Some Adaptive Contributions to Logics of Formal Inconsistency

Diderik Batens
Centre for Logic and Philosophy of Science
Ghent University, Belgium
Diderik.Batens@UGent.be

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Abstract

Some insights were gained from the study of inconsistency-adaptive logics. The aim of the present paper is to put some of these insight to work for the study of logics of formal inconsistency. The focus of attention are application contexts of the aforementioned logics and their theoretical properties in as far as they are relevant for applications. As the questions discussed are difficult but important, a serious attempt was made to make the paper concise but transparent.

1 Introduction

Logics of Formal Inconsistency, LFIs for short, exploit a typical property of Newton da Costa's $\mathbf{C}_n$-systems ($0 < n < \omega$), viz. that there is a connective that expresses consistency. The connective is explicitly definable within the $\mathbf{C}_n$-systems and the precise definition varies with n, but this matter need not concern us here. In [9], the consistency operator is studied in general, viz. in the context of a wide variety of paraconsistent logics. Many theorems are proved for classes of logics. The study is restricted to the propositional level; extending it to the predicative level involves some technical difficulties, which are studied in Section 5.

Within a paraconsistent context, consistency statements have a dramatic effect. Consider the logical symbols with their **CL** meaning, except that the negation may be paraconsistent. If A and $\neg A$ may true together, then $A \vee B$ and $\neg A$ may be true together while B is false. So Disjunctive Syllogism is invalid because the truth of the premises $A \vee B$ and $\neg A$ may result from the inconsistent behaviour of A rather than from the truth of B and $\neg A$. Put differently, $A \vee B$ and $\neg A$ entails $(A \wedge \neg A) \vee (B \wedge \neg A)$ and if the first disjunct may be true, then B may be false. A consistency statement $\circ A$ typically says that A is consistent, in other words that the first disjunct of $(A \wedge \neg A) \vee (B \wedge \neg A)$ is false and hence that B is true if the disjunction is true. All this may be summarized by the comment that $A \vee B, \neg A \nvdash B$ but $A \vee B, \neg A, \circ A \vdash B$.

Every paraconsistent logic classifies a set of classically valid rules as invalid. Which rules are so classified depends on the logic. That there is a need for considering some inconsistencies as true does obviously not entail that all inconsistencies should be considered as true. So it is sensible to state that some formulas behave consistently. Whenever we state a formula to be consistent, some classical consequences of the premises are added to the paraconsistent consequence set. So we can separate formulas that behave consistently from those that might not so behave, and in doing so we obtain a richer theory. This is what makes LFIs interesting.

Several insights gained during the study of (the metatheory of) adaptive logics are useful for understanding and mastering LFIs. Presenting these insights was the initial aim of the present paper. On the road, a few further insights on LFIs were added. Moreover, it turned out useful to add a comparison, for some application contexts, between an approach in terms of LFIs and an adaptive approach. I shall not argue that one of the approaches is superior, but rather compare some of their properties.

As realistic applications of LFIs seem to be unavoidably predicative, the predicative case is included in the present paper. Some useful comments on predicative LFI are found in Section 5.

Next, I shall restrict my attention to LFI in which there is a (primitive or decidable) *consistency connective* $\circ$ that names a consistency operator. In some LFI, the role of the consistency operator is taken on by a set of formulas $\bigcirc(A)$ each member of which is built from logical symbols and occurrences of A. As we shall see, several consistency operators may occur within a LFI **L**. Where $\circ$ is a consistency connective, $\circ A$ will be called a *consistency statement*; it states that A is consistent (or behaves consistently). Another restriction in focus is that I shall only consider paraconsistent logics that are not paracomplete.

Finally, this paper is not phrased in the terminology from [9]. That terminology is precise as well as useful, but it would turn the present paper into one that is lengthy as well as difficult to read. This is especially so because the terminology would require modifications and extensions for the predicative case. I shall also depart from the terminology where an alternative is easier for present purposes.

It seems useful to state that the metalanguage will be classical—this actually holds for all my papers and it agrees with the Brazilian tradition. Also "true" and "false" will be used as excluding each other. So an inconsistent situation is one in which a formula A is true together with its negation $\neg A$ but such a situation cannot be described by saying that A is both true and false. This convention presupposes that paraconsistent logics can be consistently described, for example in that no formula is verified as well as falsified by a model, not both $M \Vdash A$ and $M \nVdash A$, and in that no formula is a semantic consequence as well as not a semantic consequence of any premise set, not both $\Gamma \vDash A$ and $\Gamma \nvDash A$. The two conventions are essential to interpret the theorems and other metatheoretic statements in the sequel of this paper.

Another useful warning is that this paper does not contain a decent survey of adaptive logics—other papers [5] provide introductions. The essential dynamic proof theory is not even mentioned below. Yet, there will be sufficient information to make the present paper self-contained.

2 Preliminaries

Let $\mathcal{L}$ be a variable for languages, with $\mathcal{F}$ as its set of formulas and $\mathcal{W}$ as its set of closed formulas. The standard predicative language will be called $\mathcal{L}_s$, with $\mathcal{F}_s$ and $\mathcal{W}_s$ as expected. Where $\neg$ will be the standard negation, which will usually be paraconsistent, the symbol $\sim$ will always denote the classical negation—but you will be reminded. Let us impose a minimal requirement on negations.[1]

Definition 1 *A unary connective ξ is a negation in a logic* $\mathbf{L}$ *iff there are* Γ *and* A *such that* $\Gamma, A \nvdash_{\mathbf{L}} \neg A$ *and* $\Gamma, \neg A \nvdash_{\mathbf{L}} A$.

As paracomplete negations are disregarded in this paper, every **L**-model will verify either A or ξA for every closed formula A and for every paraconsistent negation ξ.

The disjunction $\vee$ will be taken to be classical except where otherwise specified. Expressions like $A(x)$ and $A(a)$ will have their usual meaning. The existential closure of A, viz. the result of prefixing A with an existential quantifier over every variable free in A, will be denoted by $\exists A$. The universal closure of A will be denoted by $\forall A$. The **L**-consequence set of Γ is $Cn_{\mathbf{L}}(\Gamma) =_{df} \{A \mid A \in \mathcal{W}; \Gamma \vdash_{\mathbf{L}} A\}$.

An easy way to define what it means that a symbol is classical goes as follows. Every logic **L** defines a two-valued *inferential semantics*, obtained by turning every true inferential statement $A_1, \ldots, A_n \vdash_{\mathbf{L}} B$ $(n \geq 0)$ into the semantic clause "if $v_M(A_1) = \ldots = v_M(A_n) = 1$, then $v_M(B) = 1$".[2] Note that the usual **CL**-semantics contains a specific clause for every logical symbol of $\mathcal{L}_s$. The *classical clause* for disjunction, for example, reads "$v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$".

Definition 2 *A logical symbol ξ is* classical *in a logic* $\mathbf{L}$ *iff extending the inferential semantics of* $\mathbf{L}$ *with the classical clause for ξ does not affect the semantic consequence relation.*

Definition 3 *A logic* $\mathbf{L}$ *is* explosive *with respect to a negation $\neg$ iff it holds that* $\Gamma, A, \neg A \vdash_{\mathbf{L}} B$.[3]

Definition 4 *A negation $\neg$ is paraconsistent in a logic* $\mathbf{L}$ *iff* $\mathbf{L}$ *is not explosive with respect to $\neg$.*

As paracomplete negations are disregarded in this paper, a **L**-model that verifies the classical negation of A falsifies A and hence verifies the paraconsistent negation of A.[4]

[1] Some paraconsistent logicians defend a specific negation as the correct one. Priest [15], for example, seems to assign this role to the negation of **LP**. Other paraconsistent logicians, for example da Costa [10], consider a multiplicity of operators as paraconsistent negations, but sometimes impose certain conditions. Often a more general outlook is taken, as for example by Béziau [8].

[2] The insight was Suszko's [16]. The resulting semantics may be ugly but is obviously adequate.

[3] The reference to Γ may be dropped for Tarski logics (reflexive, transitive, and monotonic logics).

[4] The syntactic justification refers to the complementing character of the non-paracomplete paraconsistent negation. $A, \sim A \vdash_{\mathbf{L}} \neg A$ by explosion and $\neg A, \sim A \vdash_{\mathbf{L}} \neg A$ by reflexivity. Both together entail $\sim A \vdash_{\mathbf{L}} \neg A$ in view of the is complementing character of $\neg$.

Fact 5 *Where* $\neg$ *is a paraconsistent negation in* **L** *and* $\sim$ *is a classical negation in* **L**, $\sim A \vdash_{\mathbf{L}} \neg A$.

Definition 6 *A logic is* paraconsistent *iff one of its negations is paraconsistent.*

In agreement with the announced restriction on $\bigcirc A$, the following definition is less general than the one in [9].

Definition 7 *A logic* **L** *is* gently explosive *with respect to a negation* $\neg$ *iff there is a (primitive or defined) unary connective* $\circ$ *such that* $\circ A, A \nvdash_{\mathbf{L}} B$ *and* $\circ A, \neg A \nvdash_{\mathbf{L}} B$ *hold for some* A *and* B*, whereas* $\Gamma, \circ A, A, \neg A \vdash_{\mathbf{L}} B$ *hold for all* Γ, A*, and* B.[5]

Definition 8 **L** *is a* Logic of Formal Inconsistency *iff there is a negation* $\neg$ *such that* **L** *is not explosive with respect to* $\neg$ *but is gently explosive with respect to* $\neg$.

Let $\circ$ be a *consistency connective* for $\neg$ within a LFI iff it functions as described in Definition 7.

