Towards a Dialogic Interpretation of Dynamic Proofs[1]

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ABSTRACT. The main result presented in this paper concerns a dialogic or game-theoretical interpretation of dynamic proofs. Dynamic proofs in themselves do not form a demonstration of the derivability of their last formula from a given premise set. Apart from the proof, such a demonstration requires a specific metalevel argument. In a natural and appealing form, the metalevel argument is phrased in terms of the existence of a winning strategy for the proponent.

The aforementioned point is presented in terms of an approach that is in a sense Hilbertian and anti-Tarskian: the characterization of logical inference in terms of types of proofs, rather than in terms of properties of the consequence relation.

1 Introduction

Dialogic and game theoretical semantics are fascinating enterprises. Whether a formula A is a consequence of a premise set Γ may be understood and sometimes resolved in terms of a dialogue between a Proponent, who claims that $\Gamma \vdash_{\mathbf{L}} A$, and an Opponent, who denies it. The underlying idea of the games is not who wins the dialogue or game, but whether there exists a winning strategy for one of the players. Obviously, the rules of the dialogue have to be spelled out in a precise way. One has to delineate the moves allowed to each player in view of a certain state of the dialogue, the conditions under which the Proponent (respectively Opponent) wins, and the conditions under which the Proponent (possibly the Opponent) has a winning strategy.

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One of the fascinating aspects of dialogic is that it offers an interpretation of the idea of logical consequence: it is impossible that the conclusion is false if the premises are true. This is matched to: whatever the choices made by the Opponent, the Proponent has a winning strategy.

Of course computational aspects come into play. If the logic is decidable and $\Gamma \nvdash_{\mathbf{L}} A$, the Opponent will have a winning strategy. Sometimes there is only a positive test for the consequence relation (it is semi-decidable). In that case $\Gamma \nvdash_{\mathbf{L}} A$ has to be connected to the absence of a winning strategy for the Proponent.

While dialogic is fascinating in the case of usual logics, there are other logics for which there is not even a positive test. The proofs of such logics are typically dynamic: a formula derived at some point in a proof from Γ, may be considered not to be a consequence of Γ at a later point in the proof in view of the insights in the premises that were gained by continuing the proof. The dynamics need not stop there: at a still later point, the proof may provide further insights in the premise set Γ and, as a result of this, it is possible that the formula has to be considered again as a consequence of Γ. Incidently, there are many such logics and they explicate reasoning processes that occur frequently in scientific as well as in everyday context.

It will be shown in this paper that it is natural to apply a dialogic or game theoretical approach to consequence relations defined by dynamic proofs. I shall do so by discussing the dynamic proofs I am most familiar with, viz. those of adaptive logics.

First static proofs will be introduced in Section 2 and dynamic proofs in Section 3. Although everyone can recite Hilbert's definition of a proof, the extent to which logicians rely on implicit presuppositions concerning proofs is striking. I try to repair this in these sections. Next, I shall briefly introduce adaptive logics in Section 4. This will enable me to spell out the dialogic approach for a sufficiently concrete and studied family of logics, but nevertheless to do so for a very large set of logics. The dialogues themselves are presented and discussed in Section 5. Some open problems are mentioned in Section 6.

2 Static Proofs

Let $\mathcal{L}$ be a language with a denumerable alphabet and $\mathcal{W}$ its set of closed formulas. A *logic* $\mathbf{L}$ is a function that maps every set of closed formulas to a set of closed formulas, $\mathbf{L}\colon \wp(\mathcal{W}) \rightarrow \wp(\mathcal{W})$. In other words, a logic assigns a consequence set to every set of formulas. The $\mathbf{L}$-consequence set of Γ will be denoted by $Cn_{\mathbf{L}}(\Gamma)$. Alternatively, that A is a $\mathbf{L}$-consequence of Γ is denoted by $\Gamma \vdash_{\mathbf{L}} A$. Note that $Cn_{\mathbf{L}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$.[1]

By a *rule* I shall mean a metalinguistic expression of the form "to derive A from Σ",[2] in which A is a metalinguistic formula and Σ is a recursive (or decidable) set of metalinguistic formulas. The rule "to derive A from Σ" is *finitary* iff Σ is a finite set. If Σ is empty, A is usually called an axiom schema.

A rule is *derivable* from a set $\mathcal{R}$ of rules iff it does not belong to $\mathcal{R}$ but every result of its application can be obtained by applications of rules belonging to $\mathcal{R}$. Thus the following annotated metalinguistic proof demonstrates that the commutativity of conjunction (to derive $B \wedge A$ from $A \wedge B$) is derivable from a set of rules that contains Simplification and Adjunction.[3]

1	$A \wedge B$	...
2	A	1; Simplification
3	B	1; Simplification
4	$B \wedge A$	2, 3; Adjunction

Dynamic proofs are most easily described in terms of annotated proofs. For the sake of comparison, let me describe static proofs in terms of annotated proofs as well. Lines of static annotated proofs will be composed of a line number, a formula, and a justification. The justification of a line l consists of a (possibly empty) set of line numbers N_l and the name of a rule R_l. Lines at which members of the premise set are introduced are justified by the Premise rule. Given a set $\mathcal{R}$ of rules and a list[4] of lines, a line l in the list is $\mathcal{R}$*-correct* iff (i) all members of N_l precede l in the list, (ii) $R_l \in \mathcal{R}$, and (iii) the formula of l is obtained by application of R_l to the formulas of the lines N_l.

DEFINITION 1. A (static) $\mathcal{R}$*-proof from* (the premise set) Γ is a list of $\mathcal{R}$*-correct* lines.

Where the rule "to derive A from Σ" is a member of a set $\mathcal{R}$ of rules, the rule is *redundant* (with respect to $\mathcal{R}$) iff there is a $\Sigma' \subset \Sigma$ for which "to derive A from Σ'" is derivable from $\mathcal{R}$. A set of rules $\mathcal{R}$ is *minimal* iff, for every $\mathcal{R}' \subset \mathcal{R}$, some $\mathcal{R}$-proof is not a $\mathcal{R}'$-proof.

THEOREM 2. *If $\mathcal{R}$ is a minimal set of rules that characterizes $\mathcal{R}$-proofs, then all members of $\mathcal{R}$ are finitary.*

[1] I shall use either notation as is simplest in a specific context. It is instructive to rephrase statements in one notation into the other.

[2] The characters $\mathsf{A}, \mathsf{B}, \ldots$ will be used as metametalinguistic variables for metalinguistic formulas.

[3] As presented below in the text, the demonstration presupposes static proofs, defined later in this section.

[4] A list is an enumeration of a set in which each member of the set is associated with a positive integer, which indicates its place in the list—see [10, Ch. 1].

Proof. Consider a $\mathcal{R}$-proof. In view of Definition 1, the proof is a list of lines. If a member of $\mathcal{R}$ would be non-finitary, its resulting formula would have to appear in the proof *after* the infinitely many formulas to which the rule is applied. So applying the rule would not result in a list of formulas. The theorem follows in view of the minimality of $\mathcal{R}$. ■

DEFINITION 3. A (static) $\mathcal{R}$-*proof of* A *from* Γ is a $\mathcal{R}$-proof from Γ in which A is the formula of the last line.

The *syntactic characterization of a logic* $\mathbf{L}$ is often identified with the logic itself. In this case proofs are named after the logic rather than after a set of rules characterizing the logic.

DEFINITION 4. A logic $\mathbf{L}$ *has finite and static proofs* iff there is a recursive set $\mathcal{R}$ of non-redundant rules such that $\Gamma \vdash_{\mathbf{L}} A$ iff there is a $\mathcal{R}$-proof of A from Γ.

