Coarse Deontic Logic

Frederik Van De Putte [1]

Ghent University
Blandijnberg 2, B-9000 Ghent, Belgium
frederik.vandeputte@ugent.be

Abstract

Cariani [7] has proposed a semantics for *ought* that combines two features: (i) it invalidates Inheritance in a principled manner; (ii) it allows for coarseness, which means that *ought*(φ) can be true even if there are specific ways of making φ true that are (intuitively speaking) impermissible. We present a family of multi-modal logics based on Cariani's proposal and a more recent critique by Bronfmann&Dowell [4]. We study their formal properties in detail and show how they can be translated into normal multi-modal logics. Using well-known techniques, we establish a sound and (strongly) complete axiomatization for each of these and show them to satisfy the finite model property. In addition, we compare them to existing approaches in the deontic logic literature — most notably Anglberger et al.'s logic of obligation as weakest permission and Horty's deontic stit logic.

Keywords: Deontic logic; contrastivism; modal inheritance; Ross paradox; deontic STIT logic; coarseness

1 Introduction

Contrastivism about "ought" says that claims using this modality can only be understood relative to a (usually implicit) contrast class. [2] So according to this view, "you ought to take the bus" is shorthand for "given the set of alternatives $\mathcal{A}$ under consideration, you ought to take the bus". Here $\mathcal{A}$ consists of various ways of the agent in question getting somewhere (say, the university) at a given moment.

In recent work, Cariani has proposed a formal semantics which starts from a contrastivist reading of ought [7]. This proposal is interesting for at least

[1] This paper is the full version of [30], including all meta-proofs, a translation into normal modal logic, and complexity results that were not given in that paper. Research for this paper was funded by the Flemish Research Foundation (FWO-Vlaanderen). We are indebted to Mathieu Beirlaen and three anonymous referees for incisive comments on previous versions of that paper. In addition, we thank the audience of DEON2016 and three referees for AiML2016 for critical suggestions and remarks which helped improve the present paper.

[2] See [27, footnote 1] for some key references to contrastivism in deontic logic.

two reasons. First, it gives a principled account of why Inheritance [3] fails in cases like the Ross paradox, which makes it more insightful than most existing semantics for non-normal modalities. [4] Second, it allows for what Cariani calls *coarse* ought-claims, which means that $ought(\varphi)$ can be true even if there are specific ways of making φ true that are (intuitively speaking) impermissible. [5] This unusual combination – coarseness *without* Inheritance – is possible precisely because of the way the alternatives are modeled: rather than single worlds, they are (mutually exclusive) *sets* of worlds.

In order to argue for or against Cariani's proposal, we believe that one has to study the logics obtained from it. This paper's aim is to do exactly this.

Outline In Section 2, we present Cariani's proposal, both informally and in terms of a possible-worlds semantics. We discuss the most salient properties of the resulting logic. Next, we consider variants of this semantics that are defined over the same modal language, with special attention for one variant which is inspired by a critique from [4] (Section 3).

Sections 4-6 form the technical core of the paper. In these sections we define the base logic **CDL** and show how it can be characterized as a normal modal logic. After that, we axiomatize all the extensions that were discussed in Section 3 and establish the finite model property for these systems.

In Section 7, we show how the logics relate to existing work in the deontic logic field, and where one can draw on this link in order to solve existing problems and puzzles.

Preliminaries We use $p, q, \ldots$ for arbitrary propositional variables. The boolean connectives are denoted by $\neg, \vee, \wedge, \supset, \equiv$ (only the first two are primitive) and occasionally we will use the falsum and verum constants ($\bot$, resp. $\top$). $\varphi, \psi, \ldots$ are metavariables for formulas and $\Gamma, \Delta, \ldots$ for sets of formulas. *ought* refers to operators proposed as formal counterparts of the natural language "ought". Given an expression of the type $ought(\varphi)$, φ is the *prejacent* of this formula.

2 Cariani's Semantics

In this section, we introduce and illustrate Cariani's semantics for *ought*. We first present the semantics informally in our own terms, after which we indicate the relation with Cariani's original presentation (Section 2.1). Next, we define a full-fledged formal semantics (Section 2.2) and discuss the most salient properties of the resulting logic (Section 2.3).

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[3] By *Inheritance* we mean here: from $ought(\varphi)$ and $\varphi \vdash \psi$, to infer $ought(\psi)$. This property is also often called *monotony*.

[4] Cariani [7, p. 537] remarks that such semantics are "often purely algebraic", in the sense that they just translate rules for *ought* into conditions on neighbourhood functions. Notable exceptions are the contrastive semantics for *ought* from [12,13] and the logic of "obligation as weakest permission" from [1] which we will discuss in Section 7.

[5] We explain and illustrate Cariani's notion of coarseness in Section 2.1.

2.1 Cariani's proposal, informally

Our Version To spell out the main idea behind Cariani's semantics for *ought*, we need to introduce three parameters:

(a) a set of (mutually exclusive) *alternatives* or *options* $\mathcal{A}$
(b) a set $\mathcal{B} \subseteq \mathcal{A}$ of "optimal" or "best" options
(c) a set $\mathcal{I} \subseteq \mathcal{A}$ of "impermissible" options

Each of $\mathcal{A}, \mathcal{B}$ and $\mathcal{I}$ may be thought of as settled by a "deliberative context" in which an agent α has to choose among a number of different options. In other words, our normative reasoning always starts from a deliberative question: which of the members of $\mathcal{A}$ should α choose or carry out? The level of granularity of $\mathcal{A}$ depends on the goals, values, and desires of α within the context at hand [7, p. 539].

For instance, in a context where someone is deliberating about how Lisa ought to get to the university, her options may be represented by the following set:

$$\mathcal{A}_{\mathsf{ex}} = \{\text{walk}, \text{bike}, \text{bus}, \text{car}\}$$

indicating that she may walk to the university, drive her bike, take the bus, or drive by car. Some of these options may be optimal – e.g. biking or taking the bus. Driving may well be impermissible (since she may not yet have obtained her driver's licence) and walking may be suboptimal (since given the distance, she risks getting late) but nevertheless permissible. So we have:

$$\mathcal{B}_{\mathsf{ex}} = \{\text{bike}, \text{bus}\}$$

$$\mathcal{I}_{\mathsf{ex}} = \{\text{car}\}$$

Each of the options in $\mathcal{A}_{\mathsf{ex}}$ can be carried out in many different ways; e.g. Lisa may drive her bike in a blue dress or in a green dress; she may drive her bike in a hazardous way or very cautiously. In Cariani's terms: the alternatives are *coarse-grained*. This means that they correspond to generic actions or states of affairs (propositions, sets of worlds in a Kripke-model), in contrast to maximally specific actions or states of affairs (worlds in a Kripke-model).

In light of this feature, it is tempting to interpret Cariani's options as action-types. This interpretation is suggested by Cariani himself [7, pp. 544-545]. It is also pursued in other work on deontic logic; see e.g. [1] for a concrete and worked-out example. However, one has to be careful here: depending on the case at hand, the deliberative context may be very specific – it may well be thought of as a concrete decision problem "here and now". In such a case, the actions $X \in \mathcal{A}$ are not repeatable ones like e.g. 'closing the door" or "biking to school". Also, they are not closed under any operations such as iteration, intersection, or union, as is the case with the action types in the tradition of Dynamic Logic.

Alternatively, one may insist that options are action tokens: concrete, singular events that can be of one or several types – provided that one is willing to let one such event take place in several distinct "worlds", or in combination with several distinct states of affairs. Such a view of events is made explicit i.a. by Ming Xu and Mark Brown in their work on the extension of logics of agency with action types – see e.g. [5] for a gentle introduction. It is also embraced in recent work by Horty and Pacuit [19], where they describe the choice cells in a STIT frame as action tokens, to be distinguished from action types (i.e., on their view, labels of such cells which allow for comparisons across different states). We return to the similarities with STIT logic in Section 7.2. [6]

In the remainder, we shall remain neutral regarding these interpretations; we will accordingly stick to the more neutral terms "option", "alternative", or "choice" in order to refer to members of $\mathcal{A}$.

Whatever one's preferred interpretation, the coarseness of options explains at once how it is possible, in Cariani's framework, that there are (intuitively) impermissible instances of an optimal (or permissible) alternative. Even if Lisa ought to drive her bike or take the bus, this does not imply that every way of doing so is normatively ok. Indeed, relative to a more fine-grained set of alternatives, it may turn out that some ways of driving her bike are impermissible. Mind that the framework does not explicitly represent the impermissibility of such more specific options – hence, they are only impermissible "intuitively speaking". The point is exactly that, by choosing one specific level of granularity in a certain context, we decide to leave those more specific (impermissible) options out of the picture. Once we make them explicit, the level of granularity changes, and with it the truth of any given *ought*-claim. [7]

Since options are coarse-grained, they do not fix every property of the world. Still, some propositions are fixed by taking one option rather than the other. If Lisa takes her bike, she is definitely not taking the bus or driving her car. In general, we say that an option $X \in \mathcal{A}$ *guarantees* a proposition φ iff following that option ensures that φ is the case.

We are now ready to spell out an informal version of Cariani's proposal. That is, where φ is a proposition, $ought(\varphi)$ is true (relative to $\mathcal{A}, \mathcal{B}, \mathcal{I}$) iff each of the following hold:

(i) φ is *visible*, i.e. for all $X \in \mathcal{A}$: X guarantees φ or X guarantees $\neg\varphi$
(ii) φ is *optimal*, i.e. for all $X \in \mathcal{B}$: X guarantees φ
(iii) φ is *strongly permitted*, i.e. for all $X \in \mathcal{A}$ that guarantee φ, $X \notin \mathcal{I}$.

For instance, in our example, it is true that Lisa ought to ride her bike or take the bus. It is false that she ought to ride her bike, take the bus or take the car, since taking the car is impermissible. It is equally false that she ought to ride her bike or take the bus in a green dress, since that proposition is not

[6] It is noteworthy that Cariani himself refers to [2] as the main source of inspiration for his interpretation of his semantics [7, p. 556].

[7] This of course raises the question how oughts concerning such more fine-grained $\mathcal{A}'$ relate to the coarse-grained $\mathcal{A}$ – we return to this point in Section 7.

visible.[8]

This shows us at once that Inheritance is invalid on Cariani's semantics. It is in fact blocked in two different ways – see (i) and (iii) above. As a result, also the Ross paradox is blocked: "you ought to mail the letter" may be true while "you ought to mail the letter or burn it" is false. This will either be the case because burning the letter is invisible, or if we do take it to be a visible option, because it is impermissible.

Ranking and threshold In Cariani's original proposal, instead of $\mathcal{B}$ and $\mathcal{I}$, a "ranking" of $\mathcal{A}$ is used together with a "threshold" t on that ranking. The idea is that the "best" options are those that are maximal (according to the ranking), and the impermissible ones are those that are below the threshold. Although Cariani is not very explicit about the formal properties of his ranking and threshold, it seems that his ranking is a modular pre-order, in the sense that it distinguishes different layers of "ever better" options.[9] In other words, it can be defined as a function $r : \mathcal{A} \to \mathbb{R}$, where intuitively, X is better than X' (for $X, X' \in \mathcal{A}$) iff $r(X) > r(X')$. The threshold is then simply a $t \in \mathbb{R}$, such that whenever $r(X) < t$, X is impermissible.

It is easy enough to check that, once such an r and t are given, we can obtain $\mathcal{B}$ and $\mathcal{I}$ from them as follows: (i) $\mathcal{B} = \{X \in \mathcal{A} \mid r(X) = \mathsf{max}_<(\{r(Y) \mid Y \in \mathcal{A}\})\}$, and (ii) $\mathcal{I}$ is the set of all $X \in \mathcal{A}$ such that $r(X) < t$. Hence our simplified version of Cariani's semantics is at least as general as his original version.

Given fairly weak assumptions, we can also show the converse. That is, consider an arbitrary $\langle \mathcal{A}, \mathcal{B}, \mathcal{I} \rangle$ and suppose that each of the following hold:

(D) $\mathcal{B} \neq \emptyset$
(C∩ $\mathcal{B} \cap \mathcal{I} = \emptyset$

In other words, there are best options, and every best option is permissible. Define the function $r : \mathcal{A} \to \{1, 2, 3\}$ as follows:

(1) if $X \in \mathcal{B}$, then $r(X) = 3$
(2) if $X \in \mathcal{I}$, then $r(X) = 1$
(3) if $X \in \mathcal{A} \setminus (\mathcal{B} \cup \mathcal{I})$ then $r(X) = 2$

Let $t = 2$. It can easily be checked that (i) and (ii) hold. So if we assume (D) and (C∩), the two formats are equivalent (deontically speaking).

In the current section, we will leave restrictions (D) and (C∩) aside. In Section 3.1 we consider variants of our base logic in which these restrictions are added to the semantics.

8 As the reader may note, "Lisa ought to take her bike, take the bus, or walk to the university" is also true in our example, which might strike one as odd. We return to this point in Section 3.2.

9 At least it is in all the examples he gives. Also, this seems to be presupposed by the way he uses the notion of a threshold, viz. as a single member X of $\mathcal{A}$ such that any option below X is impermissible.

2.2 The formal semantics of $\mathbf{CDL^c}$

Our language $\mathcal{L}$ is obtained by closing the set of propositional variables $\mathcal{S} = \{p, q, \ldots\}$ under the Boolean connectives and the modal operators U (necessary/holds in every possible world), A (is guaranteed by the chosen alternative), B (is best/is guaranteed by all optimal alternatives), and P (is strongly permitted).

Two comments are in place here. First, Cariani does not explicitly mention the operators U and A. However, both are fairly natural modalities in this context. U is just a global (or universal) modality – see [16] for a systematic study. A expresses the concept of being guaranteed by a given option, which Cariani uses in the semantic clause of his *ought*-operator. Moreover, adding both modalities to the language allows us to obtain a sound and complete axiomatization of the logic – see Section 4. [10]

Second, rather than taking it as primitive as Cariani does, we treat "is visible", V, as a defined operator:

$$\mathsf{V}\varphi =_{\mathsf{df}} \mathsf{U}(\mathsf{A}\varphi \vee \mathsf{A}\neg\varphi)$$

Likewise, O (Cariani's *ought*) is a defined operator:

$$\mathsf{O}\varphi =_{\mathsf{df}} \mathsf{V}\varphi \wedge \mathsf{B}\varphi \wedge \mathsf{P}\varphi$$

The following two definitions make the informal semantics from Section 2.1 exact: [11]

Definition 2.1 A $\mathbf{CDL^c}$-*frame* is a tuple $F = \langle W, \mathcal{A}, \mathcal{B}, \mathcal{I}\rangle$, where W is a non-empty set, $\mathcal{A} \in \wp(\wp(W))$ is a partition of W, $\mathcal{B} \subseteq \mathcal{A}$ is the set of *best options* in $\mathcal{A}$, and $\mathcal{I} \subseteq \mathcal{A}$ is the set of *impermissible options* in $\mathcal{A}$.

A $\mathbf{CDL^c}$-*model* M is a $\mathbf{CDL^c}$-frame $\langle W, \mathcal{A}, \mathcal{B}, \mathcal{I}\rangle$ augmented with a valuation function $v : \mathcal{S} \to \wp(W)$.

