A Paraconsistent Proof Procedure Based on Classical Logic. *Extended Abstract*

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Abstract

Apparently Ex Falso Quodlibet (or Explosion) cannot be isolated within **CL** (Classical Logic); if Explosion has to go, then so have other inference rules, for example either Addition or Disjunctive Syllogism. This certainly holds according to the standard abstract view on logic. However, as I shall show, it does not hold if a logic is defined by a procedure—a set of instructions to obtain a proof (if there is one) of a given conclusion from a given premise set.

In this paper I present a procedure $\mathbf{pCL}^-$ that defines a logic $\mathbf{CL}^-$—a function assigning a consequence set to any premise set. Anything derivable by **CL** from a consistent premise set Γ is derivable from Γ by $\mathbf{CL}^-$. If Γ is (**CL**-)inconsistent, $\mathbf{pCL}^-$ enables one to demonstrate this (by deriving a contradiction from Γ). The logic $\mathbf{CL}^-$ validates applications of Disjunctive Syllogism as well as applications of Addition. Nevertheless, this logic is paraconsistent as well as (in a specific sense) relevant.

$\mathbf{pCL}^-$ derives from an intuitively attractive proof search procedure. A characteristic semantics for $\mathbf{CL}^-$ will be presented and the central properties of the logic will be mentioned. $\mathbf{CL}^-$ shows that (and clarifies how) adherents of **CL** may obtain non-trivial consequence sets for inconsistent theories.

1 The Problem

When non-logicians discover a theory to be inconsistent, they sometimes continue to apply **CL** to it. If confronted with the argument that any statement of the language is just as much a **CL**-consequence of the theory as any other statement, they tend to consider this argument as a logicians' trick, which can be avoided.

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Logicians know that one cannot just eliminate EFQ (Ex Falso Quodlibet) from **CL**. Either Addition or Disjunctive Syllogism has also to be given up. And the trouble does not end there. In the presence of material implication, the inference of A from $\sim A \supset (B \wedge \sim B)$ has to be abandoned, and hence also either the inference of A from $\sim\sim A$ or the inference of $\sim A$ from $A \supset (B \wedge \sim B)$. In sum, EFQ is deeply entrenched in **CL**. For a long time, I considered all this as obvious. Then I made a rather stunning discovery.

The proofs of adaptive logics explicate the internal dynamics of reasoning processes—basically, the fact that inferences are withdrawn as deeper insight in the premises is gained—by attaching conditions to certain inferences. If the conditions turn out not to be fulfilled at a later stage of the proof—the addition of a line brings the proof into a new stage—the inference is withdrawn. These conditions are defeasible; when the condition turns out not to be fulfilled, the connected inference is withdrawn. This (and the importance of proof *procedures* for adaptive logics) naturally led to the idea of conditions expressing a prospective dynamics: a formula is derivable if the condition is fulfilled. Research on the matter resulted in a special kind of goal-directed proofs.

If applied to **CL**, the prospective dynamics turns out to lead to a very natural proof search method. This is a specific kind of procedure. The focus of the enterprise is not on abstract matters (such as models, semantic consequence, derivability, or the 'existence' of a proof) but on a very concrete one: the next step to be taken in an attempt to find out whether a conclusion A is derivable from a premise set Γ. A procedure of this kind is followed by people who search for proofs.

It turned out—and this was the stunning discovery—that, with respect to such a procedure, EFQ is an isolated, unnatural kind of rule, which is never applied in searching for a sensible proof for $\Gamma \vdash A$. In other words, EFQ can be eliminated from a procedural point of view, even if, for example, Addition and Disjunctive Syllogism are retained.

The proof procedure is introduced in Section 2. In Section 3 the procedure is shown to be incomplete with respect to **CL**: EFQ has to be introduced as a separate (and unnatural) entity in order to obtain a proof procedure for **CL**. Precisely this makes the incomplete procedure $\mathbf{pCL}^-$ and the logic $\mathbf{CL}^-$ defined by it interesting. In Section 4, I shall show that $\mathbf{pCL}^-$ has a sensible semantics. Some properties of $\mathbf{CL}^-$ are studied in Section 5; among them is that fact that, for all consistent Γ, $Cn_{\mathbf{CL}^-}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$, whereas a formula of the form $A \wedge \sim A$ is $\mathbf{CL}^-$-derivable (together with a natural set of further consequences) from Γ if Γ is inconsistent. Finally, I comment on the relation between $\mathbf{CL}^-$ and adaptive logics in Section 7.

Limitations of space prevent me from discussing the predicative case, but the extension is straightforward. Also, I shall only consider *finite* premise sets Γ—see R3 of [6] for the extension to infinite premise sets.