Fact 9 *Where* $\circ$ *is a* consistency connective *for* $\neg$ *in a LFI* **L**, $\nvdash_{\mathbf{L}} \circ A$ *and, for some* B, $\circ A \nvdash_{\mathbf{L}} B$.

Fact 10 *Where* $\circ$ *is a* consistency connective *for* $\neg$ *in a LFI* **L**, *(i)* $A, \neg A \vdash_{\mathbf{L}} \neg\circ A$, *(ii) where* $\sim$ *is classical negation in* **L**, $A, \neg A \vdash_{\mathbf{L}} \sim\circ A$, *(iii) where* $\wedge$ *is a non-glutty conjunction in* **L**, $\circ A \vdash_{\mathbf{L}} \neg(A \wedge \neg A)$, *and (iv) where* $\wedge$ *is a non-glutty conjunction in* **L** *and* $\sim$ *is classical negation in* **L**, $\circ A \vdash_{\mathbf{L}} \sim(A \wedge \neg A)$.

It is worth pointing out that a consistency connective $\circ$ of a LFI **L** need not be a truth-function in **L**. Every **L**-model that verifies $A \wedge \neg A$ falsifies $\circ A$, but some **L**-models may falsify both. Put differently, a **L**-model that verifies A may verify $\neg A$ and may also verify $\circ A$; it cannot verify both $\neg A$ and $\circ A$ but it can falsify both.[6] A consistency statement provides enough information to turn a specific inconsistency into triviality, but fulfils no further requirements.

Some people dislike this aspect of the approach. If $\neg$ is a paraconsistent negation, $\neg A$ is not a truth-function of A. One sensible way to understand the situation is this: if A is true, then whether $\neg A$ is true or false depends on a separate fact—separate in the sense that it is not part of the fact that causes A to be true. Note that this idea agrees with most of the two-valued semantics devised in Brazil or Belgium for paraconsistent logics. It seems more difficult, however, to argue for a notion of consistency such that $\circ A$ is not a truth-function of A and $\neg A$—whence $\neg A$ is not a truth-function of A and $\circ A$ either. If inconsistent situations are possible, then the truth-value of $\neg A$ depends on a fact independent of the one that determines A to be true. But which fact might determine the truth of $\circ A$ in case either A or $\neg A$ is false? Whether you like it or not, this is the way in which the people who devised LFI laid it out.[7] And there

[5]Here too the reference to Γ may be dropped for Tarski logics.

[6]This is the reason why the converses of the inferences mentioned in Fact 10 do not hold for all consistency connectives.

[7]The situation may have been influenced by the somewhat odd behaviour of the (defined) consistency operator $A^{(n)}$ in da Costa's $\mathbf{C}_n$ systems with $n > 1$. Another relevant consideration might have been that the consistent behaviour of a formula A on a premise set Γ, viz. that the logic does not require Γ to entail A as well as $\neg A$, should not cause $\circ A$ to be derivable from Γ. However, this danger is nonexistent even in case $\circ A$ is the suitable truth-function of A and $\neg A$.

is nothing wrong with their decision to study, with respect to a negation $\neg$ and logic **L**, the behaviour of a connective $\circ$ that, however weak or strong, is such that $\circ A, A \nvdash_{\mathbf{L}} B$, $\circ A, \neg A \nvdash_{\mathbf{L}} B$, $\neg A, A \nvdash_{\mathbf{L}} B$, and $\circ A, A, \neg A \vdash_{\mathbf{L}} B$. Needless to say some consistency connectives $\circ$ are such that $\circ A$ holds true just in case one of A and $\neg A$ is false. As I disregard paracomplete negations, this holds just in case $v_M(A) \neq v_M(\neg A)$.

Definition 11 *The connective $\circ$ is a* complementing consistency connective *for $\neg$ within a LFI **L** iff extending the inferential semantics of **L** with the clause "$v_M(\circ A) = 1$ iff $v_M(A) \neq v_M(\neg A)$" does not affect the semantic consequence relation.*

Given that I disregard paracomplete negations in the present paper, the clause may be replaced by "$v_M(\circ A) = 1$ iff ($v_M(A) = 0$ or $v_M(\neg A) = 0$)".

Fact 12 *Where **L** is a paraconsistent logic, it is possible to extend the language of **L** with a connective $\circ$ and to devise a semantics for a LFI $\mathbf{L}'$ by extending the **L**-semantics with a clause for $\circ$ in such a way that $\circ$ is a complementing consistency connective for $\neg$ in $\mathbf{L}'$.*

Whether $\mathbf{L}'$ has a Hilbert axiomatization will depend on the logical symbols of **L**. However, it is possible to syntactically characterize $\mathbf{L}'$ by extending the syntactic characterization of **L** with the rule $A, \neg A, \circ A/B$ and with two meta-rules: (i) if $\Gamma, \circ A \vdash B$ and $\Gamma, A \vdash B$, then $\Gamma \vdash B$, and (ii) if $\Gamma, \circ A \vdash B$ and $\Gamma, \neg A \vdash B$, then $\Gamma \vdash B$.

It is worthwhile to state the semantic equivalents of some of the definitions. The proof of the subsequent lemma is standard. Note that there is no need to refer to Γ in the lemma.[8]

Lemma 13 *A unary connective $\neg$ is a negation in a logic **L** iff there are **L**-models M such that $M \Vdash A$ and $M \nVdash \neg A$ and there are **L**-models M such that $M \Vdash \neg A$ and $M \nVdash A$.*

*A negation $\neg$ is paraconsistent in **L** iff there is a **L**-model M such that $M \Vdash A$ and $M \Vdash \neg A$.*

*Where $\neg$ is paraconsistent in **L**, a unary connective $\circ$ is a consistency connective for $\neg$ in **L** iff there is no non-trivial **L**-model M such that $M \Vdash A$, $M \Vdash \neg A$, and $M \Vdash \circ A$.*

*Where $\neg$ is paraconsistent in **L** and $\circ$ is a consistency connective for $\neg$ in **L**, $\circ$ is a complementing consistency connective for $\neg$ in **L** iff, for every **L**-model M, the following holds: if $M \nVdash A$ or $M \nVdash \neg A$, then $M \Vdash \circ A$.*

The qualification "non-trivial" may obviously be dropped if the semantic clauses rule out the trivial model. Thus the usual clause for the **CL**-negation rules out the trivial model and so does the clause mentioned in Definition 11. An alternative, which leads to an equally adequate semantics, is obtained by appending to that clause "or $v_M(B) = 1$ for all B".[9]

[8] A non-monotonic logic may assign to Γ a selection of models that verify all members of Γ. The lemma contains references to all **L**-models.

[9] Adding or removing the trivial model—the model verifying all closed formulas—to the set of models defined by a semantics may require that the semantic clauses are adjusted. In view of the definition of the semantic consequence relation, it is obvious that such addition or removal does not affect the consequence relation.

Lemma 14 *Where $\neg$ is paraconsistent in* **L**, $\circ$ *is a complementing consistency connective for* $\neg$ *in* **L**, *and* $\wedge$ *is classical or gappy in* **L**, $\sim A =_{df} \neg A \wedge \circ A$ *defines a classical negation.*

Proof. It is easily seen that, if the antecedent is true, no **L**-model verifies A as well as $\neg A \wedge \circ A$ and every **L**-model verifies either A or $\neg A \wedge \circ A$. ■

The following comments are meant to cause some reflection. In the presence of the complementing consistency connective $\circ$ for $\neg$ in **L**, every **L**-model agrees with one of three possibilities with respect to A, as represented in the top row of the following two tables. The LFI **L** is reduced to **CL** by the mapping that agrees with the following schema.

$\mathcal{L}_1$:	$A, \circ A$	$A, \neg A$	$\neg A, \circ A$
$\mathcal{L}_2$:	A		$\neg A$

It seems natural to read the so obtained version of **CL** as: either A is (consistently or inconsistently) true or else $\neg A$ is consistently true, but not both. However, the LFI **L** is also turned into **CL** by the mapping that agrees with the following schema.

$\mathcal{L}_1$:	$A, \circ A$	$A, \neg A$	$\neg A, \circ A$
$\mathcal{L}_2$:	A	$\neg A$	

This mapping gives us: either A is consistently true or else $\neg A$ is true, but not both. So even if the world is inconsistent, there are two ways to describe it in terms of **CL**. The first presupposes that consistent falsehood can be identified, the second that consistent truth can be located. In both cases, the transition to **CL** leads to a lack of expressive power—distinct situations are identified. If one wants to combine the paraconsistent view with the classical one in the same language, the first mapping merely requires that a new negation symbol is introduced, whereas the second mapping requires a consistently true symbol. Although most people will consider the second alternative as conceptually more difficult, both are perfectly symmetric.

An interesting insight in LFIs is that some $\circ$-free formulas establish logical relations between consistency statements. Let $\rightarrow$ be a detachable implication in a logic **L**—for all **L**-models M, if $M \Vdash A \rightarrow B$ and $M \Vdash A$, then $M \Vdash B$—but for which Modus Tollens does not hold.[10] Note that

$$A \rightarrow B, \neg B, \circ B \vdash_{\mathbf{L}} \neg A$$

holds. Indeed, no non-trivial models of the premises verify A. So all models of the premises verify $\neg A$. However,

$$A \rightarrow B, \neg B, \circ B \vdash_{\mathbf{L}} \circ A$$

also holds for complementing consistency connectives $\circ$. Indeed, if a model of the premises would verify A, it would also verify B and hence would be trivial. So the model either falsifies A or is trivial. In both cases it verifies $\circ A$.

[10]There is no reason to handle Modus Ponens and Modus Tollens on a par. The first states a property of the implication. The justification of Modus Tollens requires a reference to negation: if A is true, then B is true (by Modus Ponens); but $\neg B$ is true; so if inconsistencies are not true, then neither is A.

