Degrees of inconsistency. Carefully combining classical and paraconsistent negation.

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Abstract

This paper is devoted to combining the negation of Classical Logic (**CL**) and the negation of Graham Priest's **LP** in a way that is faithful to central properties of the combined logics.

We give a number of desiderata for a logic **L** which combines both negations. These desiderata include the following: (a) **L** should be truth functional, (b) **L** should be strictly non-explosive for the paraconsisent negation $\sim$ (i.e. if A and $\sim A$ both have a non-trivial set of consequences, then this should also be the case for the set containing both) and (c) **L** should be a conservative extension of **CL** and of **LP**. The desiderata are motivated by a particular property-theoretic perspective on paraconsistency.

Next we devise the logic **CLP**. We present an axiomatization of this logic and three semantical characterizations (a non-deterministic semantics, an infinitely valued set-theoretic semantics and an infinitely valued semantics with integer numbers as values). We prove that **CLP** is the only logic satisfying all postulated desiderata. The infinitely valued semantics of **CLP** can be seen as giving rise to an interpretation in which inconsistencies and inconsistent properties come in degrees: not every sentence which involves inconsistencies is equally inconsistent.

1 Introduction

This paper is devoted to combining the negation of Classical Logic (**CL**), here denoted by the symbol $\neg$, and the negation of Graham Priest's Logic of Paradox (**LP**, cf. [5]), here denoted by the symbol $\sim$ in a way that is faithful to central properties of the combined logics.

LP is a paraconsistent logic: its negation is inconsistency tolerant. Therefore inconsistent sets of sentences (i.e. sets of sentences from which a sentence and its negation follow) do not (always) have a trivial set of **LP**-consequences.

Semantically, this is realized by the fact that **LP** has models which verify inconsistencies (an inconsistency is a conjunction of a sentence and

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its negation). Priest himself argues that this reflects reality: according to him there are indeed inconsistencies which are true. He argues e.g. that both the liar sentence and its negation are true. Philosophers who accept the existence of true inconsistencies are called *dialetheists*. However, one does not need to be dialetheist to find the logic **LP** useful. One may also be agnostic with respect to the truth of inconsistencies. Or one may be convinced that no inconsistencies can ever be actually true but nevertheless allow them to be true in the models of one's logic in order to be able to continue reasoning with (incorrect) inconsistent theories, for which there is not yet a consistent alternative.

Tolerating inconsistencies by formalizing certain negations as paraconsistent negations, does not make the classical negation meaningless or useless. Even if some inconsistencies are possibly, actually or even necessarily true, it is still possible to distinguish the case where a sentence is true only (its negation is classically not true) from the case where it is inconsistent. A paraconsistent logic like **LP** can never express this distinction. One needs a logic with a classical negation for this task (or at least a negation which is not inconsistency tolerant). Only then one can for example express that A is true and the (paraconsistent) negation of A is (classically) not true. Being able to express classical negation can be useful for many applications of paraconsistency. In inconsistent set theory, e.g., one may want to be able to express whether a set a is not merely paraconsistently empty (for every set x not-$x \in a$ holds) but also has (classically) no members (there is no set x such that $x \in a$). Another set theoretic example may be that it is probably useful to express that membership of some basic non-paradoxical sets (e.g. the finite ordinals), is not inconsistent. Even the uncontroversial but for a dialetheist only contingently true statement 'most inconsistent statements about our physical world are false' involves some kind of classical negation, not expressible in **LP**.

In the literature many logics have been proposed which can be seen as logics that allow adding classical negation to **LP**. Examples are Brazilian paraconsistent logics (specific existing LFI's (Logics of Formal Inconsistency and Da Costa's **Cn** logics, cf. [3]) and the logic **CLuNs** described by Batens and De Clercq, cf. [2]. Although these logics are very interesting and have many applications, one can add classical negation to **LP** with the aim to fulfil other desiderata, not satisfied by the existing logics. More particularly one may look for a combination of classical and paraconsistent negation such that (i) the combining logic is an conservative extension of both **LP** and classical logic, (ii) the paraconsistent negation is still fully paraconsistent also when classical negations occur in its scope, which can be made precise as follows:

> if A and $\sim A$ both have a non-trivial set of consequences, then also $\{A, \sim A\}$ should have a non-trivial set of consequences (independent of whether $\neg$ occurs in A),

and (iii) certain unproblematic instances of the classical equivalence rule remain valid also in the scope of paraconsistent negations; the first case concerns formulas classically equivalent by the law of Double Negation: if $\neg\neg A$ occurs in a formula B, then B follows from a set of premises iff the result of substituting $\neg\neg A$ by A in B follows form these premises. The second case concerns formulas classically equivalent by the De Morgan laws: $\neg(A \wedge B)$ and $\neg A \vee \neg B$ are inter-substitutable in the same way and so are $\neg(A \vee B)$ and $\neg A \wedge \neg B$.

Many logics that combine paraconsistent and classical negation are too weak to be extensions of **LP** (Da Costa's **Cn** logics, Batens's **CLuN**—the propositional version has also been called **PI**, **mbC**, Jaśkowski's logic **D2**), other logics like **LFI1** and **CLuNs** are extensions of **LP** and a classical negation $\neg$ can be defined in them, but they do not satisfy (ii) and (iii). In these logics $\sim\neg A \vdash A$ is valid. Hence we have in these logics $\sim\neg A, \neg A \vdash B$ falsifying (ii) and moreover $\sim\neg\neg A, A \vdash B$, which entails that (iii) cannot be valid. This is because, if the equivalence rule would hold for double negations, we would have $\sim A, A \vdash B$, which is (fortunately) not the case in these logics.

In Section 2 we sketch a picture of a particular property theoretic application of a logic which combines **LP** and **CL**. It is argued there that (i), (ii) and (iii) are desirable properties for a logic which is used for such applications. After a number of preliminary definitions in Section 3, we give a number of desiderata for a logic **L** which combines both negations in such a way that it is suited for the formalization of the property theoretic picture in Section 4. In Section 5 we devise the logic **CLP** semantically by means of 3 different semantic characterizations: a non-deterministic semantics, an infinitely valued set-theoretic semantics and an infinitely valued semantics with integer numbers as values. In section 6 we present an axiomatization of this logic. In Section 7, we prove completeness of the proof theory, maximal paraconsistency of **CLP** and the main result of the paper: **CLP** is the only logic satisfying all postulated desiderata. In Section 8, an adaptive logic is defined which interprets **CLP**-premise sets as consistently as possible. Finally in Section 9, we conclude the paper by mentioning a couple of interesting related issues and questions for further research.

2 Intuitive property theoretic picture

In what follows we see naive property theory as a sort of non-mathematical primitive set theory. Given how objects relate to primitive properties, a naive property theory determines how they relate to complex properties, i.e. properties that result from applying logical operations to primitive properties. Take for example the properties 'Green' and 'Big'. Suppose one knows how the objects in a certain domain relate to being Green and to being Big. A naive property theory should determine how objects relate to complex properties composed of Big and Green, such as 'is not (both Big and not Green) or is Green'. So, a naive property theory defines basic operations on properties, such as the intersection or the union of two properties, the complement of a property, etc.

An approach to naive property theory may be considered as paraconsistent if the approach tolerates the existence of properties for which there is an overlap between the extension of the property and the extension of the negation of the property (from now we shall call the extension of the negation of a property the *co-extension* of the property). In other words, in paraconsistent logics one does not presuppose that there are no objects which are both in the extension of a property and in its co-extension. This might for example be useful for paradoxical properties or properties that occur in a scientific theory that was supposed to be consistent but turned out to be inconsistent.

Even in contexts where there is a paraconsistent concept of co-extension, it often makes a lot of sense to also have a non-paraconsistent concept of co-extension around. It greatly enhances the expressive power. Take for example the case of a paraconsistent collection theory. The Russell collection is the collection of those collections that are not a member of themselves. Consider the property 'is a member of the Russell collection' abbreviated to MemR. The collection should famously contain itself and not contain itself at the same time. And so the property MemR should allow an overlap between its extension and co-extension. This does not mean that all objects are both in the extension and in the co-extension of MemR. The empty collection, for example, is arguably not a member of itself (it has no members) and so it should only be in the extension of MemR (not in the co-extension). On the other hand, the universal collection is arguably a member of itself and should only be in the co-extension of MemR (not in the extension). So we have three categories with respect to the property MemR: there are (i) objects that are members of its extension and not of its co-extension (e.g. the empty collection), (ii) objects that are both in its extension and in its co-extension (e.g. the Russell collection itself) and (iii) objects that are in its co-extension and not in its extension (e.g. the universal collection). If one wants a theory of properties of collections which expresses these three categories, one should be able to express that something is *consistently not* in the extension or consistently not in the co-extension. Saying that the empty collection is (paraconsistently) not a member of itself does not yet express that it might not also be a member of itself (which is obviously against the whole concept of an empty collection). So, if one does not have a way to express 'consistently not', one can express that the empty collection (supposing that one does have a means the refer to the empty collection) is in the extension of MemR and that the universal collection is in the co-extension of MemR, but there is no means to express that they are (consistently) not in the overlap between the extension and the co-extension.

So it is useful to also have a concept of consistent co-extension even in paraconsistent contexts (next to the concept of paraconsistent co-extension). To avoid confusion, from now on let the *complement* of an extension be everything that is consistently not in the extension. Let P be some property. We can now unambiguously assign names to the three categories with respect to P-hood: (i) the complement of the co-extension of P (henceforth called the *consistent extension*), (ii) the overlap between the extension and the co-extension of P (henceforth called the *inconsistent extension*), and (iii) the complement of P (henceforth called the *consistent co-extension*). Now, one may wonder what the co-extension of the complement of P looks like. Given that we allow overlap between extension and co-extension (this is exactly what it means to be paraconsistent), there may be overlap between the complement and its co-extension (there is no reason why these properties should be treated differently). So in fact there should be four categories (instead of the three mentioned above). The complement (category (iii)) can be split into the 'pure' complement and the overlap between the complement and the co-extension of the complement. What we call the 'pure' complement is actually more accurately described as the complement of the co-extension of the complement of the concept, or in other words: the consistent extension of the complement. Similarly we can split the first category into two new categories: the 'pure' consistent extension of P and the overlap between the consistent extension of P and its co-extension. So we now already have distinguished 5 reasonable categories:

(1) the consistent extension of the consistent extension of P (or equiv-

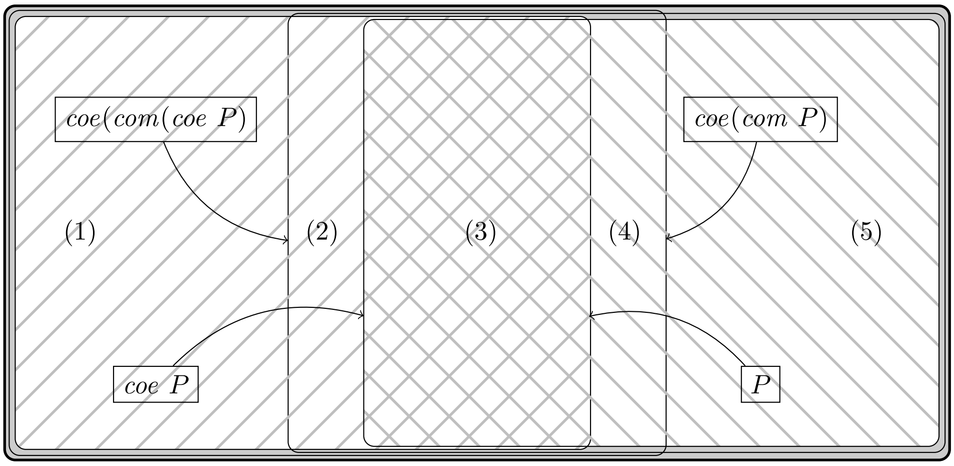

Figure 1: The five categories (1), (2), (3), (4), and (5) described on page 4. *com*(X) abbreviates the complement of X and *coe*(X) abbreviates the co-extension of X. The upward hatching denotes the extension of P, the downward hatching denotes the extension of not-P. The largest rounded rectangle denotes the universe. The grey areas are empty due to the law of excluded middle.

alently: the complement of the co-extension of the complement of the co-extension of P),

(2) the inconsistent extension of the consistent extension of P (or equivalently: the overlap between the complement of the co-extension of P and its co-extension),

(3) the inconsistent extension of P (or equivalently: the overlap between the extension of P and its co-extension),

(4) the inconsistent extension of the complement of P (or equivalently: the overlap between the complement of P and its co-extension), and

(5) the consistent extension of the complement of P (or equivalently: the complement of the co-extension of the complement of P).

Whether an object is in one of the 5 categories with respect to P-hood determines in how far they are and in how far they are not P. Objects are more consistently/unproblematically P in categories with lower numbers. More particularly, an object in (1) is more P than one in (2) because an object in (2) is still in the co-extension of the complement of co-extension of P, which means that there is still some negative 'evidence' with respect to the P-hood of the object (remark that this is not the case for the object in (1)). The objects in category (3) are as much P as they are not-P, so they are even less convincingly P than the ones in (1) and (2). Objects in categories (4) and (5) are not in the extension of P, so are definitely less convincingly P than the ones in (1)–(3). For the objects in (4), however, there is still some inconsistent information involved. Although they are in the complement of P, they are also in the co-extension of this complement, so they are still problematically not-P (there is still some positive 'evidence'). For category 5, finally, no P-related inconsistencies are involved and so these objects are unproblematically not-P. They are the least P of the 5 categories.