Incidentally, there is nothing to compare my results to (as the organizers request). All sensible paraconsistent logics known to me are based on the abstract viewpoint and for example give up either Disjunctive Syllogism or Addition, or, as in the case of inconsistency-adaptive logics, some applications of one of these rules—see [8] and [12] for systems giving up all or some applications of Addition.

2 The Proof Procedure

Some tend to exaggerate the distinction between a proof of $\Gamma \vdash A$ and a proof search procedure for $\Gamma \vdash A$. For example, in [7] (and many other publications by Hintikka and associates) trying to prove that A is derivable from Γ is compared to playing a game of chess, and a sharp distinction is drawn between the rules that define the game, called "rules of inference" in the case of logic, and the heuristic instructions than enable one to win the game. *Omnis comparatio claudicat* and this certainly does. This is not the place to discuss the matter, but, for one thing, the outcome of a game of chess is (in most cases) a checkmate king and a couple of other pieces, whereas a proof, the outcome of a proof search procedure, contains all required premises as well as all steps that lead from those premises to the conclusion. The basic distinction between a proof and a proof search procedure is twofold: (i) a proof does not contain the reasoning that led to deriving the actually displayed formulas and (ii) most proofs do not contain any traces of dead-end search paths—steps that might have resulted in deriving the conclusion from the premises but turned out unsuccessful. The reason for (ii) is that authors are expected to publish convincing proofs, and that dead-end paths do not contribute to these. The reason for (i) is that people responsible for the definition of "a proof" (basically Hilbert) took published proofs as the model, that is: proofs that show in a clear and concise way that the conclusion follows from the premises.

Part of the proof search procedure may be pushed into the proof itself. A way to do so is discussed in the present section. Suppose that one wants to find out whether there is a **CL**-proof for

$$t \vee q, p \supset (q \vee {\sim} r), r \wedge s, s \supset p \vdash q\,.$$

A sensible proof search procedure will proceed more or less as follows. The conclusion q can only be obtained from the premise $t \vee q$ or from the premise $p \supset (q \vee {\sim} r)$—$q$ is not a subformula of any other premise. In order to derive q from $t \vee q$, one needs ${\sim} t$. However, there is no premise from which ${\sim} t$ might be obtained. In order to derive q from $p \supset (q \vee {\sim} r)$, one needs first p and next r. Clearly r can be obtained from $r \wedge s$; p can only be obtained from $s \supset p$, and in order to obtain it one needs s. And s can be obtained from $r \wedge s$. So q is derivable from the premises.

This proof search process clarifies why certain steps occur in a proof for the statement displayed above. Usual proofs do not. Moreover, large parts of such processes leave no trace in usual proofs—for example, the reasoning about deriving q from $t \vee q$. It is not difficult, however, to devise a kind of proofs that contain the basic ingredients of the proof search process. I first display the proof and then offer some explanation.

1	$[q]q$	Goal	R^{14}
2	$t \vee q$	Prem	
3	$[{\sim} t]q$	2; $\vee$E	${\sim} t\vert$
4	$p \supset (q \vee {\sim} r)$	Prem	
5	$[p]q \vee {\sim} r$	4; $\supset$E	R^{11}
6	$s \supset p$	Prem	
7	$[s]p$	6; $\supset$E	R^{10}
8	$r \wedge s$	Prem	

9	s	8; $\wedge$E	
10	p	7, 9; Trans	
11	$q \vee {\sim}r$	5, 10; Trans	
12	$[r]q$	11; $\vee$E	R^{14}
13	r	8; $\wedge$E	
14	q	12, 13; Trans	

One starts by writing down the Goal of the proof: to derive q (from the premises). Where Γ is the set of premises, the derivability of $[B_1, \ldots, B_n]A$ warrants that $\Gamma \cup \{B_1, \ldots, B_n\} \vdash A$. Line 1 is obviously justified from this perspective.[1] The (prospective) *condition* of a line, viz. the sequence of the formulas between brackets,[2] obviously indicates that one should search to derive these formulas. The present proof proceeds by a "depth first" method: as soon as a condition is introduced, its first member becomes the *target*. After line 1 has been written, the target is q. As q may be obtained from the premise $t \vee q$, this premise is introduced—it obviously has no condition attached to it. As the target is still q, 2 is analysed, which gives us line 3. This states that q is derivable if ${\sim}t$ is derivable. However, ${\sim}t$ is not derivable from any premise, and hence line 3 is marked as a dead end: the goal q cannot be derived along this path. As line 3 is marked, the target is again q. One then introduces another premise from which the target might be derivable, viz. $p \supset (q \vee {\sim}r)$. Analysing the premise in line 5 makes p the target.[3] Lines 6–9 require no further comment. In line 10 transitivity is applied: if s is derivable, then so is p (line 7) and s is derivable unconditionally (line 9); so p is derivable unconditionally. At this stage, line 7 is marked as redundant. Indeed, there is no need to search for s in order to derive p because p was derived unconditionally. This makes p the target—the first formula in the condition of the last unmarked line—and as we have p in line 10, transitivity gives us $q \vee {\sim}r$ unconditionally, whence line 5 is marked as redundant and the target is again q (from line 1). Continuing thus one arrives at line 14. At this stage the conclusion is derived from the premises, which may be seen from the fact that line 1 is marked as redundant.

