Tutorial on Inconsistency-Adaptive Logics

Diderik Batens*
Centre for Logic and Philosophy of Science
Ghent University, Belgium
Diderik.Batens@UGent.be

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Abstract

This paper contains a concise introduction to a few central features of inconsistency-adaptive logics. The focus is on the aim of the program, on logics that may be useful with respect to applications, and on insights that are central for judging the importance of the research goals and the adequacy of results. Given the nature of adaptive logics, the paper may be read as a peculiar introduction to defeasible reasoning.

1 Introduction

Adaptive logics are formal logics but are not deductive logics. They do not define the meaning of logical symbols and are certainly not in competition for the title 'standard of deduction'—that is: for delineating deductively correct inferences from incorrect inferences and from non-deductive inferences. To the contrary, adaptive logics explicate reasoning processes that are typically not deductive, viz. defeasible reasoning processes.

By a *logic* I shall mean a function that assigns a consequence set to any premise set. So where $\mathcal{L}$ is a language schema, with $\mathcal{F}$ as its set of formulas and $\mathcal{W}$ as its set of closed formulas, a logic is a function $\wp(\mathcal{W}) \rightarrow \wp(\mathcal{W})$. The standard predicative language schema, viz. that of **CL** (classical logic), will be called $\mathcal{L}_s$; $\mathcal{F}_s$ its set of formulas and $\mathcal{W}_s$ its set of closed formulas.

A logic is *formal* iff its consequence relation is defined in terms of logical form. Some people identify this with the Uniform Substitution rule,[1] but that is a mistake because Uniform Substitution defines just one way in which a logic may be formal. Let me quickly spell out a different one. A language or language schema $\mathcal{L}$ will comprise one or more sets of non-logical symbols, for example sentential letters, predicative letters, letters for individual constants, etc. Consider all total functions f that map every such set to itself. Extend f to formulas, $f(A)$ being the result of replacing every non-logical symbol ξ in A

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[1]Uniform Substitution is rule of propositional logic. Predicative classical logic is traditionally axiomatized in terms of a finite set of rules and axiom schemata, rather than axioms. So no substitution rule is then required. Substitution rules in predicate logic have been studied [55] and the outcome is very instructive.

by $f(\xi)$. A logic $\mathbf{L}$ is clearly formal iff the following holds: $A_1, \ldots, A_n \vdash_{\mathbf{L}} B$ iff, for every such f, $f(A_1), \ldots, f(A_n) \vdash_{\mathbf{L}} f(B)$.

Logics may obviously be presented in very different ways. Formal logics are usually presented as sets of rules, possibly combined with the somewhat special rules that are called axioms (and axiom schemata). Apart from many types of 'axiomatizations', logics are standardly characterized by a semantics, which has a rather different function. Deductive logics are typically Tarski logics. This means that they are reflexive ($\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$),[2] transitive (if $\Delta \subseteq Cn_{\mathbf{L}}(\Gamma)$, then $Cn_{\mathbf{L}}(\Delta) \subseteq Cn_{\mathbf{L}}(\Gamma)$), and monotonic ($Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$ for all Γ'). Another interesting property, which is required if a logic has to have static proofs,[3] is compactness (if $A \in Cn_{\mathbf{L}}(\Gamma)$ then there is a finite $\Gamma' \subseteq \Gamma$ such that $A \in Cn_{\mathbf{L}}(\Gamma')$).

This paper follows several conventions that I better spell out from the start. Classical logic, **CL**, will be taken as the standard of deduction. This is a purely pragmatic decision, not a principled one. Next, all metalinguistic statements are meant in their classical sense. More specifically, the metalinguistic negation will always be classical. So where I say that A is not a **L**-consequence of Γ, I rule out that A is a **L**-consequence of Γ. Similarly, I shall use "false" in its classical sense; no statement can be true as well as false in this sense. An inconsistent situation will be one in which both A and $\neg A$ are true, not one in which A is both true and false. There is a rather deep divide between paraconsistent logicians on these matters. There are those who claim that 'the true logic' is paraconsistent and that it should always be used, in particular in its own metalanguage. Some of these even take it that classical negation is not coherent, lacks sense, and the like. Other paraconsistent logicians, with whom I side, have no objections against the classical negation or against its occurrence in the same language as a paraconsistent negation. This is related to the fact that they are pluralists, either in general or with respect to contexts. They might argue, for example, that consistent domains, like most paraconsistent logics themselves, are more adequately described by **CL** than by a paraconsistent logic.