Since $\mathcal{A}$ is a partition of W, all worlds are by definition a member of some alternative in the contrast class. In other words, we exclude the possibility that some members of W are simply irrelevant for the deontic claims that are at stake. We leave the investigation of such a possibility for another occasion.

In line with the preceding, the members of $\mathcal{A}$ are interpreted as coarse options, or as choices between general states of affairs that a given agent faces, whereas the members of W represent maximally specific states of affairs. Formulas are evaluated relative to a given $w \in W$, in accordance with Definition 2.2. This means that in general, whether or not a formula is true may depend on the option that is chosen and on the specific way it is carried out or materializes. However, for purely normative claims, this is not the case (see our discussion of the property of Uniformity in Sections 2.3 and 3.3).

Definition 2.2 Let $M = \langle W, \mathcal{A}, \mathcal{B}, \mathcal{I}, v\rangle$ be a $\mathbf{CDL^c}$-model and $w \in W$. Where $w \in W$, let X^w denote the $X \in \mathcal{A}$ such that $w \in X$.

[10] It remains an open question whether one can obtain such an axiomatization without these modalities, and with V primitive.

[11] **CDL** is shorthand for "Coarse Deontic Logic". The superscript c refers to Cariani.

(SC1) $M, w \models \varphi$ iff $w \in v(\varphi)$ for all $\varphi \in \mathcal{S}$
(SC2) $M, w \models \neg\varphi$ iff $M, w \not\models \varphi$
(SC3) $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$
(SC4) $M, w \models \mathsf{U}\varphi$ iff $M, w' \models \varphi$ for all $w' \in W$
(SC5) $M, w \models \mathsf{A}\varphi$ iff $M, w' \models \varphi$ for all $w' \in X^w$
(SC6) $M, w \models \mathsf{B}\varphi$ iff for all $X \in \mathcal{B}$, for all $v \in X$, $M, v \models \varphi$
(SC7) $M, w \models \mathsf{P}\varphi$ iff for all $X \in \mathcal{A}$ s.t. (for all $v \in X$, $M, v \models \varphi$), $X \notin \mathcal{I}$

Note that $\mathsf{V}\varphi$ means (by our definition) that at every world w in the current model, either φ is guaranteed or $\neg\varphi$ is guaranteed. Since $\mathcal{A}$ is a partition of W, this is equivalent to saying that every option either guarantees φ or guarantees $\neg\varphi$, which corresponds to Cariani's original semantics for "is visible".

The clause for P can also be rephrased in terms of the set $\mathcal{P} = \mathcal{A} \setminus \mathcal{I}$ of permissible options: $\mathsf{P}\varphi$ is true at M, w iff every option $X \in \mathcal{A}$ such that $X \subseteq |\varphi|^M$, is permissible. In other words, guaranteeing φ is sufficient for an option to be permissible – see [31] for an elaborate study of such constructions in deontic logic. However, in the remainder we stick to the formulation in terms of $\mathcal{I}$ since it will simplify our technical work in Sections 4-6.

As usual, $\Gamma \Vdash_{\mathbf{CDL^c}} \varphi$ iff for all $\mathbf{CDL^c}$-models M and every world w in the domain of M, if $M, w \models \psi$ for all $\psi \in \Gamma$, then $M, w \models \varphi$.

2.3 Properties of $\mathbf{CDL^c}$

It can be easily verified that each of U, A and B are normal modal operators in $\mathbf{CDL^c}$. In fact, both U and A are **S5**-modalities. Second, P is a non-normal but classical modality (in the sense of Chellas [8]), which means it satisfies at least replacement of equivalents. As a result, also the defined operators V and O are classical.

Now for some more distinctive properties. Each of the following hold for $\Vdash = \Vdash_{\mathbf{CDL^c}}$:

$$\mathsf{O}\varphi \nVdash \mathsf{O}(\varphi \vee \psi) \tag{1}$$

$$\mathsf{O}(\varphi \wedge \psi) \nVdash \mathsf{O}\varphi \tag{2}$$

$$\mathsf{O}\varphi, \mathsf{O}\psi \Vdash \mathsf{O}(\varphi \wedge \psi) \tag{3}$$

$$\mathsf{O}\varphi, \mathsf{O}\psi \Vdash \mathsf{O}(\varphi \vee \psi) \tag{4}$$

$$\mathsf{O}\varphi, \mathsf{P}\psi \nVdash \mathsf{O}(\varphi \vee \psi) \tag{5}$$

$$\mathsf{O}\varphi, \mathsf{P}(\varphi \vee \psi) \nVdash \mathsf{O}(\varphi \vee \psi) \tag{6}$$

$$\mathsf{O}\varphi, \mathsf{P}\psi, \mathsf{V}\psi \Vdash \mathsf{O}(\varphi \vee \psi) \tag{7}$$

$$\mathsf{V}\varphi, \mathsf{V}\psi \Vdash \mathsf{V}(\varphi \vee \psi) \tag{8}$$

$$\mathsf{V}\varphi, \mathsf{V}\psi \Vdash \mathsf{V}(\varphi \wedge \psi) \tag{9}$$

$$\Vdash \mathsf{P}(\varphi \vee \psi) \supset (\mathsf{P}\varphi \wedge \mathsf{P}\psi) \tag{10}$$

$$\Vdash \mathsf{P}\varphi \supset \mathsf{P}(\varphi \wedge \psi) \tag{11}$$

$$\nVdash (\mathsf{P}\varphi \wedge \mathsf{P}\psi) \supset \mathsf{P}(\varphi \vee \psi) \tag{12}$$

$$\Vdash (\mathsf{P}\neg\mathsf{A}\neg\varphi \wedge \mathsf{P}\neg\mathsf{A}\neg\psi) \supset \mathsf{P}(\neg\mathsf{A}\neg\varphi \vee \neg\mathsf{A}\neg\psi) \tag{13}$$

Let us comment on these properties one by one. That O does not satisfy *Inheritance* – see (1) – was already explained above. Since the logic is closed under replacement of equivalents, this also means that O does not distribute over conjunctions, cf. (2).[12] More surprisingly, *Aggregation* (3) holds for O. In a context where the possibility of deontic conflicts is omitted, this is often considered a nice feature. It follows from the fact that the three operators B, V, and P are each aggregative – see (9) for V and (11) for P. For similar reasons, *Weakening* (4) also holds in $\mathbf{CDL^c}$.

Both Aggregation and Weakening deserve our attention here. As shown in [6], these properties fail on what is perhaps the most well-known contrastive semantics for *ought*, viz. the actualist semantics from [23], which has been worked out and axiomatized by Goble [12,13].

(5) and (6) tell us that, contrary to what one might expect, neither $\mathsf{P}\psi$ nor $\mathsf{P}(\psi \vee \varphi)$ suffice in order to derive $\mathsf{O}(\varphi \vee \psi)$ from $\mathsf{O}\varphi$.[13] The reason is that neither of those propositions warrant that $\varphi \vee \psi$ is visible, which is required for $\mathsf{O}(\varphi \vee \psi)$ to hold. Only if we add $\mathsf{V}\psi$ do we obtain a restricted form of Inheritance that is $\mathbf{CDL^c}$-valid – see (7).

Together with replacement of equivalents, (10) entails that P is "downward closed": whatever is stronger than something that is permitted, is itself also permitted. To see why this is so, recall that $\mathsf{P}(\varphi \vee \psi)$ expresses that guaranteeing $\varphi \vee \psi$ implies that one is choosing a permissible option. Hence *a fortiori* guaranteeing φ (resp. ψ) is sufficient for permissibility. By the definition of O, this also means that $\mathsf{O}(\varphi \vee \psi) \Vdash_{\mathbf{CDL^c}} \mathsf{P}\varphi, \mathsf{P}\psi$: that $\varphi \vee \psi$ ought to be implies that $\varphi \vee \psi$ is strongly permitted, which in turn implies that both φ and ψ are strongly permitted. We return to this property in Section 3.2.

[12] Cariani motivates the absence of this validity in terms of the famous Professor Procrastinate case, cf. [7, p. 541].

[13] Snedegar [27, pp. 217-218] refers to Goble [14, Note 49] who rejects such a rule. However, in Goble's case, the P-operator is one of weak permission, i.e. $\mathsf{P} =_{\mathsf{df}} \neg\mathsf{O}\neg$. Besides that, Goble's main concern is to accommodate deontic conflicts, a target which Cariani explicitly rules out – as Snedegar acknowledges.

In view of (12), P is not an operator of "free choice permission" in the strict sense of [32]. To see why (12) holds, recall our example. Here, "Lisa takes the car in a green dress" ($car \wedge green$) is permissible in a vacuous way, since there is simply no option which guarantees that proposition. Likewise, "Lisa takes the car, but not in a green dress" ($car \wedge \neg green$) is permissible. However, *car* (which is equivalent to the disjunction of both propositions) is not permissible.[14]

(13) shows that for the more specific case where φ and ψ are of the form $\neg\mathsf{A}\neg\tau$, we do get the converse of (10). If it is permissible that (a) one leaves open the possibility that φ, and it is also permissible that (b) one leaves open the possibility that ψ, then it is permissible that (c) one leaves open the possibility that φ or one leaves open the possibility that ψ. Indeed, whenever (c) holds, either (a) or (b) hold and hence one is definitely taking one of the permissible options.

Other interesting validities concern the interaction between the alethic modalities U, A and the deontic modalities B, P, and O. These are of two types:

$$\text{where } \nabla \in \{\mathsf{B}, \mathsf{P}, \mathsf{O}\} : \Vdash \nabla\varphi \equiv \nabla\mathsf{A}\varphi \tag{14}$$

$$\text{where } \nabla \in \{\mathsf{B}, \mathsf{P}, \mathsf{O}\} : \Vdash \nabla\varphi \equiv \mathsf{U}\nabla\varphi \tag{15}$$

Contrast-sensitivity, (14), expresses that the deontic modalities really apply to alternatives $X \in \mathcal{A}$, rather than worlds $w \in W$. For instance, $\mathsf{B}\varphi$ can only be true if φ is true in all worlds that belong to an optimal alternative; but that is the same as saying that all optimal alternatives guarantee φ. This property is therefore essential for Cariani's constrastive approach.

Uniformity, (15), expresses that deontic claims are either settled true or settled false (to borrow terminology from [2]). It follows from the fact that $\mathcal{B}$ and $\mathcal{I}$ are independent of the world w one happens to be at in a model. We return to this property in Section 3.3.

3 Some Variants

We now consider variants of the $\mathbf{CDL^c}$-semantics and motivate each of them independently. This will be useful in Section 7, where we compare Cariani's construction to existing work in deontic logic.

3.1 Conditions (D) and (C∩)

We first return to the conditions mentioned at the end of Section 2.1. (D) corresponds to the requirement in Standard Deontic Logic that the accessibility relation is *serial*, and hence, that there is at least one "ideal" or "optimal"

[14] In view of this example, P seems to express only part of the meaning of "is permitted". A more plausible operator of (strong) permission can be defined by $\mathsf{P}^v\varphi =_{\mathsf{df}} \mathsf{P}\varphi \wedge \mathsf{V}\varphi$. Note that $(\mathsf{P}^v\varphi \wedge \mathsf{P}^v\psi) \Vdash_{\mathbf{CDL^c}} \mathsf{P}^v(\varphi \vee \psi)$, but $\mathsf{P}^v(\varphi \vee \psi) \nVdash_{\mathbf{CDL^c}} \mathsf{P}^v\varphi \wedge \mathsf{P}^v\psi$. We leave the investigation of such definable operators for future work.

world. It can be moreover easily checked that (D) is expressed by the familiar axiom schema $\mathsf{B}\varphi \supset \neg\mathsf{B}\neg\varphi$ within $\mathbf{CDL^c}$. This axiom schema (along with the failure of the T-schema, $\mathsf{B}\varphi \supset \varphi$) is traditionally seen as the distinctive feature of deontic logics.

Although it is a much debated property in the context of deontic logic in general, (D) does seem to have some intuitive power in the present context. After all, the idea is that we start from a fixed set of alternatives, one particular ranking r, and one threshold t.[15] Finiteness of $\mathcal{A}$ already entails (D). But even if we allow for a possibly infinite number of options, it seems sensible to say that we only consider finitely many of those as viable options, such that a ranking on them will always yield a non-empty set of best alternatives.

(C∩) is more difficult to interpret in the present context. It states that every best option is permissible. Interestingly, this condition is not definable in the language of $\mathbf{CDL^c}$. In fact, imposing it onto the semantics has no impact on the resulting logic, as we show in Section 5.2. This means in turn that, once we assume (D), and as far as the logic is concerned, there really is no difference between Cariani's original semantics and our reformulation of it.

3.2 Putting the threshold at optimality

Bronfman & Dowell note that Cariani's use of a set of alternatives (as a set of sets of worlds) and a ranking on them does not conflict *per se* with the standard approach in modal logic [4, p. 6]:

> [O]ne sort of value Kratzer's ordering source can take is one that would rank each world w in the modal background in terms of what an agent does in w [...] Such a ranking would have the effect of ranking all worlds in which the agent performs the same action the same. This would mean that such an ordering source would, in effect, rank options, in Cariani's sense. ([4, p. 6].)

This point is further clarified and made exact in the next section of the present paper, where we develop a simple way to translate the formal semantics from the previous section into a regular Kripke-semantics.

The distinctive feature of Cariani's semantics, according to Bronfman & Dowell, is the use of the permissibility threshold in order to block Inheritance. It is this feature that they attack.

To understand their argument, we should briefly rehearse the pragmatic defense of Inheritance for *ought*. This defense says, roughly speaking, that although affirming $ought(\varphi \vee \psi)$ is rather pointless in cases where we also know $ought(\varphi)$, the former expression is nevertheless true whenever the latter is. It is much like affirming "John is either Dutch or Italian" when we actually know that John is Dutch: not maximally helpful, but also not plainly false or mistaken. What *is* false is the Gricean implicature that follows when we *only* state $ought(\varphi \vee \psi)$, viz. that $\varphi \vee \psi$ is the most specific necessary condition for optimality.

[15] As Cariani notes, one may generalize the entire setting to cases with multiple rankings and threshold functions; that seems to be his preferred way of allowing for deontic conflicts.

Cariani rejects this defense of Inheritance, since it cannot account for the way *ought* behaves in embeddings [7, pp. 549]. Such behavior, he argues, can only be explained by the following principle:

(Implicated) *ought*$(\varphi \vee \psi)$ communicates that one has two ways of doing as one ought, viz. by making φ true or by making ψ true.

In contrast, Cariani's account seems to cover (Implicated) well: as we saw in Section 2.3, $\mathsf{O}(\varphi \vee \psi) \Vdash_{\mathbf{CDL^c}} \mathsf{P}\varphi, \mathsf{P}\psi$.

However, Bronfman & Dowell rightly remark that (Implicated) gives counterintuitive results when applied to Cariani's own semantics. That is, by taking an option that is suboptimal but permissible, the agent is also doing as (s)he ought – at least if (Implicated) holds. Let us illustrate this with our running example. The options *bus* and *bike* are the only two optimal ones. However, since *walk* is permissible, *ought*(*bus* ∨ *bike* ∨ *walk*) comes out true. But, given (Implicated), this means that by walking to the university, Lisa is doing as she ought.