The reader probably wonders why we stop at 5 categories. Could we not split category (5) into the category (5a) of the objects in the consistent extension of (5) and the category (5b) containing the ones in the inconsistent extension of (5)? Indeed we can. Can we not do the same thing for category (1)? Yes we can. We can always split up the outer categories (the most P-category and the least P-category) into more fine-grained categories (their consistent extension and their inconsistent extension). So in fact we obtain infinitely many categories of ever more consistently P-objects and infinitely many ever more consistently not-P-objects. It makes sense to order the categories in the same way as the set of positive and negative integer numbers. In this ordering, category x is greater than category y, whenever the objects in category x are more consistently P than the ones in category y. Given this ordering, one can let every category correspond to exactly one integer number, except for two special categories. Remark that we did not yet mention categories that are completely consistent. Take any positive number n. For an object in category n it holds that it is in the inconsistent extension of (the consistent extension of)^n of P. Now take any negative number $-n$. An object in category $-n$ is in the inconsistent extension of (the consistent extension of)^n of not-P. So for every category with an integer number a P-related inconsistency is involved. The objects in such categories are therefore not completely consistent. To also be able to categorize completely consistent objects, we use two categories named by infinite numbers. We use positive infinity (number ∞) to denote the category of objects that are completely consistently P and negative infinity (number $-\infty$) for the category containing the objects that are completely consistently not-P. So, now we have distinguished the categories numbered by $-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty$.

We have distinguished an infinity of categories with respect to P-hood. Let $n \in \mathbb{N}$.

- An object is in P-category n iff it is in the inconsistent extension of the (consistent extension of the)^n extension of P.
- An object is in P-category $-n$ iff it is in the inconsistent extension of the (consistent extension of the)^n co-extension of P.
- An object is in P-category ∞ iff for each $m \in \mathbb{N}$ it is in the consistent extension of the (consistent extension of the)^m extension of P.
- An object is in P-category $-\infty$ iff for each $m \in \mathbb{N}$ it is in the consistent extension of the (consistent extension of the)^m co-extension of P.

These categories are mutually exclusive and exhaustive (they define a partition on the set of objects).

Why do we only split up the outer categories into more fine-grained categories? For this question we should first say something about the intersections and unions of inconsistent properties.

Suppose we know for both property P and property Q which objects are in their respective consistent extensions, inconsistent extensions and complements. Does this information suffice to know which objects are in which of these categories with respect to the property $P \wedge Q$? Not obviously. One may argue that fresh inconsistencies can popup at any level of complexity, independent of the inconsistencies at lower levels. In such approach to paraconsistent properties it is possible that properties P and Q are perfectly consistent, but that at the same time the intersected property $P \wedge Q$ has a non-empty inconsistent extension. Such approach

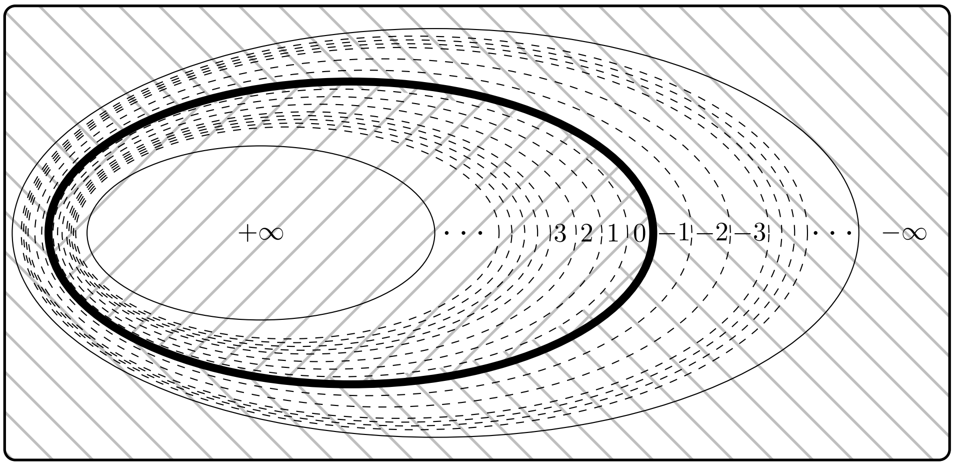

Figure 2: The infinite number of P-categories described on page 6. The upward hatching denotes the extension of P, the downward hatching denotes the extension of not-P.

may be useful in case one wants to maximally isolate inconsistencies and thus prevent them from spreading. One may want to treat them in this way for example because they are considered problematic and harmful phenomena without clear origin (for example for applications to scientific theories that were intended to be consistent but turned out or may turn out to be inconsistent). Following such an approach to paraconsistency, paraconsistent naive property theory is not very useful because the co-extension is no long an operation on the property itself. The co-extension is a separate property, only linked to the original property in the sense that each object is either in the extension of a property or in its co-extension. In this paper we choose not to follow the approach to paraconsistency. We here develop an approach towards inconsistencies in which complex properties may be inconsistent but their inconsistency originates entirely in the inconsistency of the composing parts. Choosing this road, it is perfectly possible to see the paraconsistent co-extension as a full blown operation on the property, in such a way that the naive property theory is able to determine how objects relate to complex properties even if they contain the paraconsistent co-extension.

So in our approach the question to which categories of P-hood and Q-hood an object belongs completely determines to which category of P-and-Q-hood an object belongs. More concretely we accept the law of Double Negation

DN the co-extension of the co-extension of a property is the property itself

and the laws of De Morgan for the co-extension operation

DM1 the co-extension of the union of two properties is identical to the intersection of the co-extensions of the properties.

DM2 the co-extension of the intersection of two properties is identical to the union of the co-extensions of the properties.

So let us see what the intersection comes to in this property theoretic picture. Suppose object a is in the consistent extension of P and in the consistent extension of Q. So it is in the extension of P and that of Q and therefore in the extension of the intersection of P and Q. Moreover it is not in the co-extension of P and not in the co-extension of Q either, i.e. it is not in the union of the co-extensions of P-and-Q. With DM2 this entails that it is not in the co-extension of the intersection of P and Q. We can conclude that a is in the consistent extension of P intersection Q.

So if a is in either of categories $\{1, \ldots, \infty\}$ with respect to both P and Q, then P intersection Q is in one of these categories.

More specifically if a is in P-category 1 and in Q-category 3, it is in P-and-Q-category 1. Let us explain why. If it is in P-category 1, it is also in the co-extension of the consistent extension of P. Therefore it is in the co-extension of the intersection of the consistent extensions of P and Q. But we already established that the intersection of the consistent extensions of P and of Q is identical to the consistent extension of P-and-Q. So a is also in the co-extension of the consistent extension of P-and-Q. Therefore it is in one of the P-and-Q-categories $\{-\infty, \ldots, -1, 0, 1\}$. So a must be in P-and-Q-category 1.

Similar reasoning allows us to conclude that an object is in P-and-Q-category n iff it is in P-category n_P, in Q-category n_Q and $n = \min(n_P, n_Q)$. So we have determined what the intersection of two inconsistent properties looks like. We do the same for the union of two paraconsistent properties: an object is in P-or-Q-category n iff it is in P-category n_P, in Q-category n_Q and $n = \max(n_P, n_Q)$.

But we wanted that all complex paraconsistent properties are completely determined by their their constituent parts and the way they are constructed from these. So we need to determine how the two kind of 'complements' affect the categories. If an object is in P-category n, it is in the inconsistent extension of the (consistent extension of the)^n extension of P. In which co-extension-of-P-category would this object then be? Well, given the DN law, it will be in the inconsistent extension of the (consistent extension of the)^n of the co-extension of the co-extension of P, but this is exactly the co-extension-of-P-category $-n$.

What about the complement? If object a is in P-category n, it is in the inconsistent extension of the (consistent extension of the)^n extension of P. Hence it is in the inconsistent extension of the (consistent extension of the)^n of the complement of the complement of P, and therefore also in the inconsistent extension of the (consistent extension of the)^n of the complement of the co-extension of the co-extension of the complement of P, or equivalently: in the inconsistent extension of the (consistent extension of the)$^{n+1}$ co-extension of the complement of P. Consequently, it is in the complement-of-P-category $-(n+1)$.

Let us return to the question whether it makes sense to split up our categories into further categories: a consistent extension and an inconsistent extension. Remark that there are four kind of categories. Two categories (the infinite ones) are completely consistent and so the inconsistent extension of them is per definition empty. Hence, there is no actual splitting in this case. The other categories are all characterized as the inconsistent extension of some property. So what is the consistent extension of an inconsistent extension of a property? Well, it is the complement of the co-extension of the intersection of the extension and the co-extension of the property. Applying DM2 we obtain that it is identical to the complement of the union of on the one hand the co-extension of the co-extension and on the other hand the co-extension, which is equal to the intersection of the complement of the property and the consistent extension of the property. Given the way we have defined the notions complement and consistent co-extension, they cannot have any objects in common. So this category is empty and so the category we wanted to split is identical to the inconsistent extension of it. So the splitting we suggested did not result in any actual splitting. We do not need more categories than the one mentioned before.

So if we want a reasonable formal treatment of paraconsistent properties (1) with a classical *and* a paraconsistent complement-operation and (2) such that for every complex property A, A-hood of an object a is completely determined by the B-hood of a for every primitive property B from which A is constructed, we naturally end up with infinitely many categories for A-hood for every category A instead of the 2 categories in the case of classical properties.

So we obtain something similar to fuzzy set theory as first developed by Lofti Zadeh (cf. [7]). In fuzzy set theory a set no longer divides the objects into two categories IN and OUT as in classical set theory, but there are infinitely many categories in between fully IN (real value 1) and fully OUT (real value 0). The same holds for what we describe here. Some object may be completely consistently P and some may be completely consistently not-P, with infinitely many categories in between. Even our intersection (minimum) and union (maximum) are similar to important set operations in fuzzy set theory. Of course there are important distinctions as well: our categories are necessarily discrete and fuzzy logic does not allow inconsistency: objects cannot be in a set and in the fuzzy complement of the set at the same time. This is not surprising given that we want to capture a completely different kind of properties (inconsistent ones, not vague ones).

The intuitive picture we sketched in this introduction suggests the development of a proper paraconsistent property theory or even a paraconsistent set theory. While this is indeed the long term goal of this line of research, we will in this paper only develop a propositional logic based on these ideas. All mentioned aspects of this paraconsistent property theory can already be formalized at the propositional level. Propositions correspond to properties applied to concrete objects. The operations by which we construct complex properties correspond to the logical symbols as follows: co-extension, complement, intersection and union correspond to respectively paraconsistent negation $\sim$, classical negation $\neg$, conjunction $\wedge$ and disjunction $\vee$. Actual paraconsistent set or property theories based on the presented ideas will have as their underlying logic a predicative (and possibly adaptive, cf. Section 8) version of the paraconsistent logic we are about to present.

Before getting to the actual logic, the question pops up: can we not base our property theory on an existing paraconsistent logic? The answer is no (as far as we know). There are indeed paraconsistent logics available with the four needed logical symbols and more particularly, a paraconsistent and a classical negation. An example is the logic **CLuNs** (cf. [2]). This logic is exactly like **LP** (cf. Section 3 for the definition) to which falsum ($\bot \vdash A$) and a classical implication $\supset$ is added such that $\sim(A \supset B)$ is equivalent to $A \wedge \sim B$. Classical Negation is defined as $\neg A =_{df} A \supset \bot$. This logic works exactly as one would expect (and as is described above) when there occur no classical negations within the scope of a paraconsistent negation.

However, the classical negation of **CLuNs** does not correspond to our intuitive complement operation described above but rather to what could be called *the completely consistent complement* (in our picture the completely consistent complement of any positive category or 0 is $-\infty$ and of any strictly negative category it is ∞). This results in the fact that $\sim\neg A, \neg A \vdash_{\mathbf{CLuNs}} B$ and $\neg\neg A, \sim\neg\neg A \vdash_{\mathbf{CLuNs}} B$, while of course $\sim A, A \nvdash B$. So, **CLuNs**'s classical negation actually makes the formula to which it is applied bivalent and this way the paraconsistent negation with classical negation in their scope lose their paraconsistency. This also means that classical negation in **CLuNs** is not really involutive. $\neg\neg A$ means something like the completely consistent extension of A which is essentially a bivalent proposition, while A itself may be paraconsistent and trivalent (however, remark that this distinction is not observable at the level of logical consequence: $A \vdash_{\mathbf{CLuNs}} \neg\neg A$ and $\neg\neg A \vdash_{\mathbf{CLuNs}} A$).

All this is related to the fact that the classical negation is defined by means of the **CLuNs**-implication. $\sim(A \supset B)$ is **CLuNs**-equivalent to $\sim(\neg A \vee B)$, which is then equivalent by De Morgan's laws to $\sim\neg A \wedge \sim B$. But because $\sim(A \supset B)$ is made equivalent to $A \wedge \sim B$, A and $\sim\neg A$ are equivalent in **CLuNs**. One would however expect that, since paraconsistent negation is supposed to be weaker than the classical negation, A (logically equivalent to $\neg\neg A$) entails $\sim\neg A$ but not vice versa. Reading $\neg$ here as 'completely consistent complement' makes this understandable: because $\neg A$ already has transformed A into a bivalent proposition, $\sim\neg A$ reduces to $\neg\neg A$ and from there to A. Again, remark that A and $\sim\neg A$ (although they entail each other) have a different meaning (one is trivalent and the other one is bivalent) and are not inter-substitutable: $\sim\sim\neg A, A \vdash B$ holds in **CLuNs** but obviously $\sim A, A \vdash B$ does not.

Another famous example of a logic that contains both a paraconsistent and a classical negation is the logic $\mathbf{CR}^*$ presented by Meyer and Routley in [4]. This logic does not have the strange property that the paraconsistent character of sentences is removed whenever a classical negation is applied to it. In other words in $\mathbf{CR}^*$ we have $\nvdash (\neg A \wedge \sim\neg A) \to B$ and $\nvdash (\neg\neg A \wedge \sim\neg\neg A) \to B$, where we use $\sim A$ to refer to their $\overline{A}$. Still, this logic does not suffice to formalize the kind of paraconsisitent properties characterized above. First of all, the logic is not only paraconsistent but also paracomplete (one does not have $\vdash B \to (A \vee \sim A)$), which makes the negation too weak for purely paraconsistent purposes. But this would not really be a problem as we may be able to simply add de law of excluded middle, making the logic strong enough. In that case the property theory based on $\mathbf{CR}^*$ would simply have more expressive power than the one introduced here: next to purely paraconsistent properties, one would also have properties that are both paracomplete and paraconsistent. Adding the law of excluded middle would simply exclude all the paracomplete ones.

