Adaptive $\mathbf{C}_n$ logics*

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Abstract

This paper solves an old problem: to devise decent inconsistency-adaptive logics that have the $\mathbf{C}_n$ logics as their lower limit. Two kinds of logics are presented. Those of the first kind offer a maximally consistent interpretation of the premise set in as far as this is possible in view of logical considerations. At the same time, they indicate at which points further choices may be made on extra-logical grounds. The logics of the second kind allow one to introduce those choices in a defeasible way and handle them.

1 Aim of This Paper

Both the structure of the $\mathbf{C}_n$ logics and certain statements of da Costa's seem to suggest a specific application for those logics, viz. to apply a certain stratagem—see Section 3—to theories that turned out inconsistent. Even if da Costa did not have this application in mind, the stratagem is clearly interesting and suggested by the $\mathbf{C}_n$ logics. This makes it worthwhile to develop inconsistency-adaptive logics that have the $\mathbf{C}_n$ systems as their lower limit. Indeed, the adaptive logics by themselves accomplish most of the task that is served by the stratagem. To be more precise, they accomplish that part of the task which can be accomplished in view of logical considerations.

There is a further reason to devise adaptive logics that have $\mathbf{C}_n$ logics as their lower limit—this term is explained in Section 4. It is in principle possible to do so for any paraconsistent logic. The $\mathbf{C}_n$ logics are the oldest paraconsistent logics that were presented in a direct form, that is by an axiomatic system and not by a translation. So, as one may expect, to use them as lower limit logics has been on the agenda of adaptive logicians for a long time now. The delay is caused by a technical complication.

$\mathbf{C}_n$ logics introduce dependencies between inconsistencies. Where this is the case, the flip-flop danger lurks. As we shall see in Section 5, flip-flop logics are rather uninteresting adaptive logics. Until recently, no general method was

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available for avoiding flip-flops. Today this problem is solved by semantic means. The method will be applied to the $\mathbf{C}_n$ systems in Section 6. This will enable me to formulate the inconsistency-adaptive logics in Section 7.

If the suggestion from the first paragraph is right, da Costa had the idea to allow the person applying the $\mathbf{C}_n$ logics to add consistency statements as non-logical axioms to a theory. Technically, this is made possible by the definability of classical negation within the logics. The justification for such moves is a guess, viz. that a certain contradiction is not derivable from the theory. If the guess is right, the addition seriously enriches the inconsistent theory, bringing it closer to its original intention. If the guess is wrong, however, the addition causes disaster, viz. triviality. Where this happens another technical device may be invoked: the hierarchy of $\mathbf{C}_n$ logics.

An approach in terms of adaptive logics has several advantages. First, inconsistency-adaptive logics by themselves add to a (inconsistent or consistent) theory all consistency statements that can be added on the basis of logical considerations. On top of this, combined adaptive logics enable one to add further consistency statements in a *defeasible* way. This will be discussed in Section 8. In Section 9, I shall present some further clarifying comments and a generalization of the result to logics of formal inconsistency.

2 The $\mathbf{C}_n$ Logics

The axiomatization, devised by da Costa, consists of the following elements. $\mathbf{C}_\omega$ is (predicative) positive intuitionistic logic extended with the axioms $A \vee \neg A$, $\neg\neg A \supset A$, and the rule "if $A \equiv^c B$, then $\vdash A \equiv B$", in which $A \equiv^c B$ denotes that A and B are congruent in the sense of Kleene *or* that one formula results from the other by deleting vacuous quantifiers—Kleene [18, p. 153] summarizes his definition as follows: "two formulas are congruent, if they differ only in their bound variables, and corresponding bound variables are bound by corresponding quantifiers." phrase added

In order to obtain the $\mathbf{C}_n$ logics ($1 \leq n < \omega$), we need some abbreviations. Let A° abbreviate[1] $\neg(A \wedge \neg A)$. Next, let A^1 abbreviate A°, let A^2 abbreviate $A^{\circ\circ}$, etc. Finally, let $A^{(n)}$ abbreviate $A^1 \wedge A^2 \wedge \ldots \wedge A^n$. The logic $\mathbf{C}_n$ is obtained by extending $\mathbf{C}_\omega$ with the following axioms

$$B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$$

$$(A^{(n)} \wedge B^{(n)}) \supset (A \dagger B)^{(n)} \quad \text{where } \dagger \in \{\vee, \wedge, \supset\}$$

$$\forall x(A(x))^{(n)} \supset (\forall x\, A(x))^{(n)}$$

$$\exists x(A(x))^{(n)} \supset (\exists x\, A(x))^{(n)}$$

A formula of the form $A^{(n)}$ will be called a consistency statement in $\mathbf{C}_n$. It expresses that A behaves consistently—see for example [16]—in the sense that $A^{(n)}, A, \neg A \vdash_{\mathbf{C}_n} B$ is derivable from the first displayed axiom. The other displayed axioms spread consistency statements. Incidentally, $A^{(n)} \supset (\neg A)^{(n)}$ is a theorem of each $\mathbf{C}_n$. It is also provable that $\neg^{(n)} A =_{df} \neg A \wedge A^{(n)}$ defines classical negation in $\mathbf{C}_n$.

[1] While $\neg A \wedge A$ is $\mathbf{C}_1$-equivalent to $A \wedge \neg A$, $\neg(\neg A \wedge A)$ and $\neg(A \wedge \neg A)$ are not $\mathbf{C}_1$-equivalent. Which of the latter two is taken to express the consistency of A in $\mathbf{C}_1$ is a conventional matter.