That a logic $\mathbf{L}$ has finite and static proofs has a number of interesting and easily provable consequences. Let $\mathcal{R}_{\mathbf{L}}$ be a recursive and minimal set of non-redundant rules such that there is a $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ iff $\Gamma \vdash_{\mathbf{L}} A$. Note that there is a set $\mathcal{R}_{\mathbf{L}}$ whenever $\mathbf{L}$ has finite and static proofs.

THEOREM 5. *If* $\mathbf{L}$ *has finite and static proofs, then* $\mathbf{L}$ *is Compact (if* $A \in Cn_{\mathbf{L}}(\Gamma)$ *then* $A \in Cn_{\mathbf{L}}(\Gamma')$ *for some finite* $\Gamma' \subseteq \Gamma$*).*

Proof. Suppose that the antecedent is true and that $\Gamma \vdash_{\mathbf{L}} A$. In view of Definitions 3 and 4 there is a finite $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ. So A is a consequence of finitely many members of Γ. ■

THEOREM 6. *If* $\mathbf{L}$ *has finite and static proofs, then* $\mathbf{L}$ *is Reflexive (*$\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$*).*

Proof. Suppose that the antecedent is true and that $A \in \Gamma$. By Definition 1, the list comprising a single line on which A is derived by the Premise rule is a $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ. So $A \in Cn_{\mathbf{L}}(\Gamma)$ by Definition 4. ■

THEOREM 7. *If* $\mathbf{L}$ *has finite and static proofs, then* $\mathbf{L}$ *is Transitive (if* $A \in Cn_{\mathbf{L}}(\Delta)$ *and* $\Delta \subseteq Cn_{\mathbf{L}}(\Gamma)$*, then* $A \in Cn_{\mathbf{L}}(\Gamma)$*).*

Proof. Suppose that the antecedent is true and that $A \in Cn_{\mathbf{L}}(\Delta)$ and $\Delta \subseteq Cn_{\mathbf{L}}(\Gamma)$. In view of Theorem 5, there is a finite $\Delta' \subseteq \Delta$ such that $\Delta' \subseteq Cn_{\mathbf{L}}(\Gamma)$ and $A \in Cn_{\mathbf{L}}(\Delta')$. Let $\Delta' = \{B_1, \ldots, B_n\}$. In view of Definitions 4 and 1 there is, for each B_i $(1 \leq i \leq n)$, a $\mathcal{R}_{\mathbf{L}}$-proof p_i of B_i from Γ and there is a $\mathcal{R}_{\mathbf{L}}$-proof of A from Δ'. Let p_{n+1} be the result of deleting from the latter proof all lines on which a member of Δ' is introduced by the premise rule. The list obtained by concatenating $\mathsf{p}_1, \ldots, \mathsf{p}_{n+1}$ is easily seen to be a $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ. So $A \in Cn_{\mathbf{L}}(\Gamma)$ by Definition 4. ■

THEOREM 8. *If* $\mathbf{L}$ *has finite and static proofs, then* $\mathbf{L}$ *is Monotonic* ($Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$ *for all* Γ'*).*

Proof. In view of Definition 3, every $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ is a $\mathcal{R}_{\mathbf{L}}$-proof of A from $\Gamma \cup \Gamma'$. So the theorem follows by Definition 4. ■

LEMMA 9. *If* $\mathbf{L}$ *has finite and static proofs, all possible lines that occur in* $\mathcal{R}_{\mathbf{L}}$*-proofs can be written as finite strings of a denumerable alphabet.*

Proof. This is obvious for the line number and the formulas. The justification of a line contains a finite set of line numbers (in view of Theorem 2) and the name of a rule. So all line numbers involved can be written as a finite string of a finite alphabet and as $\mathcal{R}_{\mathbf{L}}$ is a denumerable set, finite strings of a finite alphabet are sufficient to name all rules. The three elements of a line and the elements of the justification can obviously be separated by finitely many symbols. ■

So we use a denumerable alphabet to write proof lines as finite strings. Actually, if the lemma would not hold, humans would not be able to write proofs.

There is a *positive test* for a logic $\mathbf{L}$ ($\mathbf{L}$ is *semi-decidable*) iff there is a mechanical procedure that, for every decidable Γ and A, leads after finitely many steps to the answer YES iff $\Gamma \vdash_{\mathbf{L}} A$ (but may not provide an answer at any finite point if $\Gamma \nvdash_{\mathbf{L}} A$).

THEOREM 10. *If* $\mathbf{L}$ *has finite and static proofs, then there is a positive test for* $\mathbf{L}$*.*

Proof. Suppose that Γ is a decidable set of formulas and that $\Gamma \vdash_{\mathbf{L}} A$. In view of Definitions 3 and 4, there is a finite $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ.

All finite lists of finite strings of the alphabet in which proofs are written can be ordered into a list L. It is well-known (and easily seen) to be decidable whether a member of L is a $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ. As some member of L is bound to be a $\mathcal{R}_{\mathbf{L}}$-proof of A from Γ, we shall find it after finitely many steps. ■

The upshot is that logics that have finite and static proofs are logics of the usual kind. It is also the case, although not proved here, that most usual logics have finite and static proofs. Not all known logics have finite and static proofs. A well-know exception is second order logic, which is not compact. An obvious example, taken from [10, p. 283] concerns the premise set comprising second order axioms for arithmetic (roughly Peano arithmetic plus the second order axiom of mathematical induction) together with all formulas of the form $\sim c = i$ (for $i \in \{0, 0', 0'', \ldots\}$ and c a constant that is added to the language of arithmetic). This set is inconsistent and hence $0 = 0'$ is derivable from it (by second order logic), but (on the supposition that arithmetic is consistent) $0 = 0'$ is not derivable from any subset of the premises. In order to define proofs for second order logic, one might try to modify Definition 3 to the following.

DEFINITION 11. Where $\mathcal{R}$ is a set of rules, a *$\mathcal{R}$-proof of A from Γ* is a proof from Γ containing a line of which A is the formula.

Definition 1 merely requires the existence of a *list* of lines, possibly an infinite one. By changing Definition 3 to Definition 11, one might hope to allow for infinite proofs. However, this will not work, as is obvious from the above example. The application of any set of rules could only lead to deriving $0 = 0'$ from the premise set *after* all premises were introduced in the proof (because every subset of the premise set is consistent). But if all premises occur in the proof, there are no positive integers left to associate with $0 = 0'$.

There are obviously ways around this. For example, one might allow for proofs in which consequences of a (proper or improper) subset of the premise set are introduced directly, that is without the premises being introduced. This does not solve the whole problem: an infinitary rule might not enable one to derive a formula A from the premises, but might enable one to derive A from an infinite set of consequences of the premises. Rather than continuing this discussion, let us move on.

By all means, the above trouble with second order logic (and with infinitary rules in general) does not entail that one cannot consider infinite proofs in the sense of Definition 1. Although one obviously cannot write down such proofs, we may consider them and reason about them, as we shall see in Section 3.

Before leaving the matter, let me mention non-annotated proofs. These are simply lists of formulas. The easy exercise of adjusting Definitions 1–4 to non-annotated proofs is left to the reader.

3 Dynamic Proofs

Dynamic proofs may be realized in several ways. I shall present the way they grew out of the work of my research group during the last twenty years. The proofs are those of adaptive logics. It is a long term aim of the research group to characterize all forms of dynamic reasoning. Perhaps it is possible to do so in terms of adaptive logics. Otherwise new types of logics will have to be devised.