Are we able to do that? Not without also giving up on the paraconsistency of the negation. In this logic $\neg\sim A$ and $\sim\neg A$ are equivalent and inter-substitutable. From a paraconsistent, but not paracomplete, negation one would expect $\vdash \neg A \to \sim A$, (but not vice versa)—if A is not the case then $\sim A$ should be the case (it cannot be neither). However, adding this to $\mathbf{CR}^*$ has dramatic consequences. We would obtain $\vdash \neg\sim A \to \sim\sim A$ and therefore also $\vdash \neg\sim A \to A$. Moreover we would also obtain $\vdash \neg\neg A \to \sim\neg A$ and therefore also $\vdash A \to \sim\neg A$. With the inter-substitutability of $\neg\sim A$ and $\sim\neg A$, we now obtain that A, $\sim\neg A$ and $\neg\sim A$ are all inter-substitutable. On the other hand we obviously have $\vdash (\neg\sim A \wedge \sim A) \to B$ ($\neg$ is after all classical). So it turns out that giving up on the paracompleness of $\mathbf{CR}^*$ also yields $\vdash (A \wedge \sim A) \to B$, which reduces $\sim$ to a classical negation, which is obviously undesired. In other words, $\mathbf{CR}^*$ tolerates both gluts and gaps and does this adequately. However, removing the possibility of a gap immediately also removes the possibility of gluts. This logic is therefore not suited as a logic for pure paraconsistency.

In this paper we develop the logic **CLP**, which is basically **LP** plus classical negation. The resulting logic will be able to formalize the phenomenon of inconsistent properties in the presence of a classical complement relation described above. Let us call such contexts (in which one tolerates inconsistent properties but still wants the ability to express the classical set-theoretic operations on properties) *expressive inconistency-tolerant* (abbreviated to EIT) contexts.

3 Preliminaries

3.1 Truth functional semantics

We first define what a truth functional semantics is.

Definition 1 *A truth functional semantics for a language $\mathcal{L}$ is a triple $\langle V, D, F\rangle$ where V is a set (the set of truth values), $D \subset V$ (the set of designated values), and F is a set of functions $f_\ddagger : V^n \to V$, one for each n-airy connective $\ddagger$ in the language (the truth functions).*

Consider a particular language $\mathcal{L}$ and a truth functional semantics $\mathcal{TS}$ for $\mathcal{L}$ with truth values V, designated values D and truth functions $\{f_\ddagger \mid \ddagger$ is a connective in $\mathbf{L}\}$. Let $\mathcal{W}$ denote the set of formulas of $\mathcal{L}$.

Definition 2 *A $\mathcal{TS}$-*truth function *is a function $v : \mathcal{W} \to V$ for which $v(\ddagger(A_1, \ldots A_n)) = f_\ddagger(v(A_1), \ldots, v(A_n))$, for every n, every n-airy $\mathcal{L}$-connective $\ddagger$ and every set of $\mathcal{L}$-formulas $\{A_1, \ldots, A_n\}$.*

Definition 3 *$\mathcal{TS}$ defines the semantic consequence relation $\vDash_{\mathcal{TS}}: \wp(\mathcal{W}) \to \mathcal{W}$ where, for every $A \cup \{\Gamma\} \subseteq \mathcal{W}$, $\Gamma \vDash_{\mathcal{TS}} A$ iff for all $\mathcal{TS}$-truth functions v: $v(A) \in D$ whenever $v(B) \in D$ for all $B \in \Gamma$.*

Definition 4 *$\mathcal{TS}$ is* adequate *for a consequence relation $\vdash$ defined over $\mathcal{L}$ iff, for every $A \cup \{\Gamma\} \subseteq \mathcal{W}$, $\Gamma \vDash_{\mathcal{TS}} A$ iff $\Gamma \vdash A$.*

Definition 5 *A particular consequence relation $\vdash$ defined over a language $\mathcal{L}$ is truth functional iff there is a truth functional semantics for $\mathbf{L}$ which is adequate for $\vdash$.*

Definition 6 *Two formulas A and B are strongly equivalent in a truth functional semantics $\mathcal{TS}$ if $v(A) = v(B)$ for all $\mathcal{TS}$-truth functions v.*

3.2 Meta-properties of consequence relations

We list a number of meta-properties of consequence relations which we shall use later in the paper.

Definition 7 *Where $\mathcal{S}$ and $\mathcal{W}$ are respectively the set of sentential letters and the set of formulas of the language $\mathcal{L}$, the following is the law of Uniform Substitution for $\vdash$ over $\mathcal{L}$:*

(US) For all $\sigma \in \mathcal{S}$, $A, B \in \mathcal{W}$ and $\Gamma \subseteq \mathcal{W}$: if $\Gamma \vdash A$ then $\Gamma' \vdash A'$, where Γ' and A' are the result of substituting every occurrence of σ in A resp. Γ by B.

Definition 8 *A consequence relation* $\vdash$ *for a language* $\mathcal{L}$ *is* structural *iff the law of Uniform Substitution holds for* $\vdash$ *over* $\mathcal{L}$*.*

Definition 9 *A consequence relation* $\vdash$ *is* deductive *w.r.t. a binary symbol* $\rightarrow$ *iff* $\Gamma \cup \{A\} \vdash B$ *iff* $\Gamma \vdash A \rightarrow B$

Definition 10 *A consequence relation* $\vdash$ *is* monotonic *iff, for all* $\Gamma, \Delta \subseteq \mathcal{W}$ *and* $A \in \mathcal{W}$*,* $\Gamma \cup \Delta \vdash A$ *whenever* $\Gamma \vdash A$

Definition 11 *A consequence relation* $\vdash$ *is* transitive *iff, for all* $\Gamma, \Delta \subseteq \mathcal{W}$ *and* $A \in \mathcal{W}$*,* $\Gamma \cup \Delta \vdash B$ *whenever* $\Gamma \vdash A$ *and* $\Delta \cup \{A\} \vdash B$

Definition 12 *A consequence relation* $\vdash$ *is* compact *iff, for all* $\Gamma \subseteq \mathcal{W}$ *and* $A \in \mathcal{W}$*, if* $\Gamma \vdash A$ *then there is a finite* $\Delta \subseteq \Gamma$ *such that* $\Delta \vdash A$*.*

A set of formulas Γ is said to $\vdash$-explode iff $\Gamma \vdash A$ for all formulas A of the language of $\vdash$.

Definition 13 *A consequence relation* $\vdash$ *for a language* $\mathcal{L}$ *with a unary connective* $\ddagger$ *is strictly non-explosive with respect to* $\ddagger$ *iff, for every* $\mathcal{L}$*-formula,* $\{A, \ddagger A\}$ $\vdash$*-explodes only if a proper subset of* $\{A, \ddagger A\}$ $\vdash$*-explodes.*

Definition 14 *A consequence relation* $\vdash_2 \colon \wp(\mathcal{W}_1 \cup \mathcal{W}_2) \times (\mathcal{W}_1 \cup \mathcal{W}_2)$ *is a* conservative extension *of* $\vdash_1 \colon \wp(\mathcal{W}_1) \times \mathcal{W}_1$ *iff*

$$\vdash_2 \ \cap \ (\wp(\mathcal{W}_1) \times \mathcal{W}_1) \ = \ \vdash_1$$

3.3 The logics we want to combine: CL and LP

Let $\mathcal{L}_{\mathbf{CL}}$ be the propositional language with sentential letters $\mathcal{S}$, binary connectives $\vee$ and $\wedge$, unary connective $\neg$ and set of formulas $\mathcal{W}_{\mathbf{CL}}$.

Definition 15 **CL** *is the logic semantically defined by the language* $\mathcal{L}_{\mathbf{CL}}$ *and the following truth functional semantics for this language:*

$$\langle V_1, D_1, \{f_\vee, f_\wedge, f_\neg\}\rangle,$$

where $V_1 = \{0, 1\}$*,* $D_1 = \{1\}$*,* $f_\vee = \max$*,* $f_\wedge = \min$*,* $f_\neg(1) = 0$ *and* $f_\neg(0) = 1$.

We shall write $\vdash_{\mathbf{CL}}$ to denote the consequence relation for which the truth functional semantics is adequate.

Theorem 1 $\vdash_{\mathbf{CL}}$ *is structural, truth functional, deductive w.r.t.* $\supset$*, monotonic, transitive and compact.*

Let $\mathcal{L}_{\mathbf{LP}}$ be the propositional language with sentential letters $\mathcal{S}$, binary connectives $\vee$ and $\wedge$, unary connective $\sim$ and set of formulas $\mathcal{W}_{\mathbf{LP}}$.

Definition 16 **LP** *is the logic semantically defined by the language* $\mathcal{L}_{\mathbf{LP}}$ *and the following truth functional semantics for this language:*

$$\langle V_2, D_2, \{f'_\vee, f'_\wedge, f'_\sim\}\rangle,$$

where $V_2 = \{\{0\}, \{1\}, \{0, 1\}\}$*,* $D_2 = \{a \in V_2 \mid 1 \in a\}$*,* $f_\vee = \max$*,* $f_\wedge = \min$*,* $f_\sim(\{1\}) = \{0\}$*,* $f_\sim(\{0\}) = \{1\}$*, and* $f_\sim(\{0, 1\}) = \{0, 1\}$*. The members of* V_2 *are ordered as follows:* $\{0\} < \{0, 1\} < \{1\}$*.*

The truth functional semantics of **LP** defines truth functions v that attach one of three truth values $\{1\}$, $\{0\}$, or $\{1, 0\}$ to all formulas with the following restrictions:

SSLP1a $1 \in v(A \vee B)$ iff $1 \in v(A)$ or $1 \in v(B)$

SSLP1b $0 \in v(A \vee B)$ iff $0 \in v(A)$ and $0 \in v(B)$

SSLP2a $1 \in v({\sim}A)$ iff $0 \in v(A)$
SSLP2b $0 \in v({\sim}A)$ iff $1 \in v(A)$

SSLP3a $1 \in v(A \wedge B)$ iff $1 \in v(A)$ and $1 \in v(B)$
SSLP3a $0 \in v(A \wedge B)$ iff $0 \in v(A)$ or $0 \in v(B)$

We shall write $\vdash_{\mathbf{LP}}$ to denote the consequence relation for which the truth functional semantics is adequate.

Theorem 2 $\vdash_{\mathbf{LP}}$ *is structural, truth functional, monotonic, transitive, compact and strictly non-explosive w.r.t.* $\sim$.

4 How to add classical negation to LP? A list of requirements.

There are different ways to add a classical negation to a logic like **LP**. We however want a logic which fits the EIT-contexts presented in Section 2. We will now give a list of requirements a logic should satisfy in order to preserve central properties of both **CL** and LP. The particular focus on EIT-contexts determines which properties are central and which are not.

Let $\mathcal{L}_{\mathbf{CLP}}$ be the propositional language with sentential letters $\mathcal{S}$, binary connectives $\vee$ and $\wedge$, unary connectives $\neg$ and $\sim$ and set of formulas $\mathcal{W}_{\mathbf{CLP}}$. $\mathcal{L}_{\mathbf{CLP}}$ will be the language of our combined logic. Conjunction and disjunction are the same in **LP** and **CL**, so only classical negation $\neg$ needs to be added to the language of **LP**.

This language is propositional. Although first order versions have been developed for **CL** and **LP**, we will not add quantifiers or predicates to the language of our combination of **LP** and **CL**. Of course this is only the first step: to actually formalize all aspects of EIT-contexts, one obviously needs to formalize things at the first order level. Nevertheless, it is unlikely that much interesting happens moving from the propositional level to the first order level. The universal/existential quantifiers can be handled as infinite conjunctions/disjunctions. Further research should make clear whether one would be able to retain all desired meta-properties such as completeness and compactness at the first order level. This is not obvious as these properties fail in some other infinitely valued first order logics.

4.1 Inheriting basic meta-properties

Formulas B and C are called *inter-substitutable* w.r.t. a consequence relation $\vdash$ iff, for every Γ and A in which B occurs as a subformula, $\Gamma \vdash A$ iff $\Gamma' \vdash A'$, where Γ' and A' are the results of substituting at least one occurrence of B by C in Γ resp. A.

We look for a logic **L** which combines the symbols of **LP** and the symbols of **CL** is such a way that that their meaning is maximally preserved in the combination. Which aspects of their meaning is to be preserved is determined by the EIT-contexts described in Section 2. We make precise what it means that the meaning of the logical symbols of **LP** and **CL** is preserved in function of EIT-contexts by means of the notion EIT-adequateness, defined as follows.

Definition 17 *A combination* **L** *of* **LP** *and* **CL** *is* EIT-adequate *iff each of the following requirements hold:*

Req1 $\vdash_{\mathbf{L}}$ *is structural (STR), monotonic, transitive, compact, and truth functional*

Req2 $\vdash_{\mathbf{L}}$ *is a conservative extension of* $\vdash_{\mathbf{LP}}$ *(CE-LP) and of* $\vdash_{\mathbf{CL}}$ *(CE-CL)*

Req3 *The formulas of the following form are inter-substitutable w.r.t.* $\vdash_{\mathbf{L}}$*: (i)* $\neg(A\vee B)$ *and* $\neg A\wedge\neg B$*, (ii)* $\neg(A\wedge B)$ *and* $\neg A\vee\neg B$*, (iii)* $\sim(A\vee B)$ *and* $\sim A\wedge\sim B$*, (iv)* $\sim(A\wedge B)$ *and* $\sim A\vee\sim B$*, (v)* A *and* $\neg\neg A$*, and (vi)* A *and* $\sim\sim A$*.*

Req4 $\vdash_{\mathbf{L}}$ *is strictly non-explosive with respect to* $\sim$*.*

We will now argue in favour of each of these requirements.

Req1 does not require much motivation. Each of these properties holds for both **CL** and **LP**. If the combination **L** is supposed to be a logic of the same kind, devised for similar purposes, the properties better also hold for **L**. There may be good reasons to give up some of these properties. The adaptive logic developed in Section 8, for example, is not monotonic nor structural and rightly so. But logics that do not merely add a classical negation to **LP**, but also give up one of these properties, change the whole function of the logic.

