Constructive contextual modal judgments for reasoning from open assumptions

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Abstract. Dependent type theories using a structural notion of context are largely explored in their applications to programming languages, but less investigated for knowledge representation purposes. In particular, types with modalities are already used for distributed and staged computation. This paper introduces a type system extended with judgmental modalities internalizing epistemically different modes of correctness to explore a calculus of provability from refutable assumptions.

1 Introduction

Constructive logics use proofs as first-class citizens to define the notion of truth. Dependent truth is easily interpreted in a contextual reading of provability, as in Martin-Löf Type Theory.[1] In such a system one distinguishes between proposition A and judgment A *true*, justified by an appropriate proof term $a : A$. Correspondingly, contextual truth allows formulae of the form $\Gamma \vdash a : A$, where Γ is of the standard form $[x_1 : A_1, \dots, x_n : A_n]$ and a a proof of A under appropriate substitutions $[x_1/a_1 : A_1, \dots, x_n/a_n : A_n] \vdash a : A$. Hypothetical truth is thus reduced to dependent closed constructions, hypotheses are obtained by abstracting on the relevant proofs and ultimately grounded on the primitive notion of premise (known judgment). This corresponds computationally to the requirement of β-reduction for proof terms and to the evaluation of codes in a program; a connected background is the modelling of contexts from AI.[2] The modal formulation of contextual calculi is the next obvious step. Along with the standard intuitionistic translation of K and the constructive version of $S4$,[3] a weaker format to accommodate the notion of context is given by the calculus

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1 See [11], [12], [13].

2 See [14]. For a constructive semantics of contextual reasoning cf. [22].

3 See for example [20], [27], [3], [2], [1]. See [24] for an overview of the early studies on intuitionistic modal logics.

CK in [15], which provides a possible-world semantics sound and complete with respect to the natural deduction interpretation given in [6]. This natural deduction system is the most basic modal logic of contexts, with formulas $\texttt{ist}(k, A)$ that read "A is true in context k" and it satisfies a multi-modal K fragment of a Propositional Logic of Contexts. Recently, contextual modal type theories for programming languages and further research in linguistics and hardware verification have been formulated, especially to model staged and distributed computation.[4]

In the present paper, a modal type system is used to formalize epistemic processes under refutable assumptions. Our starting point is the constructive reading of the notion of truth as existence of a verification, to design a type-theorethical format for the epistemic notion of *verification under open assumptions*. The notion of truth up to refutation recalls a sensible topic for constructive logics, based on the meaning of intuitionistic negation.[5] The present paper dwells on the foundational idea that truth is admissible up to a counter-example.[6] The notion of admissible truth literally satisfies the logical concept of an assumption, a computational term which is not presented together with an appropriate β-redux. The related constructive modal type system variates on a theme first proposed in [19] and later expanded in [18]. The structure of the paper is as follows. In section 2 I provide a variant interpretation of the basic system of constructive type-theory that links hypotheses and refutable contents; in section 3 a modal type system is designed that preserves refutability. Conclusive remarks set the next steps of this research.

2 A system for proven and refutable contents

Describing realistic knowledge processes requires to explain hypotheses as contents whose truth is declared, but whose refutation is not ruled out. It corresponds to consider the needed information in a process, without providing the appropriate computational instructions.[7] The corresponding logical notion is that of an assumption. To integrate it in the constructive definition of truth, one has to justify assumed propositional contents independently from proven ones. In the following, a system for a modal type theory is developed to model such processes, including a validity relation using verification by proof terms separated from justification by proof variables.

Our syntax is the following:

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[4] See e.g. [5], [16], [17], [18].

[5] For the standard intuitionistic meaning explanation of negation, indirect proofs as *reductio ad absurdum* are standardly not admitted, whereas the usual intuitionistic absurdity rule interprets the classical *ex falso quodlibet*. See e.g. [26, p. 40].

[6] Our system presents some similarity with the notion of 'pseudo-truth' introduced in [10] as double-negated classical formulae considered reducible to intuitionistic ones.

[7] Formally, it is the same intuition behind the explanation of partial evaluation, where a function program considers part of its input code as given. Cf. [4] and [9].