Bronfman & Dowell suggest that, if one really wants to satisfy Cariani's requirement, one should put the threshold at optimality.[16] There are, technically speaking, two ways to implement this suggestion. The first is to change the semantic clause for P, such that $M, w \models \mathsf{P}\varphi$ iff, whenever $X \in \mathcal{A}$ is such that $M, w' \models \varphi$ for all $w' \in X$, then $X \in \mathcal{B}$. This means that $\mathcal{I}$ becomes superfluous in the semantics of the logic.

Secondly, one may leave the semantic clause for P unaltered, but treat $\mathcal{I}$ simply as the set of all suboptimal alternatives. This means that we impose the following frame condition on $\mathbf{CDL^c}$-models:

(C+) $\mathcal{I} = \mathcal{A} \setminus \mathcal{B}$

The advantage of this second approach – which we will follow in the remainder – is that it allows for a smooth comparison with Cariani's original proposal. Note that (C+) is equivalent to the conjunction of condition (C∩) (see Section 2.1) and the following:

(C∪) $\mathcal{I} \cup \mathcal{B} = \mathcal{A}$

Henceforth, let M be a $\mathbf{CDL^{bd}}$-model iff it is a $\mathbf{CDL^c}$-model that satisfies (C+); we denote the associated consequence relation by $\Vdash_{\mathbf{CDL^{bd}}}$.

Obviously, $\mathbf{CDL^{bd}}$ is an extension of $\mathbf{CDL^c}$. But exactly what additional validities (in our language $\mathcal{L}$) do we get from imposing this condition? Each of (1)-(12) from Section 2.3 hold also for $\Vdash \; = \; \Vdash_{\mathbf{CDL^{bd}}}$, and hence not much seems to change to the deontic part of the language.

[16] There remains a problem though. Suppose that "Lisa ought to go to the supermarket" is true. Since the semantics satisfies replacement of equivalents, it follows that "Lisa ought to either go to the supermarket and pay for whatever she buys or go to the supermarket and steal something." Given (Implicated), it follows that by going to the supermarket and stealing something, Lisa is doing as she ought. So whatever refinement one proposes of Cariani's (or Kratzer's) semantics, pragmatic factors will anyway have to be called for at some point. (This example is a variant of Hansson's "vegetarian's free lunch" [17, p. 218].)

However, once we consider the interaction with U, we do get an important additional feature: if two ought-claims are both true, their prejacents have the same truth-set within the model. Following [9], we call this *Uniqueness*:

$$\Vdash_{\mathbf{CDL^{bd}}} (\mathsf{O}\varphi \wedge \mathsf{O}\psi) \supset \mathsf{U}(\varphi \equiv \psi) \tag{16}$$

This property fails for $\mathbf{CDL^c}$ – witness our example: both *ought*(*bike*$\vee$*bus*) and *ought*(*bike* $\vee$ *bus* $\vee$ *walk*) are true, but one is obviously more specific than the other. Note that since U is a global modality, Uniqueness entails both Aggregation and Weakening for O.[17]

Even if condition (C+) is well-motivated, *Uniqueness* may be hard to swallow from the viewpoint of natural language. One morning, John may have to ensure that he gets to the office in time (p), but also that his children get to school in time (q). So *ought*(p) and *ought*(q) both seem true in this scenario. But John would be rather lucky if making q true would at once ensure that p also holds (or vice versa). The only way to avoid this strange conclusion would be to insist that actually, only *ought*($p \wedge q$) is true in this scenario, since e.g. "getting to the office in time without making sure the children are at school in time" is impermissible, as is "making sure the children are in time without getting to the office in time". But such a reply – even if perfectly consistent – seems to be at odds with our use of *ought* in natural language. We often use it simply to express (non-trivial, but) merely necessary conditions for optimality.

Two comments are in place here. First, this is not just a problem for proponents of (C+), but it is just as well a problem for any account of *ought* that combines a necessity clause (here, $\mathsf{B}\varphi$) with a sufficiency clause (here, $\mathsf{P}\varphi$).[18] That is, in many cases we seem to correctly assess *ought*(φ), even if φ is not sufficient for permissibility. So it is also a problem for Cariani's original proposal, even if *Uniqueness* fails for it.

Second, it is hard to see how this problem can be avoided, as long as one models necessity, resp. sufficiency (of a given proposition, for optimality resp. permissibility) in terms of set-theoretic inclusion. Within any such intensional semantics, this is the price to pay if we explain the failure of Inheritance in terms of a sufficiency requirement.

3.3 Rejecting Uniformity

As we just saw, there are reasons for strengthening $\mathbf{CDL^c}$ in various ways. There are however also reasons for weakening $\mathbf{CDL^c}$, in the sense that it is no longer assumed that optimality and permissibility are uniform throughout a model. That is, rather than taking $\mathcal{B}$ and $\mathcal{I}$ as sets of alternatives, one may think of them as functions, taking as their argument worlds $w \in W$ (or

[17] For aggregation: from $\mathsf{O}\varphi$ and $\mathsf{O}\psi$ we can derive $\mathsf{U}(\varphi \equiv \psi)$ by Uniqueness. Hence we can derive $\mathsf{U}(\varphi \equiv (\varphi \wedge \psi))$ by normal modal logic properties; and hence since U is a global modality, we can derive $\mathsf{O}(\varphi \wedge \psi)$. The reasoning for weakening is analogous, using $\mathsf{U}(\varphi \equiv (\varphi \vee \psi))$.

[18] This means that it is also a problem for the "logic of obligation as weakest permission" from [1] – see our discussion in Section 7.1.

alternatives $X \in \mathcal{A}$), and mapping these to sets of alternatives. This means in turn that the validities mentioned in (15) — see page 9 — are denied.

To motivate such a weakening, we can point to various arguments that have been put forth in the literature. First, from the viewpoint of game theory, it has been argued that which option for a given agent α is best, may depend on the options other agents take; hence, it will also depend on the specific world one happens to be at. See e.g. [1, Section 4.2] where this point is discussed and linked to some properties of the deontic operators.

Second, in [33], Wansing attacks specific constructions of deontic logic based on a branching-time framework, in which the truth of "obligation reports" (say, claims about what ought to be, what is best, what one ought to do, etc) depend only on the moment m of evaluation. This means that such claims are either true at all moment-history pairs m/h, or false at all m/h. In the present, more abstract framework, moments correspond to the entire set W, whereas moment/history-pairs correspond to single worlds $w \in W$.

Wansing's arguments for this claim are of two kinds. On the one hand, he writes that certain obligations are simply of such a type that they depend on future contingents. For instance, if "you ought to give the prize to the winner of this race" is true, then depending on who actually wins (say a or b), it may be true that "you ought to give the prize to a" – but this will of course not be settled true. The other argument is more intricate, as it concerns the so-called Restricted Complement thesis from [2]. As Wansing shows, this thesis together with Uniformity trivializes nested ought-claims of the type "John ought to see to it that it is forbidden for Mary to eat the cake."

Third and last, Uniformity is typically rejected by actualist theories of *ought*. In contrast to possibilists, actualists argue that what ought to be depends on what is actually the case (now or in the future), rather than on what can be (or may become) the case.[19] The temporal dimension is not explicit in the simple $\mathbf{CDL^c}$-models we considered so far. Nevertheless, the fact that we abstract from the temporal dimension in our models seems a sufficient reason to remain neutral about those properties that would become problematic, once we add time back in.

4 The Base Logic CDL

In this section and the next two, we study the formal properties of the logics that were discussed in previous sections. We start with the base logic **CDL** in this section, and show how it can be translated into a normal multi-modal logic. Extensions of the logic and complexity results are discussed in Sections 5 and 6 respectively.[20]

[19] See e.g. [21, Section 7.4.3] where the two views are briefly discussed and linked to two different notions of *ought* in stit logic. A more unified theory that encompasses both these notions is presented in [22].

[20] In this and the next two sections, we assume familiarity with basic notions of modal logic such as Sahlqvist correspondence, generated submodels, and definability of frame conditions. See e.g. [3] for background.

(CL)	any complete axiomatization of classical propositional logic
(MP)	from $\varphi, \varphi \supset \psi$ to infer ψ
(NEC_{U})	from $\vdash \varphi$, to infer $\vdash \mathsf{U}\varphi$
(K_{B})	$\mathsf{B}(\varphi \supset \psi) \supset (\mathsf{B}\varphi \supset \mathsf{B}\psi)$
(S5_{A})	all **S5**-axioms for U
(S5_{A})	all **S5**-axioms for A
(G_{A})	$\mathsf{U}\varphi \supset \mathsf{A}\varphi$
(G_{B})	$\mathsf{U}\varphi \supset \mathsf{B}\varphi$
(C_{B})	$\mathsf{B}\varphi \equiv \mathsf{BA}\varphi$
(G_{P})	$\mathsf{U}\varphi \supset \mathsf{P}\neg\varphi$
(C_{P})	$\mathsf{P}\varphi \equiv \mathsf{PA}\varphi$
(P1)	$\mathsf{P}(\varphi \vee \psi) \supset (\mathsf{P}\varphi \wedge \mathsf{P}\psi)$
(P2)	$(\mathsf{P}\neg\mathsf{A}\neg\varphi \wedge \mathsf{P}\neg\mathsf{A}\neg\psi) \supset \mathsf{P}(\neg\mathsf{A}\neg\varphi \vee \neg\mathsf{A}\neg\psi)$
(EQ_{P})	$\mathsf{U}(\varphi \equiv \psi) \supset (\mathsf{P}\varphi \equiv \mathsf{P}\psi)$

Table 1
Axiomatization of **CDL**.

4.1 Neighbourhood-semantics for CDL

We generalize Definition 2.1 from Section 2.2 as follows:

Definition 4.1 A **CDL**-*frame* is a tuple $F = \langle W, \mathcal{A}, \mathcal{B}, \mathcal{I} \rangle$, where W is a non-empty set, $\mathcal{A} \in \wp(\wp(W))$ is a partition of W, $\mathcal{B} : W \to \wp(\mathcal{A})$ maps every $w \in W$ to the set of *w-best options* in $\mathcal{A}$, and $\mathcal{I} : W \to \wp(\mathcal{A})$ maps every $w \in W$ to the set of *w-impermissible options* in $\mathcal{A}$.

The definition of a model and the semantic clauses remain the same, with the exception of the following:

(SC6') $M, w \models \mathsf{B}\varphi$ iff for all $X \in \mathcal{B}(w)$, for all $v \in X$, $M, v \models \varphi$
(SC7') $M, w \models \mathsf{P}\varphi$ iff for all $X \in \mathcal{A}$ s.t. (for all $v \in X$, $M, v \models \varphi$), $X \in \mathcal{A} \backslash \mathcal{I}(w)$

Table 2 gives a sound and (strongly) complete axiomatization of **CDL**. The first six axioms and rules in this table are standard. The axioms (G_{A}), (G_{B}) and (G_{P}) follow from the fact that U is a global modality. (C_{B}), (C_{P}), (P1) and (P2) were already discussed in Section 2.3. Finally, (EQ_{P}) is a strengthened version of replacement of equivalents for P.

We define theoremhood ($\vdash_{\mathbf{CDL}} \varphi$) in the usual way; in order to save space, syntactic consequence is defined as follows: $\Gamma \vdash_{\mathbf{CDL}} \varphi$ iff there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathbf{CDL}} (\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi$. It is a matter of routine to check that each of the axioms from Table 1 are valid in all **CDL**-models, and hence that the logic is sound. For completeness, we first introduce a translation of **CDL** into a normal multi-modal logic.

4.2 Modal Translation of CDL

Let the language $\mathcal{L}^m$ be obtained by replacing P in $\mathcal{L}$ with I. Intuitively, $\mathsf{I}\varphi$ can be read as "is guaranteed by all impermissible alternatives". We use $\mathcal{W}^m$ to denote the set of well-formed formulas in $\mathcal{L}^m$. Formulas from $\mathcal{W}^m$ are interpreted over a specific type of Kripke-models, which we now define.

Definition 4.2 A $\mathbf{CDL^m}$-*frame* is a quadruple $F = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}\rangle$, where W is a non-empty set and R_{A}, R_{B}, and R_{I} are binary relations on W that satisfy the following conditions:

(C1) R_{A} is an equivalence relation on W
(C2) where $\Box \in \{\mathsf{B}, \mathsf{I}\}$, $R_\Box \circ R_{\mathsf{A}} \subseteq R_\Box$

A $\mathbf{CDL^m}$-*model* $M = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}, V\rangle$ is a $\mathbf{CDL^m}$-frame equipped with a valuation function $v : \mathcal{S} \to \wp(W)$.

Definition 4.3 Where $M = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}, V\rangle$ is a $\mathbf{CDL^m}$-model, $w \in W$, and $\Box \in \{\mathsf{A}, \mathsf{B}, \mathsf{I}\}$:

(SC1) $M, w \models p$ iff $w \in V(p)$
(SC2) $M, w \models \neg\varphi$ iff $M, w \not\models \varphi$
(SC3) $M, w \models \varphi \vee \psi$ iff $M, w \models \varphi$ or $M, w \models \psi$
(SC4) $M, w \models \mathsf{U}\varphi$ iff $M, w' \models \varphi$ for all $w' \in W$
(SC8) $M, w \models \Box\varphi$ iff for all $w' \in R_\Box(w)$, $M, w' \models \varphi$

The semantic consequence relation $\Vdash_{\mathbf{CDL^m}}$ is defined in the standard way.

The idea behind these definitions can be explained as follows. First, the equivalence relation R_{A} corresponds to the partition $\mathcal{A}$ of W. Second, rather than speaking about best options in a direct way, we do so via a detour. That is, we treat B as a normal modality that quantifies over all worlds w that are the members of a best option. Thus we need to make sure that if w' is a member of a best option, and if w'' is a member of the same option as w', then also w'' is a member of a best option. This is nothing but the exact counterpart of the idea expressed by Bronfman & Dowell (cf. the quote on p. 10). Technically, this is done by imposing condition (C2) on the accessibility relation R_{B} that is associated with B. Third and last, I quantifies over all worlds that are members of an impermissible option. As with R_{B}, this means that R_{I} will have to satisfy condition (C2). Using I and A, we can then express the operator of strong permission from **CDL** as follows:

$$\mathsf{P}\varphi =_{\mathsf{df}} \mathsf{I}\neg\mathsf{A}\varphi$$

That is, $\mathsf{I}\neg\mathsf{A}\varphi$ iff, for all impermissible options X, there is at least one point in that option at which φ is false. This is equivalent to saying that if φ is guaranteed by an option $Y \in \mathcal{A}$, then $Y \in \mathcal{A} \setminus \mathcal{I}$.

Conversely, I can be defined within **CDL** as follows: $\mathsf{I}\varphi =_{\mathsf{df}} \mathsf{P}\neg\mathsf{A}\varphi$. The definiens expresses that all options in which φ is not guaranteed, are permissible; equivalently, whenever an option is impermissible, it guarantees φ.

4.3 Axiomatization of $\mathbf{CDL^m}$

To obtain a sound and (strongly) complete axiomatization for $\mathbf{CDL^m}$, we need to add the axioms from Table 2 to those from the first half of Table 1. Theoremhood and the derivability relation ($\vdash_{\mathbf{CDL^m}}$) are defined as before.

Theorem 4.4 $\Gamma \vdash_{\mathbf{CDL^m}} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{CDL^m}} \varphi$.