The main elements of prospective proofs occur in the example. Some formulas are derived on a prospective condition. Lines that contain a (non-empty) condition may be marked—some further useful kinds of marks are illustrated in [6] and especially in forthcoming work.

There is an algorithm for turning prospective proofs into, for example, Fitch-style proofs. This is done by (possibly inserting lines,) deleting lines and adjusting the annotation. Starting backwards, the algorithm transforms lines 11–14 as follows—I take it that the names of the Fitch-style rules are self-explanatory.

11	$q \vee {\sim}r$	5, 10; Trans	
~~12~~	$\cancel{[r]q}$	~~11; $\vee$E~~	$\cancel{\mathrm{R}^{14}}$
13	r	8; Sim	
14	q	11, 13; DS	

[1]Obviously $\Gamma \cup \{B_1, \ldots, B_n\} \vdash A$ does not warrant the derivability of $[B_1, \ldots, B_n]A$ as becomes clear from the subsequent sentence in the text. See [13] for the result of proceeding in terms of $\Gamma \cup \{B_1, \ldots, B_n\} \vdash A$ rather than $[B_1, \ldots, B_n]A$.

[2]I shall introduce some notational abuse to mix such expressions as $[\Delta]A$ and $B \in \Delta$.

[3]The result of the analysis is a formula, here $q \vee {\sim}r$, from which the target q may be derived. The formula itself depends on the condition p.

After the transformation, deleted lines and unused (not referenced) lines are removed and the line numbers adjusted. The result for the considered example is:

1	$p \supset (q \vee \sim r)$	Prem
2	$s \supset p$	Prem
3	$r \wedge s$	Prem
4	s	3; Sim
5	p	2, 4; MP
6	$q \vee \sim r$	1, 5; MP
7	r	3; Sim
8	q	6, 7; Trans

Goal, Prem, Trans, and some "formula analysing rules" were illustrated. In other cases one needs "condition analysing rules" as well as EM (Excluded Middle). The following proof for $\sim p \vee q \vdash p \supset q$ illustrates both. In still other cases conditions may contain several formulas—see [6] for examples.

1	$[p \supset q]p \supset q$	Goal	R^8
2	$[q]p \supset q$	1; C$\supset$E	R^8
3	$\sim p \vee q$	Prem	
4	$[p]q$	3; $\vee$E	$p\vert$
5	$[\sim p]p \supset q$	1; C$\supset$E	R^8
6	$[\sim q]\sim p$	3; $\vee$E	$\sim q\vert$
7	$[p]p \supset q$	2, 4; Trans	$p\vert$ R^8
8	$p \supset q$	5, 7; EM	

The (only member of the) condition of 1 is analysed in 2 and 5, and EM is applied in line 8. Actually a derivable rule allows one to skip 5–7 and to move straight from 4 to 8; but I have no room to discuss derivable rules here.

If prospective proofs are transformed (in one of several possible ways) to Fitch-style proofs, the latter may contain applications of Conditional Proof and Ex Absurdo. The prospective proofs determine which hypothesis has to be introduced and which formula should be derived from it, even if this formula is a contradiction (in the case of Ex Absurdo).

In constructing a proof for $\Gamma \vdash G$, the goal and premise rules are:

Goal   To introduce $[G]G$.

Prem   To introduce A for any $A \in \Gamma$.

Two further rules we have already met are:

Trans $$\frac{\begin{array}{c}[\Delta \cup \{B\}]A \\ [\Delta']B\end{array}}{[\Delta \cup \Delta']A}$$

EM $$\frac{\begin{array}{c}[\Delta \cup \{B\}]A \\ [\Delta' \cup \{\sim B\}]A\end{array}}{[\Delta \cup \Delta']A}$$

The formula analysing rules and the condition analysing rules may be summarized by distinguishing α-formulas from β-formulas (varying on a theme from [11]). To each formula two other formulas are assigned according to the following table:

α	α_1	α_2		β	β_1	β_2
$A \wedge B$	A	B		$\sim(A \wedge B)$	$\sim A$	$\sim B$
$A \equiv B$	$A \supset B$	$B \supset A$		$\sim(A \equiv B)$	$\sim(A \supset B)$	$\sim(B \supset A)$
$\sim(A \vee B)$	$\sim A$	$\sim B$		$A \vee B$	A	B
$\sim(A \supset B)$	A	$\sim B$		$A \supset B$	$\sim A$	B
$\sim\sim A$	A	A				

Let $*A$ denote the 'complement' of A, viz. B if A has the form $\sim B$ and $\sim A$ otherwise. The formula analysing rules for α-formulas and β-formulas are respectively:[4]

$$\frac{[\Delta]\alpha}{[\Delta]\alpha_1 \quad [\Delta]\alpha_2} \qquad\qquad \frac{[\Delta]\beta}{[\Delta \cup \{*\beta_2\}]\beta_1 \quad [\Delta \cup \{*\beta_1\}]\beta_2}$$

The condition analysing rules for α-formulas and β-formulas are respectively:

$$\frac{[\Delta \cup \{\alpha\}]A}{[\Delta \cup \{\alpha_1, \alpha_2\}]A} \qquad\qquad \frac{[\Delta \cup \{\beta\}]A}{[\Delta \cup \{\beta_1\}]A \quad [\Delta \cup \{\beta_2\}]A}$$

Certain instructions are required to characterize the proof search procedure, and hence the new kind of proof format, in terms of the above rules. Many variants are possible. I pick one that is easy to understand without much comment.

Every proof starts with an application of the Goal rule, which introduces the first target. A condition-plus-formula (the second element of a line) is derived only once in a proof.

The marks are governed by *definitions*. A line at which $[\Delta]A$ is derived is marked as redundant at a stage iff, at that stage, $[\Delta']A$ has been derived for some $\Delta' \subset \Delta$. A line at which $[A_1, \ldots, A_n]B$ is derived is marked as a dead end at a stage if, at that stage, A_1 is the target and no further step can be taken in view of it. No rules are applied on marked lines unless the instructions explicitly state so.

A premise is only introduced if the target is a *positive part* of it, which means that the target may be obtained from the premise, possibly on a non-empty condition. The positive part relation is defined recursively by the following three clauses:[5]

1. $\mathrm{pp}(A, A)$.
2. $\mathrm{pp}(A, \alpha)$ if $\mathrm{pp}(A, \alpha_1)$ or $\mathrm{pp}(A, \alpha_2)$.
3. $\mathrm{pp}(A, \beta)$ if $\mathrm{pp}(A, \beta_1)$ or $\mathrm{pp}(A, \beta_2)$.

[4] The rule to the left actually summarizes two rules: both $[\Delta]\alpha_1$ and $[\Delta]\alpha_2$ may be derived from $[\Delta]\alpha$; similarly for the rule to the right.

[5] Unlike what is done in [10] and [6], I do not introduce negative parts because this complicates the predicative case.

Formula analysing rules are only applied to lines that have a premise line in their path. Moreover, $[B_1, \ldots, B_n]A$ is only derived by a formula analysing rule if the target is a positive part of A.

Condition analysing rules are only applied to a member A of some prospective condition after all premises of which A is a positive part have been introduced in the proof.

As the transformation of prospective proofs into other types of proofs will be disregarded in the sequel of this paper, the prospective proof procedure is most easily seen as consisting of two phases. In the first phase, the above instructions are executed and the Trans rules is applied whenever Δ' is empty. If the thus defined procedure halts, one moves to phase 2. Here EM, Trans, and condition analysing rules are applied to lines that are not R-marked in order to derive the goal G on a condition Δ for which $[\Delta]G$ did not yet occur in the proof. As soon as the proof is extended in phase 2, one returns to phase 1. When phase 1 halts again, one moves back to phase 2, etc. The procedure is obviously decidable (at the propositional level).

Many variants are possible for this proof search procedure and for the resulting prospective proofs. Some of these increase the efficiency of the proofs or their transparency with respect to certain applications (especially problem solving procedures). All this cannot be discussed here. The basic point is that procedures of the kind described here lead to natural and efficient proofs.[6]

3 What Happened to EFQ?

The procedure described in the previous section assigns a well-defined consequence set to every premise set and hence defines a logic. However, this logic is not **CL**: the procedure does not lead to a proof for $p, {\sim}p \vdash q$.

Let me put this straight. The procedure leads to proofs for $p, {\sim}p \vdash p \vee q$ as well as for ${\sim}p, p \vee q \vdash q$, but not to a proof for $p, {\sim}p \vdash q$—the goal q is a dead end with respect to the premise set $\{p, {\sim}p\}$.