Maybe truth functionality is a property for which it is less obvious that it should be inherited by the combination logic. Truth functionality is however essential for EIT-contexts. For these contexts one needs a logic the connectives of which can correspond to operations on properties. The kind of inconsistent properties we considered were such that the relation between complex properties and the objects of the domain are completely determined by the relation between primitive properties and the objects.

Req2 is an uncontroversial requirement for combining logics: everything what can be derived with the combined logics should also be derivable with combination logic AND the combination logic should not add anything to the combined logics with respect to the original fragments of the language. This requirement is a necessary condition for warranting that the meaning of the connectives is preserved in the combination logic. However it is definitely not a sufficient condition as the non-triviality of the following requirements will illustrate.

The meaning of all connectives in the original logics **LP** and **CL** should be preserved when combining them. This is a requirement which cannot be made precise straightforwardly. One cannot simply use the meaning as expressed by the truth functional semantics of both **CL** and **LP** and combine them. Neither the three valued semantics of **LP** nor the two valued semantics of **CL** is straightforwardly extendable with a truth function that can express the meaning of the extra connective (resp. $\sim$ or $\neg$).

The only possible candidate seems to be the following: take the three truth values of **LP** (with the same designated ones as in **LP**) and let the **CL**-truth value 'true' correspond to both the **LP**-values $\{1\}$ and $\{0,1\}$ and 'false' to $\{0\}$. This way the shared connectives $\vee$ and $\wedge$ can get the same truth functions as in **LP**, which immediately warrants that also the truth functionality of **CL** is maintained for these connectives. Also $\sim$ can remain the same as in **LP**, because we still have the original **LP**-truth values at our disposal. But what should we do with $\neg$? It is obvious for values $\{1\}$ and $\{0,1\}$, in which case the truth function for $\neg$ simply gives $\{0\}$. But what should be the value of $f_\neg(\{0\})$? If we choose $\{1\}$, we have $f_\neg(f_\neg(\{0,1\}))=\{1\}$ and if we choose $\{0,1\}$, we have $f_\neg(f_\neg(\{1\}))=\{0,1\}$. Both choices surely preserve part of the meaning of classical negation, but they both falsify the semantic variant of the law of double negation ($\neg\neg A$ has (exactly) the same meaning as A), which is an essential part of the meaning of classical negation.

Especially in EIT-contexts, we want to maintain the usual laws for the complement operation as much as possible. The complement of the complement of a property, for example, should be identical to the property itself, just like in classical property theory. It is hard to imagine how a complement operation on properties can be anything for which the DN law is not valid. Remark that this immediately implies that we can always substitute the complement of the complement of a property by the property itself, also within more complex properties.

In general, the problem is that Req2 preserves part of the meaning of the connectives, but it cannot affect the meaning of the connective $\neg$ when it occurs in the scope of $\sim$. There are two prima facie reasonable ways to extend the meaning preservation to subformulas in the scope of **LP**-negations. The first is by means of a variant of the law of contraposition (require that if $A \vdash B$ then also $\sim B \vdash \sim A$), the second is by means of a variant of a specific instance of the the equivalence rule (if $A \dashv\vdash B$ then $\sim A \dashv\vdash \sim B$, where $A \dashv\vdash B$ abbreviates '$A \vdash B$ and $B \vdash A$'). This way, the meaning of the relation between the meaning of connectives at the outmost level is transferred to occurrences of the same connectives within the scope of an **LP**-negation.

This would indeed work nicely, were it so that these criteria indeed hold for the **LP**-negation. But they do not even hold for the logic **LP**, let alone for the combination logic. The reason why they do not hold is rather obvious. We have $q \vee \sim q \dashv\vdash_{\mathbf{LP}} p \vee \sim p$ but we do not have $\sim(p \vee \sim p) \vdash_{\mathbf{LP}} \sim(q \vee \sim q)$, because $\sim(A \vee \sim A)$ means exactly the same thing as $A \wedge \sim A$ in **LP**.

A promising, but still imprecise, sufficient condition for meaning preservation within the scope of $\sim$ is the following:

SE Formulas A strongly equivalent in the truth functional semantics of **CL** and of **LP** should maximally be inter-substitutable w.r.t. a consequence relation which combines **LP** and **CL**.

The adverb maximally is of great importance here. If it were removed, there would not be any satisfactory combination logic which is still paraconsistent. Consider the following example. Because of the meaning of classical negation, A always gets the same truth value as $A \wedge (B \vee \neg B)$ in **CL**, so A and $A \wedge (B \vee \neg B)$ are strongly equivalent in **CL** and therefore these formulas would have to be inter-substitutable in the combination logic. Unfortunately (and maybe surprisingly), this aspect of the meaning of classical negation is not compatible with the paraconsistency of the **LP**-negation. We prove that $p, \sim p \vdash q$ if one would be able to substitute A by $A \wedge (B \vee \neg B)$ (in the proof below denoted by $\star$) in a logic that combines $\sim$ and $\neg$ with properties (CE-LP), (CE-CL) and (STR).

1	p	PREM
2	$\sim p$	PREM
3	$\neg p \vee \sim\neg p$	(US) on $\vdash r \vee \sim r$ (by (CE-LP))
4	$\sim\neg p$	1,3; (US) on $\neg r, r \vee s \vdash s$ (by (CE-CL))
5	$\sim p \wedge \sim\neg p$	1,4; (US) on $r, s \vdash r \wedge s$ (by (CE-LP))
6	$\sim(p \vee \neg p)$	5; (US) on $\sim r \wedge \sim s \vdash \sim(r \vee s)$ (by (CE-LP))
7	$\sim(p \vee \neg p) \vee \sim\sim q$	6; (US) on $r \vdash r \vee s$ (by (CE-LP))
8	$\sim((p \vee \neg p) \wedge \sim q)$	7; (US) on $\sim r \vee \sim s \vdash \sim(r \wedge s)$ (by (CE-LP))
9	$\sim\sim q$	8; $\star$ ($\sim q$ same **CL**-value as $\sim q \wedge (p \vee \neg p)$)
10	q	9; $\sim\sim q \vdash q$ by (CE-LP)

For similar reasons one cannot add the following rules to the combination logic without giving up (SP): substituting $A \vee (B \wedge \neg B)$ by A (and vice versa), $A \wedge \neg A$ by $B \wedge \neg B$, or $A \vee \neg A$ by $B \vee \neg B$.

However, the situation is not hopeless. For a lot of pairs of formulas that have the same meaning in **CL**, rules can be added to the combination logic stating that the members of the pairs are inter-substitutable, without threatening paraconsistency. In fact, one can require the following:

(SE-CL) For all of the pairs (A, B) of formulas A and B that are strongly equivalent in the truth functional semantics of **LP**, C and D should be inter-substitutable in the combination logic, where C and D are the result of replacing each occurrence of $\sim$ by $\neg$ in A resp. B.

So, all **CL**-formulas that have the same meaning, except those formulas that only have the same meaning because of the classical (empty) meaning of a contradiction or a theorem, are required to receive the same meaning in the combination logic. For **LP**, strong equivalence (SE) can be valid unrestrictedly without danger of losing paraconsistency.

(SE-LP) For all of the pairs (A, B) of formulas A and B that are strongly equivalent in the truth functional semantics of **LP**, A and B should be inter-substitutable in the combination logic.

If one combines (SE-CL) and (SE-LP) for the logic **L** one obtains exactly Req3.

Req4 has to do with the preservation of the most essential aspect of the **LP**-negation: its paraconsistency. Of course we want the negation of **LP** to remain paraconsistent when we add classical negation to it. It is possible to require mere paraconsistency for the combination logic, but note that this is an extremely weak property. Paraconsistency in the weak sense only requires that there are formulas A and B such that $A, \sim A \nvdash B$. Observe that every conservative extension of **LP** is paraconsistent in this sense since $p, \sim p \nvdash q$ in every conservative extension of **LP**. There are stronger notions of paraconsistency, for example the property that for every formula A there is a formula B such that $A, \sim A \nvdash B$, or with similar results: the property that every set of formulas has at least one model. **LP** is indeed paraconsistent even in this strong sense, but this property is definitely too strong for a logic containing classical negation. If A is of the form $B \wedge \neg B$ it cannot have a model and it should entail every arbitrary formula B (independent of whether $\sim A$ is also a premise).

The EIT-contexts suggest a compromise notion of paraconsistency. In EIT-contexts there can be an overlap between the extension and the co-extension of all properties, also properties constructed by means of the complement operation. Of course, if the extension or the co-extension themselves are already empty, it makes no sense to make an overlap possible between the two. But in all other cases it makes sense to allow for overlap. This translates into the property of non-explosiveness w.r.t. the paraconsistent negation: For every formula A there is a formula B such that $A, \sim A \nvdash B$, unless for all formulas B $A \vdash B$ or $\sim A \vdash B$. This is the kind of paraconsistency which is required by Req4.

5 Semantics

We shall now define the combination logic **CLP** semantically. The language of **CLP** is $\mathcal{L}_{\mathbf{CLP}}$. We define the logic in such a way that it satisfies criteria Req1–Req4.

5.1 Indeterministic semantics

We start off with a very direct semantics for the logic which adds a classical negation to **LP**, viz. a semantics which is only based on the basic assumptions listed above.

The easiest way to present this semantics requires a notion of *strong equivalence*. This is a notion close to "has the same truth value as" or "is as (in)consistently true/false as". Regular semantic equivalence is defined as usual: A is *semantically equivalent* to B in a model M iff $M \models A$ iff $M \models B$. Strong equivalence is equivalent to regular equivalence in bivalent contexts. Already in **LP**, there is a difference between the two notions. In each **LP**-model, every inconsistent formula is semantically equivalent to every sentence which is consistently true, just because they are both verified by the model. In **LP**, formulas are strongly equivalent if they are either both inconsistent, both consistently true or both consistently false. In the context of the logic we develop in this paper, formulas are strongly equivalent in a model only if all formulas constructed in the same way with these formulas are semantically equivalent in that model. More precisely: A is strongly equivalent to B in M iff for every formula C which contains A, C is semantically equivalent to D in M, where D is the result of substituting every occurrence of A in C by B.

We ensure that the formulas that should be strongly equivalent are indeed treated the same way in all models by means of the binary relation $\approx$. We define this equivalence relation as follows.

Definition 18 *$\approx$ is an equivalence relation (transitive, symmetric and reflexive) in $\mathcal{W} \times \mathcal{W}$ such that*

SE1 $\neg(A \vee B) \approx (\neg A \wedge \neg B)$
SE2 $\neg(A \wedge B) \approx (\neg A \vee \neg B)$
SE3 $\sim(A \vee B) \approx (\sim A \wedge \sim B)$
SE4 $\sim(A \wedge B) \approx (\sim A \vee \sim B)$
SE5 $A \approx \neg\neg A$
SE6 $A \approx \sim\sim A$
SE7 *if* $A \approx B$ *then* $\neg A \approx \neg B$ *and* $\sim A \approx \sim B$
SE8 *if* $A \approx B$ *and* $C \approx D$*, then* $A \vee C \approx B \vee D$ *and* $A \wedge C \approx B \wedge D$

Remark that the relation $\approx$ is not equivalent to the relation 'strongly equivalent'. It only expresses some necessary properties of 'strong equivalence', which are nevertheless sufficient to define an indeterministic semantics for **CLP**. Why $\approx$ is sufficient can be understood as follows: it will allow us to reduce $M \models A$ for every formula A to an expression $M \models B$ where B is a formula in which no conjunctions $\wedge$ or disjunctions $\vee$ occur in the scope of negations $\neg$ and $\sim$ (by means of Theorem 3 and clause IS1). Conjunction and disjunctions out of the scope of negations behave completely classical in **CLP** whence the classical semantic clauses sufficiently determine their meaning. An example may clarify this line of reasoning. $A \vee (A \wedge B)$ should be strongly equivalent to A and so $M \models \neg\sim(A \vee (A \wedge B))$ should be equivalent to $M \models \neg\sim A$. The $\approx$-relation allows us to conclude that $\neg\sim(A \vee (A \wedge B)) \approx (\neg\sim A \vee (\neg\sim A \wedge \neg\sim B))$. Only requiring that these two formulas get the same verification status in each **CLP**-model, standard conjunction and disjunction clauses suffice to prove that $M \models \neg\sim(A \vee (A \wedge B))$ iff $M \models \neg\sim A$.

Definition 19 *Where* $\mathcal{W}^{\neg,\sim}$ *is the set of* **CLP***-formulas in which* $\neg$ *and* $\sim$ *are the only logical symbols, the set of formulas with negations pushed inwards, denoted as* $\mathcal{W}^n$*, is the smallest subset of* $\mathcal{W}_{\mathbf{CLP}}$ *such that* $\mathcal{W}^{\neg,\sim} \subset \mathcal{W}^n$ *and if* $A, B \in \mathcal{W}^n$ *then* $A \wedge B \in \mathcal{W}^n$ *and* $A \vee B \in \mathcal{W}^n$*.*

Definition 20 *Let* $\mathcal{W}^{pn}$ *be the set of formulas in* $\mathcal{W}^{\neg,\sim}$ *in which no substring* $\neg\neg$ *or* $\sim\sim$ *occurs.*

Theorem 3 *For every formula* A *there is a formula* $B \in \mathcal{W}^n$ *such that* $A \approx B$*.*

Proof. Every formula $B \in \mathcal{W}_{\mathbf{CLP}}$ can be obtained from a formula $A \in \mathcal{W}^n$ by doing nothing ($A = B$) or by substituting some primitive formulas in A by complex formulas $A_1, \dots A_n$ of the form $C \wedge D$ or $C \vee D$. Let $c(A_i) \geq 1$ denote the complexity (defined as usual as the maximal depth at which the primitive formulas occur in A_i) of A_i for $i \in \{1, \dots, A_n\}$. Let the maximal depth $md(A)$ of the occurrence of a conjunction or disjunction within the scope of a negation $\neg$ or $\sim$ of a conjunctive or disjunctive formula be defined as $md(B) = 0$ if $B \in \mathcal{W}^n$ and $md(B) = \max(c(A_1), \dots, c(A_n))$ otherwise.