A warning of a different kind is that the materials discussed in the subsequent pages have been studied at the predicative level. That I shall offer mainly propositional toy examples has a pedagogical rationale.

The last general survey paper that I wrote on adaptive logics was [19]. Meanwhile new results were and are being obtained, some of them are still unpublished. This may be as expected, but one aspect needs to be mentioned from the start. Quite a group of people have contributed to adaptive logics and have published in the field, many more than I shall mention below. While I was always eager to retain the unity of the domain, not everyone attached the same value to unification. Such a situation was obviously very useful to prevent that interesting things are left out of the picture—in principle the aim is to integrate directly or under a translation all potentially realistic first order defeasible reasoning forms. As we shall see, this integrating frame is the standard format. Little changes were introduced over the years in an attempt to make it as embracing as possible. While most were improvements or clarifications, there was one development that I now consider as misguided. In the end it resulted in the systematic introduction of a set of new symbols to any language. These new

[2] The **L**-consequence set of Γ is defined as $Cn_{\mathbf{L}}(\Gamma) =_{df} \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$.

[3] Just think about usual proofs. Every formula in the proof is a consequence of the premise set and every proof may be extended into a longer proof by applications of the rules.

symbols had their **CL**-meaning, whence they were called classical. They were added even if they duplicated existing symbols. In the second half of section 11, I shall discuss the idea of adding classical symbols and the reasons for not adding them any more today.

The present paper is by no means a summary of all available results on adaptive logics. It merely provides an introduction to the central highlights. Moreover, this paper is explicitly intended as an introduction to inconsistency-adaptive logics, viz. adaptive logics that handle inconsistency. They concern compatibility, inductive generalization, abduction, prioritized reasoning, the dynamics of discussions, belief revision, abstract argumentation theory, deontic logic, and so on. Most adaptive logics in standard format are not inconsistency-adaptive and have no connection to paraconsistency. Nevertheless, the present paper can also be read as an introduction to adaptive logics in general, with special attention to handling inconsistency and with illustrations from that domain. The reference section is not a bibliography of inconsistency-adaptive logics.

2 The Original Problem

Consider a theory T that was intended as consistent and was given **CL** as its underlying logic: $T = \langle \Gamma, \mathbf{CL} \rangle$, in which Γ is the set of non-logical axioms of T and $Cn_{\mathbf{CL}}(\Gamma)$ is the set of theorems of T, often simply called T. Suppose, however, that T turns out to be inconsistent. There are several well-documented examples of such situation, both in mathematics (Newton's infinitesimal calculus, Cantor's set theory, Frege's set theory, ...) and in the empirical sciences [30, 42, 43, 46, 50, 51, 52, 61]. Actually, it is not difficult to find more examples, especially in creative episodes, for example in scientists' notes.

What scientists do in such situations, is look for a consistent replacement for T. As history teaches, however, they do not look for a consistent replacement from scratch. To the contrary, they reason from T, trying to locate the problems in it. This reasoning obviously cannot proceed in terms of **CL** because **CL** validates Ex Falso Quodlibet: $A, \neg A \vdash_{\mathbf{CL}} B$. So the theory T, viz. its set of theorems $Cn_{\mathbf{CL}}(\Gamma)$ is trivial; it contains each and every sentence of the language. If **CL** is the criterion, all one can do is give up the theory and restart from scratch; but scientists do not do so. The upshot is that one should reason about T in terms of a paraconsistent logic, a logic that allows for non-trivial inconsistent theories. Note that any such logic has a semantics that contains inconsistent models—models that verify inconsistent sets of sentences.