It is easily seen that the $\mathbf{C}_n$ logics form a hierarchy of logics: if $m > n$ and $\Gamma \vdash_{\mathbf{C}_m} A$, then $\Gamma \vdash_{\mathbf{C}_n} A$. The logic $\mathbf{C}_\omega$ forms a limit of this hierarchy, although not a very natural one. In view of the subsequent sections, it is useful to note that an equally sensible limit is the logic $\mathbf{C}_{\overline{\omega}}$, which contains predicative positive **CL** (Classical Logic) together with with the axioms $A \vee \neg A$ and $\neg\neg A \supset A$ and the rule "if $A \equiv^c B$, then $\vdash A \equiv B$".[2] Moreover, it will be useful to have classical negation available even in $\mathbf{C}_{\overline{\omega}}$. So let us extend the language with the symbol $\check{\neg}$ and give it the meaning of classical negation (by introducing the usual axioms). Note that the standard negation, $\neg$, is still paraconsistent. Note also the difference between $\neg^{(n)}$ and $\check{\neg}$. The first is definable within the standard language and behaves like classical negation in $\mathbf{C}_n$ or, more precisely, in the logics $\mathbf{C}_1, \ldots, \mathbf{C}_n$. The second symbol does not belong to the standard language, and hence does not occur in the premises, but is added to the language for technical reasons.

Two features of the $\mathbf{C}_n$ logics may cause some wonder. First, what is the use of having classical negation, viz. the symbol $\neg^{(n)}$, definable within paraconsistent logics? Next, what is the use of the hierarchy of $\mathbf{C}_n$ logics? The subsequent section offers a possible answer.

3 A Possible Stratagem in Terms of $\mathbf{C}_n$ Logics

Suppose that a theory T has $\mathbf{C}_1$ as its underlying logic and contains the consistency statement $A^{(1)}$ (for some specific formula A). As $A, \neg A, A^{(1)} \vdash_{\mathbf{C}_1} B$, it is excluded that both A and $\neg A$ are theorems of T on penalty of triviality—this is the outlook taken in [14], where the function of $A^{(n)}$ is served by the implicitly defined $\circ A$.

The definability of classical negation in the logics $\mathbf{C}_n$ ($1 \leq n < \omega$) is a striking feature, which distinguishes these logics from most other paraconsistent logics. Moreover, one may wonder which precise purpose it is supposed to serve. Let Γ be the non-logical axioms of a theory that was intended as consistent but turns out to be inconsistent—Frege's set theory is an obvious example. As Γ was intended and believed to be consistent, it will not comprise any explicit consistency statements. So what is the use of consistency statements?

Seen from the perspective of inconsistency-adaptive logics, the answer to this question seems obvious. Suppose that $A \vee B$ and $\neg A$ are $\mathbf{C}_1$-derivable from Γ. As Γ was intended to be consistent, one would expect B to be derivable as well. But $A \vee B, \neg A \nvdash_{\mathbf{C}_1} B$. So, if A is not $\mathbf{C}_1$-derivable from Γ, one might add the consistency statement $A^{(1)}$ as a new axiom of the theory. This delivers the desired result because $A \vee B, \neg A, A^{(1)} \vdash_{\mathbf{C}_1} B$. Exactly the same situation arises if $B \supset A$ and $\neg A$ are $\mathbf{C}_1$-derivable from Γ.

The possibility to extend an inconsistent theory with consistency statements has dramatic effects: within the paraconsistent context, it leads to a theory that is drastically richer than the original theory. As a result, the extended theory approaches the theory as it was originally intended, viz. as fully consistent.

Adding consistency statements involves a danger. If one reformulates an inconsistent theory T_0 in terms of $\mathbf{C}_1$ and adds, for one or more specific A,

[2] An interesting study of limits of the hierarchy is presented in [15]. The logic $\mathbf{C}_{\overline{\omega}}$ is there called $\mathbf{C}_{min}$.

$A^{(1)}$ to T_0, it may turn out that triviality results. When this happens, one may retract the added consistency statements. There is, however, another possibility.

At this point, the use of the other $\mathbf{C}_n$ becomes apparent. We have seen that one may replace **CL** by $\mathbf{C}_1$ in an attempt to save the original ideas behind T_0. By adding consistency statements, the theory is brought closer to T_0 as originally intended. If the resulting theory T_1 turns out trivial, one may replace $\mathbf{C}_1$ by $\mathbf{C}_2$. By this move, triviality is avoided because the statements of the form $A^{(1)}$, are not consistency statements in the context of $\mathbf{C}_2$. Moreover, relying on the insights from the failed previous attempt, one may again enrich T_1 by adding consistency statements of the form $A^{(2)}$, which have the desired effect in the context of $\mathbf{C}_2$. This process may be repeated. If T_n has $\mathbf{C}_n$ as its underlying logic, comprises no statements $A^{(m)}$ for which $m > n$, and is trivial, replacing $\mathbf{C}_n$ by $\mathbf{C}_{n+1}$ restores non-triviality because no $A^{(m)}$ occurring in T_n is a consistency statement with respect to $\mathbf{C}_{n+1}$.

I do now know whether da Costa ever had such stratagem in mind. I can only note that the structure of the $\mathbf{C}_n$ systems and the definability of classical negation makes the stratagem possible and even suggests it. Some phrases used by da Costa go in the same direction. Thus he sometimes states that $\mathbf{C}_n$ logics isolate inconsistencies. In $\mathbf{C}_n$-theories he distinguishes between 'good' and 'bad' theorems, the latter being those whose negation is also a theorem. Note that, in order to take advantage of the 'good' quality of one of the former, one needs to add a consistency statement to the theory. Also da Costa notes, [16, p. 501], that $\mathbf{NF}_1$, a specific inconsistent variant of Quine's **NF**, contains elementary arithmetic and is apparently arithmetically consistent. Yet, in order that this contained arithmetic be as strong as elementary arithmetic, consistency statements will have to be added in some or other way.