It suffices to prove that, for each formula A with $md(A) = n \geq 1$, there is a formula B such that $A \approx B$ and $md(B) < n$. The transitivity of $\approx$ and a straightforward mathematical induction show that there is also a formula C such that $A \approx C$ and $md(C) = 0$, which entails what is proven here.

So suppose $md(A) = n$. The complex formulas $A_1, \dots A_n$ of the form $C \wedge D$ or $C \vee D$ are themselves each subformulas of subformulas of A of the form $A'_i = \ddagger^{1_i} \dots \ddagger^{m_i} A_i$ where $\ddagger^{j_i} \in \{\sim, \neg\}$. By applying SE1–5 one can easily prove that

$$(\ddagger^1 \dots \ddagger^k (C \vee D)) \approx (\ddagger^1 \dots \ddagger^k C \vee \ddagger^1 \dots \ddagger^k D),$$

when k is an even number and that

$$(\ddagger^1 \dots \ddagger^k (C \vee D)) \approx (\ddagger^1 \dots \ddagger^k C \wedge \ddagger^1 \dots \ddagger^k D),$$

when k is an odd number. The same is true if one replaces each $\vee$ by $\wedge$ and vice versa in the 2 latter equations. So for each $i \leq n$ there is a formula F_i such that $A'_i \approx F_i$ where $md(F_i) < md(A_i)$.

By E5 and E6 one can prove that

$A \approx B$ if $C \approx D$, where B is the result of substituting C by D in A.

Hence if one substitutes each A'_i in A by F_i, one obtains a formula B such that $A \approx B$ and $md(B) < md(A)$, which completes our proof.

■

Now we can list the indeterministic semantic clauses for **CLP**. Except the first, the clauses are all directly inherited from **CL**, **LP** or both. The first clause guarantees meaning preservation by means of the concept of strong equivalence.

IS1 If $A \approx B$, then $M \models A$ iff $M \models B$
IS2 $M \models A \vee B$ iff $M \models A$ or $M \models B$
IS3 $M \models A \wedge B$ iff $M \models A$ and $M \models B$
IS4 $M \models A$ or $M \models \sim A$
IS5 $M \models \neg A$ iff $M \not\models A$

Definition 21 *A model*[1] *M is a* **CLP***-model iff it respects clauses [IS1]–[IS5].*

Definition 22 *$M \models \Gamma$ iff $M \models A$ for each $A \in \Gamma$.*

Definition 23 *A is a semantic* **CLP***-consequence of Γ, denoted by $\Gamma \vDash_{\mathbf{CLP}} A$, iff, for each* **CLP***-model M, $M \models A$ whenever $M \models \Gamma$.*

Theorem 4 *Whether $M \models A$ holds for a* **CLP***-model M and a complex formula A is completely determined by whether $M \models (\sim\neg)^n\rho$ holds and whether $M \models (\sim\neg)^n{\sim}\rho$ holds for natural numbers n and sentential letters ρ that occur in A.*

Proof. By Theorem 3 we have $M \models A$ iff $M \models B$ for some $B \in \mathcal{W}^n$. This formula B is composed of formulas $C_1, \ldots C_m$ of the form $\ddagger^1 \ldots \ddagger^n \rho$ (when $n = 0$, $\ddagger^1 \ldots \ddagger^n$ is the empty string) by means of conjunctions and disjunctions, where n, m are natural numbers ≥ 0, ρ is a sentential letter which also occurs in A, and $\ddagger \in \{\neg, \sim\}$. If it is determined whether $M \models \ddagger^1 \ldots \ddagger^n \rho$ for each such formula $C_i = \ddagger^1 \ldots \ddagger^n \rho$, then $M \models B$ is also determined (and therefore also $M \models A$ is determined) in view of the fact that the semantic clauses IS2 and IS3 are perfectly deterministic.

So it remains to show that expressions of the form $M \models \ddagger^1 \ldots \ddagger^n \rho$ can be reduced to expressions of the form $M \vdash (\sim\neg)^k\rho$ or expressions of the form $M \vdash (\sim\neg)^k{\sim}\rho$. First we reduce $M \models \ddagger^1 \ldots \ddagger^n \rho$ to an expression of the same form in which no substring $\neg\neg$ or $\sim\sim$ occurs. SE5–7 allow us to show that when $\dagger^1 \ldots \dagger^k$ is the result of removing all substrings '$\neg\neg$' and '$\sim\sim$' from $\ddagger^1 \ldots \ddagger^n$, then $\dagger^1 \ldots \dagger^k \rho \approx \ddagger^1 \ldots \ddagger^n \rho$. IS1 moreover entails that $M \models \dagger^1 \ldots \dagger^l \rho$ iff $M \models \ddagger^1 \ldots \ddagger^n \rho$. So the first reduction is complete and we can safely assume that no double negations occur in $\ddagger^1 \ldots \ddagger^n$.

Next, suppose $\ddagger^1 = \neg$, then $M \models \ddagger^1 \ldots \ddagger^n \rho$ iff $M \not\models \ddagger^2 \ldots \ddagger^n$. Moreover, $\ddagger^2 = \sim$ otherwise $\ddagger^1\ddagger^2 = \neg\neg$ which would contradict our assumption obtained by the previous reduction. So we can reduce all expressions where $\ddagger^1 = \neg$ to expressions where $\ddagger^1 = \sim$ and safely assume that $\ddagger^1 = \sim$.

It is now clear that all expressions of the form $M \models \ddagger^1 \ldots \ddagger^n \rho$ are determined by expressions of the form $M \models (\sim\neg)^k\rho$ and expressions of the form $M \models (\sim\neg)^k{\sim}\rho$. This concludes our proof. ∎

5.2 Truth values as sets

We now have a semantics for our logic, but it is clarifying to give a semantics which is deterministic. We can do this in a similar way as the semantics of **LP** presented above: by means of sets as truth values. Applying Theorem 4 the model verification status of complex formulas can completely determined by the validity of expressions of the form $M \vdash (\sim\neg)^n\rho$ and expressions of the form $M \vdash (\sim\neg)^n{\sim}\rho$, where n is a natural number and ρ is a sentential letter. Moreover, because ${\sim}A \nvdash \neg A$ none of these expression can be reduced to each other. So, instead of only considering notions 0 (falsity) and 1 (truth) as in **LP**, we could consider infinitely many shades of truth (1 (truth), $1'$ (weak truth), $1''$ (weaker truth), $1'''$ (even weaker truth), ...) and falsity (0 (falsity), $0'$ (weak falsity), $0''$ (weaker falsity), $0'''$ (even weaker falsity), ...). Let $a^{(n)}$, where $a \in \{0, 1\}$, be defined by $a^{(0)} = a$ and $a^{(n+1)} = (a^{(n)})'$. For **LP** we have $M \vdash A$ iff

[1]It is not relevant at this point what a model in general is. One could see it as nothing but a function from formulas A to {true, false}, where the value true gives $M \models A$ and false gives $M \not\models A$.

$1 \in v(A)$ and $M \vdash \sim A$ iff $0 \in v(A)$. Similarly we could define truth value sets by means of the following translation to model verification.

$$1^{(n)} \in v_M(A) \text{ iff } M \models (\sim\neg)^{(n)} A$$
$$0^{(n)} \in v_M(A) \text{ iff } M \models (\sim\neg)^{(n)} \sim A$$

Define $t = \{1^{(n)} \mid n \in \mathbb{N}\}$, $f = \{0^{(n)} \mid n \in \mathbb{N}\}$ and $u = t \cup f$. Just like in the case of the **LP**-semantics, not every subset of all truth degrees is a truth value. In the case of **LP** the empty set is not a truth value in order to validate excluded middle by excluding the possibility that $M \not\models A$ and $M \not\models \sim A$. In the case of **CLP** validating excluded middle for $\sim$ becomes a bit more complicated. As one can see in the following definition, we need two conditions to reassure the validity of excluded middle for $\sim$.

Definition 24 *a is an element of the set of* **CLP***-truth values V iff*

- $a \subseteq u$
- $0 \in a$ *or* $1 \in a$
- *for all $b \in u$: if $b \in a$ then $b' \in a$*

The second condition literally corresponds to semantic clause IS4. The third condition corresponds to $A \models_{\mathbf{CLP}} \sim\neg A$ or $\neg\sim A \models_{\mathbf{CLP}} A$, which is also a consequence of semantic clause IS4.

Definition 25 *A* **CLP***-set assignment is a function $S : \mathcal{S} \to V$.*

Definition 26 *Given a* **CLP***-set assignment S, the* **CLP***-valuation function $\mathbf{v}_S$ for S is the function $\mathbf{v}_S : \mathcal{W}_{\mathbf{CLP}} \to V$ where*

SS0 $\mathbf{v}_S(\rho) = S(\rho)$ *for all sentential letters* ρ

SS1a $1^{(n)} \in \mathbf{v}_S(A \vee B)$ *iff* $1^{(n)} \in \mathbf{v}_S(A)$ *or* $1^n \in \mathbf{v}_S(B)$
SS1b $0^{(n)} \in \mathbf{v}_S(A \vee B)$ *iff* $0^{(n)} \in \mathbf{v}_S(A)$ *and* $0^{(n)} \in \mathbf{v}_S(B)$

SS2a $1^{(n)} \in \mathbf{v}_S(A \wedge B)$ *iff* $1^{(n)} \in \mathbf{v}_S(A)$ *and* $1^n \in \mathbf{v}_S(B)$
SS2b $0^{(n)} \in \mathbf{v}_S(A \wedge B)$ *iff* $0^{(n)} \in \mathbf{v}_S(A)$ *or* $0^n \in \mathbf{v}_S(B)$

SS3a $1^{(n)} \in \mathbf{v}_S(\sim A)$ *iff* $0^{(n)} \in \mathbf{v}_S(A)$
SS3b $0^{(n)} \in \mathbf{v}_S(\sim A)$ *iff* $1^{(n)} \in \mathbf{v}_S(A)$

SS4a *where* $n \geq 1$, $1^{(n)} \in \mathbf{v}_S(\neg A)$ *iff* $0^{(n-1)} \in \mathbf{v}_S(A)$
SS4b $1 \in \mathbf{v}_S(\neg A)$ *iff* $1 \notin \mathbf{v}_S(A)$
SS4c $0^{(n)} \in \mathbf{v}_S(\neg A)$ *iff* $1^{(n+1)} \in \mathbf{v}_S(A)$

Hence, for every **CLP**-valuation function $\mathbf{v}_S$ and every $A \in \mathcal{W}_{\mathbf{CLP}}$ the following holds (the '...' in the following expressions denote the infinite continuation of the sequence that *precedes* it):

$$\text{(a) } \mathbf{v}_S(A) = \{0, 0', 0'', 0''', \ldots\} \text{ or}$$
$$\text{(b) } \mathbf{v}_S(A) = \{1, 1', 1'', 1''', \ldots\} \text{ or}$$
$$\text{(c}n\text{) } \mathbf{v}_S(A) = \{0, 0', 0'', 0''', \ldots 1^{(n)}, 1^{(n+1)}, 1^{(n+2)}, \ldots\} \text{ or}$$
$$\text{(d}n\text{) } \mathbf{v}_S(A) = \{1, 1', 1'', 1''', \ldots 0^{(n)}, 0^{(n+1)}, 0^{(n+2)}, \ldots\}$$

Definition 27 *A* **CLP***-set assignment is said to* correspond *to a* **CLP***-model M iff for all sentential letters ρ and all natural numbers n*

C1 $1^{(n)} \in S(A)$ *iff* $M \models (\sim\neg)^{(n)}\rho$, *and*
C2 $0^{(n)} \in S(A)$ *iff* $M \models (\sim\neg)^{(n)}\sim\rho$.

Theorem 5 *If a* **CLP***-set assignment* S *corresponds to a* **CLP***-model* M*, then* $M \models A$ *iff* $1 \in \mathbf{v}_S(A)$

Proof. Suppose **CLP**-set assignment S corresponds to **CLP**-model M. We prove

$$\begin{array}{rcl} M \models (\sim\neg)^n A & \text{iff} & 1^n \in \mathbf{v}_S(A) \text{ and} \\ M \models (\sim\neg)^n {\sim} A & \text{iff} & 0^n \in \mathbf{v}_S(A) \end{array} \tag{1}$$

by a mathematical induction on the complexity of A.

Basic case. For A a sentential letter, (1) follows immediately from the supposition.