Types := $A\ type; A\ type_{inf}$;
Propositions := $A; A \wedge B; A \vee B; A \rightarrow B; A \supset B; \neg A \rightarrow \bot$;
Proof terms := $a : A; (a, b); a(b); \lambda(a(b)); <a, b>$;
Proof variables := $x : A; (x(b)); (x(b))(a)$;
Contexts := $\Gamma, x : A; \Gamma, a : A; \Box\Gamma; \Diamond\Gamma$;
Judgments := $A\ true; A\ true^*; \Gamma \vdash A\ true; \Diamond(A\ true); \Box(A\ true)$.

The language contains one basic type for categorical judgments, justified by term constructors a, b; one type for contextual judgments justified by variables x_1, x_2; identity of terms hold within *type*, as usual variables are unique in context for $type_{inf}$. Proof terms can occur both in and outside of a context, proof variables hold only in contexts. Types are typically propositions, judgments are built by declaration of truth and truth up to refutation. Semantic notions for this system rely entirely on corresponding syntactic formulations, as it is typical of the procedural semantics of Constructive Type Theory. In the following, we omit for brevity the identity rules that define Reflexivity, Simmetry and Transitivity on types.

The set of *type* judgments includes the standard constructive definition of logical connectives $\wedge, \vee$; implication $\rightarrow$ from verified formulae is in place of the usual functional abstraction and it is expressed as application of a construction of the antecendent to the construction of the consequent.[8] Universal predication abstracts from enumerable sets of equivalent constructions; existential quantification is justified by paired constructions; negation $\neg$ is introduced as implication from a type to the absurdum: if $A\ true$ is a known judgment, then one can infer that a construction for $\neg A$ is an absurdity; we use the λ-operator as a $\forall$-constructor on terms combined by application, angled parentheses $<, >$ for ordered pairs as an $\exists$-constructor. We list only introduction rules:

$$\frac{a : A}{A\ type}\ \text{Type Formation} \quad \frac{a : A}{A\ true}\ \text{Truth Definition}$$

$$\frac{a : A}{l(a) : A \vee B\ true}\ \text{Left } I\vee \quad \frac{b : B}{r(b) : A \vee B\ true}\ \text{Right } I\vee$$

$$\frac{a : A \qquad b : B}{(a, b) : A \wedge B\ true}\ I\wedge \quad \frac{a : A \qquad A\ true \vdash b : B}{a(b) : A \rightarrow B\ true}\ I \rightarrow \text{(Implication)}$$

$$\frac{a_1 : A_i, \ldots, a_n : A_i \qquad [A_i\ true] \vdash b : B \qquad \lambda((a_i(b))A, B)}{(\forall a_i : A_i)B\ type}\ I\forall$$

[8] A literal interpretation of intuitionistic implication á la Heyting which does not interpret hypothetical reasoning. The application $a(b)$ for the implication relation can be seen as a λ-term that is presented along with one of its α-terms, provided there is a bound construction given in the antecedent. This recalls ideas mentioned by Martin-Löf and the calculus of types with explicit substituitions presented in [25].

$$\frac{a_1 : A_i, \ldots, a_n : A_i \qquad [a_i : A_i] \vdash b : B \qquad (< a_i, b >, A, B)}{(\exists a_i : A_i) B \ type} \quad I\exists$$

$$\frac{a : A}{\neg A \rightarrow \bot} \quad I\bot$$

An elimination rule on the $\bot$-rule would validate a double-negation elimination, which is avoided by a non-standard extension to functional expressions. Formally, a dependent judgment is nothing else than a functional relation among expressions: if A *type* holds, then a construction of a new type B is possible by considering the latter as a family of sets over some $x : A$ such that $[x : A]B$ *type* whenever the substitution $[x/a]$ is performed.[9] A new task is to admit no explicit evaluation on such formulae, extending the system with a new type format $type_{inf}$ for *information type*. Formulas of the information type are introduced by proof variables; a judgment $A \ type_{inf}$ is justified by running a test on previous derivations such that it checks no construction for $\neg A$ *type* to be given:

$$\frac{\neg(A \rightarrow \bot)}{A \ type_{inf}} \quad \text{Informational Type formation}$$
$$\frac{A \ type_{inf} \qquad x : A}{A \ true^*} \quad \text{Hypothetical Truth Definition}$$