(G_{I})	$\mathsf{U}\varphi \supset \mathsf{I}\varphi$
(K_{I})	$\mathsf{I}(\varphi \supset \psi) \supset (\mathsf{I}\varphi \supset \mathsf{I}\psi)$
(C_{I})	$\mathsf{I}\varphi \equiv \mathsf{I}\mathsf{A}\varphi$

Table 2
Axioms for I in $\mathbf{CDL^m}$.

Proof. The proof of soundness is safely left to the reader. For (strong) completeness, note first that all the axioms are members of the Sahlqvist class. The corresponding frames are like $\mathbf{CDL^m}$-frames, with the exception that the canonical relation R^c_{U} for the operator U is not the total relation but merely an equivalence relation that satisfies the additional condition that, where $\boxplus \in \{\mathsf{B}, \mathsf{I}, \mathsf{A}\}$, $R_{\boxplus} \subseteq R_{\mathsf{U}}$ (in view of the axioms of the type ($G_{\boxplus}$). To obtain a model for a given maximally consistent set Γ, one thus has to take the generated submodel M^c_Γ in the standard way (restrict the domain to all worlds that are R^c_{U}-accessible from Γ; restrict all the accessibility relations and the valuation function accordingly). □

4.4 From CDL to CDL$^{\mathbf{m}}$ and back

We now show that **CDL** and $\mathbf{CDL^m}$ are intertranslatable. First, we define two translation functions $tr^m : \mathcal{W} \to \mathcal{W}^m$ and $tr^n : \mathcal{W}^m \to \mathcal{W}$:

Definition 4.5 Where $x \in \{n, m\}$:

$tr^x(\varphi) =_{\mathsf{df}} \varphi$ if $\varphi \in \mathcal{S}$
$tr^x(\neg\varphi) =_{\mathsf{df}} \neg tr^x(\varphi)$
$tr^x(\varphi \vee \psi) =_{\mathsf{df}} tr^x(\varphi) \vee tr^x(\psi)$
$tr^x(\Box\varphi) =_{\mathsf{df}} \Box tr^x(\varphi)$ where $\Box \in \{\mathsf{U}, \mathsf{A}, \mathsf{B}\}$
$tr^m(\mathsf{P}\varphi) =_{\mathsf{df}} \mathsf{I}\neg\mathsf{A}(tr^m(\varphi))$
$tr^n(\mathsf{I}\varphi) =_{\mathsf{df}} \mathsf{P}\neg\mathsf{A}(tr^n(\varphi))$

We lift this translation to sets in the usual way, i.e. $tr^x(\Delta) =_{\mathsf{df}} \{tr^x(\varphi) \mid \varphi \in \Delta\}$. In Appendix A, we prove each of the following:

Theorem 4.6 *Where* $\Gamma \cup \{\varphi\} \subseteq \mathcal{W}$*:* $\Gamma \vdash_{\mathbf{CDL}} \varphi$ *iff* $tr^m(\Gamma) \vdash_{\mathbf{CDL^m}} tr^m(\varphi)$.

Theorem 4.7 *Where* $\Gamma \cup \{\varphi\} \subseteq \mathcal{W}^m$*:* $\Gamma \vdash_{\mathbf{CDL^m}} \varphi$ *iff* $tr^n(\Gamma) \vdash_{\mathbf{CDL}} tr^n(\varphi)$.

Theorem 4.8 *Where* $\Gamma \cup \{\varphi\} \subseteq \mathcal{W}$*:* $\Gamma \Vdash_{\mathbf{CDL}} \varphi$ *iff* $tr^m(\Gamma) \Vdash_{\mathbf{CDL^m}} tr^m(\varphi)$.

Relying on Theorem 4.6, Theorem 4.4 and Theorem 4.8 consecutively, we immediately obtain:

Corollary 4.9 $\Gamma \vdash_{\mathbf{CDL}} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{CDL}} \varphi$.

In view of the above theorems, we can also easily derive the following counterpart of Theorem 4.8:

Theorem 4.10 *Where* $\Gamma \cup \{\varphi\} \subseteq \mathcal{W}^m$*:* $\Gamma \Vdash_{\mathbf{CDL^m}} \varphi$ *iff* $tr^n(\Gamma) \Vdash_{\mathbf{CDL}} tr^n(\varphi)$.

Proof. Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{W}^m$. We have: $\Gamma \Vdash_{\mathbf{CDL^m}} \varphi$ iff [by Theorem 4.4] $\Gamma \vdash_{\mathbf{CDL^m}} \varphi$ iff [by Theorem 4.7] $tr^n(\Gamma) \vdash_{\mathbf{CDL}} tr^n(\varphi)$ iff [by Corollary 4.9] $tr^n(\Gamma) \Vdash_{\mathbf{CDL}} tr^n(\varphi)$. □

5 Extensions of CDL

We now turn to the extensions of **CDL** that were discussed in Sections 2 and 3. It will be convenient to divide these extensions in two different groups: those obtained by imposing definable frame conditions on **CDL**-frames (Section 5.1), and the ones obtained by imposing, in addition to the definable conditions, also non-definable frame conditions (Sections 5.2).

5.1 Definable frame conditions

Table 3 lists the definable conditions on **CDL**-frames that were considered in this paper, the corresponding conditions on $\mathbf{CDL^m}$-frames, and the axioms in $\mathcal{L}^m$ that correspond to these conditions. To obtain an axiomatization of the **CDL**-extensions, it suffices to translate those axioms literally, i.e. replace each occurrence of I with $\mathsf{P}\neg\mathsf{A}$.

Where (C1), …, (Cn) are (relational) frame conditions from Table 3, M is a $\mathbf{CDL^m_{C1,\ldots,Cn}}$-*model* iff M is a $\mathbf{CDL^m}$-model and M obeys these conditions. We use $\Vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}}$ to refer to the associated semantic consequence relation. Likewise, we use $\vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}}$ to denote the syntactic consequence relation obtained by adding axioms that correspond to (C1), …, (Cn) to $\mathbf{CDL^m}$.

(U_B)	$\forall w, w' \in W$, $\mathcal{B}(w) = \mathcal{B}(w')$ $\forall w, w' \in W$, $R_\mathsf{B}(w) = R_\mathsf{B}(w')$	$\mathsf{B}\varphi \equiv \mathsf{U}\mathsf{B}\varphi$
(U_I)	$\forall w, w' \in W$, $\mathcal{I}(w) = \mathcal{I}(w')$ $\forall w, w' \in W$, $R_\mathsf{I}(w) = R_\mathsf{I}(w')$	$\mathsf{I}\varphi \equiv \mathsf{U}\mathsf{I}\varphi$
(A_B)	where $X \in \mathcal{A}$ and $w, w' \in X$, $\mathcal{B}(w) = \mathcal{B}(w')$ $\forall w, w' \in W$, if $w \in R_\mathsf{A}(w')$, then $R_\mathsf{B}(w) = R_\mathsf{B}(w')$	$\mathsf{B}\varphi \equiv \mathsf{A}\mathsf{B}\varphi$
(A_I)	where $X \in \mathcal{A}$ and $w, w' \in X$, $\mathcal{I}(w) = \mathcal{I}(w')$ $\forall w, w' \in W$, if $w \in R_\mathsf{A}(w')$, then $R_\mathsf{I}(w) = R_\mathsf{I}(w')$	$\mathsf{I}\varphi \equiv \mathsf{A}\mathsf{I}\varphi$
(D)	$\forall w \in W$, $\mathcal{B}(w) \neq \emptyset$ R_B is serial	$\mathsf{B}\varphi \supset \neg\mathsf{B}\neg\varphi$
(C$\cup$)	$\forall w \in W$, $\mathcal{B}(w) \cup \mathcal{I}(w) = \mathcal{A}$ $\forall w \in W$, $R_\mathsf{B}(w) \cup R_\mathsf{I}(w) = W$	$(\mathsf{B}\varphi \wedge \mathsf{I}\varphi) \supset \mathsf{U}\varphi$

Table 3
Definable frame conditions for $\mathbf{CDL}/\mathbf{CDL^m}$ and corresponding axioms in $\mathcal{L}^m$.

Theorem 5.1 *Where $C1, \ldots, Cn$ are frame conditions from Table 3:*

$$\Gamma \vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}} \varphi \textit{ iff } \Gamma \Vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}} \varphi$$

Proof. Again, we only give a proof sketch. Soundness is again a matter of routine. For completeness, we use a similar construction as in the proof of Theorem 4.4, again relying on Sahlqvist correspondence and the well-known technique of generated submodels. □

5.2 Non-definable frame conditions

This leaves us with two remaining frame conditions, both of which were discussed in Section 3. These conditions and their relational counterparts are given by Table 4.

(C+)	for all $w \in W$, $\mathcal{B}(w) = \mathcal{A} \setminus \mathcal{I}(w)$ for all $w \in W$, $R_{\mathsf{B}}(w) = W \setminus R_{\mathsf{I}}(w)$
(C$\cap$)	for all $w \in W$, $\mathcal{B}(w) \cap \mathcal{I}(w) = \emptyset$ for all $w \in W$, $R_{\mathsf{B}}(w) \cap R_{\mathsf{I}}(w) = \emptyset$

Table 4
Non-definable frame conditions for $\mathbf{CDL^m}$.

Frame conditions of this type were studied extensively in the context of Boolean Modal Logics and more generally, modal logics with intersection and/or complement – see e.g. [15,10,11]. We will apply similar proof techniques in the current paper in order to obtain complete axiomatizations for the logics based on these conditions.

Note that (C+) is equivalent to the conjunction of (C$\cap$) and (C$\cup$) (see Table 3). For technical reasons, we further divide the possible combinations of conditions in two: on the one hand, those with only (C$\cap$) and without (C$\cup$); on the other hand, those with both (C$\cup$) and (C$\cap$) (and hence (C+)). We will first show how the logics defined by such combinations of conditions are axiomatized. After that, we show that neither (C$\cap$), nor (C+) is definable within $\mathbf{CDL^m}$.

Extensions with (C$\cap$), without (C$\cup$) In short, imposing only condition (C$\cap$), or imposing (C$\cap$) on top of the other conditions from Table 3 except (C$\cup$), does not change anything to the semantic consequence relation of the logic. We now make this claim exact and prove it.

For the proof, we will make use of a specific kind of construction, which we call *copy-merge*. This technique originates in [10,25] and was generalized more recently in [31]. It consists in making two disjoint copies of a model M and merging them into one suitably defined model $M^\dagger$. M is then provably a bounded morphic image of $M^\dagger$.

In the remainder, we let i, j, k range over $\{1, 2\}$.

Definition 5.2 [Simple Copy-merge] Let $M = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}, V\rangle$ be a $\mathbf{CDL^m}$-model. Let W^1, W^2 be disjoint copies of W. For each $w \in W$, denote the counterpart of w in W^i by w^i.

The *simple copy-merge of* M is the model $M^\dagger = \langle W^\dagger, R^\dagger_{\mathsf{A}}, R^\dagger_{\mathsf{B}}, R^\dagger_{\mathsf{I}}, V^\dagger\rangle$, where

1. $W^\dagger = W^1 \cup W^2$
2. $R^\dagger_{\mathsf{A}} = \{(w^i, v^i) \mid (w, v) \in R_{\mathsf{A}}\}$
3. $R^\dagger_{\mathsf{B}} = \{(w^i, v^1) \mid (w, v) \in R_{\mathsf{B}}\}$
4. $R^\dagger_{\mathsf{I}} = \{(w^i, v^2) \mid (w, v) \in R_{\mathsf{I}}\}$
5. for all $\varphi \in \mathcal{S}$, $V^\dagger(\varphi) = \{w^1, w^2 \in W^\dagger \mid w \in V(\varphi)\}$

The following is shown in Appendix B.1:

Theorem 5.3 *Let $M = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}, V\rangle$ be a* $\mathbf{CDL^m}$*-model. Then each of the following hold:*

(i) If W is finite, then so is $W^\dagger$
(ii) $M^\dagger$ is a $\mathbf{CDL^m_{C\cap}}$*-model*
(iii) if M satisfies a given condition (C) from Table 3 except (C∪), then so does $M^\dagger$
(iv) $f : W^\dagger \to W$ with $f(w^i) = w$ is a bounded morphism from $M^\dagger$ to M
(v) for all $w \in W$, $i \in \{1, 2\}$ and φ, $M, w \models \varphi$ iff $M^\dagger, w^i \models \varphi$

Theorem 5.4 *Where $C1, \ldots, Cn$ are conditions from Table 3 except (C∪):* $\Gamma \Vdash_{\mathbf{CDL^m_{C\cap,C1,\ldots,Cn}}} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}} \varphi$.

Proof. Right to left holds vacuously. For left to right, suppose that $\Gamma \nVdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}} \varphi$. Hence, there is an $\mathbf{CDL^m_{C1,\ldots,Cn}}$-model M and a world w such that $M, w \models \psi$ for all $\psi \in \Gamma$ but $M, w \not\models \varphi$. Let $M^\dagger$ be the simple copy-merge of M. By the previous theorem, we can readily infer that $M^\dagger$ is a $\mathbf{CDL^m_{C\cap,C1,\ldots,Cn}}$-model and $M^\dagger, w^1 \models \psi$ for all $\psi \in \Gamma$ but $M^\dagger, w^1 \not\models \varphi$. Hence, $\Gamma \nVdash_{\mathbf{CDL^m_{C\cap,C1,\ldots,Cn}}} \varphi$ □

By Theorem 5.1, this gives us:

Corollary 5.5 *Where $C1, \ldots, Cn$ are conditions from Table 3 except (C∪):* $\Gamma \vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{CDL^m_{C\cap,C1,\ldots,Cn}}} \varphi$.

Extensions with both (C∩) and (C∪) When we consider classes of frames for which both (C∩) and (C∪) – or equivalently, (C+) – hold, it is important to note that the other frame conditions from Table 3 are no longer independent. That is, since for these frames, R_{B} is by definition the complement of R_{I}, we have the following equivalences between frame conditions:

$$(U_{\mathsf{B}}) \Leftrightarrow (U_{\mathsf{I}})$$

$$(A_{\mathsf{B}}) \Leftrightarrow (A_{\mathsf{I}})$$

Hence, the number of logically possible combinations of additional conditions drastically shrinks in the presence of (C+).[21] See Table 5 for an overview of these, and the corresponding combinations of axioms. In the remainder of this section, we will establish the soundness and completeness of each of the resulting logics.

We cannot apply the simple copy-merge technique for this purpose, since that construction does not warrant that (C∪) is satisfied for $M^\dagger$ whenever it is for M. So we need a slightly more complicated version of the technique, given by the following definition.

[21] Mind that in all **CDL**-frames, (U_{B}) implies (A_{B}) and (U_{I}) implies (A_{I}).

(U)	$(\mathsf{B}\varphi \equiv \mathsf{U}\mathsf{B}\varphi) \wedge (\mathsf{I}\varphi \equiv \mathsf{U}\mathsf{I}\varphi)$
(A)	$(\mathsf{B}\varphi \equiv \mathsf{A}\mathsf{B}\varphi) \wedge (\mathsf{I}\varphi \equiv \mathsf{A}\mathsf{I}\varphi)$
(D)	$\mathsf{B}\varphi \supset \neg\mathsf{B}\neg\varphi$

Table 5
Frame conditions and axioms for extensions of $\mathbf{CDL^m_{C+}}$.