Induction step. Suppose for all formulas B less complex than A, (1) holds (substituting A by B). There are four cases:

1. A is of the form $C \vee D$.

$$M \models (\sim\neg)^n (C \vee D)$$

iff, because $((\sim\neg)^n (C \vee D)) \approx ((\sim\neg)^n C \vee (\sim\neg)^n D)$,

$$M \models (\sim\neg)^n C \text{ or } M \models (\sim\neg)^n D$$

iff, by the induction hypothesis,

$$1^{(n)} \in \mathbf{v}_S(C) \text{ or } 1^{(n)} \in \mathbf{v}_S(D).$$

Next

$$M \models (\sim\neg)^n {\sim}(C \vee D)$$

iff, because $((\sim\neg)^n {\sim}(C \vee D)) \approx ((\sim\neg)^n {\sim} C \wedge (\sim\neg)^n {\sim} D)$,

$$M \models (\sim\neg)^n {\sim} C \text{ and } M \models (\sim\neg)^n {\sim} D$$

iff, by the induction hypothesis,

$$0^{(n)} \in \mathbf{v}_S(C) \text{ and } 0^{(n)} \in \mathbf{v}_S(D).$$

2. A is of the form $C \wedge D$. Analogous to case 1.
3. A is of the form ${\sim} C$.

$$M \models (\sim\neg)^n {\sim} C$$

iff, by the induction hypothesis,

$$0^{(n)} \in \mathbf{v}_S(C)$$

iff, by SS3a,

$$1^{(n)} \in \mathbf{v}_S({\sim} C).$$

Next,

$$M \models (\sim\neg)^n {\sim\sim} C$$

iff, because $(\sim\neg)^n {\sim\sim} C \approx (\sim\neg)^n C$,

$$M \models (\sim\neg)^n C$$

iff, by the induction hypothesis,

$$1^{(n)} \in \mathbf{v}_S(C)$$

iff, by SS3a,

$$0^{(n)} \in \mathbf{v}_S({\sim} C).$$

4. A is of the form $\neg C$. Where $n \geq 1$,

$$M \models (\sim\neg)^n \neg C$$

iff, because $(\sim\neg)^n \neg C \approx (\sim\neg)^{n-1} \sim C$,

$$M \models (\sim\neg)^{n-1} \sim C$$

iff, by the induction hypothesis,

$$0^{(n-1)} \in \mathbf{v}_S(C)$$

iff, by SS4a,

$$1^{(n)} \in \mathbf{v}_S(\neg C).$$

Next $M \models \neg C$ iff $M \not\models C$ iff, by the induction hypothesis, $1 \notin \mathbf{v}_S(C)$ iff, by SS4b, $1 \in \mathbf{v}_S(\neg C)$. Where $n \geq 1$,

$$M \models (\sim\neg)^n \sim\neg C$$

iff,

$$M \models (\sim\neg)^{n+1} C$$

iff, by the induction hypothesis,

$$1^{(n+1)} \in \mathbf{v}_S(C)$$

iff, by SS4c,

$$0^{(n)} \in \mathbf{v}_S(\neg C).$$

■

Theorem 6 *There is a one-to-one-mapping form the* **CLP**-*set assignments S to the* **CLP**-*models M*

Proof. Theorem 4 tells us that **CLP**-models are completely determined by the set Δ_M of formulas A of the form $(\sim\neg)^n \rho$ or of the form $(\sim\neg)^n \sim\rho$, where $\rho \in \mathcal{S}$ and $n \in \mathbb{N}$, for which $M \vdash A$. We need to prove that there is a one-to-one mapping g from the set $\{\Delta^M \mid M \text{ is a } \mathbf{CLP}\text{-model}\}$ to the set of **CLP**-set assignments.

Define g as the mapping from $\{\Delta^M \mid M \text{ is a } \mathbf{CLP}\text{-model}\}$ to the set of functions in $\mathcal{S} \to \wp(u)$ such that $g(\Delta) = S$ iff, for every $\rho \in \mathcal{S}$ and $n \in \mathbb{N}$,

$$(\sim\neg)^n \rho \in \Delta \text{ iff } 1^{(n)} \in S(\rho), \text{ and}$$
$$(\sim\neg)^n \sim\rho \in \Delta \text{ iff } 0^{(n)} \in S(\rho).$$

From the definition it is clear that this and the inverse mapping are both total functions. We need to prove that the image of g is the set of **CLP**-assignments.

Observe that, with respect to the formulas they verify of the form $(\sim\neg)^n \rho$ and $(\sim\neg)^n \sim\rho$, **CLP**-models are only restricted by IS4. IS4 is, for formulas of that form, equivalent to the conjunction of

$$((\sim\neg)^n \rho \in \Delta^M \text{ or } (\sim\neg)^{n-1} \rho \notin \Delta^M),$$

$$((\sim\neg)^n \sim\rho \in \Delta^M \text{ or } (\sim\neg)^{n-1} \sim\rho \notin \Delta^M), \text{ and}$$

$$(\sim\rho \in \Delta^M \text{ or } \rho \in \Delta^M)$$

for all $n \in \mathbb{N}$ and all $\rho \in \mathcal{S}$. This means that the image of g is restricted exactly by

$$(1^{(n)} \in S(\rho) \text{ or } 1^{(n-1)} \notin S(\rho)),$$

$$(0^{(n)} \in S(\rho) \text{ or } 0^{(n-1)} \notin S(\rho)), \text{ and}$$
$$(0 \in S(\rho) \text{ or } 1 \in S(\rho))$$

for all $n \in \mathbb{N}$, all $\rho \in \mathcal{S}$, where $S = g(\Delta)$, for every Δ^M and every **CLP**-model M. Consequently, in view of this restriction and the definition of V, the image of g is indeed exactly the set of functions $\mathcal{S} \to V$.

■

Remark that the one-to-one mapping defined in this proof is exactly the correspondence relation defined in Definition 27.

The latter two theorems together imply the following corollary.

Corollary 1 $\Gamma \vDash_{\mathbf{CLP}} A$ *iff, for all* **CLP***-set assignments* S, $1 \in \mathbf{v}_S(A)$ *whenever, for all* $B \in \Gamma$, $1 \in \mathbf{v}_S(B)$.

This entails that one can also use the **CLP**-set assignments to define semantic **CLP**-consequence. In other words: the deterministic set theoretic semantics we have implicitly defined in this subsection is a full blown alternative to the indeterministic semantics of the previous subsection.

5.3 Integer numbers as truth values

The set theoretic semantics turns out to be a straightforward generalization of the **LP**-set theoretic semantics. It clearly reveals the links between **LP**-truth values and **CLP**-truth values. Nevertheless, for calculations of truth values it is much more practical to translate the set theoretic values to integer numbers extended with $-\infty$ and ∞. Let $\mathbb{Z}_\infty$ abbreviate $\mathbb{Z} \cup \{-\infty, \infty\}$, where $\mathbb{Z}$ is the usual set of integer numbers. This makes it very easy to calculate the values of complex formulas from the values of primitive formulas (the **CLP**-assignment) and to check validity of formulas.

Definition 28 *A* **CLP***-integer assignment is a function* $I : \mathcal{S} \to \mathbb{Z}_\infty$.

Definition 29 *Given a* **CLP***-integer assignment* I*, the* **CLP***-integer valuation function* v_I *for* I *is the function* $v_I : \mathcal{W} \to \mathbb{Z}_\infty$ *where*

NS1 $v_I(\rho) = I(\rho)$
NS2 $v_I(\sim A) = -v_I(A)$
NS3 $v_I(\neg A) = -v_I(A) - 1$
NS4 $v_I(A \vee B) = \max(v_I(A), v_I(B))$
NS5 $v_I(A \wedge B) = \min(v_I(A), v_I(B))$

Definition 30 *A* **CLP***-integer assignment* I *is said to* correspond to *a* **CLP***-set assignement* S *iff*

CI1 $I(A) = 0$ *iff* $S(A) = t \cup f$,
CI2 *Where* $n \geq 1$*:* $I(A) = n$ *iff* $0^{(n-1)} \notin S(A)$ *and* $0^{(n)} \in S(A)$,
CI3 *Where* $n \geq 1$*:* $I(A) = -n$ *iff* $1^{(n-1)} \notin S(A)$ *and* $1^{(n)} \in S(A)$,
CI4 $I(A) = \infty$ *iff, for all* $n \in \mathbb{N}$, $0^{(n)} \notin S(A)$, *and*
CI5 $I(A) = -\infty$ *iff, for all* $n \in \mathbb{N}$, $1^{(n)} \notin S(A)$.

There is a shortcut to this correspondance. Let $\sharp(a)$ denote the number of objects in a. The reader can check the following theorem by translating the set theoretic semantic clauses into integer number clauses and by elementary calculations.

Theorem 7 *A* **CLP***-integer assignment* I *corresponds to a* **CLP***-set assignement* S *iff, for all* $A \in \mathcal{W}_{\mathbf{CLP}}$,

$$v_I(A) = \sharp(f - v_S(A)) - \sharp(t - v_S(A)).$$

Definition 31 *A* **CLP***-integer assignment* I *corresponds to a* **CLP***-model* M *iff* I *corresponds to the* **CLP***-set assignment* S *that corresponds to* M*.*

Corollary 2 *If a* **CLP***-integer assignment* I *corresponds to a* **CLP***-model* M*, then* $v_I(A) \geq 0$ *iff* $M \models A$

Corollary 3 $\Gamma \vDash_{\mathbf{CLP}} A$ *iff, for all* **CLP***-integer assignments* I*,* $v_I(A) \geq 0$ *whenever, for all* $B \in \Gamma$*,* $v_I(B) \geq 0$*.*

This entails that one can also use the **CLP**-integer assignments to define semantic **CLP**-consequence. In other words: the deterministic integer semantics we have implicitly defined in this subsection is a full blown alternative to the set theoretic semantics of the previous subsection. In other words:

Corollary 4 **CLP** *can semantically be defined as the logic with language* $\mathcal{L}_{\mathbf{CLP}}$ *and the following truth functional semantics for this language:*

$$\langle \mathbb{Z}_\infty, \{a \in \mathbb{Z}_\infty \mid a \geq 0\}, \{f_\vee, f_\wedge, f_\neg, f_\sim\}\rangle,$$

where $f_\vee = \max$*,* $f_\wedge = \min$*,* $f_\sim(a) = -a$ *and* $f_\neg(a) = -a - 1$ *for all* $a \in \mathbb{Z}_\infty$*.*

6 Proof theory

There is no finite axiomatization for **CLP** in the language $\mathcal{W}_{\mathbf{CLP}}$ itself. The reason for this is easy to see. We have $(\sim\neg)^n\sim\sim A \vDash_{\mathbf{CLP}} (\sim\neg)^n A$, for every natural number n. However, the expression $\neg\sim A \vDash_{\mathbf{CLP}} \neg\sim B$ does not follow from $A \vDash_{\mathbf{CLP}} B$ (nor from any other less complex relation between A and B). To show this, consider $A = p \wedge \sim p$ and $B = r$. The reader can check that $\neg\sim(p \wedge \sim p) \vDash_{\mathbf{CLP}} \neg\sim r$ while $p \wedge \sim p \nvDash_{\mathbf{CLP}} r$. So if one would want an axiomatization in the language itself, one needs the axiom schema $\neg(\sim\neg)^n\sim\sim A \vee (\sim\neg)^n A$ for every natural number n.

An elegant way to avoid this is extending the language with a symbol $\bowtie$, resulting in the set of formulas $\mathcal{W}^{\bowtie} = \mathcal{W}_{\mathbf{CLP}} \cup \{A \bowtie B \mid A, B \in \mathcal{W}_{\mathbf{CLP}}\}$. The added symbol $\bowtie$ is merely a proof theoretical tool. We are still only concerned with a logic for the language $\mathcal{W}_{\mathbf{CLP}}$. Consequently, in premises and conclusions of **CLP**-proofs we only accept $\mathcal{W}_{\mathbf{CLP}}$-formulas. Formulas of the extended language can occur on all lines of proofs that are not either premises or the final conclusion.

AE1 $A \bowtie A$
AE2 $\neg(A \vee B) \bowtie (\neg A \wedge \neg B)$
AE3 $\neg(A \wedge B) \bowtie (\neg A \vee \neg B)$
AE4 $\sim(A \vee B) \bowtie (\sim A \wedge \sim B)$
AE5 $\sim(A \wedge B) \bowtie (\sim A \vee \sim B)$
AE6 $A \bowtie \neg\neg A$
AE7 $A \bowtie \sim\sim A$

RE8 from $A \bowtie B$ derive $\neg A \bowtie \neg B$
RE9 from $A \bowtie B$ derive $\sim A \bowtie \sim B$
RE10 from $A \bowtie B$ and $C \bowtie D$, derive $(A \vee C) \bowtie (B \vee D)$
RE11 from $A \bowtie B$ and $C \bowtie D$, derive $(A \wedge C) \bowtie (B \wedge D)$

RE12 from $A \bowtie B$ and $B \bowtie C$, derive $A \bowtie C$
RE13 from $A \bowtie B$, derive $B \bowtie A$

D1 $A \supset B =_{df} \neg A \vee B$

AL14 $A \supset (B \supset A)$
AL15 $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
AL16 $(A \supset B) \supset (\neg B \vee \neg A)$
AL17 $\sim\! A \vee A$

RL18 From $A \bowtie B$, derive $A \supset B$
RL19 From A and $A \supset B$, derive B

D2 $\Box^z A =_{df} \begin{cases} (\neg\sim)^z A & \text{if } z \in \mathbb{N} \\ (\sim\neg)^z A & \text{if } -z \in \mathbb{N} \end{cases}$

Definition 32 *Let an instance of a rule or axiom be the $\mathcal{W}^{\bowtie}$-formula that results from universally substituting every metavariable in the axiom schema with a formula in $\mathcal{W}_{\mathbf{CLP}}$. A* **CLP***-proof from $\Gamma \subseteq \mathcal{W}_{\mathbf{CLP}}$ is a finite list of formulas in $\mathcal{W}^{\bowtie}$ such that for each element A of the list: either (i) $A \in \Gamma$, (ii) A is an instance of an axiom schema AE1–10 or AL1–4, or (iii) A is the conclusion of an instance of one of the rules RE or RL if the premises of that instance of the rule occur in the list before to A.*

The reader can easily verify that $A \bowtie B$ occurs in a **CLP** proof iff $A \approx B$.

Definition 33 *Where $\Gamma \cup \{A\} \subseteq \mathcal{W}$, $\Gamma \vdash_{\mathbf{CLP}} A$ iff A occurs in a* **CLP***-proof form Γ.*

As usual, we write $A_1, \ldots, A_n \vdash_{\mathbf{CLP}} A$ to abbreviate $\{A_1, \ldots, A_n\} \vdash_{\mathbf{CLP}} A$ and $\vdash_{\mathbf{CLP}} A$ to abbreviate $\emptyset \vdash_{\mathbf{CLP}} A$.

7 Metatheory

Theorem 8 *If $\Gamma \vdash_{\mathbf{CLP}} A$ then $\Gamma \vDash_{\mathbf{CLP}} A$*

The reader can check that all rules and axioms are valid given the **CLP**-semantics.

Theorem 9 *If $\Gamma \vDash_{\mathbf{CLP}} A$ then $\Gamma \vdash_{\mathbf{CLP}} A$*

Proof. Suppose $\Gamma \nvdash_{\mathbf{CLP}} A$. We prove that $\Gamma \nvDash_{\mathbf{CLP}} A$.