The judgment $\neg(A \rightarrow \bot)$ in the previous module of the language says that there exists no pair of constructions $< a, b >$ such that $a(b) : A \ true \rightarrow \bot$. Its combination with $type_{inf}$ formation does not imply that given $\neg(A \rightarrow \bot)$ it follows $a : A$: the latter justification is kept entirely constructive and therefore cannot be given by indirect proof. This represents an elimination rule with respect to double elimination, but not the appropriate counterpart of its classical version. The second rule says that provided A can be admitted as a $type_{inf}$, a weak truth-predicate $true^*$ (true up to refutation) is inferred by assuming a construction for A exists: it can be seen as a place-holder for ungrounded truth. On this interpretation one defines functional expressions of $type_{inf}$:

$$\frac{A \ type_{inf} \quad x : A \vdash B \ type_{inf}}{x : A \vdash B \ true^*}$$

which says that B is true up to a refutation of A. The weak truth predicate induces the standard dependent functional construction by abstraction; β-conversion provides the appropriate translation to standard dependent type formation by application:

[9] The type checking will require first well-formedness of A, secondly evaluation to a current environment for extraction of variable terms, thirdly construction for the variable in that environment, and finally evaluation of the variable and the formulation of the binding expression to a value for that environment. The generalization to multiple dependence being allowed, terms for $[x_1 : A_1, \ldots, x_n : A_n]B$ *type* are evaluated to normal forms (eventually: weak head normal forms, explicit substitutions, closures) in order the predication B *type* to be valid.

$$\frac{A\ type_{inf} \qquad x\!:\!A \vdash B\ true^{*}}{((x)b) : A \supset B\ true}\ \text{Functional Abstraction}$$

$$\frac{A\ type_{inf} \qquad x\!:\!A \vdash B\ type_{inf} \qquad a\!:\!A}{(x(b))(a) = b[a/x]\!:\!B\ type[a/x]}\ \beta\text{-conversion}$$

3 Contextual Modal Type Theory for verification and refutation

The different notions of truth are internalized in our system by the use of epistemic modalities. Previous modal versions of type theory [19] and [18] use propositional modalities to speak about dependent truth. In the present system, modalities are judgmental operators:[10] $\Box(A\ true)$ says that A is true and has no refutable conditions (either there are none, or all of them have been secured); $\Diamond(A\ true)$ says that A is true in those epistemic states where conditions are not refuted. The *type* formulas induce the strict constructive *true* predicate. If $A\ true$ holds, it also holds under refutable data being added, by definition no declaration $\neg A\ type_{inf}$ being allowed if $a : A$ is formulated. This will make A verified in *any* extension of the empty context:[11]

$$A\ true \Leftrightarrow \emptyset \vdash A\ true \Leftrightarrow \Box(A\ true).$$

Truth in context relates to expressions in the refutable protocol of the language and judgmental possibility to truth in *some* context, namely when conditions are not refuted:

$$A\ true^{*} \Leftrightarrow \Gamma \vdash A\ true \Leftrightarrow \Diamond(A\ true)$$

where Γ is inteded as containing propositions of the form $A_i\ true^{*}$. Where β-conversion applies, there is an immediate reduction to the previous case of $A\ true$ and the necessary judgment. A premise and a hypothesis rule introduce the truth predicates (both rules can have $\Gamma, \Delta = \{\emptyset\}$):

$$\frac{}{\Gamma, a\!:\!A, \Delta \vdash A\ true}\ \text{Premise Rule} \qquad \frac{}{\Gamma, x\!:\!A, \Delta \vdash A\ true^{*}}\ \text{Hypothesis Rule}$$

Definition 1 (Definition of (Local) Validity).

1. *If A true then A is valid.*
2. *If A is valid then $\Gamma \vdash A$ true, for every Γ.*
3. *If A true* then A is locally valid in view of some $\Gamma \vdash A$ true.*

[10] For more on the philosophical justification of this notion of judgmental modalities, see [23].

[11] Judgmental necessity satisfies the correlation between validity and unconditional justification, as for the system presented in [19].