Dynamic proofs are meant to characterize a logic, in other words a consequence relation. This means that the dynamics should be handled in a controlled way. It should not depend on decisions of the person constructing the proof, but on something ‘objective’. In the peculiar form of dynamic proofs considered here, this is realized as follows. First, the rules will allow one to derive some formulas on a *condition*. Next, there is a *Marking definition* which determines whether a line is *marked* or *unmarked* at some stage of the proof—stages will be defined in a precise way below. The definition proceeds in terms of the conditions and of the occurrence of certain formulas at the stage. The formula of a line is considered as derived when it is unmarked. To make the annotated proofs transparent, their lines are quadruples consisting of a line number, a formula, a justification, and a condition. The condition is a set of formulas—the expression “the formula of a line” will always refer to the second element of the line. In the next section, I shall present specific rules, which introduce and handle conditions, as well as specific marking definitions. In the present section, I use these notions in a more abstract way.

The rules and the Marking definition will be kept strictly apart. The rules determine which lines may be added to a proof and do not in any way interfere with the marks. The Marking definition determines which lines are marked at a stage and which unmarked.

As a dynamic proof proceeds, marks may come and go. So in order to describe dynamic proofs, we have to consider the relation between consecutive lists of lines. These will be identified with the stages of a proof. Let $\mathcal{R}$ denote a set of rules as before.[5]

DEFINITION 12. A *$\mathcal{R}$-stage* from Γ is a list of *$\mathcal{R}$-correct* lines.

To facilitate terminology, I shall consider the empty list as a stage of every $\mathcal{R}$-proof from every premise set.

DEFINITION 13. Where L and L' are $\mathcal{R}$-stages from Γ, L' is an *extension* of L iff all elements that occur in L occur in the same order in L'.

Normally, the line resulting from the application of a rule to members of L is appended to L. This will not be required for dynamic proofs. Indeed, we shall soon see a reason to allow for the insertion of lines between the lines of a previous stage (in such a way that the resulting list is a $\mathcal{R}$-stage from the premise set according to Definition 12).

DEFINITION 14. A (dynamic) *$\mathcal{R}$-proof* from Γ is a chain of $\mathcal{R}$-stages from Γ, the first element of which is the empty list and all other elements of which are extensions of their predecessors.

[5] I am indebted to Andrzej Wiśniewski who convinced me, several years ago, to see a proof as a chain of stages.

Note that a $\mathcal{R}$-proof from Γ may be finite or infinite. Consider the premise set $\{p_i, p_i \supset q_i \mid i \in \mathbb{N}\}$ and let Modus Ponens belong to $\mathcal{R}$. Some $\mathcal{R}$-proof contains a stage in which occur the following three lines for all $i \in \mathbb{N} - \{0\}$:

$3i-2$	p_i	Premise
$3i-1$	$p_i \supset q_i$	Premise
$3i$	q_i	3i-2, 3i-1; Modus Ponens

It is often convenient to use the following terminology. A $\mathcal{R}$-proof that has s as its last stage will also be called a proof *at stage* s. Where no confusion arises, I shall sometimes call finite stages by the number of the line that was last added to them. This will also enable me to refer to stages that are predecessors of the present stage.

The proof of Theorem 15 is obvious in view of that of Theorem 2 (which sounds identical but concerns static proofs).

THEOREM 15. *If* $\mathcal{R}$ *is a minimal set of rules that characterizes* $\mathcal{R}$*-proofs, then all members of* $\mathcal{R}$ *are finitary.*

We shall see in the next section that, in some cases, there are reasons to consider extensions of infinite stages (stages comprising infinitely many lines). Are there such extensions? Clearly, if a stage s comprises infinitely many lines, no extension of s can be obtained by appending a line to s, because this does not result in a list of formulas. Recall, however, that we have only to consider finitary rules (in view of Theorem 15). It follows that all formulas required to apply some rule occur at a finite point in s; let the last formula occur at the nth line of s. So the result of the application can be inserted between the nth and $(n+1)$th line of s. It is unimportant whether one renumbers the lines from $n+1$ on, or gives the inserted line an unusual number, say $n.1$. All that matters is that the extension of s is a list of formulas.

People who might have principled objections against the insertion of lines in a proof should realize that the result of the insertion may obviously also be obtained by appending only. In a sense, the resulting objects, viz. the stages, are identical; only their history, viz. the chain of stages, is different.

DEFINITION 16. A is *derived at stage* s of a $\mathcal{R}$-proof from Γ iff A is the formula of a line of s and this line is unmarked at stage s.

DEFINITION 17. A is $\mathcal{R}$*-derivable at a stage* from Γ iff A is derived on a line of a stage of a $\mathcal{R}$*-proof* from Γ.

A formula may be derivable at one stage and underivable at the next, or vice versa. This is typical of dynamic proofs and there is nothing wrong with it. Yet we also want to define a more stable notion of derivability to express where the dynamics leads to in the end—whether we are able to find out where the dynamics leads to is a different matter. This stable notion will be called final derivability.

DEFINITION 18. Where p is a $\mathcal{R}$-proof from Γ at stage s, p is *stable with respect to* line i iff (i) line i occurs in s and (ii) if line i is marked, respectively unmarked, at stage s, then it is marked, respectively unmarked, in all extensions of s.

The intuitive idea behind final derivability is that A is derived from Γ on an unmarked line of a stage of a $\mathcal{R}$-proof from Γ and that the proof is stable with respect to that line.

Note that the final derivability of A from Γ cannot be warranted by any proof in itself. One always needs a proof together with a *metalevel reasoning* about all possible extensions of this proof. This should be taken into account when forging a definition of final derivability.

There are A and Γ for which the following holds: only infinite $\mathcal{R}$-proofs from Γ contain an unmarked line on which A is derived *and* are stable with respect to this line. That is obviously inconvenient. Such a proof can be reasoned about, at the metalevel, but it cannot be produced. This is a good reason to look for a different approach. However, it is difficult to do so at the abstract level of the present section. So let us postpone this to the next section, in which a specific family of logics is presented. If you wonder what concrete dynamic proofs look like, you will obtain an answer there.

Before leaving this section, let me point out that static proofs are a special case of dynamic proofs. Static proofs too may be seen as chains of stages. They are just like dynamic proofs except that all lines have $\emptyset$ as their condition and that no line is ever marked. As a consequence of this, derivability at a stage coincides with final derivability.

4 Adaptive Logics

The motivation for studying adaptive logics cannot be presented here. I refer the reader for example to [4].

Nearly all known adaptive logics have been phrased in *standard format*,[6] which has major advantages as will become clear below. The format was first introduced in [3] and most extensively studied in [5], which contains details and metatheoretic proofs. Not too long from now, the best reference should be [6]. From the next paragraph on, I disregard adaptive logics that are not in standard format. So all claims on adaptive logics should be read as claims on adaptive logics in standard format (even if some claims hold

[6] The only exception is Graham Priest's $\mathbf{LP}^m$ from [11], which at the predicative level defines abnormalities with respect to models rather than with respect to the formulas verified by models. See [2] for a discussion of some odd effects.

for all adaptive logics).

While describing the standard format, I shall illustrate it with two related logics, which I shall use for examples in Section 5. The information on these logics is presented in separate paragraphs.

An adaptive logic **AL** in standard format is defined by a triple:

1. A *lower limit logic* **LLL**: a logic that has finite and static proofs, has a characteristic semantics, and contains **CL** (Classical Logic).[7]

2. A *set of abnormalities* Ω: a set of **LLL**-contingent formulas, characterized by a (possibly restricted) logical form F which contains at least one logical symbol.