Given the procedure's relation to **CL**, let us call it $\mathbf{pCL}^-$. In order to obtain a procedure **pCL** that characterizes **CL**, one needs to add EFQ, for example by the following rule:

EFQ To introduce $[{\sim}A]G$ for any $A \in \Gamma$.

The procedure then also needs to be modified. A third phase is introduced. One moves to it if phase 2 halts, extends the proof with an application of EFQ and returns to phase 1. **pCL** is shown in [6] to characterize the propositional fragment of **CL** and to be a decision method for it: $A_1, \ldots, A_n \nvdash_{\mathbf{CL}} G$ iff the procedure stops after finitely many steps without G being derived (on the empty condition).

The rule EFQ is clearly unnatural and ad hoc. From a procedural perspective EFQ is not the unavoidable result of other (apparently acceptable) rules, but has to be introduced as a separate item. Moreover the EFQ rule cannot be justified unless by explicitly requiring that $A, {\sim}A \vdash B$ should hold for all A and B. Indeed, EFQ stipulates by fiat that the goal G is derivable from the premises if the negation of a premise is derivable from the premises, in other words if

[6] A striking difference with tableau methods is that prospective search procedures avoid that useless formulas are analysed.

the premises are inconsistent. This illustrates the huge difference between the traditional abstract view on logic, which proceeds basically in terms of the semantic consequence relation, and the procedural outlook.

Some readers might be sceptical about the claims made in the previous paragraph. While it is straightforward that $p, {\sim}p \nvdash_{\mathbf{pCL}^-} q$, it is less clear that no *other* **CL**-valid inferences are lost in $\mathbf{CL}^-$. This will be shown in Section 5.

4 Semantics

There are many ways to devise a semantics for $\mathbf{CL}^-$. Only one possibility will be explored here. Where $\mathcal{W}$ denotes the set of formulas of the propositional **CL**-language, consider all partial functions $v : \mathcal{W} \mapsto \{0, 1\}$ with the following properties:

1. if $v(A) \in \{0, 1\}$ and B is a subformula of A, then $v(B), v({\sim}B) \in \{0, 1\}$
2. if $v(A \wedge B) = 1$ then $v(A) = 1$ and $v(B) = 1$.
3. if $v(A \wedge B) = 0$ then $v(A) = 0$ or $v(B) = 0$.
4. if $v(A \equiv B) = 1$ then $v(A \supset B) = 1$ and $v(B \supset A) = 1$.
5. if $v(A \equiv B) = 0$ then $v(A \supset B) = 0$ or $v(B \supset A) = 0$.
6. if $v({\sim}(A \vee B)) = 1$ then $v({\sim}A) = 1$ and $v({\sim}B) = 1$.
7. if $v({\sim}(A \vee B)) = 0$ then $v({\sim}A) = 0$ or $v({\sim}B) = 0$.
8. if $v({\sim}(A \supset B)) = 1$ then $v(A) = 1$ and $v({\sim}B) = 1$.
9. if $v({\sim}(A \supset B)) = 0$ then $v(A) = 0$ or $v({\sim}B) = 0$.
10. if $v({\sim}{\sim}A) = 1$ then $v(A) = 1$.
11. if $v({\sim}{\sim}A) = 0$ then $v(A) = 0$.
12. if $v(A \vee B) = 1$ then $v(*A) = 0$ or $v(B) = 1$.
13. if $v(A \vee B) = 1$ then $v(A) = 1$ or $v(*B) = 0$.
14. if $v(A \vee B) = 0$ then $v(A) = 0$ and $v(B) = 0$.
15. if $v(A \supset B) = 1$ then $v(A) = 0$ or $v(B) = 1$.
16. if $v(A \supset B) = 1$ then $v({\sim}A) = 1$ or $v(*B) = 0$.
17. if $v(A \supset B) = 0$ then $v({\sim}A) = 0$ and $v(B) = 0$.
18. if $v({\sim}(A \wedge B)) = 1$ then $v(A) = 0$ or $v({\sim}B) = 1$.
19. if $v({\sim}(A \wedge B)) = 1$ then $v({\sim}A) = 1$ or $v(B) = 0$.
20. if $v({\sim}(A \wedge B)) = 0$ then $v({\sim}A) = 0$ and $v({\sim}B) = 0$.
21. if $v({\sim}(A \equiv B)) = 1$ then $v((A \supset B)) = 0$ or $v({\sim}(B \supset A)) = 1$.
22. if $v({\sim}(A \equiv B)) = 1$ then $v({\sim}(A \supset B)) = 1$ or $v((B \supset A)) = 0$.

23. if $v(\sim(A \equiv B)) = 0$ then $v(\sim(A \supset B)) = 0$ and $v(\sim(B \supset A)) = 0$.