Let Δ be a non-trivial subset of $\mathcal{W}_{\mathbf{CLP}}$ such that (i) $A \notin \Delta$, (ii) $\Gamma \subseteq \Delta$, (iii) if $\Delta \vdash_{\mathbf{CLP}} B$, then $B \in \Delta$ (Δ is deductively closed), and (iv) Δ is maximal (i.e. every proper superset of Δ lacks one (i)–(iii)).

In view of the fact that **CLP** has all axioms of **CL**, one can easily see that

$$\text{for all } B \in \mathcal{W}_{\mathbf{CLP}}, \text{ either } B \in \Delta \text{ or } \neg B \in \Delta, \text{ but not both.} \qquad (2)$$

We prove that we can define a **CLP**-model M by stipulating that $M \models B$ iff $B \in \Delta$ for all $B \in \mathcal{W}_{\mathbf{CLP}}$

It suffices to prove that IS1–5 hold for the relation $A \in \Delta$ on $A \in \mathcal{W}_{\mathbf{CLP}}$ (so, substituting each $M \models A$ by $A \in \Delta$) .

IS1. Follows by (i) Δ being deductively closed, (ii) RL1, (iii) RL2, and (iv) $A \approx B$ iff there is a **CLP**-proof containing $A \bowtie B$.

IS2. Left to right. By contraposition. Suppose $A \notin \Delta$ and $B \notin \Delta$. Hence, by (2), $\neg A \in \Delta$ and $\neg B \in \Delta$. By the fact that **CLP** has all **CL**-axioms, we have $\neg A, \neg B \vdash_{\mathbf{CLP}} \neg(A \vee B)$. By Δ being deductively closed we obtain $\neg(A \vee B) \in \Delta$, whence, by (2), $A \vee B \notin \Delta$.

IS2. Right to left. Immediate in view of Δ being deductively closed, $A \vdash_{\mathbf{CLP}} A \vee B$ and $B \vdash_{\mathbf{CLP}} A \vee B$.

IS3. Left to right. Immediate in view of Δ being deductively closed, $A \wedge B \vdash_{\mathbf{CLP}} A$ and $A \wedge B \vdash_{\mathbf{CLP}} B$.

IS3. Right to left. By contraposition. Suppose $A \notin \Delta$ or $B \notin \Delta$. Hence, by (2), $\neg A \in \Delta$ or $\neg B \in \Delta$. By the fact that **CLP** has all **CL**-axioms, we have $\neg A \vdash_{\mathbf{CLP}} \neg(A \wedge B)$ and $\neg B \vdash_{\mathbf{CLP}} \neg(A \wedge B)$. By Δ being deductively closed we obtain $\neg(A \wedge B) \in \Delta$, whence, by (2), $A \wedge B \notin \Delta$.

IS4. By $\vdash A \vee {\sim} A$ and Δ being deductively closed, $A \vee {\sim} A \in \Delta$. IS4 follows by IS2.

IS5. By (2).

We now established that M is a **CLP**-model. Moreover, since $M \models \Gamma$ and $M \not\models A$, we have also shown that $\Gamma \nvdash_{\mathbf{CLP}} A$.

■

Definition 34 *An integer number function (a function in $\mathbb{Z}^r \rightarrow \mathbb{Z}$) is called* d-continuous *iff* $|f(a_1, \ldots, a_i, \ldots, a_r) - f(a_1, \ldots, a'_i, \ldots, a_r)| \leq |a_i - a'_i|$ *for all* $i \leq r$ *and all* $a_1 \ldots, a_r, a'_i \in \mathbb{Z}$.

Fact 1 *For every total d-continuous integer number n-ary function f, every $\vec{a}, \vec{b} \in \mathbb{Z}^n$, and every $c \in \mathbb{Z}$ such that $f(\vec{a}) \leq c \leq f(\vec{b})$ or $f(\vec{b}) \leq c \leq f(\vec{a})$, there is a $\vec{d} \in \mathbb{Z}^n$ such that $f(\vec{d}) = c$.*

Proof. We first show that it holds for unary d-continuous functions. The fact holds in that case in light of the fact that there cannot be 'jumps' in a d-continuous function, i.e. if one observes the graph of a function going from $(a, f(a))$ to $(b, f(b))$, where $a < b$, it can only go in a continuous (but not necessarily straight) line, i.e. without ever going from $(e, f(e))$ to $(e+1, f(e+1))$ with $|f(e+1) - f(e)| > 1$. It is easily seen that therefore each number c in between $f(a)$ and $f(b)$ is reached as the point (d, c) on the line that connects $(a, f(a))$ and $(b, f(b))$.

We now show that it also holds for n-ary functions. One starts from $\vec{a} = a_1, \ldots a_n$ and moves stepwise to $\vec{b} = b_1, \ldots b_n$ as follows: let $e_i = f(a_1, \ldots, a_{i-1}, b_i, \ldots b_n)$ for each $i \leq n$.

We prove for all e_i: for each c' in between e_1 and e_i, there is a $\vec{d^i_{c'}}$ such that $f(\vec{d^i_{c'}}) = c'$. Note that if this holds for all $i \leq n$, it holds for $i = n$ and $e_n = f(\vec{b})$. We do this by induction on i.

Basic case. If $i = 1$, we have, for each c', $c' = e_1$ and we can take $\vec{d^1_{e_1}} = \vec{a}$ because $f(\vec{a}) = e_1$.

Induction step. Suppose that for each c' in between e_0 and e_m, there is a $\vec{d^m_{c'}}$ such that $f(\vec{d^m_{c'}}) = c'$. We need to prove this for $m+1$. The difference between $e_m = f(a_1, \ldots, a_m, b_{m+1}, \ldots a_n)$ and $e_{m+1} = f(a_1, \ldots, a_{m+1}, b_{m+2}, \ldots a_n)$ is such that only one argument (the m+1-th) changes and therefore the reasoning for unary functions outlined above applies here. So for every c' in between e_{m+1} and e_m there is an appropriate $\vec{d^{m+1}_{c'}}$. The induction hypothesis states that there is also an appropriate $\vec{d^{m+1}_{c'}}$ for every c' in between e_0 and e_m. Elementary interval calculus allows us to conclude that therefore there is an appropriate $\vec{d^{m+1}_{c'}}$ for every c' in between e_0 and e_{m+1}. ■

Fact 2 *Every composition $f \circ (g_1, \ldots, g_n)$ of total d-continuous n-ary integer number functions $f, g_1, \ldots g_n$ is itself a total d-continuous n-ary integer number function.*

Proof. For every $i, j \leq n$ and every $a_1, b_1, \ldots, a_n, b_n \in \mathbb{Z}$ we have

$$|f(g_1(a_j), \ldots g_n(a_j)) - f(g_1(b_j), \ldots g_n(b_j))| \leq |g_i(a_j) - g_i(b_j)| \leq |a_j - b_j|,$$

because f and all g_i are d-continuous. By transitivity of $\leq$ this entails (with the same quantification for the variables)

$$|f(g_1(a_j), \ldots g_n(a_j)) - f(g_1(b_j), \ldots g_n(b_j))| \leq |a_j - b_j|.$$

Hence $f \circ (g_1, \ldots g_n)$ is d-continuous. ■

Definition 35 *A function in* $\mathbb{Z}_\infty^r \to \mathbb{Z}_\infty$ *is called* d-continuous *iff f restricted to arguments in* $\mathbb{Z}$ *is a total d-continuous integer number function and for all* $1 \leq i \leq r$*:*

$$\lim_{a_i \to \infty} f(a_1, \ldots, a_i, \ldots, a_r) = f(a_1, \ldots, \infty, \ldots, a_r) \text{ and}$$

$$\lim_{a_i \to -\infty} f(a_1, \ldots, a_i, \ldots, a_r) = f(a_1, \ldots, -\infty, \ldots, a_r)$$

Fact 3 *For every total d-continuous* $f \in \mathbb{Z}_\infty^r \to \mathbb{Z}_\infty$*, every* $\vec{a}, \vec{b} \in \mathbb{Z}_\infty^r$*, and every* $c \in \mathbb{Z}_\infty$ *such that* $f(\vec{a}) \leq c \leq f(\vec{b})$ *or* $f(\vec{b}) \leq c \leq f(\vec{a})$*, there is a* $\vec{d} \in \mathbb{Z}_\infty^r$ *such that* $f(\vec{d}) = c$*.*

Fact 4 *Every composition* $f \circ (g_1, \ldots, g_n)$ *of total d-continuous function* $f, g_1, \ldots g_n \in \mathbb{Z}_\infty^r \to \mathbb{Z}_\infty$ *is itself a total d-continuous function in* $\mathbb{Z}_\infty^r \to \mathbb{Z}_\infty$*.*

Lemma 1 *Every* **CLP***-truth function is d-continuous.*

Proof. Observe that every **CLP**-truth function is a composition of the primitive truth functions $f_\vee$, $f_\wedge$, $f_\neg$ and $f_\sim$. Fact 4 entails therefore that every **CLP**-truth function is d-continuous. ■

Theorem 10 **CLP** *is strictly non-explosive with respect to* $\sim$*.*

Proof. We need to prove that, for every $\mathcal{L}$-formula, $\{A, \ddagger A\}$ $\vdash$-explodes only if a proper subset of $\{A, \ddagger A\}$ $\vdash$-explodes.

We prove this by showing that whenever there are **CLP**-models M_1, M_2 such that $M_1 \models A$ and $M_2 \models {\sim}A$ then there is a model M_3 such that $M_3 \models A$ and $M_3 \models {\sim}A$.

A defines a **CLP**-truth function f_A. This function gives the truth value of A as output, given the truth values of the sentential letters that occur in A. Each **PCL**-model M defines a vector $\vec{a}_M$ of truth values, such that this vector contains one value for each sentential letter in A. We have

$$f_A(\vec{a}_{M_1}) \geq 0 \text{ and } f_A(\vec{a}_{M_2}) \leq 0.$$

Because f_A is d-continuous (by Lemma 1) and the fact that 0 is in between $f_A(\vec{a}_{M_1})$ and $f_A(\vec{a}_{M_2})$, Fact 3 teaches us that there is vector $\vec{b}$ such that $f_A(\vec{b}) = 0$. Define M_3 as the **CLP**-model that assigns the values which occur in $\vec{b}$ to the sentential letters in A and arbitrary values to the other sentential letters. Because $f_A(\vec{b}) = 0$ we obtain that $M_3 \models A$ and $M_3 \models {\sim}A$. ■

Theorem 11 $\vdash_{\mathbf{CLP}}$ *is truth functional*

Proof. Both the integer-numbers semantics and the set-theoretic semantics for **CLP** are truth functional semantics. ■

Theorem 12 $\vdash_{\mathbf{CLP}}$ *is a conservative extension of* $\vdash_{\mathbf{LP}}$

Proof. If one removes $\neg$ from the language $\mathcal{L}_{\mathbf{CLP}}$, and one replaces in the integer-numbers-semantics for **CLP** all values > 0 by $\{1\}$, all values < 0 by $\{0\}$ and 0 by $\{0, 1\}$, one obtains exactly the truth functional **LP**-semantics. ■

Theorem 13 $\vdash_{\mathbf{CLP}}$ *is a conservative extension of* $\vdash_{\mathbf{CL}}$

Proof. If one removes $\sim$ from the language $\mathcal{L}_{\mathbf{CLP}}$, and one replaces in the integer-numbers-semantics for **CLP** all values ≥ 0 by 1 and all values < 0 by 0, one obtains exactly the truth functional **CL**-semantics. ∎

Theorem 14 **CLP** *is maximally paraconsistent, i.e. for every compact, monotonic, transitive, deductive w.r.t.* $\supset$*, and structural consequence relation* $\vdash \in \wp(\mathcal{W}) \times \mathcal{W}$*, if* $\vdash$ *is stronger then* $\vdash_{\mathbf{CLP}}$ *(*$\vdash \neq \vdash_{\mathbf{CLP}}$ *and if* $\Gamma \vdash_{\mathbf{CLP}} A$ *then* $\Gamma \vdash A$*), then* $\vdash$ *is explosive w.r.t.* $\sim$ *(*$\{A, {\sim}A\} \vdash B$ *for every* $A, B \in \mathcal{W}$*.*

Proof. Suppose $\vdash$ is compact, monotonic, transitive, structural, deductive w.r.t. $\supset$ and stronger than $\vdash_{\mathbf{CLP}}$. We prove that $\vdash$ is explosive w.r.t. $\sim$.

There is a Γ, A such that $\Gamma \vdash A$ but $\Gamma \nvdash_{\mathbf{CLP}} A$.

As $\vdash$ is compact, there are $A_1, \ldots, A_n \in \Gamma$ such that $A_1, \ldots, A_n \vdash A$. Obviously, $A_1, \ldots, A_n \nvdash_{\mathbf{CLP}} A$.