A final preliminary comment concerns weird logics. There are some paraconsistent logics and some LFI that we do not want to consider because they have exceptional properties and, as far as we can see at this moment, no one is interested in them. I shall call them irregular and now explain what I mean by that. Any decent semantics presupposes a complexity ordering $<$ which is such that if $A < B$, then all non-logical symbols that occur in A also occur in B. The valuation function defines the valuation value $v_M(A)$ in terms of the assignment function and in terms of valuation values $v_M(B_1), \ldots, v_M(B_n)$ such that $B_1 < A, \ldots, B_n < A$. Some paraconsistent models M verify both A and $\neg A$ while this is not determined by the truth values of formulas less complex than $\neg A$. There is nothing wrong with this, but if the logic is *regular* there should be a model M' such that M and M' verify the same formulas, except for $\neg A$ and formulas B such that $\neg A < B$.[11] Similarly, some LFI-models M falsify a member of $\{A, \neg A\}$ but also falsify $\circ A$. Again, this is all right, but if the LFI is *regular* there should be a LFI-model M' that verifies exactly the same formulas as M except for $\circ A$ and formulas B such that $\circ A < B$. verb.

3 Derivable Disjunctions Of Contradictions

Let **L** be a LFI in which $\wedge$ and $\vee$ have their classical meaning and let us, for this section, restrict our attention to propositional premise sets. From some such sets, a set of contradictions is derivable. From others only a disjunction of contradictions is derivable. A common example of the latter is $\Gamma_1 = \{\neg p, \neg q, p \vee q, p \vee r, q \vee s, \neg t, u \vee t\}$. According to many paraconsistent logics, no contradiction is derivable from Γ_1, but a disjunction of contradictions is derivable from it, viz. $(p \wedge \neg p) \vee (q \wedge \neg q)$. Such disjunctions may count any (finite) number of disjuncts and infinitely many such disjunctions may be derivable from a premise set.

To save on terminology, I already introduce here some concepts from the metatheory of adaptive logics. As we shall see in Section 6, one of the elements of an adaptive logic is a 'set of abnormalities' Ω. Let us, for the present propositional discussion, identify abnormalities with contradictions, whence $\Omega = \{A \wedge \neg A \mid A \in \mathcal{W}\}$. A disjunction of members of Ω will be called a *Dab-formula* (a disjunction of abnormalities). In the expression $Dab(\Delta)$ (and in similar expressions), Δ is a finite subset of Ω and $Dab(\Delta)$ is the disjunction of the members of Δ. If $\Gamma \vdash_{\mathbf{L}} Dab(\Delta)$, we shall say that $Dab(\Delta)$ is a *Dab-consequence* of Γ.[12] Finally, consider a semantic notion. Where M is a **L**-model, define $Ab(M) = \{A \in \Omega \mid M \Vdash A\}$—the set of abnormalities verified by M, in some papers called the 'abnormal part' of M.

As the disjunction is classical, Addition holds. So if a premise set has a *Dab*-consequence, then it has infinitely many different *Dab*-consequences. In the sequel I shall need minimal *Dab*-consequences of a premise set. Where $Dab(\Delta)$ is a *Dab*-consequence of Γ, $Dab(\Delta)$ is a *minimal Dab-consequence of* Γ iff there is no $\Delta' \subset \Delta$ such that $Dab(\Delta')$ is a *Dab*-consequence of Γ.

If $Dab(\Delta)$ is a minimal *Dab*-consequence of Γ and $A \wedge \neg A \in \Delta$, then $Dab(\Delta -$

[11] This regularity requirement is stronger than the requirement for being a negation in the sense of Lemma 13. If S contains all **CL**-models as well as the trivial model, $\neg$ is a negation but the regularity requirement is not fulfilled.

[12] There is no need to add "with respect to **L**" as *Dab*-consequences of Γ will always be considered for a specific logic.

$\{A \wedge \neg A\}$) is a minimal *Dab*-consequence of $\Gamma \cup \{\circ A\}$. Put differently, extending Γ with consistency statements may result in *Dab*-consequences that contain less disjuncts. The reader who frowns at the "may" should consider that extending Γ_1 with $\circ t$ does not have any effect on the minimal *Dab*-consequences of Γ_1.[13]

The previous paragraph hides an interesting insight. Instead of spelling it out here, I save it for Section 7 where its consequences can be highlighted.

4 A Logical Boundary

A theory may be seen (on the statement view) as a couple $T = \langle \Gamma, \mathbf{L} \rangle$ in which Γ is a set of non-logical axioms and **L** is a logic. Adding consistency statements to T only makes sense if at least one negation of **L** is paraconsistent and provided the consistency statements pertain to such a negation. The decision to add consistency statements to T is extra-logical. It is a decision to extend T with new non-logical theorems by strengthening a certain statement in a specific way. This is clearly extra-logical with respect to **L**. Strengthening A to $A \wedge \circ A$, or $\neg A$ to $\neg A \wedge \circ A$, may be justified by a general consistency presumption, but not if $A \wedge \neg A$ is a disjunct of a minimal *Dab*-consequence of Γ.

Although the decision is extra-logical, there are logical constraints. If the non-logical axioms are $\Gamma_2 = \{p, q, \neg p \vee r, \neg q \vee s, \neg q\}$, then adding $\circ q$ causes triviality, whereas adding $\circ p$ does not. If the non-logical axioms are $\Gamma_1 = \{\neg p, \neg q, p \vee q, p \vee r, q \vee s, \neg t, u \vee t\}$, then neither adding $\circ p$ nor adding $\circ q$ causes triviality, but adding both does. In general there are, for every inconsistent theory T, sets of consistency statements such that adding all members of the set to T causes triviality but adding all but one members does not.[14] Only a very limited number of sensible people will judge that extra-logical reasons may outweigh reasons to avoid triviality. So in constructing LFI-theories, one should mind the *triviality danger.*

As soon as this is agreed upon, a further question surfaces: Given a paraconsistent theory, which are the maximal sets of consistency statements that can be added to it? This may be termed the *maximality question.* Reconsider the premise set Γ_1. It is possible that one has no good (extra-logical) reason to prefer adding $\circ p$ to adding $\circ q$ and *vice versa.* In that case, opting for one of the extensions seems unjustifiable. Of course several alternatives are still open. One may simply not add either consistency statement. One may consider and study the two extended theories without choosing between them, for example in the hope that this may lead to a good reason to prefer one decision over the other. One may also extend the theory with the (classical or gappy) disjunction $\circ p \vee \circ q$. This will cause $r \vee s$ rather than one of its disjuncts to be a theorem.

It is not in general desirable that one tries to obtain a theory to which no

[13] The set of minimal *Dab*-consequences obviously depends on the logic. For some paraconsistent logics, like **CLuN** mentioned in a subsequent section, $(p \wedge \neg p) \vee (q \wedge \neg q)$ is the only *Dab*-consequence of Γ_1. Other paraconsistent logics assign infinitely many *Dab*-consequences to Γ_1. Still, I cannot picture any formal paraconsistent logic for which $\circ t$ has an effect on the minimal *Dab*-consequences of Γ_1. This is weaker than what is claimed in the text, but I shall buy you a beer if you show my imagination lacking at this point and that is stronger than what is said in the text.

[14] To be more precise, this is the case for some (not necessarily all) sets $\{A_1, \ldots, A_n\}$ such that $(A_1 \wedge \neg A_1) \vee \ldots \vee (A_n \wedge \neg A_n)$ is a minimal *Dab*-consequence of the non-logical axioms of the theory.

further consistency statements can be added. After all, a person who devises a theory is free to organize it along his or her preferences. Theories are judged in view of what they state and in view of the way in which they 'react' to other knowledge. That an otherwise good theory does not contain a maximal set of consistency statements is at best a theoretical problem. Nevertheless, it is useful to solve the maximality question, viz. to study the maximal sets of consistency statements that extend a premise set without causing triviality. A set of consistency statements non-trivially extends the considered theory iff it is a subset of one of those maximal sets.

5 Predicative Consistency Statements

The transition from propositional LFI to predicative ones is not completely obvious and some definitions from Section 2 have to be adjusted, for example Definition 7. The matter is important in view of realistic applications.

The main technical difficulty concerns the typical predicative consistency statement. Indeed,

$$\exists(A \wedge \neg A) \vdash_{\mathbf{CL}} B \tag{1}$$

whereas, for many paraconsistent logics $\mathbf{L}$, there are A such that $\exists(A \wedge \neg A) \nvdash_{\mathbf{L}} B \wedge \neg B$ holds for all $B \in \mathcal{W}$—for example, $\exists x(Px \wedge \neg Px) \nvdash_{\mathbf{CLuN}} B \wedge \neg B$.[15] In view of this situation, if a predicative LFI $\mathbf{L}$ has a suitable conjunction,[16] then it needs a formula X that functions as the consistency statement for A, viz.

$$X, \exists(A \wedge \neg A) \vdash_{\mathbf{L}} B \tag{2}$$

or perhaps

$$\exists(X \wedge A \wedge \neg A) \vdash_{\mathbf{L}} B\,. \tag{3}$$

In the presence of the **CL**-negation $\sim$, it obviously holds that

$$\sim\exists(A \wedge \neg A), \exists(A \wedge \neg A) \vdash_{\mathbf{L}} B\,, \tag{4}$$

equivalently

$$\forall(\sim A \vee \sim\neg A), \exists(A \wedge \neg A) \vdash_{\mathbf{L}} B\,, \tag{5}$$

and it also holds that

$$\exists((\sim A \vee \sim\neg A) \wedge A \wedge \neg A) \vdash_{\mathbf{L}} B\,. \tag{6}$$

Both $\sim\exists(A \wedge \neg A)$ and $\forall(\sim A \vee \sim\neg A)$ are complementing consistency operators for $\neg$. Similarly, for the option corresponding to (3), the 'internal' consistency operator $\sim A \vee \sim\neg A$ from (6) is complementing. However, in line with the way in which the consistency operator is introduced at the propositional level, one should also consider consistency operators that are not complementing. So we

[15] The predicative logic **CLuN**, first introduced in [2], is the predicative extension of the propositional **PI** from [1]. The latter extended with a suitable axiom for a consistency operator is the LFI **mbC**—see Definition 42 of [9].