It is useful to make a little excursion at this point because many people underestimate the difficulties arising in inconsistent situations. Time and again, people argue that one should figure out where the inconsistency resides and next modify the theory in such a way that the inconsistency disappears. They apparently think that it is easy to separate the consistent parts of a theory from the inconsistencies. Next, if they are very uninformed, they will think that one may choose one half of the inconsistency (or inconsistencies) and add that to the consistent part. If they are a bit better informed, they will realize that a conceptual shift may very well be required, that the new consistent theory should only contain the important statements from the consistent parts, or even a good approximation of them, and should only contain an approximation of one of the 'halves' of the inconsistencies. What is wrong with this reasoning, even with the sophisticated version, is that it is in general impossible to identify the consistent parts of a predicative theory. There is no general positive test for consistency. Being a consistent set of predicative statements is *not* semi-decidable. The set of consistent subsets of a set of predicative statements is *not* semi-recursive. So there is no systematic method, no Turing machine, that is able to identify an arbitrary consistent set as consistent, independent of the number of steps that one allows the Turing machine (or the person who applies the method) to take. So the reasoning from an inconsistent theory can only be explicated in terms of a paraconsistent logic.

Moving from **CL** to a paraconsistent logic has some drastic consequences. Not only Ex Falso Quodlibet, but many other rules are invalidated. Which rules will be invalidated will depend on the chosen paraconsistent logic. If one chooses a compact Tarski logic in which negation is paraconsistent but in which all other logical symbols have the same meaning as in **CL**, then Disjunctive Syllogism and several other rules are definitely invalidated. Incidentally, the *weakest* compact Tarski logic in which negation is paraconsistent but not paracomplete[4] and in which all other logical symbols have their **CL**-meaning is **CLuN**, to which I return in Section 3.

Let us first have a look at Disjunctive Syllogism (or rather at one of its forms), for example $A \vee B, \neg A/B$. Reasoning about the classical semantics one shows: if $A \vee B$ and $\neg A$ are true, then B is true. Here is one version of the reasoning.

1	$A \vee B$ and $\neg A$ are true	supposition
2	$A \vee B$ is true	from 1
3	$\neg A$ is true	from 1
4	A is true or B is true	from 2
5	A is false	from 3
6	B is true	from 4 and 5

Reasoning about the paraconsistent semantic leads to a very different result because 5 is not derivable from 3. Indeed, both A and $\neg A$ may be true in a paraconsistent model. If that is the case, however, then both $A \vee B$ and $\neg A$ are true even if B is false. So there are models in which $A \vee B$ and $\neg A$ are true and B is false.

Remember that we were considering **CLuN** and paraconsistent extensions of it. We have seen that Disjunctive Syllogism is invalid in **CLuN**. Moreover, as Addition (in particular the variant $A/A \vee B$) is valid, extending **CLuN** with Disjunctive Syllogism would make Ex Falso Quodlibet derivable, whence we would be back at **CL**. Other **CL**-rules are also invalid in **CLuN**, but **CLuN** may be extended with them. Double Negation is among those rules, for example the axiom $\neg\neg A \supset A$ and also its converse. If A is false, $\neg A$ is bound to be true, but $\neg\neg A$ may still be true also. So some paraconsistent models verify $\neg\neg A$ and falsify A. Although $\neg\neg A \supset A$ is invalid in **CLuN**, extending **CLuN** with it results in a paraconsistent logic. This holds for many **CL**-theorems, for example $\neg(\neg A \wedge \neg B) \supset (A \vee B)$. However, extending **CLuN** with several such **CL**-theorems may again result in **CL**.

[4]A logic **L** is paracomplete (with respect to a negation $\neg$) iff some A may false together with its negation $\neg A$; syntactically: iff there are Γ, A and B such that $\Gamma, A \vdash_{\mathbf{L}} B$ and $\Gamma, \neg A \vdash_{\mathbf{L}} B$, but $\Gamma \nvdash_{\mathbf{L}} B$.

3 Paraconsistent Tarski Logics

The basic paraconsistent logic **CLuN** was already mentioned in the previous section. It is obtained in two steps. First, full positive logic $\mathbf{CL}^+$ is retained.[5] So if $A_1, \ldots, A_n \vdash_{\mathbf{CL}^+} B$ then $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$. Next, for the negation, Excluded Middle ($\vdash A \vee \neg A$, which is contextually equivalent to $\vdash (A \supset \neg A) \supset \neg A$) is retained, but Ex Falso Quodlibet is not. In the context of $\mathbf{CL}^+$, Excluded Middle together with Ex Falso Quodlibet define the classical negation.