Modalities are internalized by appropriate formation rules from categorical and hypothetical judgments:

$$\frac{a:A}{\Box(A\ true)}\ \Box\text{-Formation} \qquad \frac{x:A}{\Diamond(A\ true)}\ \Diamond\text{-Formation}$$

The inference to truth is valid only where verified assumptions are used. To this aim, modalities are now generalized to contextual formulas. We shall refer to $\Box\Gamma$ as Γ containing only valid assumptions (premises):

Definition 2 (Necessitation Context). *For any context* Γ, $\Box\Gamma$ *is given by* $\bigcup\{\Box A\ true \mid\ \textit{for all}\ A \in \Gamma\}$.

Correspondingly, a context is 'normal' when not every assumption it contains has been verified:[12]

Definition 3 (Normal Context). *For any context* Γ, $\Diamond\Gamma$ *is given by* $\bigcup\{\circ A\ true \mid \circ = \{\Box, \Diamond\}$ *and* $\Diamond A\ true$ *for at least one* $A \in \Gamma\}$.

The introduction of judgmental $\Box$ is allowed under verification of judgments in context, its elimination rule induces a valid proposition:[13]

$$\frac{\Gamma \vdash A\ true}{\Box\Gamma \vdash \Box(A\ true)}\ I\Box \qquad \frac{\Box\Gamma \vdash \Box(A\ true) \qquad \Delta, a:A \vdash b:B}{\Gamma, \Delta \vdash B\ true}\ E\Box$$

where $\Box\Gamma$ iff $[x_i/a_i] : A_i, \forall A_i \in \Gamma$, as by Definition 2. Local validity is in turn defined by introduction and elimation rules for the $\Diamond$-operator:

$$\frac{\Gamma, x:A \vdash B\ true^*}{\Box\Gamma, \Diamond(A\ true) \vdash \Diamond(B\ true)}\ I\Diamond \qquad \frac{\Gamma, \Delta \vdash A\ true^* \quad \Box\Gamma, \Diamond(A\ true) \vdash \Diamond(B\ true)}{\Gamma, \Delta \vdash B\ true^*}\ E\Diamond$$

The introduction rule shows the dependency of possible contents from refutable conditions, the corresponding elimination uses this information to infer further possible knowledge under the condition expressed by Definition 3.

Substitution of variables by constants is as usual indicated by $[x/a]B$ as the substitution of occurrences of x in B by a; it is crucial in our system to give the relation between verification and truth; the modal version shows that term substitution satisfies the inclusion of $\Diamond$ in $\Box$:

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[12] In various literature in modal logic, *Necessitation* and *Normal Context* are usually called *Global* and *Local Context*. This distinction is crucial for preserving the problem of derivability under assumption in modal languages and involve the validity of the Deduction Theorem, see [8]. I have strenghtened here the reasoning, by obtaining modal judgments (rather than formulae) from the preservation/verification of assumptions. Cf. [7].

[13] This is the crucial difference with the system introduced in [19], where $\Box A$ expresses validity but it can be introduced under hypotheses. In the comparison with the system in [6], the obvious similarity is that the therein contained modality $\Box_k$ satisfies the same principle of our $I\Box$, namely it builds-in the substituitions needed for formulas in contexts. On the other hand, the propositional format does not require any $\Diamond$ operator, its role being syntactically satisfied by standard contexts.

Theorem 1 (Substitution on terms).

1. *If* $\Gamma, x{:}A, \Delta \vdash B\ true^*$ *and* $\Gamma, \Delta \vdash a{:}A$, *then* $\Gamma, \Delta \vdash [x/a]B\ true$.
2. *If* $\Box\Gamma, \Diamond(A\ true), \Box\Delta \vdash \Diamond(B\ true)$ *and* $\Box\Gamma, \Box\Delta \vdash \Box(A\ true)$, *then* $\Box\Gamma, \Box\Delta \vdash \Box(B\ true)$.