3. An *adaptive strategy*: Reliability or Minimal Abnormality.[8]

Examples Two related adaptive logics are $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$. Their lower limit logic is **CLuN** (*C*lassical *L*ogic allowing for gl*u*ts with respect to *N*egation), viz. full positive **CL** with $(A \supset {\sim}A) \supset {\sim}A$ added as the only axiom for the standard negation, and extended[9] with classical negation $\check{\sim}$—see note 7. All other standard symbols have the same meaning as the classical symbols. While $A \vee {\sim}A$ is a **CLuN**-theorem, $A \wedge {\sim}A$ is **CLuN**-contingent. The set of abnormalities Ω comprises all formulas of the form $\exists(A \wedge {\sim}A)$ (the existential closure of $A \wedge {\sim}A$). The strategies are respectively Reliability and Minimal Abnormality (as the superscripts reveal).

Extending **LLL** with an axiom that declares all abnormalities logically false results in the *upper limit logic* **ULL**. If a premise set Γ does not require that any abnormalities are true, the **AL**-consequences of Γ are identical to its **ULL**-consequences. If the premise set requires some abnormalities to be true, the **AL**-consequence set is stronger than the **LLL**-consequence set (except for border cases) and is weaker than the **ULL**-consequence set.

Examples The upper limit logic of $\mathbf{CLuN}^r$ and of $\mathbf{CLuN}^m$ is **CL**.

[7]That **LLL** contains **CL** is realized by adding classical logical symbols (those having the same meaning as in **CL**) to the language. These will be written as $\check{\sim}$, $\check{\vee}$, $\check{\exists}$, etc. The classical symbols have mainly a technical use and are not meant to occur in the premises or conclusions of standard applications.

[8]Strategies are ways to cope with derivable disjunctions of abnormalities. The effects of Reliability and Minimal Abnormality become clear below in the text. Other strategies than Reliability and Minimal Abnormality were developed mainly in order to characterize consequence relations from the literature in terms of an adaptive logic. All those strategies can be reduced to Reliability or Minimal Abnormality under a translation.

[9]Suitable axioms are $(A \supset \check{\sim}A) \supset \check{\sim}A$ and $A \supset (\check{\sim}A \supset B)$.

In the expression $Dab(\Delta)$, Δ will always be a finite subset of Ω, and $Dab(\Delta)$ will denote the *classical* disjunction (see note 7) of the members of Δ. $Dab(\Delta)$ is called a *Dab*-formula (a disjunction of abnormalities).

Adaptive logics have dynamic proofs in the sense of the previous section. The rules of adaptive logics in standard format are defined in terms of the lower limit logic. Where

$$A \qquad \Delta$$

abbreviates that A occurs in the proof on the condition Δ, the (generic) rules are:

PREM If $A \in \Gamma$:

$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Theta)$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

We need some technicalities in preparation of the marking definitions. $Dab(\Delta)$ is a *minimal Dab-formula* at stage s of an **AL**-proof iff $Dab(\Delta)$ has been derived at that stage on the condition $\emptyset$ whereas there is no $\Delta' \subset \Delta$ for which $Dab(\Delta')$ has been derived on the condition $\emptyset$. A *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ is a set that contains an element out of each member of Σ. A *minimal choice set* of Σ is a choice set of Σ of which no proper subset is a choice set of Σ. Consider a proof from Γ at stage s and let $Dab(\Delta_1)$, $\ldots$, $Dab(\Delta_n)$ be the minimal *Dab*-formulas at that stage. $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$ and $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \ldots, \Delta_n\}$.

DEFINITION 19. Marking for Reliability: Line i is marked at stage s iff, where Δ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.

DEFINITION 20. Marking for Minimal Abnormality: Line i is marked at stage s iff, where A is derived on the condition Δ at line i, (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.

This reads more easily: where A is derived on the condition Δ on line i, line i is *unmarked* at stage s iff (i) there is a $\varphi \in \Phi_s(\Gamma)$ for which $\varphi \cap \Delta = \emptyset$

and (ii) for every $\varphi \in \Phi_s(\Gamma)$, there is a line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.

The sense of the marking definitions (and their relation to the semantics) is studied in other papers, for example [5], and cannot be discussed here.

What I can do here, now that we are considering more peculiar logics, viz. adaptive logics in standard format, is to illustrate the different approach to final derivability, as promised in the previous section.

As we have seen in the previous section, final derivability is established by a proof and a metalevel reasoning. The existence of the proof should of course not be established at the metalevel. So the proof should be finite.

I now present two definitions, show them to be adequate, and add some comments. In the definitions, "proof" obviously refers to an **AL**-proof, or rather to a proof defined in terms of the generic rules Prem, RU and RC.[10]

DEFINITION 21. A is *finally derived* on line i of an **AL**-proof from Γ at stage s iff (i) A is the formula of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.

DEFINITION 22. $\Gamma \vdash_{\mathbf{AL}} A$ (A is *finally* **AL**-*derivable* from Γ) iff A is finally derived on a line of a proof from Γ.

Let me first show that this definition is adequate with respect to the intuitive understanding of final derivability—see the previous section. To do so we need some preparation. $Dab(\Delta)$ is a *minimal Dab-consequence of* Γ iff $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta')$. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-consequences of Γ, $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$ and $\Phi(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \Delta_2, \ldots\}$.[11]

Let $\mathbf{AL}^r$ and $\mathbf{AL}^m$ be adaptive logics the third element of which is Reliability, respectively Minimal Abnormality. Theorems 23 and 24 are proved as Theorems 6 and 8 in [5].[12]

THEOREM 23. $\Gamma \vdash_{\mathbf{AL}^r} A$ *iff there is a (finite)* $\Delta \subset \Omega$ *for which* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ *and* $\Delta \cap U(\Gamma) = \emptyset$.

THEOREM 24. $\Gamma \vdash_{\mathbf{AL}^m} A$ *iff, for every* $\varphi \in \Phi(\Gamma)$*, there is a* $\Delta \subset \Omega$ *such that* $\Delta \cap \varphi = \emptyset$ *and* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$.

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[10] The set of rules generated by the three generic rules contains redundant rules and is obviously not minimal. However, this is irrelevant for the point I am making below in the text.

[11] It is useful to compare the definition of $U(\Gamma)$ with that of $U_s(\Gamma)$ and to compare the definition of $\Phi(\Gamma)$ with that of $\Phi_s(\Gamma)$. In each case, the latter set is an estimate of the former depending on the insights provided by the proof at a stage.

[12] In the theorems, I write $\check{\vee}$ because the standard disjunction might be abnormal in the specific adaptive logic. In the adaptive logics $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$, the standard negation is classical and hence $\check{\vee}$ can be safely replaced by $\vee$.

There are only countably many minimal *Dab*-consequences of Γ, say $\{Dab(\Delta_1), Dab(\Delta_2), \ldots\}$. For each of these, there is a finite **AL**-proof, say p_i, in which $Dab(\Delta_i)$ is derived on the condition $\emptyset$. The resulting set $\{\mathsf{p}_1, \mathsf{p}_2, \ldots\}$ is countable.