That **CLuN** contains (all inferences of) $\mathbf{CL}^+$ warrants that, for example, $\neg p \vdash_{\mathbf{CLuN}} q \supset (\neg p \wedge q)$ because $A \vdash_{\mathbf{CL}^+} B \supset (A \wedge B)$. This is because $\mathbf{CL}^+$ theorem schemata hold for all formulas, formulas of the form $\neg A$ included. However, $\mathbf{CL}^+$ does not have any effect *within* such formulas, in other words within the scope of a negation symbol. As a result of this, Replacement of Equivalents is invalid: $\vdash_{\mathbf{CLuN}} p \equiv (p \wedge p)$ and $\vdash_{\mathbf{CLuN}} p \equiv p$ but $\nvdash_{\mathbf{CLuN}} \neg p \equiv \neg(p \wedge p)$. For the same reason, Replacement of Identicals is invalid: $a = b \vdash_{\mathbf{CLuN}} Pa \equiv Pb$ but $a = b \nvdash_{\mathbf{CLuN}} \neg Pa \equiv \neg Pb$. It is easy to extend **CLuN** with Replacement of Identicals.

In the previous section, I referred several times to **CLuN**-models. The reader may wonder what these models precisely look like. For all that was said until now, the **CLuN**-semantics is *indeterministic*. Excluded Middle is retained, $v_M(\neg A) = 1$ whenever $v_M(A) = 0$, but the converse obviously cannot hold because, if it did, Ex Falso Quodlibet would be valid. It is not difficult to restore determinism and the method is interesting because it can be applied rather generally. Two functions play an important role in connection with models. The assignment v is part of the model itself: $M = \langle D, v \rangle$.[6] The assignment fixes the 'meaning' of non-logical symbols. Next, the valuation v_M fixes the 'meaning' of logical symbols. A decent semantics presupposes a complexity ordering $<$ which is such that if $A < B$, then all non-logical symbols that occur in A also occur in B. If the semantics is deterministic, the valuation function defines the valuation value $v_M(A)$ in terms of the assignment function and in terms of valuation values $v_M(B_1), \ldots, v_M(B_n)$ such that $B_1 < A, \ldots, B_n < A$. So every valuation value $v_M(A)$ is a function of assignment values of formulas B such that $B < A$ and of non-logical symbols that occur in those B. Actually, a deterministic semantics is the standard. If two models are identical $M = \langle D, v \rangle = \langle D', v' \rangle = M'$, whence $D = D'$ and $v = v'$, then they better verify the same formulas. If they don't, then we should describe a semantics in terms of model variants rather than models. Nevertheless, indeterministic semantic systems have been around for more than thirty years, never caused any confusion, and were the subject of several interesting systematic studies [3, 4, 5, 6].

The official deterministic semantics for **CLuN** is obtained from the indeterministic one by replacing the clause "if $v_M(A) = 0$, then $v_M(\neg A) = 1$" by

$$v_M(\neg A) = 1 \text{ iff } v_M(A) = 0 \text{ or } v(\neg A) = 1\,.$$

Obviously, for this to work, v needs to assign a value to formulas of the form $\neg A$. Note that $v_M(\neg A)$ is still not a function of $v_M(A)$ in the deterministic **CLuN**-semantics. Determinism does not entail truth-functionality.

[5] $\mathbf{CL}^+$ is like **CL** except that axioms and rules (in the semantics clauses) in which occurs a negation sign are removed.

[6] Names and notation may obviously be different and the model may be more complex.

A useful observation is the following. Precisely because, in the two-valued semantics of paraconsistent logics, $v_M(\neg A)$ is not a function of $v_M(A)$, the truth-value of $\neg A$ depends on information not contained in the truth-value of A. Information of this type must naturally be conveyed by the assignment v. Indeed, a model itself, viz. $M = \langle D, v\rangle$, represents a possible situation (or possible state of the world, etc.), whereas the valuation describes the conventions by which we define logical symbols in order to build complex statements—formulas at the schematic level—that enable us to describe the situation. So all information should obviously come from the model itself—the situation, the world, or however you prefer to call it. Moreover, in order to handle not only negation gluts, viz. inconsistencies, but gluts and gaps with respect to any logical symbol, one better lets the assignment map every formula of the language to the set of truth values $\{0, 1\}$.[7]