[16] Suitable are a classical conjunction or a gappy one. Glutty conjunctions have to be considered contextually because they allow for models that verify a conjunction and falsify one of the conjuncts. While such models are clearly abnormal with respect to **CL** and many other logics, it depends on further properties whether a consistency operator should handle this. See for example [7] on gluts and gaps of all kinds.

want to allow that the X in (2) is weaker than the first formula in (5) and that the X in (3) is a weaker than the open formula $\sim A \vee \sim\neg A$ in (6).

A little reflection readily reveals the road to be taken. Instead of explicitly defining a consistency operator by the definiens $\sim A \vee \sim\neg A$, we should replace this expression in (5) and (6) by $\circ A$ in which $\circ$ is any propositional consistency connective—remember the comment on Definition 7.

For the option corresponding to (2), this results in universally closed consistency statements,

$$\forall \circ A, \exists(A \wedge \neg A) \vdash_{\mathbf{L}} B\,, \tag{7}$$

supposing that the universal quantifier is classical. However weak the consistency connective, the consistency statement cannot warrant that $\exists(A \wedge \neg A)$ results in triviality unless $\circ A$ holds true independent of the way in which the free variables in A are mapped on the model's domain. Precisely this is warranted by $\forall \circ A$. If no free variables occur in A, there are no quantifiers in (7), whence it reduces to the desired propositional property

$$\circ A, A, \neg A \vdash_{\mathbf{L}} B \tag{8}$$

provided the conjunction is not glutty.

For the option corresponding to (3) the possibly open formula $\circ A$ will do. However, this option does not seem very attractive, neither with respect to LFI properly nor with respect to adaptive LFI. Consider indeed $\Gamma_3 = \{\exists x Px, \forall(Qx \vee \neg Px)\}$ and let $\mathbf{L}$ be a logic in which conjunction, disjunction and the quantifiers behave classically, whence $\Gamma_3 \vdash_{\mathbf{L}} \exists x(Qx \vee (Px \wedge \neg Px))$. In order that $\exists x Qx$ be $\mathbf{L}$-derivable, it is obviously not sufficient to add $\exists x \circ Px$; we need to add at least $\exists x(Px \wedge \circ Px)$. In other words, we have to state that some object has property P and is consistent in this respect.

The situation is easily misleading. Indeed, $\exists x(Px \wedge \circ Px)$ cannot be seen as a consistency statement because it also contains the information $\exists x Px$, which is not part of the meaning of $\exists x \circ Px$. One might think that $\exists x(Px \wedge \circ Px)$ may be seen as a specification of the premise $\exists x Px$, as the addition 'under the quantifier' that the x which has property P has this property in a consistent way. This, however is mistaken. Consider indeed $\Gamma_4 = \{\exists x Px, \forall(Qx \vee \neg Px), \exists x(Px \wedge \neg Px)\}$. If we extend Γ_4 with $\exists x(Px \wedge \circ Px)$, then, just as in the case of Γ_3, $\exists x Qx$ is derivable. So it is quite obvious that $\exists x(Px \wedge \circ Px)$ cannot be seen as a specification of the premise $\exists x Px$ for the simple reason that, in the extended Γ_4, some x have property P in a consistent way whereas other x have it in an inconsistent way. Here is a different way of stating the matter: given that conjunction, disjunction and the quantifiers were presumed to be classical, the set of consequences of Γ_4 coincides with the set of consequences of $\Gamma_5 = \{\forall(Qx \vee \neg Px), \exists x(Px \wedge \neg Px)\}$. So $\exists x(Px \wedge \circ Px)$ is new information, viz. that an object has property P in a consistent way, and is not the specification that an object known to have property P has this property in a consistent way.

What precedes shows that formulas containing a consistency statement that is 'internal' in the sense of (3) introduces new information. It easily follows, however, that these are not consistency statements at all. The correct rendering of (3), implemented as $\exists(\circ A \wedge A \wedge \neg A) \vdash_{\mathbf{L}} B$ is

$$\exists(\circ A \wedge A \wedge \neg A), \exists(A \wedge \neg A) \vdash_{\mathbf{L}} B$$

because $\exists(\circ A \wedge A \wedge \neg A)$ constitutes new information and does not specify the statement $\exists(A \wedge \neg A)$. Note also that $\exists(\circ A \wedge A \wedge \neg A)$ itself is not a consistency operator. This is obvious from

$$\exists(\circ A \wedge A \wedge \neg A) \vdash_{\mathbf{L}} B$$

and Definition 7.

The preceding considerations, so you might think, show that the option corresponding to (3) leads to trouble and that the option corresponding to (2) is the right one. But that is wrong too. Extending Γ_4 (or Γ_5) with $\forall x \circ Px$ results in triviality. This, however, does not mean that the consistency connective does not allow one to extend Γ_4 or Γ_5 in such a way that $\exists x Qx$ is a consequence. Indeed, as we have seen, extending those premise sets with $\exists x(Px \wedge \circ Px)$ does the job.

Even more astonishing might be that the option corresponding to (2) is by no means exhausted by what was said before. Although $\forall \circ A$ seems to be the regular form of the predicative consistency statement, it sometimes also pays to add $\exists \circ A$. Consider indeed the premise set $\Gamma_6 = \{\forall x(Px \supset Qx), \exists x(Qx \wedge \neg Qx)\}$ and the LFI **mbC**. Extending Γ_6 with $\forall x \circ Qx$ results in triviality, but extending it with $\exists x \circ Qx$ does not. Moreover, it extends the **mbC**-consequence set with $\exists x((Qx \wedge \circ Qx) \vee (\neg Qx \wedge \circ Qx))$ and hence also with $\exists x((Qx \wedge \circ Qx) \vee (\neg Px \wedge \circ Px))$. By the same reasoning, if $\Gamma_7 = \{\forall x(Px \supset Qx), \forall x(Rx \supset \neg Qx), \exists x(Qx \wedge \neg Qx)\}$ is extended with $\exists x \circ Qx$, then its **mbC**-consequence set is extended with $\exists x((Qx \wedge \circ Qx) \vee (\neg Qx \wedge \circ Qx))$ and hence also with $\exists x((\neg Rx \wedge \circ Rx) \vee (\neg Px \wedge \circ Px))$.

Allow me to stress that what precedes is by no means a criticism of the LFI programme. I just want to point out that the transition from propositional LFIs to predicative LFIs is not obvious. Indeed, one of the oddities is that no logical form can function in general as *the* predicative consistency statement.

The only further noteworthy comment at the predicative level concerns decidability matters. Many propositional LFIs **L** assign a recursive consequence set $Cn_{\mathbf{L}}(\Gamma)$ to every finite premise set Γ. So it is decidable whether the **L**-consequence set of a finite propositional Γ is trivial. For infinite but decidable premise sets Γ, $Cn_{\mathbf{L}}(\Gamma)$ is only semi-decidable. By moving to the predicative level, $Cn_{\mathbf{L}}(\Gamma)$ is only semi-decidable even for most finite Γ. So it is in general only semi-decidable whether $Cn_{\mathbf{L}}(\Gamma)$ is trivial.

6 A Few Adaptive Basics

Adaptive logics are defined as triples consisting of (i) a lower limit logic **LLL**: a logic that has static proofs,[17] (ii) a set of abnormalities Ω: a set of closed formulas, characterized by a possibly restricted logical form,[18] and (iii) an adaptive strategy (as clarified below).

In this section, I consider the question what adaptive LFI should look like. By an adaptive LFI I mean an inconsistency-adaptive logic that has a LFI as lower limit and that enables one to derive consistency statements that, once derived, play the typical LFI-role.

[17]For present purposes, this may be identified with a compact Tarski logic.

[18]The set Ω may comprise formulas of the form $\exists(A \wedge \neg A)$. If A is any formula, the form is unrestricted; if A is required to be an atomic formula, the form is restricted.

The intuitive idea behind Ω is that it contains the formulas that are presumed to be false unless and until the premises require them to be true. The precise meaning of the latter expression depends on the strategy—only two strategies will be given some attention in this paper—and on the (classical) disjunctions of abnormalities derivable by **LLL** from the premise set. As is the case for many Tarski logics, many LFI may be combined with different strategies and with different sets of abnormalities to obtain a multiplicity of adaptive logics. Where the lower limit logic **LLL** is a LFI, two hints help to avoid inadequate sets Ω. First, abnormalities should be **LLL**-contingent. Next, Ω should be such that the adaptive LFI maximally approaches the **CL**-consequence set without being trivial and without involving choices that are arbitrary from a logical point of view. This second hint requires some explanation.