LEMMA 25. *If* $\Gamma \vdash_{\mathbf{AL}^r} A$*, then there is an* $\mathbf{AL}^r$*-proof from* Γ *in which* A *is derived on an unmarked line and that is stable with respect to that line.*

Proof. Suppose that $\Gamma \vdash_{\mathbf{AL}^r} A$. By Theorem 23 there is a (finite) $\Delta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. As **LLL** has finite and static proofs, there is a finite $\mathbf{AL}^r$-proof in which $A \check{\vee} Dab(\Delta)$ is derived on the condition $\emptyset$. From this A is derived on the condition $Dab(\Delta)$ (in one step by RC), say on line i. Let this be an $\mathbf{AL}^r$-proof at the finite stage s and call this proof p_0.

Consider the proof p' of which the last stage, call it s', is the concatenation $\langle \mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_2, \ldots\rangle$. As all minimal *Dab*-consequences of Γ have been derived on the condition $\emptyset$ in s', $U_{s'}(\Gamma) = U(\Gamma)$. As $\Delta \cap U(\Gamma) = \emptyset$, line i is unmarked. Moreover, as all minimal *Dab*-consequences of Γ have been derived on the condition $\emptyset$ in s', line i will be unmarked in every extension of s'. So p' is stable with respect to line i. ■

For some Γ, $\Phi(\Gamma)$ is uncountable. However, the set of Δ such that, for some $\varphi \in \Phi(\Gamma)$, $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$, is a countable set—each of *these* Δ is a finite set of formulas. Moreover, for each such Δ, there is a finite proof of $A \check{\vee} Dab(\Delta)$. Let $\{\mathsf{p'}_1, \mathsf{p'}_2, \ldots\}$ be the countable set of these proofs. The proof of Lemma 26 proceeds exactly as that of Lemma 25, except that we now define p' as a proof that has as stage s' the concatenation $\langle \mathsf{p}_1, \mathsf{p'}_1, \mathsf{p}_2, \mathsf{p'}_2, \ldots\rangle$, which warrants that $\Phi_{s'}(\Gamma) = \Phi(\Gamma)$ and that, for every extension s^* of s', $\Phi_{s^*}(\Gamma) = \Phi(\Gamma)$.

LEMMA 26. *If* $\Gamma \vdash_{\mathbf{AL}^m} A$*, then there is an* $\mathbf{AL}^m$*-proof from* Γ *in which* A *is derived on an unmarked line and that is stable with respect to that line.*

Whether the third element of an adaptive logic is Reliability or Minimal Abnormality, the following lemma holds.

LEMMA 27. *If* A *is derived on an unmarked line of an* **AL***-proof from* Γ *that is stable with respect to that line, then* $\Gamma \vdash_{\mathbf{AL}} A$

Proof. Suppose that the antecedent is true. As the unmarked line on which A is derived will not be marked in any extension of the proof, A is finally **AL**-derived in this proof. ■

THEOREM 28. $\Gamma \vdash_{\mathbf{AL}} A$ *iff* A *is derived on an unmarked line of an* **AL***-proof from* Γ *that is stable with respect to that line.*

Proof. Immediate in view of Lemmas 25, 26, and 27. ■

Having established that Definition 22 is adequate, let me show that the matter is actually much simpler for Reliability than for Minimal Abnormality. Let an **AL**-proof be *finite* iff each stage of the proof is a finite list of formulas.

THEOREM 29. *If the strategy is Reliability, Definitions 21 and 22 are still adequate if the proof and all extensions mentioned in Definition 21 are finite.*

Proof. Case 1: $\Gamma \vdash_{\mathbf{AL}^r} A$. In view of Theorem 23 there is a (finite) $\Delta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. As **LLL** has finite and static proofs, there is a finite **AL**-proof from Γ in which A is derived on the condition $Dab(\Delta)$. Let A be so derived on line i of the proof and let s be the last stage of this proof. Consider a finite extension s' of s in which line i is marked. There are at most finitely many Θ for which $Dab(\Theta)$ is a minimal Dab-formula of s' and $\Theta \cap \Delta \neq \emptyset$. For each of these Θ, there is a minimal Dab-consequence $Dab(\Delta_i)$ of Γ such that $\Delta_i \subset \Theta$ and $\Delta \cap \Delta_i = \emptyset$. Append the proof $\mathsf{p_i}$ of each of these Δ_i to s' and let the result be s''. The stage s'' counts finitely many lines and $\Delta \cap U_{s''}(\Gamma) = \emptyset$.

Case 2: $\Gamma \nvdash_{\mathbf{AL}^r} A$. In view of Theorem 23 it holds for all $\Delta \subset \Omega$ that $\Delta \cap U(\Gamma) \neq \emptyset$ if $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} Dab(\Delta)$. Suppose that A has been derived on the condition Δ on a line, say i, of a finite $\mathbf{AL}^r$-proof from Γ and that the last stage of this proof is s. It follows that there is a minimal Dab-consequence $Dab(\Theta)$ of Γ for which $\Theta \cap \Delta \neq \emptyset$. As $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta)$, Θ can be derived on the condition $\emptyset$ in a finite extension s' of s and there is no extension of s' in which line i is unmarked. ■

5 The Dialogues

As the dialogue I want to propose is somewhat unusual, let me first say a few words about usual dialogues. It is not difficult to define these, first for **CLuN**, and next for $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$. Tableau methods presented in [8] and [9] form a good start. The tableau methods may even be simplified by extending the language with classical negation, whence there is no need for signed formulas.

Adaptive logics do not have theorems of their own. If theorems are defined by $\emptyset \vdash A$, then the theorems of the adaptive logic, for example $\mathbf{CLuN}^r$, are identical to those of its upper limit logic, in the example **CL**. If theorems are defined by "for all Γ, $\Gamma \vdash A$", then the theorems of the adaptive logic are identical to those of its lower limit logic, in the example **CLuN**—obviously all theorems of the lower limit logic are theorems of the upper limit logic. This means that one cannot define the adaptive consequence relation in terms of theorems, but that dialogues for the consequence relation should be devised. So one will have to adjust the description of a dialogue from, for example, [12] or [13], and there will be a few peculiarities, to which I return briefly in Section 6.

Let us now turn to the unusual dialogues I announced. The proponent claims that $\Gamma \vdash_{\mathbf{AL}} A$ and the opponent denies this. We let the proponent and opponent construct a proof together, laying down specific rules about the moves they can make. The proponent starts. If, at the end of the dialogue, A is derived in the proof, the proponent wins; otherwise the opponent wins.

This kind of dialogue is completely silly if the logic has static proofs. If the proponent has to make a point in her first move, all she can do is produce a proof of the conclusion from the premises. After this, the opponent can only recognize that he lost.

The situation is dramatically different for logics that have dynamic proofs. If the conclusion A is not derivable from the premises Γ by the lower limit logic, then the proponent can only derive it on a non-empty condition. We have seen that the resulting proof does not constitute a demonstration of $\Gamma \vdash_{\mathbf{AL}} A$. Actually, no proof forms such a demonstration. So it seems natural to construct a demonstration of $\Gamma \vdash_{\mathbf{AL}} A$ as a dialogue between a proponent, who tries to show that A is finally derivable but has to defend herself against moves of the opponent. Let me first comment on the natural character of the approach.

First a comparison. Every logician is acquainted with the situation in which he or she tries to find out whether a formal system has a certain property. If one is convinced that the property holds, one will attempt to prove so. If one does not find the proof, this very fact will undermine the conviction. At some point one will become convinced that the property does not hold and one will try to find a counterexample—often insights from the failing proof will indicate in which direction to look for a counterexample. If, in turn, one fails to produce a counterexample, this may induce one to look again for a proof, etc. The alternating phases may be seen as a dialogue between a proponent and an opponent.