Incidentally, the view on models presented in the previous paragraph throws some doubt on claims to the effect that classical negation is not a sensible logical operator, among other things because it would be tonk-like. Unless a different approach to logic and models is elaborated, such claims seem not to refer to the situation or world, but to the way in which we handle language. If that is so, one wonders why a modification to our logical operators (for example banning classical negation) is more legitimate than modifying the way in which we handle language.[8]

As already suggested in the previous section, several **CL**-theorems (as well as the corresponding rules) are lost in **CLuN**. Moreover, some of these are such that if **CLuN** is extended with them, even separately, then Ex Falso Quodlibet is derivable, whence we are back to **CL**, or Ex Falso Quodlibet Falsum ($A, \neg A \vdash \neg B$) is derivable, whence we are back to something almost as explosive as **CL**. Disjunctive Syllogism is such a rule. Other examples of such rules are (full) Contraposition, Modus Tollens, Reductio ad Absurdum, and Replacement of Equivalents. Let me illustrate the matter for Modus Tollens. In view of $A \vdash_{\mathbf{CLuN}} B \supset A$ and reflexivity, $B \supset A, \neg A \in Cn_{\mathbf{CLuN}}(\{A, \neg A\})$. So extending **CLuN** with Modus Tollens results in $A, \neg A \vdash_{\mathbf{CLuN}} \neg B$ in view of transitivity.

As was also suggested in the preceding section, some **CL**-theorems and **CL**-rules are invalid in **CLuN**, but adding them (separately) to **CLuN** results in a richer paraconsistent logic. Among the striking examples are $\neg\neg A/A$; de Morgan properties; $A, \neg A \vdash B$ for non-atomic A; Replacement of Identicals; and so on. Note that some combinations of such **CL**-theorems and **CL**-rules still result in the validity of Ex Falso Quodlibet or of Ex Falso Quodlibet Falsum.

It still seems useful to mention a result from an almost 35 years old publication [7]. There is an infinity of logics between the propositional fragments of **CLuN** and **CL**. These logics form a mesh. Some of them are maximally paraconsistent in that every extension of them is either propositional **CL** or the trivial logic **Tr**, characterized by $\Gamma \vdash_{\mathbf{Tr}} A$, in other words $Cn_{\mathbf{Tr}}(\Gamma) = \mathcal{W}$. Many

[7] Take conjunction as an example. The clause allowing for gluts: $v_M(A \wedge B) = 1$ iff ($v_M(A) = 1$ and $v_M(B) = 1$) or $v(A \wedge B) = 1$; the one allowing for gaps: $v_M(A \wedge B) = 1$ iff ($v_M(A) = 1$ and $v_M(B) = 1$) and $v(A \wedge B) = 1$; the one allowing for both: $v_M(A \wedge B) = v(A \wedge B)$.

[8] I heard the claim that restricting the formation rules of natural language so as to classify "this sentence is false" as non-grammatical is illegitimate because the sentence is 'perfect English'. I also heard the claim that invalidating Disjunctive Syllogism is illegitimate because this reasoning form is 'perfectly sound'.

propositional paraconsistent logics have a place in this mesh—exceptions are extensions of **CLuN** that validate non-**CL**-theorems like $\neg(A \supset \neg A)$.[9] Other paraconsistent logics are fragments of logics in this mesh, for example Priest's **LP**, which has no detachable implication. Other paraconsistent propositional logics are obviously not within the mesh, for example relevant logics, modal paraconsistent logics, logics that display other gluts or gaps, and so on.

An example of a maximal paraconsistent logic is the propositional fragment of a logic which is called **CLuNs** in Ghent because Schütte [58] was the first to describe that propositional fragment. **CLuNs**, fragments of it, and slight variants of it were heavily studied and are known under many names [1, 2, 7, 25, 32, 34, 35, 36, 37, 38, 39, 56, 60]. **CLuNs** is obtained by extending **CLuN** with axiom schemas to 'drive negations inwards' as well as with an axiom schema that restores Replacement of Identicals: $\neg\neg A \equiv A$, $\neg(A \supset B) \equiv (A \wedge \neg B)$, $\neg(A \wedge B) \equiv (\neg A \vee \neg B)$, $\neg(A \vee B) \equiv (\neg A \wedge \neg B)$, $\neg(A \equiv B) \equiv ((A \vee B) \wedge (\neg A \vee \neg B))$, $\neg(\forall\alpha)A \equiv (\exists\alpha)\neg A$, $\neg(\exists\alpha)A \equiv (\forall\alpha)\neg A$, and $\alpha = \beta \supset (A \supset B)$, in which B is obtained by replacing in A an occurrence of α by β. **CLuNs** has a nice two-valued semantics and several other semantic systems, among which a three-valued one, are adequate for it. I refer the reader elsewhere [25] for this. Priest's **LP** is obtained from **CLuNs** by removing the axioms and semantic clauses for implication and equivalence and defining the symbols in a non-detachable way: $A \supset B =_{df} \neg A \vee B$ and $A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$.