It actually is worthwhile to comment on the notions of 'good' and 'bad' theorems of a theory. Actually, several such notions were introduced over the years. So let me distinguish between them by adding subscripts. In [16], a theorem A of a theory $\langle \Gamma, \mathbf{C}_A \rangle$ is called good$_1$ if $A \in \mathit{Cons}_{\mathbf{C_1}}(\Gamma)$ and $\neg A \notin \mathit{Cons}_{\mathbf{C_1}}(\Gamma)$ and is called bad$_1$ if $A, \neg A \in \mathit{Cons}_{\mathbf{C_1}}(\Gamma)$.[3] In [17], A is called good$_2$ if $A^{(1)} \in \mathit{Cons}_{\mathbf{C_1}}(\Gamma)$ and bad$_2$ if $A^{(1)} \notin \mathit{Cons}_{\mathbf{C_1}}(\Gamma)$. Consider a premise set $\Gamma = \{\neg p \vee q, p, r \vee s, \neg r, r\}$. Obviously, p is good$_1$, r is bad$_1$, and both are bad$_2$. There obviously are still different notions: A is good$_3$ if $\Gamma \cup \{A^{(1)}\}$ is non-trivial and A is bad$_3$ if $\Gamma \cup \{A^{(1)}\}$ is trivial. With respect to the aforementioned premise set, p is good$_3$, and r is bad$_3$. That neither of all these notions is as significant as they might seem to be is obvious from the following premise set: $\Gamma = \{q, \neg p \vee \neg q, p, r \vee s, r\}$. With respect to this Γ, p, q, and r are all good$_1$, bad$_2$, and good$_3$. Yet, $\Gamma \cup \{p^{(1)}, r^{(1)}\}$ and $\Gamma \cup \{q^{(1)}, r^{(1)}\}$ are non-trivial, whereas $\Gamma \cup \{p^{(1)}, q^{(1)}\}$ is trivial. So it seems advisable to define good and bad with respect to sets, and this is exactly the outlook taken by the adaptive approach, as we shall see below.

4 Adaptive Logics

Several introductions to adaptive logics are available, for example [5, 6, 9, 13]. So I shall be very brief here. Adaptive logics 'interpret' a premise set 'as nor-

[3]As usual, $\mathit{Cons}_{\mathbf{L}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$. I write *Cons* instead of the more common *Cn* or *C* for the sake of readability in the present context.

mally as possible' with respect to some standard of normality. In particular, inconsistency-adaptive logics 'interpret' a premise set 'as consistently as possible'. It is worth mentioning that, while inconsistencies may be naturally seen as abnormalities with respect to a classical framework, standard of normality is a the general notion which is technical in nature, and can only be given a philosophical interpretation with respect to a specific application context.

If adaptive logics are tailored from the $\mathbf{C}_n$ logics, these logics—let us call them $\mathbf{C}_n{}^m$—are inconsistency-adaptive. They should have a manifold of properties, among which the following. First, the adaptive logics should extend the $\mathbf{C}_n$ logics: $Cons_{\mathbf{C_n}}(\Gamma) \subseteq Cons_{\mathbf{C_n}^\mathbf{m}}(\Gamma)$. If Γ is non-trivial, $Cons_{\mathbf{C_n}^\mathbf{m}}(\Gamma)$ should be non-trivial and, border cases aside, the extension should be proper. Next, $Cons_{\mathbf{C_n}^\mathbf{m}}(\Gamma)$ should be a fixed point, $Cons_{\mathbf{C_n}^\mathbf{m}}(\Gamma) = Cons_{\mathbf{C_n}^\mathbf{m}}(Cons_{\mathbf{C_n}^\mathbf{m}}(\Gamma))$, and should be closed under $\mathbf{C}_n$, $Cons_{\mathbf{C_n}^\mathbf{m}}(\Gamma) = Cons_{\mathbf{C_n}}(Cons_{\mathbf{C_n}^\mathbf{m}}(\Gamma))$.

It is worth noting that inconsistency-adaptive logics are seen as corrective by some and as ampliative by others. If your standard of deduction is **CL** (or another explosive logic) and the theory $T = \langle \Gamma, \mathbf{CL} \rangle$ was meant as consistent, but turns out to be inconsistent and hence trivial, you will want to move to a theory $T' = \langle \Gamma, \mathbf{AP} \rangle$ that interprets Γ *as consistently as possible*, viz. as close as possible to the original intention. This is typically a corrective attitude: the standard of deduction leads to disaster and, for the time being and possibly in preparation of an improved consistent theory, one moves on to an approximation of the original theory that is non-trivial, locates the inconsistencies, but approximates the original theory in as far as possible. The approximation requires that inconsistencies are taken to be false unless and until proven otherwise. Some people, for example dialetheists like Graham Priest, take the standard of deduction to be some paraconsistent logic **P** but agree that most inconsistencies are false and hence can be considered as false unless and until proven true. So their original theory is $T = \langle \Gamma, \mathbf{P} \rangle$, but, relying on the extra-logical consideration that most inconsistencies are false, they upgrade to $T' = \langle \Gamma, \mathbf{AP} \rangle$, in which **AP** is an inconsistency-adaptive logic. This is typically an ampliative attitude: the inconsistency-adaptive logic delivers a stronger consequence set that the standard of deduction.

Whatever the position taken, adaptive logics are not competitors for the standard of deduction. They are formal systems characterizing defeasible reasoning forms; they are instruments, formally characterized methods, and the like. Note, incidentally, that there are also people (like me) who do not believe in the existence of a (global) standard of deduction.

Let us now briefly look at the technicalities involved in adaptive logics. An adaptive logic **AL** (in standard format) is a triple:

- a *lower limit logic* **LLL**: a compact Tarski logic
- a *set of abnormalities* Ω: a set of formulas characterized by a (possibly restricted) logical form
- a *strategy*: Reliability, Minimal Abnormality, ...

Every adaptive logic defines an upper limit logic **ULL**, which is a Tarski logic obtained by extending the lower limit logic with an axiom or rule that trivializes abnormalities. Semantically the **ULL** models are the **LLL** models that verify no abnormality. Note that **ULL** extends **LLL** with some further rules, requiring that *all* abnormalities are false, whereas **AL** extends **LLL** with certain *applications* of **ULL**-rules, requiring that as many abnormalities are false as the premises permit.