Let us now look more closely at adaptive logics. The idea is that abnormalities are presupposed to be false, unless and until proven otherwise.[13] So two different aims should be realized in a well-directed proof: to establish the conclusion on some condition and to establish that the condition is safe—in the case of Reliability, this means that no member of the condition

[13]The expression is taken from the oldest paper on adaptive logics, [1]. It is obviously vague if disjunctions of abnormalities (*Dab*-formulas) are derivable on the condition $\emptyset$. The strategy removes the vagueness.

is unreliable; in the case of Minimal Abnormality, it means that the condition does not overlap with a minimal choice set of all *Dab*-consequences of the premises *and* that, for each such minimal choice set φ, the conclusion can be derived on a condition that does not overlap with φ. So it is indeed natural to see this as a dialogue in which the proponent first establishes the conclusion on some condition, next the opponent tries to show that the condition is unsafe, next the proponent tries to reestablish the safety of the condition, and so on. Some variant dialogues are possible. They will be considered in some detail below.

Although no dynamic proof will establish that a conclusion is finally derived from a premise set, the metalevel reasoning that is required next to the proof can be seen in dialogic or game theoretic terms: the conclusion is finally derivable iff the proponent can uphold it against every possible attack.

It seems to me that this is at the heart of all forms of defeasible reasoning: that one establishes a conclusion on some condition, and next considers the safety of the condition.

Let us have a look at some possible types of dialogues. Given the differences between the two strategies, I shall consider them in turn and start with *Reliability*.

Stability with respect to a line One may devise a dialogue in terms of proofs that are stable with respect to some line. First the proponent establishes the conclusion on some condition on an unmarked line, say line l, of a (finite or infinite) proof. Next, the opponent may extend the proof. The opponent wins if he produces an extension in which line l is marked; otherwise the proponent wins. The proponent has a winning strategy iff she can produce a proof that warrants her winning.

This approach is all right, but requires that the proponent sometimes starts off by producing an infinite proof. Consider the premise set $\Gamma_1 = \{p \vee q, {\sim}q, (q \wedge {\sim}q) \vee (r_i \wedge {\sim}r_i), (q \wedge {\sim}q) \supset (r_i \wedge {\sim}r_i) \mid i \in \mathbb{N}\}$ and let the proponent aim at establishing $\Gamma_1 \vdash_{\mathbf{CLuN}^r} q$. Consider a finite proof, produced by the proponent, that starts off with

1	$p \vee q$	Premise	
2	${\sim}q$	Premise	
3	p	1, 2; RC	$\{q \wedge {\sim}q\}$

and moreover contains a (forcibly finite) number of lists of lines that have the following form

j	$(q \wedge {\sim}q) \vee (r_i \wedge {\sim}r_i)$	Premise
$j+1$	$(q \wedge {\sim}q) \supset (r_i \wedge {\sim}r_i)$	Premise
$j+2$	$r_i \wedge {\sim}r_i$	j, j+1; RU $\emptyset$

Clearly, line 3 is unmarked in this proof. However, if the opponent extends the proof with the line

$l \qquad (q \wedge {\sim}q) \vee (r_k \wedge {\sim}r_k) \quad$ Premise

for a r_k that does not yet occur in the proof, then line 3 is marked. So the proponent looses. Of course, she should have a winning strategy, because $\Gamma_1 \vdash_{\mathbf{CLuN}^r} q$. And indeed there is one, but it requires that she introduces all premises and all connected lines $j+1$ and $j+2$, which means that she should produce an infinite proof in her first move. This is not handy. Infinite proofs cannot be produced, but should be handled by a metalevel reasoning. It would be more attractive if at least the first move in the dialogue would be a proof that can actually be produced. Moreover, the type of dialogue is not very attractive because the outcome fully depends on the first move of the proponent. She has a winning strategy iff she is able to produce a proof, as her first step, to which the opponent has no possible reaction.

Incidentally, some readers might balk at the artificiality of the premise set Γ_1. It is indeed hard to imagine real life applications in which the depicted complication would arise. Nevertheless, describing logics, one should consider all possible complications, whether they are artificial or not.

Many turns In her first move, the proponent produces a finite proof in which the conclusion is derived from the premises on a condition Δ on an unmarked line, say line l. Next, the opponent may try to show that Δ is unreliable by producing a finite extension of the proof. If the opponent's move is successful, line l is marked at the last stage of the extended proof. The proponent may react by trying to finitely extend the proof in such a way that line l is unmarked. And so on. The proponent has a winning strategy iff she is able to answer every move of the opponent, viz. iff she is able to extend every new extension in such a way that line l is unmarked.

This set up does not require that infinite proofs or infinite extensions are produced. It does not exclude, however, that each player interferes infinitely many times. Consider again Γ_1 and let the dialogue be about the question whether $\Gamma_1 \vdash_{\mathbf{CLuN}^r} q$—we know that the correct answer is positive. And indeed, the proponent has a winning strategy: if the proof starts off with lines 1–3 (displayed before), she is able to extend every extension produced by the opponent in such a way that line l is unmarked. Yet, the opponent can react after every finite number of moves. That this set up is adequate is a (slightly trivial) consequence of Theorem 29.

That the dialogue may go on forever does not constitute much of an objection. In usual dialogues, for example establishing **CL**-validity, the dialogue may go on forever if the conclusion is **CL**-invalid. Here the matter is just a trifle more complicated: an infinite dialogue may result even if the conclusion is an adaptive consequence of the premises. This results immediately from the absence of a positive test for final derivability. In the end, the proponent will either win or loose, but we may be unable to find out what the result in the end will be. This, however, is unavoidable in the case of defeasible reasoning that is not artificially restricted to decidable fragments.[14]

POP The proponent starts by producing a finite proof in which the conclusion is derived from the premises on a condition Δ on an unmarked line, say line l. Next, the opponent may try to show that Δ is unreliable by producing a finite extension. If the opponent is successful, line l is marked at the last stage of the extended proof. The proponent may react by trying to finitely extend the proof in such a way that line l is unmarked. The proponent wins the dialogue if line l is unmarked after she extended the extension produced by the opponent. The proponent has a winning strategy iff there is a proof, which she should produce in her first move, that she can defend against every possible move of the opponent. Note that the proof is a finite proof in which the conclusion is finally derived according to Definition 22.

The only difference with the previous dialogue is that the opponent can interfere only once. That the set up is adequate is established by Theorem 29. Given this, the many turns dialogue is needlessly postponing a decision and is needlessly complicating the notion of a winning strategy.

Calling premises The proponent starts by producing a finite proof in which the conclusion is derived from the premises on a condition Δ on an unmarked line, say line l. At this point, the opponent delineates a finite set Γ' of premises, but does not extend the proof at this point. The proponent finitely extends her proof, introducing whatever premises she wants. Next, the opponent extends the extension, introducing as premises only members of Γ' (but possibly relying on premises introduced by the proponent). The proponent wins the dialogue if line l is unmarked after the opponent extended the extension; otherwise the opponent wins. The proponent has a winning strategy iff there is a proof, which she should produce in her first move, that she can defend against every possible move of the opponent.

This type of dialogue is a variant on the previous one. It illustrates that, given the initial proof, winning the dialogue depends only on the **LLL**-derivability of *Dab*-formulas. If the conclusion is $\mathbf{AL}^r$-derivable from the premises and $\Delta \cap U(\Gamma) = \emptyset$, then there is a finite Γ'' for which $\Gamma' \subseteq \Gamma'' \subseteq \Gamma$

[14] If the premises and conclusion belong to a **CL**-decidable fragment of the language (this entails that the premise set is finite), then the dialogue will be finite. This follows from a forthcoming result on the embedding of (full predicative) **CLuN** into **CL**—for the result on the propositional case see [7].

and $\Delta \cap U(\Gamma'') = \emptyset$.