Several paraconsistent logics having been described, we may now return to the original problem and phrase things in a more precise way.

4 The Original Problem Revisited

We considered a $T = \langle \Gamma, \mathbf{CL} \rangle$ that turned out inconsistent. T itself is obviously too strong, viz. trivial, to offer a sensible view on 'what T was intended to be'. But we know a way to avoid triviality: replace **CL** by a paraconsistent logic. So let us pick **CLuN**, or any other paraconsistent Tarski logic. For nearly all sensible Γ, $T' = \langle \Gamma, \mathbf{CLuN} \rangle$ offers a non-trivial interpretation of 'what T was intended to be'. A little reflection reveals, however that this T' is too weak.

A toy example will be helpful. Specify the Γ in T to be $\Gamma_1 = \{p, q, \neg p \vee r, \neg q \vee s, \neg q\}$. Note that $\Gamma \nvdash_{\mathbf{CLuN}} s$ and $\Gamma \nvdash_{\mathbf{CLuN}} r$. However, there seems to be a clear difference between p and q. Intuitively speaking, Γ_1 obviously requires that q behaves inconsistently but does not require that p behaves inconsistently. However, and this is interesting, **CLuN** leads to exactly the same insight. Indeed, $\Gamma_1 \vdash_{\mathbf{CLuN}} q \wedge \neg q$ whereas $\Gamma_1 \vdash_{\mathbf{CLuN}} p$ but $\Gamma_1 \nvdash_{\mathbf{CLuN}} \neg p$. Let us see whether something interesting can be done with the help of this apparently interesting distinction.

As p and $\neg p \vee r$ are T-theorems, r was intended as a T-theorem. Similarly, as q and $\neg q \vee s$ are T-theorems, s was *intended* as a T-theorem. However, s better be not a T-theorem. Indeed, intuitively and by **CLuN**, q and $\neg q \vee A$ are T-theorems for every A. So if, relying q, we obtain the conclusion s from $\neg q \vee s$, then, by exactly the same move we obtain the conclusion A from $\neg q \vee A$. The justification for deriving s justifies deriving every formula A because $\neg q \vee A$ is just as much a **CLuN** consequence of Γ_1 as is $\neg q \vee s$. In other words, this kind of reasoning leads to triviality. The matter is very different in the case of

[9] This formula is **CL**-equivalent to A but not **CLuN**-equivalent to it.

r. Indeed, r can be a T-theorem. Relying on p one obtains the conclusion r from$\neg p \vee r$ and there is no other formula of the form $\neg p \vee A$ to which the same move might sensibly be applied.[10] A different way to phrase the matter is by saying that applications of Disjunctive Syllogism of which q is the minor result in triviality, but that applications of Disjunctive Syllogism of which p is the minor do not result in triviality. The reason for the difference is clear: Γ_1 requires q to behave inconsistently, but does not require p to behave inconsistently.

One might take that the preceding paragraphs led to the following insight: what was intended as a T-theorem and can be retained as a T-theorem, should be retained as a T-theorem. Alas, this will not do. Consider another toy example for the non-logical axioms: $\Gamma_2 = \{\neg p, \neg q, p \vee r, q \vee s, \neg t, u \vee t, p \vee q\}$. Clearly r was intended as a theorem and indeed it can be retained. However, then q, which was also intended as a theorem, should by the same reasoning also be retained. Moreover, if q is retained, then so is $q \vee A$ for every formula A. So, although s was also intended as a theorem, it cannot be retained because, relying on $\neg q$ we cannot only obtain s from $q \vee s$, but we can obtain every formula A from $q \vee A$.