Because $\vdash$ and $\vdash_{\mathbf{CLP}}$ are both deductive w.r.t. $\supset$ and are both extensions of **CL**, we have $\vdash (A_1 \wedge \ldots, \wedge A_n) \supset A$ and moreover $\nvdash_{\mathbf{CLP}} (A_1 \wedge \ldots, \wedge A_n) \supset A$, otherwise $\Gamma \vdash_{\mathbf{CLP}} A$ would hold. Let C abbreviate $(A_1 \wedge \ldots, \wedge A_n) \supset A$

The following rules are valid in **CLP** and its extensions: (1) Distributivity of $\vee$ and $\wedge$ and of $\wedge$ and $\vee$, (2) De Morgan laws for $\sim$ and $\neg$, and (3) from E derive F, where F is the result of deleting substrings $\neg\neg$ and $\sim\sim$ from E. Given the validity of these rules we can show the following in the same way as the way one shows that each **CL**-formula is **CL**-equivalent to its conjunctive normal form: there is a formula C' such that C is equivalent to C' in both $\vdash_{\mathbf{CLP}}$ and $\vdash$ (i.e. $C \vdash C'$, $C' \vdash C$, $C \vdash_{\mathbf{CLP}} C'$, and $C' \vdash_{\mathbf{CLP}} C$) and such that C' is a conjunction of disjunctions of members of $\mathcal{W}^{pn}$. We obtain that

$$\vdash C' \text{ and } \nvdash_{\mathbf{CLP}} C'.$$

There is a conjunct D of C, which is a disjunction of formulas of the form $\Box^z B$ or $\Box^z {\sim} B$, where $z \in \mathbb{Z}$ and B is a literal, such that

$$\vdash D \text{ and } \nvdash_{\mathbf{CLP}} D.$$

Observe that $\Box^{z_1} B \vdash_{\mathbf{CLP}} \Box^{z_2} B$ whenever $z_1 \geq z_2$. Consequently, $F \vee \Box^{z_1} B \vee \ldots \vee \Box^{z_n} B$ is equivalent to $F \vee \Box^{\min\{z_i | i \leq n\}} B$ both in $\vdash$ and in $\vdash_{\mathbf{CLP}}$. This entails that D is, both in $\vdash_{\mathbf{CLP}}$ and in $\vdash$, equivalent to a disjunction D', such that: for every sentential letter ρ that occurs in D, D' is of the form $E_1 \vee B \vee E_2$ ($E_1 \vee$ and $\vee E_2$ can also denote the empty string), where (i) ρ does not occur in E_1 or in E_2 and (ii) $B = \Box^{z_1^\rho} {\sim}\rho \vee \Box^{z_2^\rho} \rho$ or $B = \Box^{z_3^\rho} \rho$ or $B = \Box^{z_4^\rho} {\sim}\rho$. Given that $\nvdash_{\mathbf{CLP}} B$ and $\vdash_{\mathbf{CLP}} \Box^z A \vee \Box^{-z+1} A$ for every $A \in \mathcal{W}_{\mathbf{CLP}}$, we obtain

$$z_2^\rho > -z_1^\rho + 1 \text{ for every } \rho. \tag{3}$$

Let D'' be the result of substituting in D' every $\Box^{z_2^\rho} \rho$ by $\Box^{-z_1^\rho + 1} \rho$. Given (3), the fact that $\Box^n A \vdash_{\mathbf{CLP}} A$ and that $\vdash$ extends $\vdash_{\mathbf{CLP}}$ entails that $D' \vdash D''$. Now, let D''' be the result of substituting each sentential letter ρ in D'' by

$$\begin{array}{ll} \Box^{z_1^\rho} p & \text{if } \rho \text{ occurs in } \Box^{z_1^\rho} {\sim}\rho \vee \Box^{-z_1^\rho + 1} \rho \\ \Box^{-z_3^\rho} \neg(p \wedge {\sim}p) & \text{if } \rho \text{ occurs in } \Box^{z_3^\rho} \rho \\ {\sim}\Box^{-z_4^\rho} \neg(p \wedge {\sim}p) & \text{if } \rho \text{ occurs in } \Box^{z_4^\rho} {\sim}\rho. \end{array} \tag{4}$$

This is a uniform substitution on D'' and so, by the structurality of $\vdash$, we have $\vdash D'''$. D''' is a disjunction of formulas of one of the next three forms:

$$\Box^z{\sim}\Box^z p \vee \Box^{-z+1}\Box^z p$$

$$\Box^z{\sim}{\sim}\Box^{-z}\neg(p \wedge {\sim}p)$$

$$\Box^z\Box^{-z}\neg(p \wedge {\sim}p)$$

because $\Box^{z_1}\Box^{z_2}F \vdash_{\mathbf{CLP}} \Box^{z_1+z_2}F$ and ${\sim}\Box^z A \vdash_{\mathbf{CLP}} \Box^{-z}{\sim}A$, the latter two imply $\neg(p \wedge {\sim}p)$ and the first form implies ${\sim}p \vee \neg{\sim}p$. By De Morgan laws all 3 forms imply $\neg(p \wedge {\sim}p)$, whence

$$\vdash \neg({\sim}p \wedge p)$$

whence, by the structurality of $\vdash$, for every formula F

$$\vdash \neg({\sim}F \wedge F).$$

Because $\vdash$ extends $\vdash_{\mathbf{CLP}}$, the latter equation entails

$${\sim}F, F \vdash G$$

for every formula F, G ■

We have not only proven that Req1–4 hold for **CLP**, Theorem 14 implies that every consequence relation stronger than $\vdash_{\mathbf{CLP}}$ lacks Req3. Moreover, the fact that all rules and axioms of the proof theory of **CLP** can directly be derived by Req1 and Req4, entails that every consequence relation that respects Req1 and Req4 is at least as strong as **CLP**. Consequently, we now have enough information to conclude that

Corollary 5 $\vdash_{\mathbf{CLP}}$ *is the only consequence relation which adds a classical negation to* **LP** *satisfying requirements Req1–4.*

8 Adaptive logic

For many purposes it is useful to consider not just the formulas verified by all paraconsistent models of a set of premises, but also the ones that are verified (only) by those models that are minimally inconistent. This makes it for example possible to treat the paraconsistent negation $\sim$ as a classical negation, whenever there is no need to treat it paraconsistently. In this picture properties can have overlap with their co-extension but only if this is required by the background theory on pain of triviality.

To formalize such an 'only go paraconsistent when really necessary'-account, one may use a logic in the adaptive logic programme. This is not the place to give a philosophical or technical introduction to adaptive logics, nor is it useful to list their basic properties. But it is nevertheless useful to define an adaptive logic based on **CLP**, because it illuminates an interesting feature of **CLP**.

We define the adaptive logic **ACLP** as a logic within the standard format of Lexicographic Adaptive Logics. This format is a generalization of the Standard Format of adaptive logic defined in [1]. It is developed in [6], where also a proof theory and the metatheory are given for every logic defined within the format. The format extends the Standard Format in the sense that it allows for prioritization within the logic itself (Standard Format adaptive logics have to be combined in order to enable prioritization).

What kind of prioritization is enabled? Well, adaptive logics give us the formulas verified by all least abnormal models of a set of premises. However, Standard Format abnormality of models determines which abnormal formulas are verified by the model and which are not. In many applications, not every abnormal formula is equally problematic. If we are able to consider more and less abnormal formulas, we are also able to make a finer ordering among the models with respect to their abnormality. Hence we are able to make a more narrow selection of least abnormal models and so obtain more adaptive logic consequences.

Definition 36 *Lexicographic Adaptive Logic.*
A Lexicographic Adaptive Logic is a triple consisting of a Lower Limit Logic (the **LLL***), a sequence of sets of abnormalities ($\langle \Omega_i \rangle_{i \in \mathbb{N}}$) and a strategy (here we only consider Minimal Abnormality)*

Definition 37 *The lexicographic ordering $\sqsubseteq$ of* **LLL***-models w.r.t. a sequence of sets of abnormalities $\langle \Omega_i \rangle_{i \in \mathbb{N}}$*
$M_1 \sqsubseteq M_2$ *iff* $\langle Ab_0(M_1), Ab_1(M_1), \ldots \rangle \sqsubseteq \langle Ab_0(M_2), Ab_1(M_2), \ldots \rangle$, *where*

- $Ab_i(M) =_{df} \{A \in \Omega_i \mid M \models A\}$ *for* **LLL***-models M and*
- $\langle \Delta_0^1, \Delta_1^1, \ldots \rangle \sqsubseteq \langle \Delta_0^2, \Delta_1^2, \ldots \rangle$ *iff there is a $k \geq 0$ such that*
  1. *for every $i < k$, $\Delta_i^1 = \Delta_i^2$*
  2. $\Delta_k^1 \subseteq \Delta_k^2$,

Definition 38 *Semantics of Lexicographic ALs* **LAL** *with Minimal Abnormality strategy.*
$\Gamma \models_{\mathbf{LAL}} A$ *iff $M \models A$ for all $\sqsubseteq$-minimal elements of $\{M$ is an* **LLL***-model $\mid M \models \Gamma\}$*

Definition 39 *The lexicographic adaptive logic* **ACLP***.*

- *The Lower Limit Logic is* **CLP***.*
- *Where $\mathbb{L}$ is the set of literals (sentential letters and $\sim$-negations of sentential letters), the sequence of sets of abnormalities of* **AP** *is*

$$\langle \{\Box^i A \wedge \sim\Box^i A \mid A \in \mathcal{S}\} \rangle_{i \in \mathbb{N}}.$$

- *The strategy of the logic is the Minimal Abnormality strategy.*

Of course, one can also construct a Standard Format adaptive logic based on **CLP** which minimizes those **CLP**-models which are plainly minimally inconsistent, without making any distinction between inconsistencies (they are all equally abnormal). In this case the **LLL** is still **CLP**, the strategy still Minimal Abnormality, but the sequence of sets of abnormalities is reduced to one member: $\langle \{\Box^i A \wedge \sim\Box^i A \mid A \in \mathbb{L}, i \in \mathbb{N}\} \rangle$. In case we want to define this logic correctly within the Standard Format, we replace the one-membered sequence of sets of abnormalities by the one member. With this Standard Format adaptive logic the consequence set of a consistent premise set (a set from which no **CLP**-abnormalities follow) is the set of all **CLP**′-consequences of the premise set[2], where **CLP**′ is exactly like **CLP**, except that $\sim$ behaves classically (realized e.g. by adding $(A \wedge \sim A) \supset B$ as an axiom). This is as it should be: selecting the most consistent models of consistent premises comes to selecting exactly the consistent models. In all these models $\sim$ behaves classically.

[2]This can be proven by showing that $\Gamma \vdash_{\mathbf{CLP}'} B$ iff $\Gamma \cup \{\neg(\Box^i A \wedge \sim\Box^i A) \mid A \in \mathcal{S}, i \in \mathbb{N}\} \vdash_{\mathbf{CLP}} B$, using Theorem 3.

So it is possible to construct an adaptive logic without prioritization. Why then does it make sense to define a prioritized adaptive logic for adaptive **CLP**-reasoning?

To explain this, we first have to fix a philosophical interpretation for the degrees of truth and the so obtained degrees of inconsistency. In the property theoretic picture we sketched in Section 2, we saw that inconsistencies can be involved with respect to the question whether a certain object has a certain property, although it is not in the overlap between the property and its co-extension. It may for example be the case that it is in the consistent extension of the property, but also in the co-extension of the consistent extension of the property. So the property holds consistently for the object, but an inconsistency nevertheless pops up with regards to the co-extension of its consistent extension. One may say that a model in which the property behaves this way w.r.t. the object is more consistent (and less abnormal) than exactly the same model but in which the property is explicitly inconsistent w.r.t. the object.

The integer number truth values of **CLP** can be read as formalizing the amount of positive/negative evidence one has for inconsistent statements. By negative evidence for A I mean evidence in favour of $\sim A$ and by positive evidence I mean evidence in favour of A. If the value is 0, the positive evidence is just as valuable as the negative evidence. For a statement with value 4 we have more positive evidence than a statement with value 2 but less a statement with value 6. Statements with value ∞ and $-\infty$ are absolutely reliably consistent, i.e. there is no negative resp. positive evidence for them. Observe that there are no finite premise sets the **CLP**-models of which can only assign infinite (∞ or $-\infty$) values for some sentential letter. Incidentally, this property can be interpreted as corroborating the reasonable thesis that no finite knowlegde base can ever express that its sentences are absolutely consistent.

So, given this interpretation, we have more positive or negative evidence for primitive sentences far from 0 than for sentences closer to 0. A sentence may be considered as more abnormal if the inconsistencies involved take away the strength of the positive or negative evidence for the sentence. By the same normality standards, we may want to consider primitive sentences with a truth value far from 0 as more normal than primitive sentences with a truth value closer to 0. If one accepts this picture in which some abnormalities are more abnormal than others, one needs to go for a prioritized adaptive logic, such as the lexicographic adaptive logic defined in this section.

9 A related result, further research and conclusion

This logic can easily be turned into its dual: a logic which adds a classical negation to the three valued Strong Kleene logic. To do this, take the (integer valued) truth functional semantics of **CLP**, but replace the set of designated values $\{a \in \mathbb{Z}_\infty \mid a \geq 0\}$ by $\{a \in \mathbb{Z}_\infty \mid a > 0\}$.

In the proof theory we used the connective $\bowtie$ as a mere proof theoretic device which does not occur in premises nor in conclusions of **CLP**-proofs. The question immediately pops up what would happen if $\bowtie$ would be added to the actual language of **CLP**. It turns out that the resulting logic is very interesting, especially with respect to a conditional $A \to B =_{df} A \bowtie (A \wedge B)$. This conditional seems to express an interesting relation between sentences A and B: 'B is at least as true as A', or 'if A were true (and if the relation between A and B remained the same), then B would be true'. In that sense, the conditional has a counterfactual gist to it. It is moreover interesting from an algebraic point of view. In a forthcoming paper, **CLP** plus $\rightarrow$ is axiomatized and different semantical characterizations are proposed, together with the most important metatheoretic properties.

To wrap things up: we have defined an infinitely valued paraconsistent logic **CLP** which adds classical negation to **LP**. **CLP** turns out to be the only logic which satisfies certain desiderata dictated by a property theoretic perspective on the combination of classical and paraconsistent negation. We can therefore conclude that this perspective requires a degree theoretic account of paraconsistency.

References

[1] Diderik Batens. A universal logic approach to adaptive logics. *Logica Universalis*, 1:221–242, 2007.

[2] Diderik Batens and Kristof De Clercq. A rich paraconsistent extension of full positive logic. *Logique et Analyse*, 185–188:227–257, 2004. Appeared 2005.

[3] Walter A. Carnielli, Marcelo E. Coniglio, and João Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 14, pages 1–93. Springer, Springer, 2007.

[4] Robert K. Meyer and Richard Routley. Classical relevant logics ii. *Studia Logica*, 33(2):183–194, 1974.

[5] Graham Priest. The logic of paradox. *Journal of Philosophical Logic*, 8:219–241, 1979.

[6] Frederik Van De Putte and Christian Straßer. Extending the standard format of adaptive logics to the prioritized case. *Logique Et Analyse*, 120, 2012.

[7] L.A. Zadeh. Fuzzy sets. *Information Control*, 8:338–353, 1965.