Definition 5.6 [Strict Copy-merge] Let $M = \langle W, R_\mathsf{A}, R_\mathsf{B}, R_\mathsf{I}, V\rangle$ be a $\mathbf{CDL^m_{C\cup}}$-model. Let W^1, W^2 be disjoint copies of W. For each $w \in W$, denote the counterpart of w in W^i by w^i.

The *strict copy-merge of* M is the model $M^\ddagger = \langle W^\ddagger, R^\ddagger_\mathsf{A}, R^\ddagger_\mathsf{B}, R^\ddagger_\mathsf{I}, V^\ddagger\rangle$, where

1. $W^\ddagger = W^1 \cup W^2$
2. $R^\ddagger_\mathsf{A} = \{(w^i, v^i) \mid (w,v) \in R_\mathsf{A}\}$
3. $R^\ddagger_\mathsf{B} = \{(w^i, v^1) \mid (w,v) \in R_\mathsf{B}\} \cup \{(w^i, v^2) \mid (w,v) \in R_\mathsf{B} \setminus R_\mathsf{I}\}$
4. $R^\ddagger_\mathsf{I} = \{(w^i, v^2) \mid (w,v) \in R_\mathsf{I}\} \cup \{(w^i, v^2) \mid (w,v) \in R_\mathsf{I} \setminus R_\mathsf{B}\}$
5. for all $\varphi \in \mathcal{S}$, $V^\ddagger(\varphi) = \{w^1, w^2 \in W^\ddagger \mid w \in V(\varphi)\}$

The proof of the following is now a matter of routine – see Appendix B.2:

Theorem 5.7 *Let* $M = \langle W, R_\mathsf{A}, R_\mathsf{B}, R_\mathsf{I}, V\rangle$ *be a* $\mathbf{CDL^m_{C\cup}}$*-model. Then each of the following hold:*

(i) *If* W *is finite, then so is* $W^\dagger$
(ii) $M^\ddagger$ *is a* $\mathbf{CDL^m_{C+}}$*-model*
(iii) *if* M *satisfies a given condition (C) from Table 5, then so does* $M^\ddagger$
(iv) $f : W^\ddagger \to W$ *with* $f(w^i) = w$ *is a bounded morphism from* $M^\ddagger$ *to* M
(v) *for all* $w \in W$, $i \in \{1,2\}$ *and* φ, $M, w \models \varphi$ *iff* $M^\ddagger, w^i \models \varphi$

Theorem 5.8 *Where* $C1, \ldots, Cn$ *are frame conditions from Table 5:*
$\Gamma \Vdash_{\mathbf{CDL^m_{C+,C1,\ldots,Cn}}} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}} \varphi$.

Proof. Analogous to the proof of Theorem 5.4. □

Corollary 5.9 *Where* $C1, \ldots, Cn$ *are conditions from Table 5:*
$\Gamma \vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}} \varphi$ *iff* $\Gamma \Vdash_{\mathbf{CDL^m_{C+,C1,\ldots,Cn}}} \varphi$.

Non-definability of (C+) and (C∩) Using the copy-merge techniques defined above, we now briefly show that neither (C+) nor (C∩) are definable within $\mathbf{CDL^m}$.

Theorem 5.10 *There is no* φ *such that* φ *is globally valid on a* $\mathbf{CDL^m}$*-frame* F *iff* F *satisfies (C∩).*

Proof. Assume, for contradiction, that there is such a φ. Clearly, not all $\mathbf{CDL^m}$-frames satisfy (C∩). Hence, there is a $\mathbf{CDL^m}$-model $M = \langle W, R_\mathsf{A}, R_\mathsf{B}, R_\mathsf{I}, V\rangle$ such that $M, w \not\models \varphi$ for some $w \in W$. Let F be the frame associated with M. Where $M^\dagger$ is a simple copy-merge of M, let $F^\dagger$ be the frame associated with $M^\dagger$. By Theorem 5.3.(v), $M^\dagger, w^1 \not\models \varphi$. However, note that $F^\dagger$ satisfies (C∩). But this contradicts the assumption that φ is globally valid on $\mathbf{CDL^m}$-frames that satisfy (C∩). □

Theorem 5.11 *There is no φ such that φ is valid on a $\mathbf{CDL^m}$-frame F iff F satisfies (C+).*

Proof. Analogous to the proof of Theorem 5.10; simply replace † with ‡ and replace Theorem 5.3.(v) with Theorem 5.7.(v). □

6 Finite Model Property and Decidability

In this section, we briefly show that the logics that were introduced in this paper are all decidable. [22] In view of our axiomatizations and Theorem 5.1, it suffices to establish the finite model property for all $\mathbf{CDL^m}$-extensions mentioned in that theorem, in order to show that they are decidable. We do this in the usual way, viz. by filtration. The full proof can be found in Appendix C; here we just show how the filtration is defined exactly.

Definition 6.1 [Filter Set] Let $\mathsf{sf}(\varphi)$ be the set of all subformulas of φ. Let $F^-(\varphi) = \mathsf{sf}(\varphi) \cup \{\mathsf{BA}\psi, \mathsf{IA}\psi, \mathsf{A}\psi \mid \psi \in \mathsf{sf}(\varphi)\}$ and let $F(\varphi)$ be smallest set Ψ such that $F^-(\varphi) \subset \Psi$ and Ψ is closed under A and $\neg$.

Definition 6.2 [Filtration] Let $M = \langle W, R_\mathsf{A}, R_\mathsf{B}, R_\mathsf{I}, V\rangle$ be an arbitrary $\mathbf{CDL^m}$-model and let $\varphi \in \mathcal{W}^m$. For all $w \in W$, let $|w| = \{w' \in W \mid$ for all $\psi \in F(\varphi), M, w \models \psi$ iff $M, w' \models \psi\}$. The *filtration of M through φ* is $M^f = \langle W^f, R^f_\mathsf{A}, R^f_\mathsf{B}, R^f_\mathsf{I}, V^f\rangle$, where

1. $W^f = \{|w| \mid w \in W\}$
2. where $\Box \in \{\mathsf{A}, \mathsf{B}, \mathsf{I}\}$,
   $R^f_\Box = \{(|w|, |u|) \mid$ for all $\Box\psi \in F(\varphi)$, if $M, w \models \Box\psi$ then $M, u \models \psi\}$

3a. for all $\varphi \in \mathcal{S} - F(\varphi)$, $V^f(\varphi) = W^f$

3b. for all $\varphi \in \mathcal{S} \cap F(\varphi)$, $V^f(\varphi) = \{|w| \in W^f \mid M, w \models \varphi\}$

Theorem 6.3 *Let M^f be the filtration of M through φ. Then each of the following hold:*

(i) M^f *is a* $\mathbf{CDL^m}$*-model*

(ii) *the domain* W^f *of* M^f *is finite*

(iii) *for all* $\psi \in F(\varphi)$, $M, w \models \psi$ *iff* $M^f, |w| \models \psi$

(iv) *if* M *satisfies a given condition (C) from Table 3, then so does* M^f

In view of this theorem, for all the definable frame conditions, we obtain the finite model property. Applying Theorems 5.3.(i) and 5.7.(i), we can generalize this to include also the non-definable frame conditions:

Corollary 6.4 (Finite Model Property) *Let $C1, \ldots, Cn$ be any of the frame conditions studied in Section 5. If $\nVdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}} \varphi$, then there is a finite $\mathbf{CDL^m_{C1,\ldots,Cn}}$-model M and w in the domain of M such that $M, w \not\models \varphi$.*

Corollary 6.5 (Decidability) *Let $C1, \ldots, Cn$ be any of the frame conditions studied in Section 5. Then $\Vdash_{\mathbf{CDL^m_{C1,\ldots,Cn}}}$ is decidable.*

[22] An investigation into more specific lower and upper bounds on their complexity is left for future work.

7 Related Work

CDL and its extensions bear close resemblances to existing work in deontic logic. In fact, leaving some specific modeling choices aside, one could say that they are just a combination of two well-known constructions in the field. We only explain both of these here in a nutshell; a more detailed comparison of the respective logics is left for future work.

7.1 Deontic necessity and sufficiency

Combining a notion of necessity and sufficiency for modeling *ought* was proposed fairly recently in [1,26] under the name "obligation as weakest permission". The idea is that what one ought to do is that which is implied by every strongly permitted proposition, where a proposition is strongly permitted iff it is sufficient for optimality. The resulting *ought*-operator satisfies the same basic properties as our O in $\mathbf{CDL_{C+}}$ does – Uniqueness, and hence also Aggregation and Weakening. Likewise, it does not satisfy Inheritance, Uniformity, and the rule of Necessitation.

In [31], richer logics are studied in which both deontic necessity and sufficiency are expressible, which can be traced back to an extended abstract by van Benthem [29]. As shown in [9, Section 3], the deontic action logic from [28] is a fragment of van Benthem's system, and hence belongs to the same family of logics.

The main difference between the aforementioned logics and the **CDL**-family is that in the semantics of the former, we speak about the optimality (permissibility) of single worlds (which are thought of as action-tokens), whereas the latter speak about sets of worlds or "options" as we called them. As a result, we can also express the additional condition that φ is visible whenever $ought(\varphi)$ is true. Also, because of this feature, the logic of obligation as weakest permission and its relatives do not allow for coarseness.

7.2 Deontic stit logics

Deontic logics for optimal actions, which are conceived as *sets* of worlds, are at least as old as Horty's [20], which he further developed in his influential book [21].[23] Roughly speaking, $ought_\alpha(\varphi)$ is true at a world w in a model of Horty's most basic semantics if and only if α sees to it that φ whenever it takes one of its best options at w.

Horty distinguishes between two ways to determine what the best options are; one is called *dominance act utilitarianism* and satisfies Uniformity; the other is called *orthodox act utilitarianism* and invalidates Uniformity.[24] Both satisfy the (D)-axiom (see Table 3). Horty's $ought_\alpha$-operators are hence much

[23] In Horty's stit-based semantics, the points of evaluation are moment-history pairs rather than worlds, and the sets of worlds are rather sets of histories. There is however a one-to-one correspondence between such models and more regular Kripke-models – see e.g. [18, Section 2.1].

[24] In [22], Horty proposes a way to unify both accounts and hence overcome semantic ambiguity w.r.t. "the right action(s)".

like the B-operator of $\mathbf{CDL}_{\mathbf{D},\mathbf{U}_\mathsf{B}}$ (resp. $\mathbf{CDL}_\mathbf{D}$), with the obvious difference that they refer explicitly to an agent or group of agents. Horty's systems lack an operator for strong permission (our P).

Apart from the usual benefits – the transfer of insights and results from one system to the other –, there is one particular sense in which this link can be highly useful. In [27], Snedegar considers the problem of *coarsening inferences*, i.e. inferences that involve sets of alternatives that differ in their degree of coarseness. Snedegar's question then is: how do ought-claims relative to $\mathcal{A}$ relate to ought-claims relative to a finer partition $\mathcal{A}'$?

In view of the preceding, this question is analogous to asking how the obligations of a group of agents relate to the obligations of subgroups of that group, within the framework of deontic stit-logic. [25] Indeed, in the most common versions of stit-logic that can handle group agency, the alternatives that are available to the group correspond exactly to a partition that refines the partition that represents the alternatives available to a subgroup.

8 Summary and Outlook

The main contribution of this paper consisted in the formal study of different variants of Cariani's semantics for *ought*. Spelling out these variants in turn allowed us to point at links with existing work in deontic logic, most particularly the logic of obligation as weakest permission and deontic stit logic.

Many issues remain unsettled, such as a more exact comparison of these systems. As explained, the link with deontic stit logic suggests possible solutions to the problem of coarsening inferences; in future work we want to study this relation in more detail. Also, it is an open question whether deontic stit logic can be enriched with an operator for strong permission, and in particular, how such an operator will behave for group obligations.

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APPENDIX

A The Modal Translation

In the remainder we freely rely on the fact that replacement of equivalents (RE) is valid in **CDL** and $\mathbf{CDL^m}$. We also rely on the fact that (RE) is derivable within the axiom systems of both logics. We shall moreover rely on the fact that the modalities $\mathsf{U}, \mathsf{A}, \mathsf{B}$ and I satisfy the well-known properties of Inheritance and Aggregation (since they are normal modalities) in both logics:

(Inheritance) From $\Box\varphi$ and $\varphi \vdash \psi$, to infer $\Box\psi$
(Aggregation) From $\Box\varphi$ and $\Box\psi$, to infer $\Box(\varphi \wedge \psi)$

A.1 Some Lemmas

Before we can prove our main theorems, we need to establish two lemmas that concern the combination of tr^n and tr^m.

Lemma A.1 *Each of the following holds:*

(i) $\Vdash_{\mathbf{CDL}} \mathsf{P}\varphi \equiv \mathsf{PA}\varphi$
(ii) $\Vdash_{\mathbf{CDL^m}} \mathsf{I}\varphi \equiv \mathsf{IA}\varphi$
(iii) $\Vdash_{\mathbf{CDL}} \mathsf{P}\varphi \equiv \mathsf{P}\neg\mathsf{A}\neg\mathsf{A}\varphi$
(iv) $\Vdash_{\mathbf{CDL^m}} \mathsf{I}\varphi \equiv \mathsf{I}\neg\mathsf{A}\neg\mathsf{A}\varphi$
(v) $\vdash_{\mathbf{CDL}} \mathsf{P}\varphi \equiv \mathsf{P}\neg\mathsf{A}\neg\mathsf{A}\varphi$
(vi) $\vdash_{\mathbf{CDL^m}} \mathsf{I}\varphi \equiv \mathsf{I}\neg\mathsf{A}\neg\mathsf{A}\varphi$

Proof. *Ad (i).* Let M be an arbitrary **CDL**-model. We have: $M, w \models \mathsf{P}\varphi$ iff [by (SC7)] for all $X \in \mathcal{A}$ such that (for all $v \in X$, $M, v \models \varphi$), $X \notin \mathcal{I}(w)$ iff [by (SC5)] for all $X \in \mathcal{A}$ such that (for all $v \in X$, $M, v \models \mathsf{A}\varphi$), $X \notin \mathcal{I}(w)$ iff [by (SC7)] $M, w \models \mathsf{PA}\varphi$.

Ad (ii). Immediate in view of frame conditions (C1) and (C2) on $\mathbf{CDL^m}$-models.

Ad (iii) and (iv). Immediate in view of items (i) and (ii), the fact that A is an **S5**-modality in both semantics (and hence $\mathsf{A}\varphi$ and $\neg\mathsf{A}\neg\mathsf{A}\varphi$ are equivalent) and (RE).

Ad (v) and (vi). Immediate in view of the axiom schemas (C_{P}), resp. (C_{I}), (RE), and the fact that A satisfies the **S5**-axioms in both logics. □

Let tr^{m+n} be the sequential composition of tr^m and tr^n: $tr^{m+n}(\varphi) =_{\mathsf{df}} tr^n(tr^m(\varphi))$ for all φ. Likewise, let tr^{n+m} be the sequential composition of tr^n and tr^m.