Let me quickly show that this type of dialogue is adequate. Two cases have to be considered. *Case 1:* $\Delta \cap U(\Gamma) = \emptyset$. Consider the finitely many minimal *Dab*-formulas, $Dab(\Theta_1), \ldots, Dab(\Theta_n)$, that are **LLL**-derivable from Γ'. If $\Delta \cap U(\Gamma') \neq \emptyset$, then there are finitely many $\Delta_1, \ldots, \Delta_m \in \Omega$ such that (i) each of $Dab(\Delta_1), \ldots, Dab(\Delta_m)$ is **LLL**-derivable from Γ, (ii) for every Θ_i $(1 \leq i \leq n)$ there is a Δ_j $(1 \leq j \leq m)$ such that $\Delta_j \subseteq \Theta_i$ and (iii) $\Delta \cap (\Delta_1 \cup \ldots \cup \Delta_m) = \emptyset$. As **LLL** is compact (in view of Theorem 5), all of these Δ_j are derivable from a finite subset of Γ. So the proponent can choose as Γ'' the union of this finite subset of Γ and (the finite set) Γ' and derive all minimal *Dab*-consequences of Γ''. *Case 2:* $\Delta \cap U(\Gamma) \neq \emptyset$. So there is a minimal *Dab*-consequence of Γ, say $Dab(\Theta)$, for which $\Delta \cap \Theta \neq \emptyset$. By the compactness of **LLL**, $Dab(\Theta)$ is **LLL**-derivable from a finite premise set. The opponent chooses this set as Γ'. Whatever the reaction of the proponent, the opponent introduces the members of Γ' and derives $Dab(\Theta)$ on the condition $\emptyset$. This causes line l to be marked.

More dialogue types may be possible, but those described before are sufficient to make the point I was trying to make. So let us now move on to the *Minimal Abnormality* strategy.

In general, Minimal Abnormality requires more complex proofs than Reliability. For some Γ and A, A can only be derived on an unmarked line if A is derived on several conditions (and hence on several lines). Here is a simple example. Let $\Gamma_2 = \{p \vee q, \sim p, \sim q, p \vee r, q \vee r\}$.

1	$\sim p$	Premise	$\emptyset$
2	$p \vee r$	Premise	$\emptyset$
3	r	1, 2; RC	$\{p \wedge \sim p\}$
4	$p \vee q$	Premise	$\emptyset$
5	$\sim q$	Premise	$\emptyset$
6	$(p \wedge \sim p) \vee (q \wedge \sim q)$	1, 4, 5; RU	$\emptyset$
7	$q \vee r$	Premise	$\emptyset$
8	r	5, 7; RC	$\{q \wedge \sim q\}$

Line 3 is unmarked at stages 3–5 of the proof, marked at stage 6 and 7, and again unmarked at stage 8. At stage 8, the proof is stable with respect to line 3. Note that both lines 3 and 8 would be marked if Reliability were the strategy.

The proof illustrates an interesting point. If Minimal Abnormality is the strategy, the 'defense' of the proponent against an 'attack' by the opponent is different from what it is in the case of Reliability. Indeed, the proponent has not only to derive *Dab*-formulas in order to show that some of the *Dab*-formulas in the opponents attack are not minimal. The opponent should also derive the intended conclusion on several conditions. For example, if line 8 is absent from the previous proof, line 3 is marked.

Stability with respect to a line This dialogue is identical to its namesake for Reliability. The problems are also the same: in some cases the only winning strategy for the proponent requires that she produces an infinite proof in her first move. The trouble is adequately illustrated by the dialogue for $\Gamma_1 \vdash_{\mathbf{CLuN}^m} q$.

Many turns This dialogue is identical to its namesake for Reliability, except that not all restrictions on the finiteness of the proof and its extensions can be upheld. Actually, several complications should be considered.

Let $\Gamma_3 = \{(p_i \wedge {\sim}p_i) \vee (p_j \wedge {\sim}p_j) \mid i \neq j; i, j \in \mathbb{N}\} \cup \{q \vee (p_i \wedge {\sim}p_i) \mid i \in \mathbb{N}\}$. As $\Phi(\Gamma_3) = \{\{p_i \wedge {\sim}p_i \mid i \in \mathbb{N}\} - \{p_j \wedge {\sim}p_j\} \mid j \in \mathbb{N}\}$, it is easily seen (in view of Theorem 24) that $\Gamma_3 \vdash_{\mathbf{CLuN}^m} q$ (because q can be derived on the condition $\{p_j \wedge {\sim}p_j\}$ for every $j \in \mathbb{N}$). This seems to work fine with a finite proof and finite extensions. The proponent starts off with, for example, the proof

1	$q \vee (p_0 \wedge {\sim}p_0)$	Premise	$\emptyset$
2	q	1; RC	$\{p_0 \wedge {\sim}p_0\}$

after which the opponent offers a finite reply, an extension of 1–2 in which line 2 is marked. There are infinitely many such extensions. All that is required for line 2 to be marked is that, where s is the last stage of the extension, there is a $\varphi \in \Phi_s(\Gamma_3)$ for which $p_0 \wedge {\sim}p_0 \in \varphi$. A simple example is the extension of 1–2 with the following single line.

3	$(p_0 \wedge {\sim}p_0) \vee (p_1 \wedge {\sim}p_1)$	Premise	$\emptyset$

To every such extension—let it count l lines, the proponent has an adequate reply. As the extension is finite, at most finitely many letters p_i occur in it. So the opponent can simply pick a p_i that does not occur in the extension and add the lines:

$l+1$	$q \vee (p_i \wedge {\sim}p_i)$	Premise	$\emptyset$
$l+2$	q	$l+1$; RC	$\{p_i \wedge {\sim}p_i\}$

As p_i does not occur up to line l, $p_i \wedge {\sim}p_i$ is not a member of any $\varphi \in \Phi_l(\Gamma_3)$ and hence line $l+2$ is unmarked. Moreover, as some $\varphi \in \Phi_l(\Gamma_3)$ are bound not to contain $p_0 \wedge {\sim}p_0$, line 2 is unmarked. So all seems well: $\Gamma_3 \vdash_{\mathbf{CLuN}^m} q$ and the proponent has a reply to every attack of the opponent to 1–2.

However, consider $\Gamma_4 = \{(p_i \wedge {\sim}p_i) \vee (p_j \wedge {\sim}p_j) \mid i \neq j; i, j \in \mathbb{N}\} \cup \{q \vee (p_i \wedge {\sim}p_i) \mid i \in \mathbb{N} - \{0\}\}$—so $\Gamma_4 = \Gamma_3 - \{q \vee (p_0 \wedge {\sim}p_0)\}$. As $\Phi(\Gamma_4) = \{\{p_i \wedge {\sim}p_i \mid i \in \mathbb{N}\} - \{p_j \wedge {\sim}p_j\} \mid j \in \mathbb{N}\}$, we now have (in view of Theorem 24) that

$\Gamma_4 \nvdash_{\mathbf{CLuN}^m} q$ (because q cannot be derived on the condition $\{p_0 \wedge \sim p_0\}$). The only point at which the proponent turns out to loose the game is after all premises of the form $(p_i \wedge \sim p_i) \vee (p_j \wedge \sim p_j)$ have been introduced. As there is no line on which q is derived on the condition $\{p_0 \wedge \sim p_0\}$, all lines on which q is derived are marked at this stage, call it s, because every condition Δ on which q has been derived, overlaps with the $\varphi \in \Phi_s(\Gamma_4) = \Phi(\Gamma_4)$ for which $\varphi = \{\{p_i \wedge \sim p_i \mid i \in \mathbb{N}\} - \{p_0 \wedge \sim p_0\}\}$.