An example of a specific adaptive logic in standard format is $\mathbf{C_1}^m$, viz.

- *lower limit logic*: $\mathbf{C}_1$
- *set of abnormalities* $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}\}$
- *strategy*: Minimal Abnormality

The need for a strategy is best illustrated by an example. Let the premise set be $\{\neg p, \neg q, p \vee r, q \vee s, p \vee q\}$. From this $(p \wedge \neg p) \vee (q \wedge \neg q)$ is $\mathbf{C}_1$-derivable. So we need to decide in which way this disjunction of abnormalities will affect our maximally normal 'interpretation' of the premise set.

The standard format provides the adaptive logic with (i) a dynamic proof theory—the systematic study of this is available in [10], (ii) a preferential semantics, and (iii) most of the metatheory. The dynamic proofs consist of lines that have a *condition* attached to them, are marked or unmarked (in function of their condition and of formulas derived at other lines), and are governed by rules (to add lines) and a *marking definition* (to settle which lines are marked at a stage of the proof). The preferential semantics selects lower limit models of the premise set. The metatheory includes soundness, completeness, and many properties (cautious monotonicity, cautious transitivity, ...).

Incidentally, the aim of the adaptive logic program is to characterize *all* forms of defeasible reasoning by (combinations of) adaptive logics in standard format, possibly under a translation.

5 Adaptive Logics and Flip-Flops

The stratagem described in Section 3 requires human interference, viz. adding consistency statements. Precisely this is avoided by adaptive logics. In the previous section, I introduced $\mathbf{C_1}^m$. What does this logic come to? The $\mathbf{C_1}^m$-consequence set of a premise set Γ offers a maximally consistent interpretation of Γ. In semantic terms, the $\mathbf{C_1}^m$-consequence set of Γ comprises the formulas that are true in all minimally abnormal $\mathbf{C}_1$-models of Γ. Where M is a $\mathbf{C}_1$-model, let $Ab(M)$ be the set of the abnormalities, that is members of Ω, that are verified by M. A $\mathbf{C}_1$-model M of Γ is a *minimal abnormal* model of Γ iff there is no $\mathbf{C}_1$-model M' of Γ for which $Ab(M') \subset Ab(M)$. If a $\mathbf{C}_1$-model M verifies $A \vee B$ as well as $\neg A$, and does not verify $A \wedge \neg A$, then M verifies B. So, if no minimal abnormal $\mathbf{C}_1$-model of Γ verifies $A \wedge \neg A$, then the $\mathbf{C_1}^m$-consequence set of Γ is identical to the $\mathbf{C_1}^m$-consequence set of $\Gamma \cup \{A^{(1)}\}$. Seen from the stratagem from Section 3, one might see the $\mathbf{C_1}^m$-consequence set of Γ as the $\mathbf{C}_1$-consequence set of $\Gamma \cup \Gamma'$, in which Γ' is the set of all consistency statements that can be added to Γ without resulting in triviality. This description needs to be a refined, as we shall see in Section 8, but for the time being it will do.

Although an inconsistency-adaptive logic with a $\mathbf{C}_n$ logic as its lower limit seems to an attractive alternative for the 'handwork' required by the stratagem, there is a problem. The $\mathbf{C}_n$ logics introduce dependencies between contradictions. For example, $\neg A \wedge \neg\neg A$ entails $A \wedge \neg A$ in view of the axiom $\neg\neg A \supset A$. If there are such dependencies, there is a particular difficulty for defining the set of abnormalities. If this set is not defined in a sufficiently restrictive way, a flip-flop logic results. This demands some explanation.

A flip-flop logic is an adaptive logic, but a rather uninteresting one.[4] A flip-flop **L** displays the following behaviour. If Γ has models M for which $Ab(M) = \emptyset$ (so Γ is 'normal'), **L** behaves like its upper limit logic—in the present context this is **CL**. If, however, $Ab(M) \neq \emptyset$ for all models M of Γ, then **L** behaves like its lower limit logic—in the present case $\mathbf{C}_1$. Obviously, this is rather uninteresting. Suppose we apply $\mathbf{C}_1$ to the premise set $\{\neg p, p \vee r, \neg q \wedge p, q \vee s\}$. The reason for going adaptive is that we want to derive s because $q \wedge \neg q$ is not $\mathbf{C}_1$-derivable from the premises, while avoiding the consequence r because $p \wedge \neg p$ is $\mathbf{C}_1$-derivable from the premises. A flip-flop logic, however, will assign to that premise set exactly the same consequences as its lower limit, $\mathbf{C}_1$, and hence will not deliver s.

Let us consider some well known examples to illustrate the problem. The logic **CLuN** comprises full positive logic together with excluded middle.[5] The suitable set of abnormalities for an adaptive logic that has **CLuN** as its lower limit is $\{\exists(A \wedge \neg A) \mid A \in \mathcal{F}\}$, in which $\mathcal{F}$ is the set of open and closed formulas of the standard predicative language and $\exists(A \wedge \neg A)$ is the existential closure of $A \wedge \neg A$. The logic **CLuNs** (see [11]), which actually is the most popular paraconsistent logic (under sundry names), is obtained from **CLuN** by adding both directions of double negation, de Morgan theorems, and similar axioms to push negation inwards. If an adaptive logic has **CLuNs** as its lower limit and the set of abnormalities is defined as $\{\exists(A \wedge \neg A) \mid A \in \mathcal{F}\}$, a flip-flop results. A decent adaptive logic is obtained by restricting this set to $\{\exists(A \wedge \neg A) \mid A \in \mathcal{F}^a\}$ in which $\mathcal{F}^a$ is the set of (open and closed) atomic (or primitive) formulas—those in which occurs no logical symbol except possibly for identity.