That may seem all right at first sight, but it is not. If you take a closer look at Γ_2, you will see that p and q are strictly on a par. The reasoning in the preceding paragraph relied on the consistent behaviour of p to derive s and q and hence to find out that q behaves inconsistently. However, one may just as well start off by relying on the consistent behaviour of q to obtain s as well as p and hence to find out that p behaves inconsistently. So the insight mentioned at the outset of the previous paragraph should be corrected. Here is the correct version: what was intended as a T-theorem and can be retained as a T-theorem *in view of a systematic and formal account,* should be retained as a T-theorem. A little reflection on the part of the reader will readily reveal that neither r nor s can be retained as consequences of Γ_2, but that u can be so retained.

What is the upshot? We want to replace T by a consistent theory. Obviously, there is no point in devising a consistent replacement for a trivial theory. Moreover, T', in which **CL** is replaced by **CLuN** will be non-trivial for most Γ, but is clearly too weak. However, for most Γ one may strengthen T' by adding certain instances of applications of **CL**-rules that are **CLuN**-invalid. These instances of applications may be added to T' in view of the fact that a systematic distinction can be made between formulas that behave consistently with respect to Γ and others that do not. In this way one obtains T "in its full richness, except for the pernicious consequences of its inconsistency"; one obtains an 'interpretation' of T that is as consistent as possible, and also as much as possible in agreement with the intention behind T.

Of course the matter should still be made precise. This will be done in the next section, but a central clue is the following:

$$\neg A, A \vee B \nvdash_{\mathbf{CLuN}} B \text{ but } \neg A, A \vee B \vdash_{\mathbf{CLuN}} B \vee (A \wedge \neg A)\,.$$

In view of this, one may consider formulas of the form $A \wedge \neg A$ as false, unless and until proven otherwise—unless it turns out that the premises do not permit to consider them as false on systematic grounds. In the first toy example Γ_1

[10] As q is **CLuN**-derivable from the premises, so is $\neg p \vee q$. However, relying on p to repeat the move described in the text delivers a formula that was already derivable, viz. q. The same story may be retold for every **CLuN**-consequence of Γ_1 and each time the move will be harmless because nothing new will come out of it.

requires that $q \wedge \neg q$ is true, but not that $p \wedge \neg p$ is true: $\Gamma_1 \vdash_{\mathbf{CLuN}} q \wedge \neg q$ whereas $\Gamma_1 \nvdash_{\mathbf{CLuN}} p \wedge \neg p$. Relying on the presumed falsehood of $p \wedge \neg p$, we may take r to be true. The second toy example shows that the matter is slightly more complicated: $\Gamma_2 \vdash_{\mathbf{CLuN}} (p \wedge \neg p) \vee (p \wedge \neg p)$ whereas neither $\Gamma_2 \vdash_{\mathbf{CLuN}} p \wedge \neg p$ nor $\Gamma_2 \vdash_{\mathbf{CLuN}} p \wedge \neg p$. We shall deal with this in the next section.

In order to avoid circularity, it is essential to distinguish between **CLuN**-consequences of a premise set and defeasible consequences derived in view of **CLuN**-consequences. Which formulas behave consistently with respect to a given premise set, will typically be decided in terms of the **CLuN**-consequences of Γ.

5 Dynamic Proofs

Dynamic proofs are a typical feature of adaptive logics. The logics were 'discovered' in terms of the proofs. In the first paper written on the topic [9], not the first published, only a rather clumsy semantics was available. The semantics for what became later known as the Minimal Abnormality strategy was described in an article [8] that was written six years later but published earlier. A decent semantics for the Reliability strategy appears only in [11]. Dynamic proofs are also typical for adaptive logics because nearly no other approaches to defeasible reasoning present proofs and certainly not proofs that resemble Hilbert proofs. A theoretic account of static proofs as well as dynamic proofs, which turn out to be a generalization of the former, is published [20]; a more extensive account is available on the web [23, §4.7].

Let us, very naively, have a look at some examples of dynamic proofs. More precise definitions follow in Section 7, but obtaining a clear and intuitive insight may be more important for the reader. Let us start with a dynamic proof from Γ_1. First have a look at stage 7 of the proof—a stage is a sequence of lines; think about stage 0 as the empty sequence and let the addition of a line to stage n result in stage $n+1$.