Fact A.2 *Each of the following holds, where $x, y \in \{n, m\}$ and $x \neq y$:*

1. $tr^{x+y}(\varphi) = \varphi$ *if* $\varphi \in \mathcal{S}$
2. $tr^{x+y}(\neg\varphi) = \neg tr^{x+y}(\varphi)$

3. $tr^{x+y}(\varphi \vee \psi) = tr^{x+y}(\varphi) \vee tr^{x+y}(\psi)$
4. $tr^{x+y}(\Box\varphi) = \Box tr^{x+y}(\varphi)$ *where* $\Box \in \{\mathsf{U}, \mathsf{A}, \mathsf{B}\}$
5. $tr^{m+n}(\mathsf{P}\varphi) = \mathsf{P}\neg\mathsf{A}\neg\mathsf{A}(tr^{m+n}(\varphi))$
6. $tr^{n+m}(\mathsf{I}\varphi) = \mathsf{I}\neg\mathsf{A}\neg\mathsf{A}(tr^{n+m}(\varphi))$

Lemma A.3 *Where* $\varphi \in \mathcal{W}$*:* $\Vdash_{\mathbf{CDL}} \varphi \equiv tr^n(tr^m(\varphi))$ *and* $\vdash_{\mathbf{CDL}} \varphi \equiv tr^n(tr^m(\varphi))$*.*

Proof. By induction on the complexity of φ. We show the reasoning for $\Vdash_{\mathbf{CDL}}$:
Base case: $\varphi \in \mathcal{S}$. Then $tr^n(tr^m(\varphi)) = \varphi$ and hence the property holds vacuously.
Induction step: we distinguish three cases:

Case 1: $\varphi = \psi_1 \vee \psi_2$. By Fact A.2, $tr^{m+n}(\varphi) = tr^{m+n}(\psi_1) \vee tr^{m+n}(\psi_1)$. By the IH, $\Vdash_{\mathbf{CDL}} tr^{m+n}(\psi_1) \equiv \psi_1$ and $\Vdash_{\mathbf{CDL}} tr^{m+n}(\psi_2) \equiv \psi_2$; hence $\Vdash_{\mathbf{CDL}} tr^{m+n}(\varphi) \equiv \varphi$ by CL-properties and (RE).

Case 2: $\varphi = \Box\psi$ where $\Box \in \{\neg, \mathsf{U}, \mathsf{A}, \mathsf{B}\}$. By Fact A.2, $tr^{m+n}(\varphi) = \Box tr^{m+n}(\psi)$. By the IH, $\Vdash_{\mathbf{CDL}} tr^{m+n}(\psi) \equiv \psi$. By (RE), $\Vdash_{\mathbf{CDL}} tr^{m+n}(\varphi) \equiv \Box\psi$ and hence $\Vdash_{\mathbf{CDL}} tr^{m+n}(\varphi) \equiv \varphi$.

Case 3: $\varphi = \mathsf{P}\psi$. By Fact A.2, $tr^{m+n}(\varphi) = \mathsf{P}\neg\mathsf{A}\neg\mathsf{A}(tr^{m+n}(\psi))$. By Lemma A.1.(iii), $\Vdash_{\mathbf{CDL}} tr^{m+n}(\varphi) \equiv \mathsf{P}(tr^{m+n}(\psi))$. Hence by the IH, $\Vdash_{\mathbf{CDL}} tr^{m+n}(\varphi) \equiv \mathsf{P}(\psi)$.

For $\vdash_{\mathbf{CDL}}$, the reasoning is completely analogous; just rely on item (v) of Lemma A.1 instead of item (iii). □

Lemma A.4 *Where* $\varphi \in \mathcal{W}^m$*, each of the following hold:*

$\vdash_{\mathbf{CDL^m}} \varphi \equiv tr^m(tr^n(\varphi))$
$\Vdash_{\mathbf{CDL^m}} \varphi \equiv tr^m(tr^n(\varphi))$.

Proof. Analogous to the preceding lemma, relying on Fact A.2 and items (iv) and (vi) of Lemma A.1. □

A.2 The Relation between $\vdash_{\mathbf{CDL}}$ and $\vdash_{\mathbf{CDL^m}}$

Lemma A.5 *Where* $\varphi \in \mathcal{W}$*: if* $\vdash_{\mathbf{CDL}} \varphi$*, then* $\vdash_{\mathbf{CDL^m}} tr^m(\varphi)$*.*

Proof. By an induction on the length of the derivation of φ in **CDL**. Since **CDL** and **CDL$^{\mathbf{m}}$** use the same rules, it suffices to prove that for every **CDL**-axiom φ, $\vdash_{\mathbf{CDL^m}} tr^m(\varphi)$. In view of the overlap between both axiom systems, this further reduces to the claim that every instance in $\mathcal{W}^m$ of the following schemas is a **CDL$^{\mathbf{m}}$**-theorem:

($\mathrm{G}^m_{\mathsf{P}}$)	$\mathsf{U}\varphi \supset \mathsf{I}\neg\mathsf{A}\neg\varphi$
($\mathrm{C}^m_{\mathsf{P}}$)	$\mathsf{I}\neg\mathsf{A}\varphi \equiv \mathsf{I}\neg\mathsf{A}\mathsf{A}\varphi$
($\mathrm{P1}^m$)	$\mathsf{I}\neg\mathsf{A}(\varphi \vee \psi) \supset (\mathsf{I}\neg\mathsf{A}\varphi \wedge \mathsf{I}\neg\mathsf{A}\psi)$
($\mathrm{P2}^m$)	$(\mathsf{I}\neg\mathsf{A}\neg\mathsf{A}\neg\varphi \wedge \mathsf{I}\neg\mathsf{A}\neg\mathsf{A}\neg\psi) \supset \mathsf{I}\neg\mathsf{A}(\neg\mathsf{A}\neg\varphi \vee \neg\mathsf{A}\neg\psi)$
($\mathrm{EQ}^m_{\mathsf{P}}$)	$\mathsf{U}(\varphi \equiv \psi) \supset (\mathsf{I}\neg\mathsf{A}\varphi \equiv \mathsf{I}\neg\mathsf{A}\psi)$

Ad(G^m_{P}) Suppose $\mathsf{U}\varphi$. By (G_{I}), this entails $\mathsf{I}\varphi$. By the (T)-axiom for A, $\varphi \vdash_{\mathbf{CDL^m}} \neg\mathsf{A}\neg\varphi$. Hence by Inheritance for I, we can derive $\mathsf{I}\neg\mathsf{A}\neg\varphi$.
Ad(C^m_{P}) Immediate in view of (RE) for I and the fact that A is an **S5**-modality.

Ad (P1^m): this follows immediately since $\neg\mathsf{A}(\varphi \vee \psi) \vdash_{\mathbf{CDL^m}} \neg\mathsf{A}\varphi$, $\neg\mathsf{A}(\varphi \vee \psi) \vdash_{\mathbf{CDL^m}} \neg\mathsf{A}\psi$, and by Inheritance for I.
Ad(P2^m): Suppose $\mathsf{I}\neg\mathsf{A}\neg\mathsf{A}\neg\varphi \wedge \mathsf{I}\neg\mathsf{A}\neg\mathsf{A}\neg\psi$. By **S5**-properties and (RE) for I, we can derive: $\mathsf{IA}\neg\varphi \wedge \mathsf{IA}\neg\psi$. By aggregation for I and A, we can derive $\mathsf{IA}(\neg\varphi \wedge \neg\psi)$. Again by (RE), we get $\mathsf{IA}\neg(\varphi \vee \psi)$. By **S5**-properties again, this entails $\mathsf{I}\neg\mathsf{A}\neg\mathsf{A}\neg(\varphi \vee \psi)$. By (RE) and modal logic properties, $\mathsf{I}\neg\mathsf{A}(\neg\mathsf{A}\neg\varphi \vee \neg\mathsf{A}\neg\psi)$.
Ad (EQ$^m_{\mathsf{P}}$): Suppose $\mathsf{U}(\varphi \equiv \psi)$. It suffices to prove that $\mathsf{I}\neg\mathsf{A}\varphi \supset \mathsf{I}\neg\mathsf{A}\psi$; hence, suppose moreover that $\mathsf{I}\neg\mathsf{A}\varphi$. By the first supposition and the (4)-axiom for U, we can derive $\mathsf{UU}(\varphi \equiv \psi)$. Hence, by ($\mathrm{G}_{\mathsf{A}}$) and Inheritance for U, $\mathsf{UA}(\varphi \equiv \psi)$. Again by Inheritance and **S5**-properties, this yields $\mathsf{U}(\neg\mathsf{A}\varphi \supset \neg\mathsf{A}\psi)$. By ($\mathrm{G}_{\mathsf{I}}$) we can derive $\mathsf{I}(\neg\mathsf{A}\varphi \supset \neg\mathsf{A}\psi)$, and hence by the (K)-axiom for I and our second supposition, we can derive $\mathsf{I}\neg\mathsf{A}\psi$. □

Lemma A.6 *Where $\varphi \in \mathcal{W}^m$: if $\vdash_{\mathbf{CDL^m}} \varphi$, then $\vdash_{\mathbf{CDL}} tr^n(\varphi)$.*

Proof. Similar to the proof of the previous lemma. It suffices to show that each instance in $\mathcal{W}$ of the following schemas is a **CDL**-theorem:

($\mathrm{K}^n_{\mathsf{I}}$) $\mathsf{P}\neg\mathsf{A}(\varphi \supset \psi) \supset (\mathsf{P}\neg\mathsf{A}\varphi \supset \mathsf{P}\neg\mathsf{A}\psi)$
($\mathrm{G}^n_{\mathsf{I}}$) $\mathsf{U}\varphi \supset \mathsf{P}\neg\mathsf{A}\varphi$
($\mathrm{C}^n_{\mathsf{I}}$) $\mathsf{P}\neg\mathsf{A}\varphi \equiv \mathsf{P}\neg\mathsf{AA}\varphi$

Ad(K$^n_{\mathsf{I}}$): Suppose $\mathsf{P}\neg\mathsf{A}(\varphi \supset \psi)$ and $\mathsf{P}\neg\mathsf{A}\varphi$. By (RE), we can derive $\mathsf{P}\neg\mathsf{A}\neg\neg(\varphi \supset \psi)$ and $\mathsf{P}\neg\mathsf{A}\neg\neg\varphi$. By (P2), this gives us $\mathsf{P}(\neg\mathsf{A}\neg\neg(\varphi \supset \psi) \vee \neg\mathsf{A}\neg\neg\varphi)$. By double negation elimination, this gives us ($\star$) $\mathsf{P}(\neg\mathsf{A}(\varphi \supset \psi) \vee \neg\mathsf{A}\varphi)$.

Suppose that $\neg\mathsf{A}\psi$. By normal modal logic properties for A, this entails $\neg\mathsf{A}(\varphi \supset \psi) \vee \neg\mathsf{A}\varphi$. Hence,

$$\vdash_{\mathbf{CDL}} (\neg\mathsf{A}(\varphi \supset \psi) \vee \neg\mathsf{A}\varphi) \equiv (\neg\mathsf{A}(\varphi \supset \psi) \vee \neg\mathsf{A}\varphi \vee \neg\mathsf{A}\psi)$$

By ($\star$) and (RE), we can thus derive $\mathsf{P}(\neg\mathsf{A}(\varphi \supset \psi) \vee \neg\mathsf{A}\varphi \vee \neg\mathsf{A}\psi)$. Finally, by (MP) and (P1), this yields $\mathsf{P}\neg\mathsf{A}\psi$.
Ad(G$^n_{\mathsf{I}}$): Suppose $\mathsf{U}\varphi$. Hence, $\mathsf{UU}\varphi$ and by Inheritance and (G_{A}), $\mathsf{UA}\varphi$. By (G_{P}), this entails $\mathsf{P}\neg\mathsf{A}\varphi$.
Ad(C$^n_{\mathsf{I}}$): Immediate in view of (RE) for P and the fact that A is an **S5**-modality. □

Lemma A.7 *Where $\varphi \in \mathcal{W}$: $\vdash_{\mathbf{CDL}} \varphi$ iff $\vdash_{\mathbf{CDL^m}} tr^m(\varphi)$.*

Proof. The left-right direction is Lemma A.5. For the other direction, suppose that $\nvdash_{\mathbf{CDL}} \varphi$. By Lemma A.3, $\nvdash_{\mathbf{CDL}} tr^n(tr^m(\varphi))$. Hence by Lemma A.6, $\nvdash_{\mathbf{CDL^m}} tr^m(\varphi)$. □

Lemma A.8 *Where $\varphi \in \mathcal{W}^m$: $\vdash_{\mathbf{CDL^m}} \varphi$ iff $\vdash_{\mathbf{CDL}} tr^n(\varphi)$.*

Proof. The left-right direction is Lemma A.6. For the other direction, suppose that $\nvdash_{\mathbf{CDL^m}} \varphi$. By Lemma A.4, $\nvdash_{\mathbf{CDL^m}} tr^m(tr^n(\varphi))$. Hence by Lemma A.5, $\nvdash_{\mathbf{CDL}} tr^n(\varphi)$. □

Proof of Theorem 4.6. $\Gamma \vdash_{\mathbf{CDL}} \varphi$ iff [by the definition of $\vdash_{\mathbf{CDL}}$] there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathbf{CDL}} (\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi$ iff [by Lemma A.7] there are

$\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathbf{CDL^m}} tr^m((\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi)$ iff [by the definition of tr^m] there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathbf{CDL^m}} (tr^m(\psi_1) \wedge \ldots \wedge tr^m(\psi_n)) \supset tr^m(\varphi)$ iff [by the Definition of $\vdash_{\mathbf{CDL^m}}$ and tr^m] $tr^m(\Gamma) \vdash_{\mathbf{CDL^m}} tr^m(\varphi)$.

Proof of Theorem 4.7. $\Gamma \vdash_{\mathbf{CDL^m}} \varphi$ iff [by the definition of $\vdash_{\mathbf{CDL^m}}$] there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathbf{CDL^m}} (\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi$ iff [by Lemma A.8] there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathbf{CDL}} tr^n((\psi_1 \wedge \ldots \wedge \psi_n) \supset \varphi)$ iff [by the definition of tr^n] there are $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_{\mathbf{CDL}} (tr^n(\psi_1) \wedge \ldots \wedge tr^n(\psi_n)) \supset tr^n(\varphi)$ iff [by the Definition of $\vdash_{\mathbf{CDL}}$ and tr^n] $tr^n(\Gamma) \vdash_{\mathbf{CDL^m}} tr^n(\varphi)$.

A.3 Proof of Theorem 4.8

In order to prove Theorem 4.8, we first define two operations – one on **CDL**-models, another on $\mathbf{CDL^m}$-models.