As $\mathbf{C}_1$ spreads inconsistencies, one might fear that the logic ${\mathbf{C}_1}^m$ from Section 4 is a flip-flop. So the question is whether, in order to avoid this, the set of abnormalities should be restricted if $\mathbf{C}_1$ is the lower limit. I failed to obtain an answer for many years, but today the problem is solved because a general criterion has been devised. I apply this criterion to $\mathbf{C}_1$ in the next section.

6 The Semantic Criterion

The criterion ties up the abnormalities to the occurrence of gluts (and possibly gaps) in an indeterministic semantics—see [1, 2] on indeterministic semantics. There are certain restrictions on the indeterministic semantics, but the easiest approach is to present a semantics for $\mathbf{C}_1$ and to explain it. So here we go.

The semantics is provably characteristic for $\mathbf{C}_1$. In order to avoid assigning values to open formulas (containing free occurrences of members of the set $\mathcal{V}$ of MODIFIED FROM HERE

[4] Astounding as it may seem, some flip-flops have interesting application contexts, for example in the context of inductive generalization, in case one wants to completely reject certain background theories that are contradicted by the data.

[5] The propositional fragment of **CLuN** was first studied in [3] (with the name **PI**). It is a basic paraconsistent logic, and is taken as the basis for constructing the basic logic of formal inconsistency, **mbC**, in [14]. The predicative version was first presented in [4]. Later the Ghent group standardly extended **CLuN** with classical negation. Replacement of Identicals is invalid in **CLuN**: it does not apply within the scope of a negation. Obviously, **CLuN** can be extended with Replacement of Identicals. Replacement of Equivalents is also invalid in **CLuN**, as in many other paraconsistent logics.

variables), I extend the standard predicative language $\mathcal{L}$ to a pseudo-language $\mathcal{L}_{\mathcal{O}}$ by adding, next to the set of individual constants $\mathcal{C}$, a set $\mathcal{O}$ of pseudo-constants. $\mathcal{O}$ should have the cardinality of the largest set, which is the largest domain of the models considered. Let $\mathcal{S}$ be the set of sentential letters, $\mathcal{P}^r$ the set of predicative letters of rank r, $\mathcal{W}$ the set of closed formulas of $\mathcal{L}$, $\mathcal{W}_{\mathcal{O}}$ the set of closed formulas of $\mathcal{L}_{\mathcal{O}}$, and $\mathcal{W}_{\mathcal{O}}^a$ the set of atomic formulas in $\mathcal{W}_{\mathcal{O}}$.

In a model $M = \langle D, v\rangle$, D is the domain and v the assignment function defined by:

C1 $v{:}\,\mathcal{W}_{\mathcal{O}} \to \{0,1\}$
C2 $v{:}\,\mathcal{C} \cup \mathcal{O} \to D$ (where $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$)
C3 $v{:}\,\mathcal{P}^r \to \wp(D^r)$

Clause C1 assigns a truth value to every closed formula. Only the values assigned to sentential letters matter for the present semantics. The other values are important to turn the indeterministic semantics into a deterministic one, an exercise that will be skipped in the present paper. The restriction in clause C2 ensures that every member of D is named by a constant or pseudo-constant.[6] In C3, $\mathcal{P}^r$ is the set of predicative letters of rank r and $\wp(D^r)$ is the power set of the r-th Cartesian product of D.

The pre-valuation $v_M : \mathcal{W}_{\mathcal{O}} \to \{t, u, f\}$, with t (true) and u (glut) designated, is characterized by the following tables:

Where $A \in \mathcal{S}$:

$v(A)$	A
1	t
0	f

Where $\alpha_1, \ldots, \alpha_r \in \mathcal{C} \cup \mathcal{O}$ and $\pi \in \mathcal{P}^r$:

$\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle, v(\pi)$	$\pi\alpha_1 \ldots \alpha_r$
$\in$	t
$\notin$	f

Where $\alpha, \beta \in \mathcal{C} \cup \mathcal{O}$:

$v(\alpha), v(\beta)$	$\alpha = \beta$
$=$	t
$\neq$	f

Where $A \in \mathcal{W}_{\mathcal{O}}^a$:

A	$\neg A$
t	$[f, u]$
f	t

Where $A * B$ is not of the form $C \wedge \neg C$:

$A * B$	$A^{(1)}$	$B^{(1)}$	$\neg(A * B)$
t	t	t	f
t	(other)		$[f, u]$
f	(any)		t

Where $\mathsf{Q} \in \{\forall, \exists\}$ and $\alpha \in \mathcal{V}$:

$\mathsf{Q}\alpha A(\alpha)$	$\{v_M(A(\beta)^{(1)}) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$	$\neg\mathsf{Q}\alpha A(\alpha)$
t	$\{t\}$	f
t	(other)	$[f, u]$
f	(any)	t

[6] So some models are ω-incomplete with respect to the standard language, but the extended language allows for a transparent handling of the quantifiers.

The other tables apply to all members of $\mathcal{W}_{\mathcal{O}}$:

$\neg A$	$\neg\neg A$
t	f
u	$[f,u]$
f	t

$\neg A$	$A^{(1)}$
t	t
u	f
f	t

$\wedge$	t	u	f
t	t	t	f
u	t	t	f
f	f	f	f

$\vee$	t	u	f
t	t	t	t
u	t	t	t
f	t	t	f

$\supset$	t	u	f
t	t	t	f
u	t	t	f
f	t	t	t

$\equiv$	t	u	f
t	t	t	f
u	t	t	f
f	f	f	t

$\{v_M(A(\alpha)) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$	$\forall\alpha A(\alpha)$	$\exists\alpha A(\alpha)$
$\in \wp\{t,u\}$	t	t
$\{f\}$	f	f
(other)	f	t

A pre-valuation v_M is a valuation iff $v_M(A) = v_M(B)$ whenever $A \equiv^c B$ (see Section 2). This means that, at points where the semantics is indeterministic, the same choice was made for A and B. If some pre-valuation assigns the value t to all members of a set of formulas, then so does some valuation.