24. if $v(\sim A) = 0$ then $v(A) = 1$.

Definition 1 *v is* a valuation for $A_1, \ldots, A_n \vdash B$ *iff* $v(A_1) = \ldots = v(A_n) = 1$ *and* $v(B) \in \{0, 1\}$.

Definition 2 *$A_1, \ldots, A_n \models B$ (B is a semantic consequence of $A_1, \ldots, A_n$) iff all valuations for $A_1, \ldots, A_n \vdash B$ verify B.*

The semantics may be transformed into a more usual one, for example a three-valued one, but space prevents me to discuss this.

The proof procedure is sound and complete with respect to the semantics. In the present context, an expression $[A_1, \ldots, A_n]B$ will be read as "either one of the A_i is false or B is true".

Theorem 1 *If $[A_1, \ldots, A_n]B$ is derived in a* **pCL**$^-$*-proof for $\Gamma \vdash G$, then $v(A_1) = 0$ or $\ldots$ or $v(A_n) = 0$ or $v(B) = 1$ for every v that is a valuation for $\Gamma \vdash G$.*

Outline of the proof. Let v be a valuation for $\Gamma \vdash G$. The proof proceeds by an obvious induction on the length of the prospective proof. The basis is formed by the rules Goal and Prem. The Goal rule introduces $[G]G$. As $v(G) \in \{0, 1\}$, $v(G) = 1$ or $v(G) = 0$. The Prem rule introduces a $C \in \Gamma$, and $C \in \Gamma$ warrants that $v(C) = 1$.

For the induction step we have to consider formula analysing rules, condition analysing rules, Trans and EM. Consider first $\wedge$E. If $[\Delta]A \wedge B$ occurs in the proof, then $v(C) = 0$ for some $C \in \Delta$ or $v(A \wedge B) = 1$. So, in view of clause 2 of the semantics, $v(C) = 0$ for some $C \in \Delta$ or $v(A) = 1$, which justifies the derivation of $[\Delta]A$, and $v(C) = 0$ for some $C \in \Delta$ or $v(B) = 1$, which justifies the derivation of $[\Delta]B$. All other formula analysing rules are justified in an analogous way. Next consider C$\wedge$E. If $[\Delta \cup \{A \wedge B\}]D$ occurs in the proof, then $v(C) = 0$ for some $C \in \Delta$ or $v(A \wedge B) = 0$ or $v(D) = 1$. In view of clause 3 of the semantics, it follows that $v(C) = 0$ for some $C \in \Delta$ or $v(A) = 0$ or $v(B) = 0$ or $v(D) = 1$, which justifies the derivation of $[\Delta \cup \{A, B\}]D$. All other condition analysing rules are justified similarly. EM is justified by clause 24 of the semantics and the justification of Trans is obvious. ■

Corollary 1 *If G is derived in a* **pCL**$^-$*-proof for $\Gamma \vdash G$, then $\Gamma \models_{\mathbf{CL}^-} G$. (Soundness)*

A prospective proof is said to halt if no instruction enables one to add a further line or if its first line is marked (and hence its Goal was derived). Given that I consider only finite premise sets in this paper, a prospective proof for $\Gamma \vdash G$ obviously halts after finitely many steps. So the following establishes completeness.

Theorem 2 *If a prospective proof for $\Gamma \vdash G$ halts without G begin derived, then $\Gamma \not\models_{\mathbf{CL}^-} G$. (Completeness)*

The proof of this theorem is not difficult to understand but requires some three pages. The proof is skipped here because it may easily be adapted from the completeness proof in [6]. The main difference is that Lemma 5, which there relies on EFQ, is now a direct consequence of the definition of the semantic consequence relation.

5 Some Properties and a Comment

The derivability relation defined by $\mathbf{pCL}^-$ is reflexive (if $A \in \Gamma$, then $G \vdash A$) and monotonic (if $\Gamma \vdash A$ then $\Gamma \cup \Delta \vdash A$) but non-transitive (possibly $\Gamma \vdash A$ and $A \vdash C$ but $\Gamma \nvdash C$)—for example $p, {\sim}p \vdash (p \vee q) \wedge {\sim}p$, $(p \vee q) \wedge {\sim}p \vdash q$, and $p, {\sim}p \nvdash q$.

There is a specific sense in which $\mathbf{CL}^-$ is relevant. The involved notion of relevance is not exactly the one propagated by relevant logicians but comes very close to it (variable sharing, relevance indices in Fitch-style proofs, ...). This deserves attention, but space does not permit.