Where $M = \langle W, \mathcal{A}, \mathcal{B}, \mathcal{I}, V \rangle$ is a **CDL**-model, let $M^m = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}, V \rangle$ be such that (i) for all $w \in W$, $R_{\mathsf{A}}(w) = \{w' \in X \mid X \in \mathcal{A} \text{ and } w \in X\}$; (ii) for all $w \in W$, $R_{\mathsf{B}}(w) = \bigcup \mathcal{B}(w)$; and for all $w \in W$, $R_{\mathsf{I}}(w) = \bigcup \mathcal{I}(w)$. We have:

Lemma A.9 *If M is a **CDL**-model, then (i) M^m is an $\mathbf{CDL^m}$-model and (ii) for all $\varphi \in \mathcal{W}$, $M^m, w \models tr^m(\varphi)$ iff $M, w \models \varphi$.*

Proof. Suppose the antecedent holds. *Ad (i).* Condition (C1) follows immediately from the fact that $\mathcal{A}$ is a partition of W and the definition of M^m. For condition (C2) we only prove the case where $\Box = \mathsf{B}$ (the other one is completely analogous). Suppose that $w' \in R_{\mathsf{B}}(w)$ and $w'' \in R_{\mathsf{A}}(w')$. By the definition of M^m, there is an $X \in \mathcal{B}(w)$ (and hence $X \in \mathcal{A}$) such that $w' \in X$ and there is a $Y \in \mathcal{A}$ such that $w', w'' \in Y$. Since $\mathcal{A}$ is a partition, it follows that $X = Y$ and hence also $w'' \in R_{\mathsf{B}}(w)$ by the definition of M^m.

Ad (ii). We prove this by induction on the complexity of φ. The base case ($\varphi \in \mathcal{S}$) and the induction step for the connectives and $\varphi = \mathsf{U}\psi$ are safely left to the reader. This leaves us with three cases:

- $\varphi = \mathsf{A}\psi$ We have: $M^m, w \models tr^m(\mathsf{A}\psi)$ iff [by the definition of tr^m] $M^m, w \models \mathsf{A}(tr^m(\psi))$ iff [by (SC8)] for all $w' \in R_{\mathsf{A}}(w)$, $M^m, w' \models tr^m(\psi)$ iff [by the IH] for all $w' \in R_{\mathsf{A}}(w)$, $M, w' \models \psi$ iff [by the construction of R_{A}] where $X \in \mathcal{A}$ is such that $w \in X$, $M, w' \models \psi$ for all $w' \in X$ iff [by (SC5)] $M, w \models \mathsf{A}\psi$.
- $\varphi = \mathsf{B}\psi$ We have: $M^m, w \models tr^m(\mathsf{B}\psi)$ iff [by the definition of tr^m] $M^m, w \models \mathsf{B}(tr^m(\psi))$ iff [by (SC8)] for all $w' \in R_{\mathsf{B}}(w)$, $M^m, w' \models tr^m(\psi)$ iff [by the IH] for all $w' \in R_{\mathsf{B}}(w)$, $M, w' \models \psi$ iff [by the construction of R_{B}, and in view of (C2)] for all $X \in \mathcal{B}(w)$, for all $w' \in X$, $M, w' \models \psi$ iff [by (SC6)] $M, w \models \mathsf{B}\psi$.
- $\varphi = \mathsf{P}\psi$ We have: $M^m, w \models tr^m(\mathsf{P}\psi)$ iff [by the definition of tr^m] $M^m, w \models \mathsf{I}\neg\mathsf{A}(tr^m(\psi))$ iff [by (SC8), applied twice] for all $w' \in R_{\mathsf{I}}(w)$, there is a $w'' \in R_{\mathsf{A}}(w')$ such that $M^m, w'' \not\models tr^m(\psi)$ iff [by the (IH)] for all $w' \in R_{\mathsf{I}}(w)$, there is a $w'' \in R_{\mathsf{A}}(w')$ such that $M, w'' \not\models \psi$ iff [by the construction of R_{I} and R_{A} and by (C2)] for all $X \in \mathcal{I}(w)$, there is a $w'' \in X$ such that $M, w'' \not\models \psi$ iff for all $X \in \mathcal{A}$, if ($M, w'' \models \psi$ for all $w'' \in X$), then $X \notin \mathcal{I}(w)$ iff [by (SC7)] $M, w \models \mathsf{P}\psi$.

□

Where $M = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}, V\rangle$ is an $\mathbf{CDL^m}$-model, let $M^n = \langle W, \mathcal{A}, \mathcal{B}, \mathcal{I}, V\rangle$ be such that $\mathcal{A} = \{\{w' \in R_{\mathsf{A}}(w)\} \mid w \in W\}$, for all $w \in W$, $\mathcal{B}(w) = \{X \in \mathcal{A} \mid X \subseteq R_{\mathsf{B}}(w)\}$ and for all $w \in W$, $\mathcal{I}(w) = \{X \in \mathcal{A} \mid X \subseteq R_{\mathsf{I}}(w)\}$.

Lemma A.10 *If M is a $\mathbf{CDL^m}$-model, then (i) M^n is a $\mathbf{CDL}$-model and (ii) for all $\varphi \in \mathcal{W}$, $M^n, w \models \varphi$ iff $M, w \models tr^m(\varphi)$.*

Proof. Suppose the antecedent holds. *Ad (i).* It suffices to check that $\mathcal{A}$ is a partition of W; this however follows at once from condition (C1).

Ad (ii). We prove this by induction on the complexity of φ. The base case ($\varphi \in \mathcal{S}$) and the induction step for the connectives and $\varphi = \mathsf{U}\psi$ are safely left to the reader. This leaves us with three cases:

$\varphi = \mathsf{A}\psi$ We have: $M, w \models tr^m(\mathsf{A}\psi)$ iff [by the definition of tr^m] $M, w \models \mathsf{A}(tr^m(\psi))$ iff [by (SC8)] for all $w' \in R_{\mathsf{A}}(w)$, $M, w' \models tr^m(\psi)$ iff [by the IH] for all $w' \in R_{\mathsf{A}}(w)$, $M^n, w' \models \psi$ iff [by the construction of $\mathcal{A}$] where $X \in \mathcal{A}$ is such that $w \in X$, $M^n, w'' \models \psi$ for all $w'' \in X$ iff [by (SC5)] $M^n, w \models \mathsf{A}\psi$.

$\varphi = \mathsf{B}\psi$ We have: $M, w \models tr^m(\mathsf{B}\psi)$ iff [by the definition of tr^m] $M, w \models \mathsf{B}(tr^m(\psi))$ iff [by (SC8)] for all $w' \in R_{\mathsf{B}}(w)$, $M, w' \models tr^m(\psi)$ iff [by the IH] for all $w' \in R_{\mathsf{B}}(w)$, $M^n, w' \models \psi$ iff [by the construction of $\mathcal{B}$ and in view of (C2)] for all $X \in \mathcal{B}(w)$, for all $w' \in X$, $M^n, w' \models \psi$ iff [by (SC6)] $M^n, w \models \mathsf{B}\psi$.

$\varphi = \mathsf{P}\psi$ We have: $M, w \models tr^m(\mathsf{P}\psi)$ iff [by the definition of tr^m] $M, w \models \mathsf{I}\neg\mathsf{A}(tr^m(\psi))$ iff [by (SC8), applied twice] for all $w' \in R_{\mathsf{I}}(w)$, there is a $w'' \in R_{\mathsf{A}}(w')$ such that $M, w'' \not\models tr^m(\psi)$ iff [by the (IH)] for all $w' \in R_{\mathsf{I}}(w)$, there is a $w'' \in R_{\mathsf{A}}(w')$ such that $M^n, w'' \not\models \psi$ iff [by the construction of $\mathcal{A}$ and $\mathcal{I}$] for all $X \in \mathcal{I}(w)$, there is a $w'' \in X$ such that $M^n, w'' \not\models \psi$ iff for all $X \in \mathcal{A}$, if ($M^n, w'' \models \psi$ for all $w'' \in X$), then $X \notin \mathcal{I}(w)$ iff [by (SC7)] $M^n, w \models \mathsf{P}\psi$.

□

Proof of Theorem 4.8. ($\Rightarrow$) Suppose $tr^m(\Gamma) \nVdash_{\mathbf{CDL^m}} tr^m(\varphi)$. Hence, there is an $\mathbf{CDL^m}$-model M and a world w such that $M, w \models tr^m(\psi)$ for all $\psi \in \Gamma$ and $M, w \not\models tr^m(\varphi)$. By Lemma A.10, $M^n, w \models \psi$ for all $\psi \in \Gamma$ and $M^n, w \not\models \varphi$. It follows that $\Gamma \nVdash_{\mathbf{CDL}} \varphi$. ($\Leftarrow$) Analogous, just use Lemma A.9 instead of Lemma A.10.

B Non-Definable Frame Conditions

B.1 Proof of Theorem 5.3

For the sake of readibility, we first restate Theorem 5.3 and the definition that precedes it.

Let $M = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}, V\rangle$ be a $\mathbf{CDL^m}$-model. Let W^1, W^2 be disjoint copies of W. For each $w \in W$, denote the counterpart of w in W^i by w^i. The *simple copy-merge of M* is the model $M^\dagger = \langle W^\dagger, R^\dagger_{\mathsf{A}}, R^\dagger_{\mathsf{B}}, R^\dagger_{\mathsf{I}}, V^\dagger\rangle$, where

1. $W^\dagger = W^1 \cup W^2$
2. $R^\dagger_{\mathsf{A}} = \{(w^i, v^i) \mid (w,v) \in R_{\mathsf{A}}\}$
3. $R^\dagger_{\mathsf{B}} = \{(w^i, v^1) \mid (w,v) \in R_{\mathsf{B}}\}$
4. $R^\dagger_{\mathsf{I}} = \{(w^i, v^2) \mid (w,v) \in R_{\mathsf{I}}\}$
5. for all $\varphi \in \mathcal{S}$, $V^\dagger(\varphi) = \{w^1, w^2 \in W^\dagger \mid w \in V(\varphi)\}$

According to the theorem, each of the following hold:

(i) If W is finite, then so is $W^\dagger$
(ii) $M^\dagger$ is a $\mathbf{CDL}^{\mathbf{m}}_{\mathsf{C}\cap}$-model
(iii) if M satisfies a given condition (C) from Table 3 except (C∪), then so does $M^\dagger$
(iv) $f : W^\dagger \to W$ with $f(w^i) = w$ is a bounded morphism from $M^\dagger$ to M
(v) for all $w \in W$, $i \in \{1,2\}$ and φ, $M, w \models \varphi$ iff $M^\dagger, w^i \models \varphi$

Proof. *Ad (i).* Trivial.

Ad (ii). It suffices to show that $M^\dagger$ satisfies (C1), (C2), and (C∩). (C1) follows immediately from the fact that M satisfies (C1) and the construction. (C∩) also follows immediately by the construction. For (C2), we only consider the case $\Box = \mathsf{B}$ (the case for I is completely analogous). Suppose that $(w^i, v^j) \in R^\dagger_{\mathsf{B}}$ and $(v^j, u^k) \in R^\dagger_{\mathsf{A}}$. By the construction, $j = 1 = k$, $(w,v) \in R_{\mathsf{B}}$, and $(v,u) \in R_{\mathsf{A}}$. Since (C2) holds for M, $(w,u) \in R_{\mathsf{B}}$. Hence, $(w^i, u^1) \in R^\dagger_{\mathsf{B}}$.

Ad (iii). First, suppose that (C) = (A_{B}) is satisfied by M. Let $w^i, v^j \in W^\dagger$ be such that $w^i \in R_{\mathsf{A}}(v^j)$ and suppose that $u^k \in R^\dagger_{\mathsf{B}}(w^i)$; it suffices to prove that $u^k \in R^\dagger_{\mathsf{B}}(v^j)$. By the construction, $k = 1$ and $u \in R_{\mathsf{B}}(w)$. Also by the construction, $w \in R_{\mathsf{A}}(v)$. Since (A_{B}) is satisfied by M, $u \in R_{\mathsf{B}}(v)$. Hence, by the construction, $u^1 \in R^\dagger_{\mathsf{B}}(v^j)$.

The cases for (C) = (U_{I}), (C) = (U_{B}) and (C) = (A_{I}) are either analogous or (for the U-variants) simpler.

For (C) = (D), suppose that $w^i \in W^\dagger$. Hence $R_{\mathsf{B}}(w) \neq \emptyset$. Let $v \in R_{\mathsf{B}}(w)$. Hence, $v^1 \in R^\dagger_{\mathsf{B}}(w^i)$.

Ad (iv). It suffices to prove each of the following:

1. where $i \in \{1,2\}$, w and w^i verify the same propositional variables
2. for all $\Box \in \{\mathsf{A}, \mathsf{B}, \mathsf{I}\}$: if $R_\Box(w,u)$ then there are i, j such that $R^\dagger_\Box(w^i, u^j)$
3. for all $\Box \in \{\mathsf{A}, \mathsf{B}, \mathsf{I}\}$: if $R^\dagger_\Box(w^i, u^j)$, then $R_\Box(w,u)$
4. $w^1, w^2 \in W^\dagger$ iff $w \in W$ (by definition)

(1.) follows immediately in view of the construction. For (2.), we distinguish three cases: if $\Box = \mathsf{A}$, put $i = j$; if $\Box = \mathsf{B}$, put $j = 1$; and if $\Box = \mathsf{I}$ put $j = 2$. (3.) and (4.) are again immediate in view of the construction.

Ad (v). Immediate in view of (iv). □

B.2 Proof of Theorem 5.7

Again, we start by restating the relevant definition and theorem.

Let $M = \langle W, R_{\mathsf{A}}, R_{\mathsf{B}}, R_{\mathsf{I}}, V\rangle$ be a $\mathbf{CDL}^{\mathbf{m}}_{\mathsf{C}\cup}$-model. Let W^1, W^2 be disjoint copies of W. For each $w \in W$, denote the counterpart of w in W^i by w^i. The *strict copy-merge of* M is the model $M^\ddagger = \langle W^\ddagger, R^\ddagger_{\mathsf{A}}, R^\ddagger_{\mathsf{B}}, R^\ddagger_{\mathsf{I}}, V^\ddagger\rangle$, where

1. $W^\ddagger = W^1 \cup W^2$
2. $R^\ddagger_{\mathsf{A}} = \{(w^i, v^i) \mid (w,v) \in R_{\mathsf{A}}\}$
3. $R^\ddagger_{\mathsf{B}} = \{(w^i, v^1) \mid (w,v) \in R_{\mathsf{B}}\} \cup \{(w^i, v^2) \mid (w,v) \in R_{\mathsf{B}} \setminus R_{\mathsf{I}}\}$
4. $R^\ddagger_{\mathsf{I}} = \{(w^i, v^2) \mid (w,v) \in R_{\mathsf{I}}\} \cup \{(w^i, v^2) \mid (w,v) \in R_{\mathsf{I}} \setminus R_{\mathsf{B}}\}$
5. for all $\varphi \in \mathcal{S}$, $V^\ddagger(\varphi) = \{w^1, w^2 \in W^\ddagger \mid w \in V(\varphi)\}$

We need to establish each of the following:

(i) If W is finite, then so is $W^\dagger$
(ii) $M^\ddagger$ is a $\mathbf{CDL}^{\mathbf{m}}_{\mathsf{C}+}$-model
(iii) if M satisfies a given condition (C) from Table 5, then so does $M^\ddagger$
(iv) $f : W^\ddagger \to W$ with $f(w^i) = w$ is a bounded morphism from $M^\ddagger$ to M
(v) for all $w \in W$, $i \in \{1,2\}$ and φ, $M, w \models \varphi$ iff $M^\ddagger, w^i \models \varphi$

Proof. *Ad (i).* Trivial.