The expression $[f,u]$ indicates that the value may be f or u—this is an indeterministic semantics. Note that $\emptyset$ is not a possible value on the first line of the table for the quantifiers. The "(other)" on the third line of that table abbreviates sets that contain a designated as well as an non-designated value. In the table for $\neg(A*B)$, "(other)" means that either $A^{(1)}$ or $B^{(1)}$ does not have the value t (they cannot have the value u); "(any)" indicates that the values of $A^{(1)}$ or $B^{(1)}$ do not matter—this line summarizes four lines. In the table for $\neg\mathsf{Q}\alpha A(\alpha)$, "(other)" means that some $A(\beta)^{(1)}$ does not have the value t; "(any)" indicates that the values of the $A(\beta)^{(1)}$ do not matter. UP TO HERE

The pre-valuation and the valuation assign a value to all closed formulas of $\mathcal{L}$, which is what we are interested in. Validity and semantic consequence are defined as usual.

This particular three-valued indeterministic semantics is constructed from the two-valued one. Typical for this semantics is that the value u is only assigned where a glut *originates* (in comparison to **CL**). Thus if both A and B have a designated value, the truth of $A \wedge B$ agrees with **CL** *at this point* and hence is not a glut. For example, if all of A, $\neg A$ and $\neg(A \wedge \neg A)$ have a designated value, then $v_M(\neg A) = u$, $v_M(A \wedge \neg A) = t$, $v_M(\neg(A \wedge \neg A)) = u$, and $v_M((A \wedge \neg A) \wedge \neg(A \wedge \neg A)) = t$.

Transforming the above semantics to any logic $\mathbf{C}_n$ $(n < \omega)$ is an easy exercise left to the reader—the formulation of the tables for $\mathbf{C}_1$ and the plot described in the previous paragraph indicate the road. For $\mathbf{C}_{\varpi}$ (see Section 2), one replaces the tables for negation by the left and middle table below, and adds the table to the right below for the (added) classical negation:

A	$\neg A$
t	$[f,u]$
f	t

$\neg A$	$\neg\neg A$
t	f
u	$[f,u]$
f	t

A	$\check{\neg} A$
t	f
u	f
f	t

We have seen that the abnormalities of inconsistency-adaptive logics have the form $\exists(A \wedge \neg A)$. It can be shown, in terms of this particular type of indeterministic three-valued semantics, that an inconsistency-adaptive logic is not a flip-flop if the abnormalities are restricted to the case where (i) $A \wedge \neg A$ is an abnormality if its truth requires $\neg A$ to have the value u and (ii) $\exists(A \wedge \neg A)$ is an abnormality if it requires that there is an instance $B \wedge \neg B$ (obtained by systematically replacing every free variable in $A \wedge \neg A$ by a constant or pseudo-constant) for which $\neg B$ has the value u. This solves the flip-flop problem for the $\mathbf{C}_n$ logics: defining the set of abnormalities as $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}\}$ does not cause a flip-flop. Some of the abnormalities are logically impossible in $\mathbf{C}_1$, but that does not cause any trouble.

7 The Inconsistency-Adaptive Logics

The result is even simpler than one might expect. The inconsistency-adaptive logic ${\mathbf{C}_n}^m$ from Section 7 is not a flip-flop. The result may be generalized. The following adaptive logics are not flip-flops: ${\mathbf{C}_n}^m$ ($1 \leq n \leq \omega$) defined as the triple[7] consisting of (i) $\mathbf{C}_n$, (ii) $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}\}$, and (iii) Minimal Abnormality. So there is no need to vary Ω for any ${\mathbf{C}_n}^m$ logic. Given that these logics are in standard format, their proof theory and semantics are at once defined, the soundness and completeness of the proof theory with respect to the semantics is warranted (in view of the soundness and completeness of the lower limit logics with respect to their semantics), and most metatheoretic properties of the logics are known (because they follow from the standard format).

A striking specific feature reveals itself. If a theory was meant to be consistent and its underlying logic is explosive, then, as we have seen before, it will not contain any consistency statement $A^{(n)}$. In this case all ${\mathbf{C}_n}^m$ ($n \in \{1, 2, \ldots, \overline{\omega}\}$) assign the same consequence set to Γ. In general, if Γ is not $\mathbf{C}_k$-trivial, then all ${\mathbf{C}_n}^m$ ($n \in \{k, k{+}1, \ldots, \overline{\omega}\}$) assign the same consequence set to Γ. The astonishing result is that ${\mathbf{C}_{\overline{\omega}}}^m$ can be used for all premise sets. So while adaptive logics involve some complexity problems, they avoid the complication of the hierarchy of $\mathbf{C}_n$ logics that is needed for implementing the stratagem.

For the proof theory, semantics, etc., I refer to [9]. Here I can at best present a simple propositional ${\mathbf{C}_{\overline{\omega}}}^m$-proof. Let the premise set be $\{\neg\neg p, \neg q, \neg p, p \vee r, q \vee s\}$.

1	$\neg\neg p$	Prem	
2	$\neg q$	Prem	
3	$\neg p$	Prem	
4	$p \vee r$	Prem	
5	$q \vee s$	Prem	
6	r	3, 4; RC	$\{p \wedge \neg p\}$ $\checkmark^{8}$
7	s	2, 5; RC	$\{q \wedge \neg q\}$
8	$p \wedge \neg p$	1, 3; RU	$\emptyset$

The superscripted number 8 on line 6 indicates that the mark is added at stage 8 of the proof, that is immediately after line 8 was added. In whichever way the proof (from these premises) is extended, the marks of lines 1-8 are stable from

[7] I only consider the Minimal Abnormality strategy for lack of space; the result generalizes, for example, to Reliability.

this point on (the marked line remains marked and the unmarked ones remain unmarked). So s is a final consequence of the premise set and r is not (because line 6 is marked).