Adding $\circ p$ rather than $\circ q$ to $\Gamma_1 = \{\neg p, \neg q, p \vee q, p \vee r, q \vee s, \neg t, u \vee t\}$ is an obvious example of a logically arbitrary choice. The choice may obviously be justified by non-logical preferences. So there is nothing wrong when a person applying a LFI chooses to add one consistency statement rather than another. Adaptive logics, however, whether their lower limit logic is a LFI or not, cannot make such choices. They may add consistency statements to premise sets, but only in a logically symmetric way—more detailed insights follow in this section. Adaptive LFIs interpret premise sets as consistently as possible in the following sense: if a consistency statement is not in the adaptive consequence set of Γ, then adding the statement either leads to the trivial consequence set or involves a logically arbitrary choice. In the specific case where Γ is a *normal* premise set, viz. one that has **CL**-models, the adaptive consequence set of Γ should be identical to its **CL**-consequence set. That **CL** is chosen as the upper limit logic[19] is a decision taken by the people who devised LFI. A neat comparison requires that a consistency operator is added to the language of **CL** and that it is explicitly or implicitly defined in such a way that $\circ A$ is a **CL**-theorem—an obvious choice is $\circ A =_{df} \neg(A \wedge \neg A)$.

In many inconsistency-adaptive logics, the set of abnormalities is

$$\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{X}\}$$

in which $\exists$ and $\wedge$ are classical[20] and $\mathcal{X}$ is the set of open and closed formulas of the standard predicative language or a subset of it, often the set of atomic formulas—note that the logical form is then restricted.[21] The need for the existential closure is obvious by the reasoning from Section 5.

In combination with a LFI as lower limit logic, Ω will only lead to an adaptive logic that maximally adds consistency statements if $\circ$ is complementing. A different choice, which will function well for any consistency connective $\circ$, is

$$\Omega = \{\exists\neg\circ A \mid A \in \mathcal{X}\} \tag{9}$$

[19]The *upper limit logic* **ULL** is obtained by extending **LLL** with a rule that causes all abnormalities to entail triviality.

[20]If one of those symbols would be glutty or gappy, the of abnormalities would need to contain members that describe to gluts or gaps in the the existential quantifier and the conjunction in order to handle the situation in an adequate way. See [7] for more information.

[21]The absence of the restriction may cause the adaptive logic to be a flip-flop, which means that the adaptive consequence set reduces to the lower limit consequence set whenever the premise set is abnormal.

in which $\exists$ and $\mathcal{X}$ are as before and, as agreed, the negation is not gappy.[22] The central desirable feature is that the falsehood of the abnormality should entail the truth of the corresponding consistency statement. And indeed, if the quantifiers are classical and the negation is not gappy, then $\forall\circ A$ is true whenever $\exists\neg\circ A$ is false.

For most paraconsistent logics **L** and premise sets Γ, it holds, first, that $Cn_{\mathbf{L}}(\Gamma)$ is inconsistent iff $Cn_{\mathbf{CL}}(\Gamma)$ is so and, next, that classical disjunctions of abnormalities are **L**-derivable from Γ while none of the disjuncts is. The second property is what interest us here. An obvious example is again $\Gamma_1 = \{\neg p, \neg q, p \vee q, p \vee r, q \vee s, \neg t, u \vee t\}$. When the logic is **CLuN**, or nearly any other sensible paraconsistent logic, $(p \wedge \neg p) \vee (q \wedge \neg q)$ is derivable from Γ_1 but neither disjunct is.

A few definitions were already hinted at in Section 3 but are repeated here in a more precise setting. By a *Dab-formula* I shall mean a classical disjunction of abnormalities, *including* the border case where there is only one disjunct. In expressions like $Dab(\Delta)$, Δ is a finite subset of Ω and $Dab(\Delta)$ is the classical disjunction of the members of Δ. $Dab(\Delta)$ is a *Dab*-consequence of Γ iff $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$. $Dab(\Delta)$ is a *minimal Dab-consequence of* Γ iff it is a *Dab*-consequence of Γ and no $\Delta' \subset \Delta$ is such that $Dab(\Delta')$ is a *Dab*-consequence of Γ. A *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ is a set that contains one element out of each member of Σ. A *minimal choice set* of Σ is a choice set of Σ of which no proper subset is a choice set of Σ.

Definition 15 *Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, … are the minimal Dab-consequences of Γ, $\Phi(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \Delta_2, \ldots\}$.*[23]

Definition 16 *Where M is a* **LLL***-model, $Ab(M) = \{A \mid A \in \Omega;\ M \Vdash A\}$.*

Let $\mathbf{AL}^m$ denote the adaptive logic defined by a given **LLL**, an Ω, and the Minimal Abnormality strategy. Let $\mathcal{M}_\Gamma^{\mathbf{LLL}}$ be the set of all **LLL**-models of Γ and let $\mathcal{M}_\Gamma^m$ be the set of all minimally abnormal models of Γ as defined below.

Definition 17 *$M \in \mathcal{M}_\Gamma^m$ (M is a minimally abnormal model of Γ) iff $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ and no $M' \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ is such that $Ab(M') \subset Ab(M)$.*

Definition 18 *$\Gamma \vDash_{\mathbf{AL}^m} A$ iff $M \Vdash A$ for all $M \in \mathcal{M}_\Gamma^m$.*

Note that there are no $\mathbf{AL}^m$-models, but only $\mathbf{AL}^m$-models *of* a set Γ. Theorem 19, proven as Lemma 4 in [5], establishes an important relation between the semantics and the syntactic level; it actually plays an essential role in the proof that the dynamic proof theory of $\mathbf{AL}^m$ is sound and complete with respect to the $\mathbf{AL}^m$-semantics.

Theorem 19 *If Γ has* **LLL***-models, then $\varphi \in \Phi(\Gamma)$ iff $\varphi = Ab(M)$ for some $M \in \mathcal{M}_\Gamma^m$.*

The following corollary holds in view of Theorem 19 and Definition 18 (and holds vacuously in case $\mathcal{M}_\Gamma^{\mathbf{LLL}} = \emptyset$):

[22] This Ω will also be adequate for some combinations of non-classical quantifiers, but that need not concern us in the present paper.

[23] If Γ has no *Dab*-consequences, $\Phi(\Gamma) = \{\emptyset\}$; if Γ has no **LLL**-models, $\Phi(\Gamma) = \{\Omega\}$; $\Phi(\Gamma) \neq \emptyset$ always holds.

Corollary 20 $\Gamma \vDash_{\mathbf{AL}^m} A$ *iff* $M \Vdash A$ *for all* $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *for which* $Ab(M) \in \Phi(\Gamma)$.

While $\mathbf{AL}^m$ follows the Minimal Abnormality strategy, some adaptive logics follow the related Normal Selections strategy—this strategy was first invoked in [3]; the generic name of these adaptive logics is $\mathbf{AL}^n$.

Definition 21 $\Gamma \vDash_{\mathbf{AL}^n} A$ *iff, for some* $M \in \mathcal{M}_\Gamma^m$, $M' \Vdash A$ *for all* $M' \in \mathcal{M}_\Gamma^m$ *for which* $Ab(M') = Ab(M)$.

So $\Gamma \vDash_{\mathbf{AL}^n} A$ iff A is verified by all members of a set of minimally abnormal models of Γ that verify the same set of abnormalities.

If $M' \in \mathcal{M}_\Gamma^m$ and $Ab(M') = Ab(M)$ then M' verifies $\Gamma \cup Ab(M)$; so $M' \in \mathcal{M}_{\Gamma \cup Ab(M)}^{\mathbf{LLL}}$. If M' were not a minimally abnormal model of $\Gamma \cup Ab(M)$, then it would not be a minimally abnormal model of Γ in view of Definition 18. So $M' \in \mathcal{M}_{\Gamma \cup Ab(M)}^m$. In view of Theorem 19, this amounts to:

Corollary 22 $\Gamma \vDash_{\mathbf{AL}^n} A$ *iff, for some* $\varphi \in \Phi(\Gamma)$, $M \Vdash A$ *for all* $M \in \mathcal{M}_{\Gamma \cup \varphi}^m$.

The following theorem, proven as Theorem 5 in [5], is mentioned for future reference.

Theorem 23 *If* $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^m$, *then there is a* $M' \in \mathcal{M}_\Gamma^m$ *such that* $Ab(M') \subset Ab(M)$. *(Strong Reassurance for Minimal Abnormality.)*

The property fundamentally expresses that there are no infinite sequences of models of Γ such that every model is less abnormal than its predecessor. Other names for the property are Smoothness and Stopperedness.

7 Back To LFI

Information from the previous section will be put to use here. I shall only consider adaptive logics that have a LFI as **LLL** and Ω as in (9). Let $\Xi = \{\forall \circ A \mid A \in \mathcal{X}\}$—actually, for reasons that become clear later in this section, one may read $\mathcal{X}$ as $\mathcal{F}$.

Fact 24 *Where* $\Delta \subseteq \Xi$, $Cn_{\mathbf{LLL}}(\Gamma \cup \Delta)$ *is not trivial iff some member of* $\mathcal{M}_\Gamma^{\mathbf{LLL}}$ *verifies all members of* Δ.

As I have to make a decision anyway, I take $\forall \circ A$ to be the official predicative consistency statement and I take the notion of regularity from Section 2 upgraded accordingly.

Fact 25 $\Delta \subseteq \Xi$ *is* maximal *with respect to* Γ *and* **LLL** *iff a member of* $\mathcal{M}_\Gamma^{\mathbf{LLL}}$ *verifies all members of* Δ *and no member of* $\mathcal{M}_\Gamma^{\mathbf{LLL}}$ *verifies all members of* Δ *and moreover some members of* $\Xi - \Delta$.

Note that these facts are independent of the question whether $\circ$ is complementing or not. If M does not verify both A and $\neg A$ but nevertheless falsifies $\circ A$, then, by the regularity of **LLL**, a different **LLL**-model of the premises, say M', verifies the same formulas as M, except for $\circ A$ and formulas B such that $\circ A < B$.