Let us define that a $\mathbf{CL}^-$-valuation v is consistent iff $\{A \mid v(A) = 1\}$ is verified by a $\mathbf{CL}$-valuation, and that Γ is $\mathbf{CL}^-$-consistent iff it is verified by a consistent $\mathbf{CL}^-$-valuation. Each of the following are provable:

1. If Γ has $\mathbf{CL}$-valuations and v is one of them (hence $\mathsf{v}(A) = 1$ for all $A \in \Gamma$), then, for every A, there is a $\mathbf{CL}^-$-valuation v for $\Gamma \vdash A$ that agrees with v where v is defined—if $v(B)$ is defined, then $v(B) = \mathsf{v}(B)$.

2. Hence, if Γ has $\mathbf{CL}$-valuations, then, for all A, there are consistent $\mathbf{CL}^-$-valuations for $\Gamma \vdash A$.

3. If Γ has no $\mathbf{CL}$-valuations, then, for every A, all $\mathbf{CL}^-$-valuations v for $\Gamma \vdash A$ are inconsistent.

4. Γ is $\mathbf{CL}^-$-consistent iff it is $\mathbf{CL}$-consistent.

5. If Γ is inconsistent, then there is an A such that $\Gamma \vdash_{\mathbf{pCL}^-} A \wedge {\sim}A$

6. For some inconsistent Γ, $Cn_{\mathbf{CL}^-}(\Gamma)$ is not trivial.

7. If Γ is consistent, then, for all A and for all $\mathbf{CL}^-$-valuations v for $\Gamma \vdash A$, there is a $\mathbf{CL}$-valuation v such that, for all B, if $v(B) \in \{0, 1\}$ and $\mathsf{v}(B) = 1$, then $v(B) = 1$.

8. If Γ is consistent, then $Cn_{\mathbf{CL}^-}(\Gamma) = Cn_{\mathbf{CL}}(\Gamma)$.

9. If restricted to consistent premise sets, the intended domain of application of classical logic, $\mathbf{pCL}^-$-derivability is transitive.

10. If restricted to consistent premise sets, $\mathbf{pCL}^-$-derivability is sound and complete with respect to the $\mathbf{CL}$-semantics.

The inventors of $\mathbf{CL}$ attempted to formulate a logical system that provides an explication for actually published proofs, mainly in mathematics. Imagine that the ideology of the time had been different, and that the stress had not been on abstract aspects (models, a derivability relation fulfilling certain properties, ...) but on describing an inferential procedure. As inconsistent premise sets were taken to be false, none of the available proofs was intended to apply to inconsistent premise sets. Whence there was no need for including EFQ in 'classical logic'. Worse, to include EFQ would have been nonsensical because it is nonsensical to derive an arbitrary statement from a premise set that is known to be inconsistent. So, if the ideology of the time had been different, the inventors of classical logic would have come up with $\mathbf{pCL}^-$ rather than with $\mathbf{CL}$. The non-transitivity of $\mathbf{pCL}^-$-derivability would not have been objectionable because of 9. If, however, a premise set Γ had turned out inconsistent, $\mathbf{pCL}^-$ would have located the inconsistencies in Γ, whereas **CL** fails to do so. In sum, classical logic would have been paraconsistent as well as relevant (in a specific sense); EFQ would have been considered invalid; theories that turned out inconsistent would have been considered false but not necessarily trivial, and coming across them might have led to a very different view on the properties of the derivability relation. The history of relevant logic and of paraconsistent logic would have been completely different, and so on.

6 Tableaux and Deduction Systems

One of the referees complained that it is not clear from the paper whether $\mathbf{pCL}^-$ can be defined through a tableau system, an axiomatic system, a natural deduction system, and the like. Actually many fascinating aspects of $\mathbf{pCL}^-$ cannot be spelled out in the present paper, but let me briefly comment on the referee's points.

That $\mathbf{pCL}^-$ is defined by a tableau system is obvious from the semantics. A signed system is easily outlined as follows. A tableau for $A_1, \ldots, A_n \vDash_{\mathbf{pCL}^-} B$ is started by the sequence of signed formulas $TA_1, \ldots, TA_n, FB$. The tableau rules are simply 'translated' from clauses 2–24 of the semantics. Thus clause 2 and clause 12 are respectively translated by (following conventions from [11]):

$$\begin{array}{c} TA \wedge B \\ \hline TA \\ TB \end{array} \qquad \begin{array}{c} TA \vee B \\ \hline F{*}A \quad | \quad TB \end{array}$$

It is easily seen that the tableau for $p, {\sim}p \vDash_{\mathbf{pCL}^-} {\sim}q$ has one branch only, that it consists of Tp, $T{\sim}p$, $F{\sim}q$ and Tq, and hence that the branch is open.

