may go a little fast. Suppose that $P\psi, \Box\psi \in \Delta$. Hence, $\Box(\psi \equiv \top) \in \Delta$. Hence by (EQ$_P$), $P\top \in \Delta$ and hence by (Taut-Perm), $O\top \in \Delta$. Finally, by (EQ$_O$), $O\psi \in \Delta$.

[47] We can prove by standard means that there is such a Θ – this is usually called "the existence lemma". The same applies mutatis mutandis to the Θ that is used in case (b).

(c) There is a τ such that $\mathsf{S}\tau \in \Delta$, $\tau \in \Theta$. It follows that $\Box(\tau \supset \psi) \in \Delta$ and hence $\tau \supset \psi \in \Theta$ and hence $\psi \in \Theta$ — contradiction.
(d) For all τ such that $O\tau \in \Delta$, $\tau \in \Theta$. But then $\psi \in \Theta$ — contradiction again.

So we have shown that $M^+, \Delta^i \models \mathsf{N}\psi$ and hence $M^+, \Delta^i \models O\psi$. □

The extension of the above proof to cover the additional frame conditions, resp. axioms, is straightforward. For completeness, it suffices to check that whenever the axioms are added, the resulting canonical model will be one that satisfies the associated frame conditions. For soundness, it suffices to check that the axioms are valid whenever the conditions are in place.