The adaptive logics solve certain problems that may arise if the stratagem from Section 3 is applied. Thus even the infinite set of consistency statements $\{\neg(A \wedge \neg A) \mid A \in \mathcal{F} - \{p\}\}$ is insufficient to obtain by $\mathbf{C}_n$ all $\mathbf{C}_\omega{}^m$-final consequences of the original premise set (an example is $\forall x Px \supset (\exists x \neg Px \supset r)$). If a non-recursive set of minimal disjunctions of contradictions[8] is $\mathbf{C}_n$-derivable from the premise set, no recursive set of consistency statements can be added to the original premises to obtain by a $\mathbf{C}_n$ logic the consequences that $\mathbf{C}_\omega{}^m$ delivers from the original premise set.

8 Defeasible Guesses

While the inconsistency-adaptive logics $\mathbf{C}_n{}^m$ do in most respects better than the stratagem outlined in Section 3, there is one respect in which they do worse. The $\mathbf{C}_n{}^m$ offer a maximally consistent interpretation of the premises, but only in as far as *logical considerations* allow for a justified choice. This may be illustrated by the premise set $\{\neg p, \neg q, p \vee r, q \vee s, p \vee q\}$. Note that $(p \wedge \neg p) \vee (q \wedge \neg q)$ is $\mathbf{C}_{\overline{\omega}}$-derivable from these premises. The upshot is that $r \vee s$ is a $\mathbf{C}_{\overline{\omega}}{}^m$-consequence of the premises. However, neither $p^{(1)}$ nor $q^{(1)}$ is derivable, which is related to the fact that no logical considerations enable one to prefer one over the other.

Suppose that $(p \wedge \neg p) \vee (q \wedge \neg q)$, $(p \wedge \neg p) \vee (r \wedge \neg r)$, and $s \wedge \neg s$ are $\mathbf{C}_{\overline{\omega}}$-derivable from a premise set, but that neither $p \wedge \neg p$ nor $q \wedge \neg q$ nor $r \wedge \neg r$ have been so derived. The premises then apparently inform us that either both $p \wedge \neg p$ and $s \wedge \neg s$ are true or that all of $q \wedge \neg q$, $r \wedge \neg r$, and $s \wedge \neg s$ are true. The inconsistency-adaptive logic cannot possibly 'chose' between both possibilities. The person that applies the logic might, however, have a reason to make a choice. Thus both q and r may concern well-entrenched properties that may be taken to behave consistently. In this case, the person applying the stratagem would add the new premises $\neg(q \wedge \neg q)$ and $\neg(r \wedge \neg r)$ in the context of $\mathbf{C}_1$. So she would obtain a theory that is more consistent than the one provided by $\mathbf{C}_{\overline{\omega}}{}^m$. The logic cannot make this choice, because the reasons for making it are extra-logical.

The choice involves a danger. If later $q \wedge \neg q$ would turn out to be $\mathbf{C}_1$-derivable from the premise set, the resulting theory would be $\mathbf{C}_1$-trivial and one would have to move to $\mathbf{C}_2$ in order to make another try.

However, there is a way to eat your cake and still have it, viz. by replacing the adaptive $\mathbf{C}_{\overline{\omega}}{}^m$ by a specific combined adaptive logic. Indeed, this logic retains all the $\mathbf{C}_{\overline{\omega}}{}^m$-consequences, allows one to add consistency statements on extra-logical grounds, but circumvents the danger because the consistency statements are introduced in a defeasible way. In other words, a combined adaptive logic allows one to make choices that will have no effect if they would run one into triviality in the context of $\mathbf{C}_{\overline{\omega}}$.

Two important remarks are in place at this point. First when does it make sense to defeasibly introduce a consistency statement? Given the result on $\mathbf{C}_{\overline{\omega}}$, the answer is obvious: where one has reasons to believe that a choice can be made. If one has derived a disjunction of abnormalities that is, by

[8] The role played by minimal disjunctions of abnormalities becomes clear in the next section.

present insights, minimal, it makes sense to eliminate some (obviously not all) of the disjuncts as not abnormal. In other words, if one has (unconditionally) derived a disjunction of (existentially quantified) contradictions, and no stronger disjunction has been derived, then one may posit, preferably on good grounds, that some of the disjuncts are false. This agrees nicely with Wiśniewski's erotetic logic—see, for example, [19, 20, 21]. A set of declarative statements *generates* a question if the disjunction of the direct answers to the question is derivable from the statements whereas no direct answer is derivable.

The second remark is that if one wants to consider defeasible consistency statements, one better introduces them in a prioritized way. So if one has found that $(p \wedge \neg p) \vee (q \wedge \neg q) \vee (r \wedge \neg r)$ is $\mathbf{C}_{\overline{\omega}}$-derivable from the premise set, one may want to stipulate that both $p \wedge \neg p$ and $r \wedge \neg r$ are false. Yet, it is possible that one is more certain about the falsehood of $p \wedge \neg p$ than about that of $r \wedge \neg r$. So it is intuitively appealing to defeasibly reject $p \wedge \neg p$ in a stronger way than $r \wedge \neg r$. Note that, if it would turn out at a later point that $(p \wedge \neg p) \vee (r \wedge \neg r)$ is $\mathbf{C}_{\overline{\omega}}$-derivable from the premise set, then $p \wedge \neg p$ would still be taken to be false whereas $r \wedge \neg r$ would be taken to be true.

Let $!A$ abbreviate $\exists(A \wedge \neg A)$, whence $!!A$ abbreviates $\exists(A \wedge \neg A) \wedge \neg\exists(A \wedge \neg A)$, etc. Next let $!^n A$ abbreviate whatever is abbreviated by n exclamation marks followed by A. Finally, let $¡^n A$ abbreviate $\neg !^1 A \wedge \neg !^2 A \wedge \ldots \wedge \neg !^n A$.