On the standard definitions of "theorem" (either $\emptyset \vdash A$ or $\Gamma \vdash A$ for all Γ) $\mathbf{pCL}^-$ has the same theorems as **CL**. So theorems do not discriminate between the two systems. Deductive systems do discriminate between them. However, given the non-transitivity of $\mathbf{pCL}^-$-derivability—see the first paragraph of Section 5—deductive systems defining $\mathbf{pCL}^-$ contain some unusual restrictions. Let me offer a glance at (what I think to be) the simplest approach to Fitch-style proofs (on the depth first approach). The required complication is that one needs a target set for each stage of a proof, where a next stage is obtained by either adding a new line to the proof or analysing a formula in the target set—see below. At stage 1 of the proof attempt for $\Gamma \vdash A$, the target set is $\{A\}$. At each stage, a Fitch-style rule can only be applied if a member of the target set is a positive part of the formula introduced by the rule. If no Fitch-style rule can be applied, then a β-formula in the proof is 'analysed'; if no formula can be analysed, a member of the target set is 'analysed'. Analysing a β-formula in the proof means this: if A is in the target set and is a positive part of β_1 (β_2) then $*\beta_2$ ($*\beta_1$) is added to the target set. Analysing a member of the target set means that some α is replaced in the target set by α_1 and α_2 or that some β is replaced in the target set by β_1 and β_2. To make such proofs more efficient, one needs to remove dead-end members from the target set, thus matching the marking definition in Section 2. From a computational point of view it is simpler to construct a proof by the procedure from Section 2 and to transform it to a Fitch-style proof as explained in that section.

Such Fitch-style proof shed light on the freshman's reluctance to apply Addition and other 'non-informative' rules[7] (in which an arbitrary subformula appears in the conclusion). The reluctance is well known to teachers of logic. It can easily be overcome in cases where $A \vee B$ is derived from A in order to obtain C by Modus Ponens from $(A \vee B) \supset C$ and $A \vee B$. However, the freshman will keep complaining about the proof of B from A and $\sim A$ in which $A \vee B$ is derived from A and next B is derived from $A \vee B$ and $\sim A$. And the freshman's complain makes sense: precisely this application of Addition is impossible in view of the restriction on the Fitch-style proofs for $\mathbf{pCL}^-$.

7 Relation to Inconsistency-Adaptive Logics

Is $\mathbf{CL}^-$ the suitable logic for handling inconsistent premise sets? The answer will obviously depend on the involved aim. In many cases one wants to interpret a premise set as consistently as possible—see [1], [3], [4] or [9] for sundry reasons to do so. This includes the case in which one wants to study the inconsistent 'theory' in order to devise a consistent replacement for it. In such cases, one does not want to derive A from $A \vee B$ and $\sim B$ in the presence of B, even if $A \vee B$ is a premise. So the right approach to such applications is not provided by $\mathbf{CL}^-$ but by an inconsistency-adaptive logic—see the same papers.[8]

One should not conclude that $\mathbf{CL}^-$ is a useless system. $\mathbf{CL}^-$ explicates proofs from the pre-**CL** era just as good as **CL** itself. $\mathbf{CL}^-$ justifies apparently naive views that do not reduce inconsistency to triviality. All this challenges the classical logician's claim that sound reasoning forces one to reduce inconsistency to triviality.[9]

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[2] Diderik Batens. Blocks. The clue to dynamic aspects of logic. *Logique et Analyse*, 150–152:285–328, 1995. Appeared 1997.

[3] Diderik Batens. Inconsistency-adaptive logics. In Ewa Orłowska, editor, *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*, pages 445–472. Physica Verlag (Springer), Heidelberg, New York, 1999.

[4] Diderik Batens. A survey of inconsistency-adaptive logics. In Batens et al. [5], pages 49–73.

[7] See [2] for an exact definition of a non-informative rule.

[8] This naturally raises the question whether there is an inconsistency-adaptive logic that has $\mathbf{CL}^-$ as its lower limit and **CL** as its upper limit. The adaptive logic would then validate certain *applications* of EFQ, depending on the specific premise sets. However, no inconsistency-adaptive logic of the kind seems to exist. So, apparently, even from an adaptive point of view, either all applications of EFQ have to be accepted, or all have to be rejected.

[9] Unpublished papers mentioned in the reference section (and many others) are available from the internet address `http://logica.rug.ac.be/centrum/writings/`.

[5] Diderik Batens, Chris Mortensen, Graham Priest, and Jean Paul Van Bendegem, editors. *Frontiers of Paraconsistent Logic*. Research Studies Press, Baldock, UK, 2000.

[6] Diderik Batens and Dagmar Provijn. Pushing the search paths in the proofs. A study in proof heuristics. *Logique et Analyse*, in print.

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[13] Andrzej Wiśniewski. Socratic proofs. 2003.