Theorem 26 *$\Delta \subseteq \Xi$ is* maximal *with respect to Γ and* **LLL** *iff there is a $M \in \mathcal{M}^m_\Gamma$ such that (i) if $A \in \Delta$, then $M \Vdash A$ and (ii) if $A \in \Xi - \Delta$, then $M \nVdash A$.*

Proof. $\Rightarrow$ Suppose that $\Delta \subseteq \Xi$ is maximal with respect to Γ and **LLL** but that there is no $M \in \mathcal{M}^m_\Gamma$ such that (i) and (ii) are fulfilled. In view of Fact 25, (1) some $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ verifies all members of Δ and no members of $\Xi - \Delta$, and (2) no $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ verifies all members of Δ as well as some members of $\Xi - \Delta$. (2) entails that (3) no $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ falsifies $\exists\neg\circ A$ whenever $\forall\circ A \in \Delta$ and moreover falsifies $\exists\neg\circ A$ for some $\forall\circ A \in \Xi - \Delta$.

By the regularity of **LLL**, (1) entails that there is a $M' \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ that verifies $\forall\circ A$ iff $\forall\circ A \in \Delta$ and that verifies $\exists\neg\circ A$ iff $\forall\circ A \in \Xi - \Delta$. So (i) and (ii) hold for M' and M' falsifies $\exists\neg\circ A$ iff $\forall\circ A \in \Delta$. But then $M' \in \mathcal{M}^m_\Gamma$ in view of (3).

$\Leftarrow$ Suppose that (1) (i) and (ii) hold for $M \in \mathcal{M}^m_\Gamma$, but that (2) $\Delta \subseteq \Xi$ is not maximal with respect to Γ and **LLL**. (1) entails that (3) $M \in \mathcal{M}^m_\Gamma$ and M falsifies all members of $\Xi-\Delta$ and hence verifies $\exists\neg\circ A$ whenever $\forall\circ A \in \Xi-\Delta$. By Fact 25, (2) entails that some $M' \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ verifies all members of Δ as well as some members of $\Xi-\Delta$. By the regularity of **LLL**, (4) some $M'' \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ verifies all members of Δ as well as some members of $\Xi - \Delta$, falsifies $\exists\neg\circ A$ whenever $\forall\circ A \in \Delta$ and moreover falsifies at least one $\exists\neg\circ A$ for which $\forall\circ A \in \Xi - \Delta$. But then $Ab(M'') \subset Ab(M)$, which is impossible (3). ■

verb.

Corollary 27 *$\Delta \subseteq \Xi$ is* maximal *with respect to Γ and* **LLL** *iff $\{\exists\neg\circ A \mid \forall\circ A \in \Delta\} \in \Phi(\Gamma)$.*

Theorem 26 relates minimally abnormal models to maximal sets of consistency statements. But what about maximal consistent models? Consider a LFI **LLL**, a premise set Γ, and a $\Delta \subseteq \Xi$ that is maximal with respect to Γ and **LLL**. Suppose that $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ verifies all members of Δ but that $M \notin \mathcal{M}^m_\Gamma$. In view of Theorem 26, this can only mean that there is a formula B such that M verifies $\forall\circ B \in \Delta$ but also verifies $\exists\neg\circ B$, whereas some $M' \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ falsifies $\exists\neg\circ A$ as well as $\neg\forall\circ A$ for all $\forall\circ A \in \Delta$, and hence falsifies $\exists\neg\circ B$. In view of Theorem 23, it follows that some $M'' \in \mathcal{M}^m_\Gamma$ falsifies $\exists\neg\circ A$ as well as $\neg\forall\circ A$ for all $\forall\circ A \in \Delta$, and hence falsifies $\exists\neg\circ B$ as well as $\neg\forall\circ B$. But this is impossible. Indeed, as $M'' \in \mathcal{M}^m_\Gamma$, it falsifies $\neg\circ\forall\circ B$ and hence verifies $\circ\forall\circ B$. But $\circ\forall\circ B \notin \Delta$, because M falsifies it and $\Delta \subseteq \Xi$ that is maximal with respect to Γ and **LLL**.

Corollary 28 *$M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ verifies a $\Delta \subseteq \Xi$ that is maximal with respect to Γ and* **LLL** *iff $M \in \mathcal{M}^m_\Gamma$.*

Consider again the premise set $\Gamma_1 = \{\neg p, \neg q, p \vee q, p \vee r, q \vee s, \neg t, u \vee t\}$ and the LFI **mbC**—see footnote 15—in which conjunction and disjunction are classical. It is easily seen that $\Phi(\Gamma_1) = \{\{p\}, \{q\}\}$. Γ_1 may be extended with two different kinds of consistency statements. Every extension with a $\forall\circ A$ for which $\exists\neg\circ A \notin \bigcup\Phi(\Gamma_1)$ may be called a *consistency reclaim.* Such extensions are completely harmless. One may add as many consistency reclaims to Γ_1 as one desires and one may add them all together. Extending Γ_1 with a $\forall\circ A$ for which $\exists\neg\circ A \in \bigcup\Phi(\Gamma_1)$ may be called a *consistency decision.* Consistency decisions cannot always be combined—extending Γ_1 with both $\circ p$ and $\circ q$ results in triviality. Insights from adaptive logics teach us that consistency decisions may be combined iff the corresponding abnormalities belong to the same $\varphi \in \Phi(\Gamma)$. Needless to say, Γ_1 is an utterly simple toy example, but Corollary 27 shows that the matter holds generally.

Fact 29 *If* Γ *is not* **LLL**-*trivial,*[24] $\Delta \subseteq \Xi$, *and* $\exists\neg\circ A \notin \bigcup\Phi(\Gamma)$ *whenever* $\forall\circ A \in \Delta$, *then a* $\Gamma \cup \Delta$ *is not* **LLL**-*trivial. (Consistency reclaims)*

Fact 30 *If* Γ *is not* **LLL**-*trivial,* $\Delta \subseteq \Xi$, *and* $\exists\neg\circ A \in \bigcup\Phi(\Gamma)$ *whenever* $\forall\circ A \in \Delta$, *then a* $\Gamma \cup \Delta$ *is not* **LLL**-*trivial iff there is a* $\varphi \in \Phi(\Gamma)$ *such that* $\exists\neg\circ A \in \varphi$ *whenever* $\forall\circ A \in \Delta$. *(Consistency decisions)*

Theorem 31 $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$ *iff* $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$. *(***AL** *is Dab-conservative with respect to* **LLL**/*Immunity .)*

This theorem, proven as Theorem 10 in [5], shows that, if Γ is extended into Γ' by a (finite or infinite) set of consistency reclaims, then $\Phi(\Gamma') = \Phi(\Gamma)$. It follows that if a non-**LLL**-trivial Γ is extended by any set of consistency reclaims combined with any set of coherent consistence decisions—coherent in that they refer to the same $\varphi \in \Phi(\Gamma)$—then the resulting set is not **LLL**-trivial.

As we shall see in the next section, an inconsistency-adaptive logic restricts itself to extending a premise set with the full set of consistency reclaims—some extreme cases aside, this set is always infinite. Some consistency reclaims have an obvious effect—adding $\circ t$ to Γ_1 makes u derivable. For others, the gain is merely **CL**-theorems and combinations of them with already derivable formulas. Thus if $\circ v$ is added to Γ_1 and the LFI is **mbC**, then the following formulas are derivable among many others: $(v \wedge \neg v) \supset w$ and $(\neg t \wedge v) \vee (v \supset (w \wedge \neg w))$.

This is probably the best place to insert a brief comment on flip-flop adaptive logics. By minimizing abnormalities, adaptive logics interpret premise sets as much as possible in agreement with **ULL**, which is **CL** for most inconsistency-adaptive logics. However, some adaptive logics **AL** display the following odd behaviour. If the premise set Γ has no *Dab*-consequences, the **AL**-consequence set of Γ is identical to its **ULL**-consequence set. This is as it should be and as it is for all adaptive logics. However, if Γ has *Dab*-consequences, then the **AL**-consequence set of Γ is identical to its **LLL**-consequence set. This is obviously not all right. More correctly, it is nearly never what one wants. Usually one wants to isolate the unavoidable abnormalities and to consider at least some abnormalities as false.

Some adaptive logics are flip-flops because the combination of a lower limit logic with a specific consequence set causes, in case Γ has *Dab*-consequences, all models of Γ to be minimally abnormal models of Γ. What this means in terms of consistency statements is that consistency reclaims are impossible while consistency decisions are possible. This looks odd from an adaptive point of view, but not from the viewpoint of LFIs. While the distinction between consistency reclaims and consistency decisions is heuristically and computationally interesting for a person applying a LFI, there is no reason for this person, ar least for most applications, to restrict the addition of consistency statements to consistency reclaims. The target of inconsistency-adaptive logics is the set of formulas verified by every minimally abnormal model of the premises, the target of a person applying a LFI is the set of formulas verified by every model of the premises that falsifies a specific set of abnormalities.

[24]Where a logic **L** is defined over $\mathcal{L}$, a set Γ is **L**-trivial iff $Cn_{\mathbf{L}}(\Gamma) = \mathcal{W}$.

8 Comparing Application Contexts

The last statement of the previous section identifies a central difference between LFIs and inconsistency-adaptive logics. Some more differences are worth being highlighted. One of them is that LFIs are Tarski logics, are deductive logics, and have recursively enumerable consequence sets, whereas inconsistency-adaptive logics are defeasible, are formal characterizations of methods, and have very complex consequence sets—up to Π^1_1 for Minimal Abnormality—see [17, 14]. Another difference is that LFIs typically require ingenuity. The person who applies the logics should select consistency statements in order to extend the initial inconsistent theory T and to strengthen it with applications of **CL**-rules that are not generally valid. Inconsistency-adaptive logics do not depend on human decisions for their applications. They strengthen the initial T with the applications of the aforementioned **CL**-rules that are justified on logical verb.
grounds—by way of comparison: consistency reclaims are so justified while consistence decisions are not. Some consequence sets that are too complex to be reached by human ingenuity can nevertheless be defined in adaptive terms.