The guesses that are introduced as new premises have the form $¡^n A$. The priority assigned to a guess is directly proportional to n. To handle the guesses, we consider a combined adaptive logic, viz. a specific combination of a set of adaptive logics. For each of the latter, the lower limit is $\mathbf{C}_{\overline{\omega}}$ and the strategy is Minimal Abnormality. The sets of abnormalities are defined by $\Omega^i = \{!^i A \mid A \in \mathcal{F}\}$. Let the resulting adaptive logics be called $\mathbf{C}_{\overline{\omega}}{}^{mi}$, in which i determines Ω^i. Note that $\mathbf{C}_{\overline{\omega}}{}^{m1}$ is identical to $\mathbf{C}_{\overline{\omega}}{}^{m}$ from Section 7.

The consequence set of the combined adaptive logic of level n is identical to $Cn_{\mathbf{C}_{\overline{\omega}}{}^{m1}}(Cn_{\mathbf{C}_{\overline{\omega}}{}^{m2}}(\ldots(Cn_{\mathbf{C}_{\overline{\omega}}{}^{mn}}(\Gamma))\ldots))$. In semantic terms, the combined logic is easiest described as follows. From the $\mathbf{C}_{\overline{\omega}}$-models of a premise set Γ, it first selects the minimal abnormal models with respect to Ω^n,[9] from these the minimal abnormal models with respect to Ω^{n-1}, and so on up to Ω^1. The proof theory of the combined logic has an interesting property: the rules of all combining logics may be applied together. At every stage, the marking definition of the combined logic proceeds first in terms of minimal disjunctions of Ω^n-abnormalities that have been derived in the proof on the condition $\emptyset$, next in terms of the unmarked minimal disjunctions of Ω^{n-1}-abnormalities that have been derived in the proof on a condition that comprises at most members of Ω^n, next in terms of the unmarked minimal disjunctions of Ω^{n-2}-abnormalities that have been derived in the proof on a condition that comprises at most members of $\Omega^n \cup \Omega^{n-1}$, etc.—the marking definition is identical to Definition 13 of [8].

Note that a guess $¡^n A$ may be strengthened by introducing a new premise $¡^m A$ with $m > n$. This means that guesses may be corrected, where it is desirable, in view of insights obtained from an ongoing proof.

A small digression is in place at this point. If no free variable occurs in A, $¡^n A$ is identical to $A^{(n)}$. So at the propositional level, the story may be told in terms of the original consistency statements of the $\mathbf{C}_n$ logics. This is a most astonishing fact. The original construction forged by da Costa contained the

[9] The value of n is determined by the premises.

means required, at the propositional level, to handle defeasible guesses and to overrule them if there are reasons for doing so. According to the stratagem, the logic has to be replaced, for example $\mathbf{C}_n$ has to be replaced by $\mathbf{C}_{n+1}$ in order to overrule the earlier guess. The combined adaptive logic, to the contrary, is the same throughout the whole process. Apart from this and apart from the absence of the existential closure in da Costa's $A^{(n)}$, ¡$^n A$ and $A^{(n)}$ serve the same function and are identical where no free variables occur in A.

9 Some Concluding Remarks

All by itself, $\mathbf{C}_{\overline{\varpi}}{}^m$ *restores consistency* where logical reasons permit and indicates the road to obtain further consistency. The combined adaptive logic *handles* attempts to make the result even more consistent, reveals points at which choices may be made, and prevents the enterprise from running into triviality.

That the adaptive logics have $\mathbf{C}_{\overline{\varpi}}$ as their lower limit seems to make classical negation unavailable. As was described in Section 2, this may be repaired by introducing classical negation as a symbol, $\check{\neg}$, that does not belong to the standard language and hence does not occur in the premises—actually, this has been the standard adaptive approach for many years now. The upshot is that, if only one of A and $\neg A$ is a member of the adaptive consequence set, then so will be one of $\check{\neg}\neg A$ and $\check{\neg} A$.

The adaptive logic $\mathbf{C}_{\overline{\varpi}}{}^m$ defines a maximal consistent interpretation of a premise set Γ. The price to pay is obviously that the $\mathbf{C}_{\overline{\varpi}}{}^m$-consequence set of Γ is not in general decidable; there even is no positive test for it (the set is not recursively enumerable). This is unavoidable at the predicate level[10] because there is no positive test for consistency.

There are some consolations. The first is that there are proof procedures that form criteria for final derivability.[11] If the procedure is applied to some Γ and A and it stops, it answers the question whether A is finally derivable from Γ. Moreover, if a finite proof establishes that A is finally derivable from Γ—see [12] for a more precise formulation—then the procedure will stop with that answer.

A very different consolation is that the introduction of defeasible guesses may circumvent the problem even where final derivability cannot be established. If the user feels to have a sufficient insight in the studied theory she will often try to phrase a consistent replacement. In order to do so, the logic should be able to isolate inconsistencies and should *in principle* be able to locate all inconsistencies. But even an incomplete analysis may permit one to attain a consistent replacement. An illustration is that many (apparently) consistent replacements for Frege's set theory were formulated before the Curry paradox was discovered. This paradox apparently does not affect any of those set theories.

The results presented in this paper may obviously be generalized to all logics of formal inconsistency in the sense of [14]. Given such a logic $\mathbf{L}$, an adaptive logic $\mathbf{AL}$ is articulated as follows. Take $\mathbf{L}$ as the lower limit logic. Formulate a two-valued indeterministic semantics for $\mathbf{L}$ and turn it into a 3-valued or 4-valued

[10] At the propositional level adaptive logics are decidable in the same sense as **CL** is. The same holds for certain fragments of the predicative logic.

[11] The procedure for Reliability was presented in [7]; that for Minimal Abnormality was studied by Peter Verdée (paper soon forthcoming).

indeterministic semantics with the values t, f, u, and a as described in Section 6. On the basis of the insights gained from this, define $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}^x\}$, in which x is the restriction required for avoiding a flip-flop. Finally, chose an adaptive strategy. The resulting logic restores consistency where this is possible on the basis of logical considerations; it indicates how further consistency may be obtained. From this logic, one defined the combined adaptive logic along the lines followed in Section 8. The combined logic handles attempts to obtain an even more consistent result.

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