Another difficulty concerns the first move by the proponent, viz. the original proof. Consider the premise set $\Gamma_5 = \{(p_i \wedge \sim p_i) \vee (p_j \wedge \sim p_j) \mid i, j \in \mathbb{N}; i \neq j\} \cup \{(\bigwedge\{(p_j \wedge \sim p_j) \vee (p_k \wedge \sim p_k) \mid j, k \in \{0, \ldots, i+2\}; j < k\}) \supset (q \vee (p_i \wedge \sim p_i)) \mid i \in \mathbb{N}\}$. Incidentally, the shortest member of the second 'part' of the premise set is $(((p_0 \wedge \sim p_0) \vee (p_1 \wedge \sim p_1)) \wedge ((p_0 \wedge \sim p_0) \vee (p_2 \wedge \sim p_2)) \wedge ((p_1 \wedge \sim p_1) \vee (p_2 \wedge \sim p_2))) \supset (q \vee (p_0 \wedge \sim p_0))$. Note that $\Phi(\Gamma_5) = \Phi(\Gamma_3) = \{\{p_i \wedge \sim p_i \mid i \in \mathbb{N}\} - \{p_j \wedge \sim p_j\} \mid j \in \mathbb{N}\}$. Moreover, in a proof from Γ_5, q is derivable on the condition $\{p_i \wedge \sim p_i\}$ for every $i \in \mathbb{N}$. So the proponent should have a winning strategy in this case. However, in every finite stage of a proof from Γ_5, all lines on which q are derived are marked. Indeed, the premises are prepared in such a way that, in order to derive q on some condition $\{p_i \wedge \sim p_i\}$, say on line l, one has to introduce first a number of premises that are *Dab*-formulas and that cause line l to be marked.[15] So it seems that we have to allow the proponent to produce, as her first move, an infinite stage of a proof. Recall indeed that q should be derived on an unmarked line at this stage.

The premise set Γ_5 is clearly prepared in an artificial way. So I am glad and grateful that Peter Verdée and Kristof De Clercq found a way around the requirement that the first move of the proponent may end up in an infinite stage. In Section 4, it was said that a *Dab*-formula is the *classical* disjunction of the members of a finite set of abnormalities. As the standard disjunction in **CLuN** has the same meaning as the classical disjunction, I have neglected the distinction in the examples. However, by strictly keeping to the requirement that the disjunction should be classical, the proponent can produce a *finite* proof from Γ_5 in which q is derived on some condition but that does not contain *any Dab*-formula. So the proponent may introduce $(p_0 \wedge \sim p_0) \vee (p_1 \wedge \sim p_1)$, but this is not a *Dab*-formula. It is up to the opponent to derive the *Dab*-formula $(p_0 \wedge \sim p_0) \check{\vee} (p_1 \wedge \sim p_1)$. And this is sensible also from a philosophical point of view. All the proponent should do in her first move is to derive the conclusion

[15] The attentive reader will have remarked that, for example in order to derive q on the condition $\{p_0 \wedge \sim p_0\}$, there is no need to introduce all disjuncts of the antecedent of $(((p_0 \wedge \sim p_0) \vee (p_1 \wedge \sim p_1)) \wedge ((p_0 \wedge \sim p_0) \vee (p_2 \wedge \sim p_2)) \wedge ((p_1 \wedge \sim p_1) \vee (p_2 \wedge \sim p_2))) \supset (q \vee (p_0 \wedge \sim p_0))$. Nevertheless, the claim made in the text is correct.

on some condition that will allow her to win the dialogue. Pointing out *Dab*-formulas that cause the line on which the conclusion is derived to be marked or unmarked is a task that belongs to the subsequent moves.

The upshot is that, for this type of dialogue, we can keep the proof as well as the extensions finite, but we should allow the dialogue to go on infinitely in order to do justice to the opponent.

POP The long discussion of the previous dialogue type gives us at once the insights required for describing this type. The dialogue is identical to its namesake for Reliability, except that the extension of the proof and the extension of the extension should be allowed to be infinite. This is not too bad. As was remarked before, the existence or absence of a winning strategy for the proponent has to be established at the metalevel anyway.

That the set up is adequate is established by Theorem 29 and by the insight concerning the requirement that *Dab*-formulas are strictly defined in terms of classical disjunctions.

Calling premises This dialogue type is also identical to that for Reliability, except that the opponent is allowed to delineate an infinite set of premises and that, after this, the proponent is allowed to produce an infinite extension of her proof.

The demonstration that this type of dialogue is adequate has slightly to be adjusted for Minimal Abnormality. Possibly the proponent has to derive infinitely many minimal *Dab*-formulas in her extension, which is not a problem as their set is denumerable.

Given the absence of a positive test (in general), the computational complexity of adaptive logics is even greater than that of classical (predicative) logic. This does not prevent one, however, from describing dialogue types and to show them adequate (which I summarily did).

6 In Conclusion

The main point I tried to make was that it is natural to understand final derivability in dynamic proofs in terms of dialogues: roughly, that a formula is finally derivable from a premise set iff there is a derivation of the formula that can be defended against every attack. Of course the allowed moves had to be made precise. I presented different types of dialogues for adaptive logics in order to show that some variation is possible. Some type may be more attractive than another, either with respect to its philosophical interpretation or from a computational point of view.

An interesting open question concerns the combination of the types of dialogues described in the previous section with more usual dialogues. Put differently, it would be interesting to know what remains of the different moves described above if the proponent and opponent are given the usual dialogic means, viz. not proofs and extensions of proofs, but attacks on and defenses of formulas. Clearly the attacks and defenses can be most easily defined by studying the semantics of adaptive logics (which I had to skip in the present paper). Such dialogues seem to have some interesting aspects. Consider for example a usual dialogue corresponding to the POP dialogue described in the previous section. In a first phase the proponent will try to establish the conclusion on some condition. In a second phase, the opponent will try to establish that there is a selected (reliable, respectively minimally abnormal) model in which the conclusion is false because a member of the condition is true. In the third phase, the proponent tries to show that the constructed model is not a selected one. Another interesting aspect are the restrictions on the introduction of atomic formulas. It seems natural to keep the restriction that the proponent cannot introduce literals in the phase in which she attempts to derive the conclusion. In the phase in which the opponent is attempting to establish abnormalities (that jointly correspond to abnormalities in a selected model), I surmise that the restriction should be adjusted in such a way that only the proponent can introduce abnormalities.

A very different open problem concerns dynamic proofs. It seems unproblematic to define final derivability in terms of a proof that is stable with respect to a certain line (on which the conclusion has been derived). We have seen that Definitions 21 and 22 present a more attractive way to characterize final derivability. It is unclear, however, whether this characterization is adequate, in the sense of Theorem 28, for all logics that have dynamic proofs. It should not be too difficult to delineate the set or sets of conditions on the set of rules $\mathcal{R}$ that warrant that the characterization in terms of Definitions 21 and 22 are adequate. Such a result would solve a problem which is now approached in a piecemeal way, namely whether all logics having dynamic proofs can be characterized, possibly under a translation, by an adaptive logic.

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