The typical intended application context of LFIs is to phrase a theory $T = \langle \Gamma, \mathbf{L} \rangle$, in which **L** is a LFI and T is an inconsistent theory. Whether the theory was devised as inconsistent, or is the result of a failed attempt to formulate a consistent theory does not seem to be important. What is important it that the people devising T want it to be richer than a paraconsistent logic without definable consistency operator can define. They want T to be non-explosive but at the same time want T to contain the result of certain applications of **CL**-rules that are not validated by **L**. This is realized by adding consistency statements.

Given the LFI-theories that have been formulated in the da Costa tradition, I think it is fair to say that those theories basically came into being by starting from a T_0 that does not contain any consistency statements and stepwise extend it to T_1, T_2, etc. by adding a consistency statement at points where the available version T_i is judged to be too weak. Now and then, an extension will have turned out trivial, but then one may retract and remove a previously added consistency statement. This is perfectly all right; it is the way in which theories come into being in general. The only specific feature here is that T_0 is separated from the subsequent additions of consistency statements. Even if the addition and removal of consistency statements goes hand in hand with other additions and removals during the genesis of a theory, the steps that handle consistency statements are so specific and unusual that we may conceptually separate them from the other steps.

Are inconsistency-adaptive logics able to play the same role. Not quite. Given a premise set Γ, an adaptive LFI defines, all by itself, a consequence set of Γ that contains all **LLL**-consequences and moreover contains all consistency statements that are obtained by consistency reclaims. Incidentally, I write "all by itself" because the persons that apply the adaptive LFI do not need to add any consistency statements as premises. Comparing this to LFI for the typical intended application context of LFI, it seems that the adaptive approach does too much as well as not enough. It does too much by adding all consistency statements obtained from consistency reclaims. People applying the original LFI certainly do not do this and even cannot do this because the added set need not even be semi-recursive. Nevertheless, it is hard to see that anyone would object to consistency reclaims. If T can safely be extended with a consistency statement, if is can safely be so extended irrespective of the other consistency statements that are added to T[25] then what possible objection might one have to this extension? As announced, the adaptive approach does not add enough. Indeed, an adaptive LFI does not add any consistency statements obtained from consistency decisions.

One obviously might combine an adaptive LFI with a sequence of consistency decisions, just as in the application of the plain LFI. Another possibility is to apply a LFI and the Ω from (9) with the All Selections strategy, which is an obvious variant to the Normal Selections strategy. The resulting adaptive logic is somewhat unorthodox in that its consequence sets are sets of sets. Describing the approach here would take too much space. Moreover, the approach is somewhat arduous in that the adaptive logic defines all possible theories obtained by extending a premise set Γ with a $\Delta \subseteq \Xi$ that is maximal with respect to Γ and **LLL**.

There is, however, a further possibility. While inconsistency-adaptive logics handle inconsistency, other adaptive logics serve other purposes, for example defeasibly extend a set of statements (or formulas) with further entities, all on a par or in agreement with a preference ranking. One way to implement this is to add consistency statements preceded by a 'plausibility operator' $\Diamond$, which is governed by a modal logic, for example T. This lower limit logic may be combined, for example, with a set of abnormalities that (for the present application) have the form $\Diamond\forall\circ A \wedge \neg\forall\circ A$. Note that $\Diamond\forall\circ A$ entails $\forall\circ A \vee (\Diamond\forall\circ A \wedge \neg\forall\circ A)$. So if all minimally abnormal models of the premises falsify the abnormality $\Diamond\forall\circ A \wedge \neg\forall\circ A$, then $\forall\circ A$ is an adaptive consequence; otherwise it is not. The effect is that some plausible consistency statements function as actual consistency statements, whereas others remain merely plausible. The effect may be enhanced by formulas that contain several diamonds. $\Diamond\forall\circ A$ expresses that $\forall\circ A$ has the highest plausibility, $\Diamond\Diamond\forall\circ B$ that $\forall\circ B$ has the next highest plausibility, and so on. I refer to [4] for a general description of this approach and for a related approach.

The logic that handles the plausibility-ordered consistency statements has some interesting features. The persons applying the logic have to decide to which premise set they apply it—so they have to fix which plausibility is attached to a consistency statement and the result functions as a premise. Once the premises are fixed, however, the adaptive logic defines the consequence set and does not demand ingenuity from the persons applying the logic. Moreover, adding consistency statements with a certain plausibility attached to them is a safe way to proceed. It never leads to triviality. If $\Diamond\forall\circ A$ is a premise, but adding $\forall\circ A$ would result in triviality, then $\forall\circ A$ will not be a member of the adaptive consequence set. As adaptive logics are reflexive, $\Diamond\forall\circ A$ will still be in the consequence set. But that is harmless anyway.

Allow me to repeat that it is not my intention to defend the adaptive approach or to attack the LFI approach. I am merely comparing. The computational problems are in principle similar for both approaches because they depend on the problem that has to be solved. Where a person applying a LFI is unable to choose the right set of consistency statements, a person following the adaptive approach will presumably be unable to figure out which consis-

[25] If consistency decisions do not trivialise the theory, consistency reclaims do not either; if consistency decisions trivialise the theory, consistency reclaims cannot make that situation worse.

tency statements are in the adaptive consequence set. The consequence set is well-defined, which is clearly a very positive feature, but that does not make it available; its computational complexity may be just too high. Next, the fact that the adaptive approach eliminates the triviality danger should not be overestimated. The road through mistaken theories may very well be more efficient than the safe road, provided we reach the destination. Nevertheless, studying several approaches to the same problems may result in deeper insights and in improving one or both of the approaches.

The initial application context of inconsistency-adaptive logics was that a theory was intended as consistent and was given **CL** as its underlying logic, but later turned out to be inconsistent. Inconsistency-adaptive logics were devised with the aim to handle such situations by identifying and localizing the (minimal disjunctions of) inconsistencies present in a theory in the aforesaid situation. The idea was to devise a general means to 'interpret' such a theory as consistently as possible, viz. in such a way that, on the one hand, it is not trivial and, on the other hand, it maximally approaches the original intention (of those who devised the theory). The so obtained non-trivial and 'minimally inconsistent' theory was never meant as the ultimate goal. It is merely an intermediate goal on the road to consistency: once the inconsistencies in the theory are located and isolated, one may try to remove them. Forging of a consistent replacement, however, is not a logical matter. The central decisions require empirical considerations or conceptual considerations, and very often deep conceptual changes. Logics may guide this process, they may locate the interesting questions and their interrelations, they may dismiss proposals as inadequate, etc., but logics are unable to define the process.

Later inconsistency-adaptive logics turned out to have a second interesting application context. Especially in view of the twentieth century changes in the orthodoxy in mathematics, it turned out that inconsistency-adaptive and other adaptive mathematical theories have certain advantages over traditional **CL**-theories and, more generally, semi-recursive theories. Not too much was published until now [6, 18, 19], but even that seems to open interesting perspectives.

It seems to me that LFIs are not the right tools for any of the described application contexts of inconsistency-adaptive logics. For many a premise set or theory the set of consistency statements that need to be added is not only infinite but not even semi-recursive. Moreover, the work on adaptive theories in particular is mainly important from a theoretical point of view because it enables one to obtain sensible knowledge about well-defined but computationally complex sets. It might be hoped, however, that people committed towards LFIs would not be convinced by such arguments and would try to devise an approach for the typical application contexts of inconsistency-adaptive logics. Again, the interplay between competing approaches may lead to deeper insights as well as to new techniques.

9 Some Comments In Conclusion

I hope to have shown that the study of LFIs may benefit from insights gained in adaptive logics. The converse also holds but was not the topic of this paper. The apparently weak or less elegant features of other approaches allow one to discover weak or less elegant features of one's own approach. Given this and given that so much more can be said on the topic of this paper, I shall, by way of conclusion, mention some more results and insights from the adaptive side in the hope that LFI scholars will either locate flaws in my claims or will discover ways to profit from the insights and integrate the results. Before doing so, allow me to refer to the work on adaptive extensions of Jaśkowski's logics [11, 12, 13], which might provide new links between LFI and Jaśkowski's logics.

Several techniques were developed to obtain criteria for final derivability within adaptive logics. Especially techniques in terms of prospective procedures seem to be transparent and promising and they are available in a single paper [20]. It seems likely that these techniques may be rephrased in terms of LFIs in order to cope with the triviality danger and the maximality question. Moreover, this transfer may lead to new insights and improved techniques.

In a similar vein, several arguments were developed in connection with adaptive logics in order to justify acting on the insights offered by a dynamic proof stage, even if one realizes that, given the defeasible character of the logics, these insights may be overruled in the future. Note that such arguments concern a specific form of acting under uncertainty. LFIs being deductive, they do not have to face difficulties related to defeasibility. Nevertheless they face related problems, summarized before as the triviality danger and the maximality question. Even if someone is not interested in sets of consistency statements that are maximal with respect to a given premise set and LFI, consistency reclaims are always selected from infinitely many possibilities and consistency decisions are not only selections, but may moreover cause triviality. At the predicative level, the set of minimal *Dab*-consequences of a decidable premise set Γ is only semi-recursive. What seems to be a consistency reclaim may later turn out to have been a consistency decision; disjuncts of a *Dab*-consequence may turn up again as disjuncts of later derived different *Dab*-consequences; *Dab*-consequences that count more than one disjunct may turn out not to be minimal in view of later derived *Dab*-consequences. So our present estimate of $\Phi(\Gamma)$—the estimate is defined in terms of a stage s of a dynamic proof and is called $\Phi_s(\Gamma)$—may be very different from $\Phi(\Gamma)$. But precisely our estimate of $\Phi(\Gamma)$ is our guide for consistency reclaims and consistency decisions; see also Facts 29 and 30. As far as I can see, it is our only guide.