Ad (ii). That $M^\ddagger$ satisfies (C1) and (C+) is immediate in view of the construction and the fact that M satisfies (C∪). For (C2), we only consider the case for $\Box = \mathsf{B}$ (the reasoning for I is analogous).[26] Suppose that $(w^i, v^j) \in R^\ddagger_{\mathsf{B}}$ and $(v^j, u^k) \in R^\ddagger_{\mathsf{A}}$. By the construction, $j = k$, $(w,v) \in R_{\mathsf{B}}$ and $(v,u) \in R_{\mathsf{A}}$. Since (C2) holds for M, $(w,u) \in R_{\mathsf{B}}$. Case 1: $j = k = 1$. Hence also $k = 1$. By the construction, $(w^i, u^1) \in R^\ddagger_{\mathsf{B}}$. Case 2: $j = k = 2$. By the construction and the supposition, $(w,v) \in R_{\mathsf{B}} \setminus R_{\mathsf{I}}$. Again by (C2) for M, $(w,u) \in R_{\mathsf{B}} \setminus R_{\mathsf{I}}$. Hence by the construction, $(w^i, u^2) \in R^\ddagger_{\mathsf{B}}$.

Ad (iii). First, suppose that (C) = (A_{B}) is satisfied by M. Let $w^i, v^j \in W^\ddagger$ be such that $w^i \in R_{\mathsf{A}}(v^j)$ and suppose that $u^k \in R^\ddagger_{\mathsf{B}}(w^i)$; it suffices to prove that $u^k \in R^\ddagger_{\mathsf{B}}(v^j)$. By the construction, $u \in R_{\mathsf{B}}(w)$ and $w \in R_{\mathsf{A}}(v)$. Since (A_{B}) is satisfied by M, $u \in R_{\mathsf{B}}(v)$. We distinguish two cases:

($k = 1$) By the construction, $u^1 \in R^\dagger_{\mathsf{B}}(v^j)$.
($k = 2$) By the construction, $u \in R_{\mathsf{B}}(w) \setminus R_{\mathsf{I}}(w)$. By ($\mathrm{A}_{\mathsf{B}}$) and (C+) for M, also $u \in R_{\mathsf{B}}(v) \setminus R_{\mathsf{I}}(v)$. By the construction, $u^2 \in R^\ddagger_{\mathsf{B}}(v^j)$.

The cases for (C) = (U_{I}), (C) = (U_{B}) and (C) = (A_{I}) are either analogous or (for the U-variants) simpler.

For (C) = (D), suppose that $w^i \in W^\ddagger$. Hence $R_{\mathsf{B}}(w) \neq \emptyset$. Let $v \in R_{\mathsf{B}}(w)$. Hence, $v^1 \in R^\ddagger_{\mathsf{B}}(w^i)$.

Ad (iv) and (v). Analogous to the proof of Theorem 5.3.(iv). □

C Filtration Theorem

As in previous appendices, we first restate the relevant definitions:

Let $\mathsf{sf}(\varphi)$ be the set of all subformulas of φ. Let $F^-(\varphi) = \mathsf{sf}(\varphi) \cup \{\mathsf{BA}\psi, \mathsf{IA}\psi, \mathsf{A}\psi \mid \psi \in \mathsf{sf}(\varphi)\}$ and let $F(\varphi)$ be smallest set Ψ such that

[26] In fact, given (C+), (C2) for $\Box = \mathsf{B}$ is equivalent to (C2) for $\Box = \mathsf{I}$. That is, since $R_{\mathsf{B}}(w)$ is a union of cells in the partition induced by R_{A}, and since $R_{\mathsf{I}}(w) = W \setminus R_{\mathsf{B}}(w)$, also $R_{\mathsf{I}}(w)$ is a union of such cells.

$F^-(\varphi) \subset \Psi$ and Ψ is closed under A and $\neg$. $F(\varphi)$ is called the *filter set* for φ.

Let $M = \langle W, R_\mathsf{A}, R_\mathsf{B}, R_\mathsf{I}, V\rangle$ be an arbitrary **CDL^m**-model and let $\varphi \in \mathcal{W}^m$. For all $w \in W$, let $|w| = \{w' \in W \mid$ for all $\psi \in F(\varphi), M, w \models \psi$ iff $M, w' \models \psi\}$. The *filtration of* M *through* φ is $M^f = \langle W^f, R^f_\mathsf{A}, R^f_\mathsf{B}, R^f_\mathsf{I}, V^f\rangle$, where

1. $W^f = \{|w| \mid w \in W\}$
2. where $\Box \in \{\mathsf{A}, \mathsf{B}, \mathsf{I}\}$,
   $R^f_\Box = \{(|w|, |u|) \mid$ for all $\Box\psi \in F(\varphi)$, if $M, w \models \Box\psi$ then $M, u \models \psi\}$

3a. for all $\varphi \in \mathcal{S} - F(\varphi)$, $V^f(\varphi) = W^f$

3b. for all $\varphi \in \mathcal{S} \cap F(\varphi)$, $V^f(\varphi) = \{|w| \in W^f \mid M, w \models \varphi\}$

Let M^f be the filtration of M through a given formula φ. We need to prove that each of the following hold:

(i) M^f is a **CDL^m**-model

(ii) the domain W^f of M^f is finite

(iii) for all $\psi \in F(\varphi)$, $M, w \models \psi$ iff $M^f, |w| \models \psi$

(iv) if M satisfies a given condition (C) from Table 3, then so does M^f

Proof. *Ad (i).* We need to prove that (C1) and (C2) are satisfied. For (C1), we need to prove three things:

1. R^f_A is reflexive. Let $\mathsf{A}\psi \in F(\varphi)$ and suppose that $M, w \models \mathsf{A}\psi$. Since R_A is reflexive, $M, w \models \psi$. Hence, $|w| \in R_\mathsf{A}(|w|)$.
2. R^f_A is symmetric. Suppose that $(|w|, |u|) \in R^f_\mathsf{A}$. Hence, there are $w' \in |w|$ and $u' \in |u|$ such that $(\star)$ for all $\mathsf{A}\psi \in F(\varphi)$, if $M, w' \models \mathsf{A}\psi$ then $M, u' \models \psi$. Let $\mathsf{A}\psi \in F(\varphi)$ be arbitrary such that $M, u' \models \mathsf{A}\psi$. Assume that $M, w' \not\models \psi$. Hence $M, w' \not\models \mathsf{A}\psi$ and hence (by **S5**-properties) $M, w' \models \mathsf{A}\neg\mathsf{A}\psi$. Since $\mathsf{A}\neg\mathsf{A}\psi \in F(\varphi)$, $M, u' \models \neg\mathsf{A}\psi$ — a contradiction. Hence $M, w' \models \psi$ and, since ψ was arbitrary, $(|u|, |w|) \in R^f_\mathsf{A}$.
3. R^f_A is transitive. Suppose that $(|w|, |v|), (|v|, |u|) \in R^f_\mathsf{A}$. Hence there is a $w' \in |w|, v', v'' \in |v|, u'' \in |u|$ such that, for all $\mathsf{A}\psi \in F(\varphi)$: (a) if $M, w' \models \mathsf{A}\psi$ then $M, v' \models \psi$ and (b) if $M, v'' \models \mathsf{A}\psi$ then $M, u'' \models \psi$. Suppose now that $M, w' \models \mathsf{A}\psi$ for an arbitrary $\mathsf{A}\psi \in F(\varphi)$. By the transitivity of R_A, $M, w' \models \mathsf{AA}\psi$. Note that $\mathsf{AA}\psi \in F(\varphi)$. So by (a), $M, v' \models \mathsf{A}\psi$. Since $v', v'' \in |v|$, $M, v'' \models \mathsf{A}\psi$ and hence $M, v'' \models \mathsf{A}\psi$. By (b), $M, u'' \models \psi$.

For (C2), suppose that $(|w|, |v|) \in R^f_\mathsf{B}$ and $(|v|, |u|) \in R^f_\mathsf{A}$. Hence there is a $w' \in |w|, v', v'' \in |v|, u'' \in |u|$ such that, (a) for all $\mathsf{B}\psi \in F(\varphi)$: if $M, w' \models \mathsf{B}\psi$ then $M, v' \models \psi$ and (b) for all $\mathsf{A}\psi \in F(\varphi)$, if $M, v'' \models \mathsf{A}\psi$ then $M, u'' \models \psi$. Let $\mathsf{B}\psi \in F(\varphi)$ be such that $M, w' \models \mathsf{B}\psi$. Note that, in view of the definition of $F(\varphi)$, $\mathsf{B}\psi \in F^-(\varphi)$.

Case 1: $\psi = \mathsf{A}\tau$ for a $\tau \in F(\varphi)$. Hence, $M, w' \models \mathsf{BA}\tau$ and hence $M, v' \models \mathsf{A}\tau$. By **S5**-properties, $M, v' \models \mathsf{AA}\tau$. Since $v', v'' \in |v|$ and $\mathsf{AA}\tau \in F(\varphi)$, also $M, v'' \models \mathsf{AA}\tau$ and so by (b), $M, u'' \models \mathsf{A}\tau$. So $M, u'' \models \psi$.

Case 2: ψ is not of the form $\mathsf{A}\tau$. By the construction of $F^-(\varphi)$, $\mathsf{BA}\psi \in F^-(\varphi)$ and hence $\mathsf{BA}\psi \in F(\varphi)$. Since (C2) holds for M, $M, w' \models \mathsf{BA}\psi$. Thus, we can apply the same reasoning as in the previous case (replacing τ with ψ) and derive that $M, u'' \models \mathsf{A}\psi$, and hence also that $M, u'' \models \psi$.

Ad (ii). Immediate in view of the observation that $F(\varphi)$ contains only finitely many distinct formulas $\psi_1, \ldots, \psi_n$ that are not $\mathbf{CDL^m}$-equivalent. This follows itself from the fact that $F^-(\varphi)$ is finite and that A is an **S5**-modality.

Ad (iii). By a standard induction on the complexity of ψ. We only give the induction step for the modal operators: [27]

Case 1: $\psi = \Box\tau$ for $\Box \in \{\mathsf{A}, \mathsf{B}, \mathsf{I}\}$. ($\Rightarrow$) Suppose that $M^f, |w| \not\models \Box\tau$. Hence there is a $|v| \in R^f_\Box(|w|)$ such that $M^f, |v| \not\models \tau$. Since $\tau \in F(\varphi)$ and by the IH, $M, v \not\models \tau$. However, by the construction, there is a $v' \in |v|$ and $w' \in |w|$ such that, for all $\Box\epsilon \in F(\varphi)$, if $M, w' \models \Box\epsilon$ then $M, v' \models \epsilon$. Since $\tau \in F(\varphi)$, $M, v' \not\models \tau$ and hence $M, w' \not\models \Box\tau$. Hence since $\Box\tau \in F(\varphi)$, $M, w \not\models \Box\tau$.

($\Leftarrow$) Suppose that $M, w \not\models \Box\tau$. Hence there is a $v \in R_\Box(w)$ such that $M, v \not\models \tau$. Note that for all ϵ, if $M, w \models \Box\epsilon$, then $M, v \models \epsilon$. Hence $|v| \in R^f_\Box(|w|)$ by the construction. By the IH, $M^f, |v| \not\models \tau$ and hence $M^f, |w| \not\models \Box\tau$.

Case 2: $\psi = \mathsf{U}\tau$. We have: $M, w \models \mathsf{U}\tau$ iff [by (SC4)] for all $w' \in W$, $M, w' \models \tau$ iff [by the IH] for all $|w'| \in W^f$, $M^f, |w'| \models \tau$ iff [by (SC4)] $M^f, |w| \models \mathsf{U}\tau$.

Ad (iv). We consider four cases: (U_B), (A_B), (D), and (C$\cup$); the proofs for the remaining two conditions are analogous to the first two. For each of these conditions, we suppose that M satisfies them and prove that M^f does so as well.

(U_B) Suppose that $(|w|, |v|) \in R^f_\mathsf{B}$. Hence, there are $w' \in |w|$ and $v' \in |v|$ such that ($\star$) for all $\mathsf{B}\psi \in F(\varphi)$, if $M, w' \models \mathsf{B}\psi$ then $M, v' \models \psi$. Let $|u| \in W^f$ be arbitrary. Suppose that $M, u \models \mathsf{B}\psi$ for a $\psi \in F(\varphi)$. Since M satisfies (U_B), $M, u \models \mathsf{UB}\psi$ and hence $M, w' \models \mathsf{B}\psi$. By ($\star$) $M, v' \models \psi$. Hence, $(|u|, |v|) \in R^f_\mathsf{B}$.

(A_B) Suppose that $(|w|, |v|) \in R^f_\mathsf{B}$ and $(|w|, |u|) \in R^f_\mathsf{A}$. So there are w', v', w'', u'' such that (a) for all $\mathsf{B}\psi \in F(\varphi)$, if $M, w' \models \mathsf{B}\psi$ then $M, v' \models \psi$ and (b) for all $\mathsf{A}\psi \in F(\varphi)$, if $M, w'' \models \mathsf{A}\psi$ then $M, u'' \models \psi$. From (b) we can derive that (b') for all $\mathsf{A}\psi \in F(\varphi)$, if $M, u'' \models \mathsf{A}\psi$ then $M, w'' \models \psi$ (see our proof for the symmetry of R^f_A above).

Let $\mathsf{B}\psi \in F(\varphi)$ be arbitrary and suppose that $M, u \models \mathsf{B}\psi$ Since $B\psi \in F(\varphi)$, also $M, u'' \models \mathsf{B}\psi$. Since M satisfies (A_B), $M, u'' \models \mathsf{AB}\psi$. By (b') and since $\mathsf{AB}\psi \in F(\varphi)$, $M, w'' \models \mathsf{B}\psi$. Since $w', w'' \in |w|$ and $\mathsf{B}\psi \in F(\varphi)$, also $M, w' \models \mathsf{B}\psi$ and hence by (a), $M, v' \models \psi$. Since $\mathsf{B}\psi$ was arbitrary, we can infer that $(|u|, |v|) \in R^f_\mathsf{B}$.

(D) Let $|w| \in W^f$ be arbitrary. Since (D) holds for M, $R_\mathsf{B}(w) \neq \emptyset$. Let $v \in R_\mathsf{B}(w)$. Note that, for all $\mathsf{B}\psi \in F(\varphi)$ such that $M, w \models \mathsf{B}\psi$, $M, v \models \psi$.

[27] Here and below, IH abbreviates "induction hypothesis".

It follows that $|v| \in R^f_{\mathsf{B}}(|w|)$.

(C∪) Assume that (C∪) fails for M^f. Let $|w|, |v| \in W^f$ be such that $(|w|, |v|) \notin R^f_{\mathsf{B}}$ and $(|w|, |v|) \notin R^f_{\mathsf{I}}$. Hence (a) there is a $\mathsf{B}\psi_1 \in F(\varphi)$ such that $M, w \models \mathsf{B}\psi_1$ and $M, v \not\models \psi_1$ and (b) there is an $\mathsf{I}\psi_2 \in F(\varphi)$ such that $M, w \models \mathsf{I}\psi_2$ and $M, v \not\models \psi_2$. It follows that $M, w \models \mathsf{B}(\psi_1 \vee \psi_2)$ and $M, w \models \mathsf{I}(\psi_1 \vee \psi_2)$ and $M, v \not\models \psi_1 \vee \psi_2$. But then $v \in W \setminus (R_{\mathsf{B}}(w) \cup R_{\mathsf{I}}(w))$, which contradicts the supposition that (C∪) holds for M.

□