1. is proven by induction on the first given derivation, using the Hypothesis Rule and the inclusion of $B\ true^*$ in $B\ true$ by implication from validity to truth in (any) context: from the second premise all occurrences of A are declared *type*, in particular those in $B\ true^*$ by β-conversion; provided the latter is derived by Γ, Δ by the Hypothesis Rule with assumption $x : A$ and no additional assumptions are given, then $B\ true$ follows as valid in any extension of Γ, Δ. 2. is proven by induction on the first given derivation: it obtains from an occurrence of $x{:}A$ by $I\Diamond$; by the second premise and the equivalence of $\Diamond(A\ true)$ and $A\ true^*$ as truth in some context, one obtains $B\ true$ by β-conversion on $A\ true^*$ and replacement of its occurrences in B via $E\Box$; by $I\Box$ one finally obtains $\Box(B\ true)$. $\Box$

β-reduction and η-expansion, i.e. local inversion of modal rules hold; theorem 1 is crucial to this aim together with the structural properties of our system:

Theorem 2 (Weakening). *The inference systems satisfies Weakening:*

1. *If* $\Gamma \vdash B\ true$, *then* $\Gamma, a{:}A \vdash B\ true$.
2. *If* $\Gamma \vdash B\ true^*$, *then* $\Gamma, x{:}A \vdash B\ true^*$.
3. *If* $\Box\Gamma \vdash \Box(B\ true)$, *then* $\Box\Gamma, \Box(A\ true) \vdash \Box(B\ true)$.
4. *If* $\Diamond\Gamma \vdash \Diamond(B\ true)$, *then* $\Diamond\Gamma, \Diamond(A\ true) \vdash \Diamond(B\ true)$.

The proofs go by induction on derivations: in 1. uses the Premise Rule; in 2. uses the Hypothesis Rule; in 3. uses $I\Box$, in 4. uses $I\Diamond$.

Theorem 3 (Contraction). *The inference system satisfies Contraction:*

1. *If* $\Gamma, a_1{:}A, a_2{:}A \vdash B\ true$, *then* $\Gamma, a{:}A \vdash [a_1 \approx a_2/a]B\ true$.
2. *If* $\Gamma, x_1{:}A, x_2{:}A \vdash B\ true^*$, *then* $\Gamma, x{:}A \vdash [x_1 \approx x_2/x]B\ true^*$.
3. *If* $\Box\Gamma, a_1{:}A, a_2{:}A \vdash \Box(B\ true)$, *then* $\Box\Gamma, \Box(A\ true) \vdash \Box(B\ true)$.
4. *If* $\Box\Gamma, x_1{:}A, x_2{:}A \vdash \Diamond(B\ true)$, *then* $\Box\Gamma, \Diamond(A\ true) \vdash \Diamond(B\ true)$.

Again by induction on derivations: Refleflexivity and Symmetry for proof terms in 1.; unicity of proof variables for $type_{inf}$ in 2.; in addition Truth Definition and $I\Box$ for 3.; Hypothetical Truth Definition and $I\Diamond$ for 4..

Theorem 4 (Exchange). *The inference system satisfies Exchange:*

1. *If* $\Gamma, a_1{:}A, a_2{:}A \vdash B\ true$, *then* $\Gamma, a_2{:}A, a_1{:}A \vdash B\ true$.
2. *If* $\Gamma, x_1{:}A, x_2{:}A \vdash B\ true^*$, *then* $\Gamma, x_2{:}A, x_1{:}A \vdash B\ true^*$.
3. *If* $\Box\Gamma, a_1{:}A, a_2{:}A \vdash \Box(B\ true)$, *then* $\Box\Gamma, a_2{:}A, a_1{:}A \vdash \Box(B\ true)$.
4. *If* $\Box\Gamma, x_1{:}A, x_2{:}A \vdash \Diamond(B\ true)$, *then* $\Box\Gamma, x_2{:}A, x_1{:}A \vdash \Diamond(B\ true)$.

Again by induction and using the same properties of proof terms and variables as for Contraction.

Local inversion of modal rules is finally shown. Soundness by local reduction on $\Box(A\ true)$:

$$\dfrac{\dfrac{\begin{array}{c}D_1\\ \Gamma \vdash A\ true\end{array}}{\Box\Gamma \vdash \Box(A\ true)}\, I\Box \qquad \begin{array}{c}E\\ \Delta, a{:}A \vdash b{:}B\end{array}}{\Gamma, \Delta \vdash B\ true}\, E\Box \quad \Rightarrow_{Redex} \quad \begin{array}{c}D_2\\ \Gamma, \Delta \vdash B\ true\end{array}$$