Consider a paraconsistent logic **L** defined over a language $\mathcal{L}$ and suppose that disjunction and conjunction are classical and that no consistency connective is definable. Let $\mathcal{L}+$ be obtained from $\mathcal{L}$ by adding the symbol $\circ$ and let **L**+ result from extending **L** with the rule $\circ A, A, \neg A/B$. Next define $\mathbf{L}+^m$ by combining **L**+ with the Ω from (9) and the Minimal Abnormality strategy. The adaptive consequence set of a premise set Γ may contain consistency statements—all those that correspond to a consistency reclaim—and actually also disjunctions of consistency statements that are not themselves derivable. Moreover, these formulas will 'have an effect' on the adaptive consequences that belong to the initial language $\mathcal{L}$. Consider again the simplistic $\Gamma_1 = \{\neg p, \neg q, p \vee q, p \vee r, q \vee s, \neg t, u \vee t\}$ and let **L** be **CLuN** (or its propositional fragment **PI**). The $\mathbf{CLuN}+^m$-consequence set will contain $\circ p \vee \circ q$, as well as $\circ A$ for every sentential letter A different from p and from q.[26] Next, 'in line with' the presence of those consistency statements, the consequence will also contain u, $r \vee s$, $t \supset A$ for all formulas A, as well as

[26] Actually for all formulas not in the set $\{p, q\}$, but never mind.

infinitely more formulas from $\mathcal{L}$.

However, there is a little puzzle here. Suppose that we do not extend the language and logic, but proceed in terms of $\mathcal{L}$ and $\mathbf{L}^m$, the latter combining $\mathbf{L}$ with $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{X}\}$.[27] There is a rather easy proof that, if $\Gamma \subseteq \mathcal{W}$, then $Cn_{\mathbf{L}+^m}(\Gamma) \cap \mathcal{W} = Cn_{\mathbf{L}^m}(\Gamma)$. So a consistency connective does not seem to have much use with respect to consistency statements that correspond to consistency reclaims. However, roughly the same holds true for consistency decisions. If these are handled in terms of plausibilities (as explained in the previous section), and the language is not extended with a consistency operator, one may still take formulas of the form $\Diamond\neg(A\wedge\neg A)\wedge(A\wedge\neg A)$ as abnormalities. I did not spell out the proof that this adaptive logic gives us the same formulas in $\mathcal{W}$ as the adaptive logic from the previous section. However, it seems extremely likely that there is such a proof. I phrased this point as a challenge in the hope that LFI scholars will show me wrong.

My final comment concerns the concentration on consistency. Remember the initial application context of inconsistency-adaptive logics: a theory that was intended to be consistent but turns out inconsistent. My claim was that inconsistency-adaptive logics interpret such theory in a way that is maximally consistent and that the resulting adaptive theory may be taken as a starting point for devising a consistent replacement of the initial theory. It has turned out, however, that many theories may serve as the desired starting point. On the one hand, lower limit logics, sets of abnormalities, and strategies may be varied. But there is more. Inconsistencies may be seen as negation gluts. One may also consider negation gaps (both A and $\neg A$ false) and combinations of gluts and gaps. The same may be repeated for all other logical symbols. Moreover, non-logical symbols may be ambiguous. As was argued elsewhere [7], many inconsistent theories come out non-trivial if handled by logics that do not allow for negation gluts but allow for negation gaps, or for other types of gluts or gaps, or for ambiguities, or for combinations of the things mentioned. Next, gaps and gluts and ambiguities may be minimized, all at once or in a certain order. Each of these choices lead, for some inconsistent theories, to a desired starting point. All such starting points are in principle on a par. The idea that the only way out is minimizing inconsistency is just a prejudice.

All those gluts and gaps leave ample room for variants and combinations. Let me here just point to one such combination in connection with inconsistency [7, §4], the other logical symbols being kept classical. Instead of considering a complex inconsistency like $(p \vee q) \wedge \neg(p \vee q)$ as a single abnormality, one might consider three abnormalities instead: $p \wedge \neg(p \vee q)$, $q \wedge \neg(p \vee q)$, and $(p \vee q) \wedge \neg(p \vee q)$. In a sense, the first two offer a possible cause for the occurrence of the contradiction. The first and second abnormalities entail the third, but not vice versa. By minimizing all three abnormalities, one obtains a different selection of (for example **CLuN**-)models than when one minimizes only contradictions. This paragraph only sketches the vague idea in terms of an example, but a systematic approach was published.

What is common to all the cases just discussed is that the considered abnormalities are not matched by consistency statements. Take for example a conjunction glut—that $A \vee B$ is true while A is false or B is false. No consistency statement can eliminate it or make it to cause triviality. Similarly, some

[27] The reasons for the $\mathcal{X}$ is as in (9); it would be tiresome to make this more precise here.

consistency statements reduce all three abnormalities from the previous paragraph to triviality, but if $(p \vee q) \wedge \neg(p \vee q)$ is true anyway and the logic is **mbC**, no consistency statement can rule out for example $p \wedge \neg(p \vee q)$ without causing triviality.[28] So the challenge to LFI scholars is to devise and study operators that eliminate gluts and gaps that are not inconsistencies. The fact that all gluts and gaps *surface* as inconsistencies shows that, in some cases, inconsistency may be merely the symptom rather than the actual disease.

The diversity of approaches within the paraconsistent community has been overwhelming from early on. We should not strive for unification. Actually we will not strive for unification for most of us are pluralists. Yet we may continue to learn from each other—au choc des idées jaillit la lumière. This is why I hope that some comments from this paper may arouse interest of some LFI scholars and perhaps even of a few others.

References

[1] Diderik Batens. Paraconsistent extensional propositional logics. *Logique et Analyse*, 90–91:195–234, 1980.

[2] Diderik Batens. Inconsistency-adaptive logics. In Ewa Orłowska, editor, *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*, pages 445–472. Physica Verlag (Springer), Heidelberg, New York, 1999.

[3] Diderik Batens. Towards the unification of inconsistency handling mechanisms. *Logic and Logical Philosophy*, 8:5–31, 2000. Appeared 2002.

[4] Diderik Batens. Narrowing down suspicion in inconsistent premise sets. In Jacek Malinowski and Andrzej Pietruszczak, editors, *Essays in Logic and Ontology*, volume 91 of *Poznań Studies in the Philosophy of the Sciences and the Humanities*, pages 185–209. Rodopi, Amsterdam/New York, 2006.

[5] Diderik Batens. A universal logic approach to adaptive logics. *Logica Universalis*, 1:221–242, 2007.

[6] Diderik Batens. The consistency of Peano Arithmetic. A defeasible perspective. In Patrick Allo and Bart Van Kerkhove, editors, *Modestly Radical or Radically Modest. Festschrift for Jean Paul van Bendegem on the Occasion of His 60th Birthday*, volume 24 (sometimes 22) of *Tributes*, pages 11–59. College Publications, London, 2014.

[7] Diderik Batens. Spoiled for choice? *Journal of Logic and Computation*, in print. doi:10.1093/logcom/ext019, 1913.

[8] Jean-Yves Béziau. S5 is a paraconsistent logic and so is first-order classical logic. *Logical Investigations*, 9:301–309, 2002.

[9] Walter A. Carnielli, Marcelo E. Coniglio, and João Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 14, pages 1–93. Springer, 2007.

[28] If you frown here, realize that $\neg p$ is not a **mbC**-consequence of $\neg(p \vee q)$.

[10] Newton C.A. da Costa. On the theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic*, 15:497–510, 1974.

[11] Marek Nasieniewski. *Logiki adaptujące sprzeczność (Inconsistency adapting logics)*. Phd thesis, Chair of Logic, N. Copernicus University, Toruń, Poland, 2002.

[12] Marek Nasieniewski. An adaptive logic based on Jaśkowski's logic $\mathbf{D}_2$. *Logique et Analyse*, 185–188:287–304, 2004. Appeared 2005.

[13] Marek Nasieniewski. *Wprowadzenie do logik adaptywnych*. Wydawnictwo Naukowe, Universytetu Mikołaja Kopernika, Toruń, 2008.

[14] Sergei P. Odintsov and Stanislav O. Speranski. Computability issues for adaptive logics in multi-consequence standard format. *Studia Logica*, 101(6):1237–1262, 2013. doi:10.1007/s11225-013-9531-2.

[15] Graham Priest. *In Contradiction. A Study of the Transconsistent*. Oxford University Press, Oxford, 2006. Second expanded edition (first edition 1987).

[16] Roman Suszko. The Fregean axiom and Polish mathematical logic in the 1920s. *Studia Logica*, 36:377–380, 1977.

[17] Peter Verdée. Adaptive logics using the minimal abnormality strategy are Π^1_1-complex. *Synthese*, 167:93–104, 2009.

[18] Peter Verdée. Strong, universal and provably non-trivial set theory by means of adaptive logic. *Logic Journal of the IGPL*, 21:108–125, 2012.

[19] Peter Verdée. Non-monotonic set theory as a pragmatic foundation of mathematics. *Foundations of Science*, 18:655–680, 2013.

[20] Peter Verdée. A proof procedure for adaptive logics. *Logic Journal of the IGPL*, 21:743–766, 2013. doi:10.1093/jigpal/jzs046.