D_2 is obtained from D_1 and E in terms of the Premise Rule: a proof term for A is induced from Γ in D_1, in turn providing a proof term for B in E. In computational terms, this rule formalizes β-reduction of B (value) with respect to all occurrences of its procedures (codes) in A. Completeness by local expansion on $\Box(A\ true)$:

$$\begin{array}{c}D_1\\ \Box\Gamma \vdash \Box(A\ true)\end{array} \quad \Rightarrow_{Exp} \quad \dfrac{\begin{array}{c}D_2\\ \Box\Gamma \vdash \Box(A\ true)\end{array} \qquad \dfrac{\dfrac{}{\Gamma, a{:}A \vdash A\ true}\,\text{Prem}}{\Box\Gamma, a{:}A \vdash \Box(A\ true)}\, I\Box}{\Gamma, \vdash A\ true}\, E\Box$$

By this expansion one shows how $E\Box$ provides all the information needed to reconstruct $\Box(A\ true)$. Computationally, it reconstructs the value on code A.[14] Soundness by local reduction on $\Diamond(A\ true)$:

$$\dfrac{\dfrac{\begin{array}{c}D_1\\ \Gamma, x{:}A \vdash B\ true^*\end{array}}{\Box\Gamma, \Diamond(A\ true) \vdash \Diamond(B\ true)}\, I\Diamond \qquad \begin{array}{c}E\\ \Gamma, \Delta \vdash A\ true^*\end{array}}{\Gamma, \Delta \vdash B\ true^*}\, E\Diamond \Rightarrow_{Redex}$$

$$\begin{array}{c}D_2\\ \Gamma, \Delta \vdash B\ true^*\end{array}$$

Derivation D_2 is justified by the Hypothesis Rule: by E, Γ, Δ in reduced form will contain at least one formula of $type_{inf}$, which justifies the $true^*$ predicate in D_2. Computationally, this reduction formalizes the naming of codes that are presented partially evaluated to program B. Finally, completeness by local expansion on $\Diamond(A\ true)$:

$$\begin{array}{c}D_1\\ \Diamond\Gamma \vdash \Diamond(A\ true) \Rightarrow_{Exp}\end{array}$$

$$\dfrac{\begin{array}{c}D_2\\ \Diamond\Gamma \vdash \Diamond(A\ true)\end{array} \qquad \dfrac{\dfrac{}{\Gamma, x{:}A \vdash A\ true^*}\,\text{Hyp}}{\Diamond\Gamma, \Diamond(A\ true) \vdash \Diamond(A\ true)}\, I\Diamond}{\Gamma, \vdash A\ true^*}\, E\Diamond$$

This expansion shows how to reconstruct all the information needed to formulate $\Diamond(A\ true)$, as a partial evaluation of program A.

Model of this dependent types system is a weakening of the truth-values model.[15] Our truth-functional model considers its types as pairs $A = [a, \rightarrow]$,

[14] This formulation of the $\Box$-rules does not violate the meaning of hypotheses, as it is the case with the rules for necessity in [21]. On the other hand, given Definition 2, a side condition on multiple simultaneous substitutions is unavoidable, see [3].

[15] The latter is given by the category of contexts as the poset $\{1, 0\}$ that satisfies inhabitness by at most one element and intensional identity types.

where a is the verification term and $\rightarrow$ the corresponding evaluation function. This generates the due types:

- $A = [a, \rightarrow] = \{1\}$ if $x \rightarrow a = 1$ and $A : type = 1$
- $A = [a, \rightarrow] = \emptyset$ if $x \rightarrow a = undefined$ and $A : type_{inf} = 1$
- $A = [a, \rightarrow] = \{0\}$ if $x \rightarrow a = 0$ and $A : type = 0$

Including the models for $type_{inf}$ the relation on the set of proposition can be undefined, hence it preserves only symmetricity on the standard models and the partition is no longer satisfied by not guaranteeing inhabitness.[16]

4 Conclusions

Our modal type system allows refutable truths in a constructive setting, for which a main application will be the modeling of knowledge processes with embedded communication processes intended as refutable contents in a distributed or staged format. By introducing refutable contents one can formalize information updates and retraction functions. The comparison with staged and distributed processing is completed by indexing of local processes. The extension to multi-modalities and to a multi-conclusion inference relation is the next obvious step for multi-agents and multi-source contextual modal type-theory.

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