1	p	Prem	$\emptyset$
2	q	Prem	$\emptyset$
3	$\neg p \vee r$	Prem	$\emptyset$
4	$\neg q \vee s$	Prem	$\emptyset$
5	$\neg q$	Prem	$\emptyset$
6	r	1, 3; RC	$\{p \wedge \neg p\}$
7	s	2, 4; RC	$\{q \wedge \neg q\}$

So the premises were introduced and next two conditional steps were taken. Line 6 informs us that r is derivable on the condition that $p \wedge \neg p$ is false and line 7 that s is derivable on the condition that $q \wedge \neg q$ is false. Incidentally, a line with a non-empty condition corresponds nicely and directly with a line from a static proof—in the present case a Hilbert-style **CLuN**-proof. The condition, Δ, of a line is always a finite set of contradictions. Where a line of the dynamic proof contains a line at which A is derived on the condition Δ, the corresponding static **CLuN**-proof contains a line at which $A \vee \bigvee(\Delta)$ is derived—as expected, $\bigvee(\Delta)$ is the disjunction of the members of Δ. So in a sense stage 7 of this dynamic proof is nothing but a static proof in disguise. Note that the rule applied at lines 6 and 7 is called RC (conditional rule) because, as explained, a formula $A \vee \bigvee(\Delta)$ is **CLuN**-derivable from previous members of the proof, but Δ is pushed into the condition.

The way in which dynamics is introduced appears from the continuation of the proof. I do not repeat 1–5, which merely introduce the premises.

6	r	1, 3; RC	$\{p \wedge \neg p\}$	
7	s	2, 4; RC	$\{q \wedge \neg q\}$	✓
8	$q \wedge \neg q$	2, 5; RU	$\emptyset$	

At stage 8 of the proof, $q \wedge \neg q$ is unconditionally derived, viz. at line 8. So the supposition of line 7, viz. that $\{q \wedge \neg q\}$ is false, cannot be upheld. As a result, line 7 is marked, which means that its formula is considered as not derived from the premise set Γ_1.[11] Incidentally, the rule applied at line 8 is called RU (unconditional rule) because (the formula of) 8 is a **CLuN**-consequence of (the formulas of) 2 and 5.

So the dynamics is controlled by marks. Which lines are marked or unmarked is decided by a marking *definition*, which is typical for a strategy. More information on this follows in Section 7. For now, it is important that the reader understands why line 7 is marked and other lines are unmarked. As far as this specific proof stage is concerned, nothing interesting happens when the proof is continued. No mark will be removed or added to any of these 8 lines.[12] Incidentally, the only line that might become marked is line 6. The formulas derived on lines with an empty condition are **CLuN**-consequences of the premises. These are the stable consequences of the premise set. The marks pertain to the supplementary, defeasible consequences of the premise set.

How can I be so sure that the marks of lines 1–8 will not be changed in an extension of the proof from Γ_1? The example is propositional and propositional **CLuN** is decidable in the same sense as propositional **CL**. It is easy enough to prove that $q \wedge \neg q$ is the only contradiction that is **CLuN**-derivable from Γ_1.[13] Beware. As is the case for **CL**, only some fragments of **CLuN** are decidable. So arguing that a predicative proof is stable with respect to certain lines will often be much more complicated than in the present case.

Before we proceed, allow me to summarize that the two components governing dynamic proofs are rules (of inference) and the marking definition. The rules are applied at will by the people who devise the proof—if they are smart, they will follow a certain heuristics. As we shall see, the marking definition operates independently of any human intervention. In view of the stage of the proof, the marking definition determines which lines are marked.

When we consider more examples, a little complication will catch our attention. Here is a dynamic proof from $\Gamma_2 = \{\neg p, \neg q, p \vee r, q \vee s, \neg t, u \vee t, p \vee q\}$.

1	$\neg p$	PREM	$\emptyset$
2	$\neg q$	PREM	$\emptyset$
3	$p \vee r$	PREM	$\emptyset$
4	$q \vee s$	PREM	$\emptyset$

[11] Do not read the "not derived" as "not derivable". Indeed, a formula may be derivable in several ways from the same premise set.

[12] A more accurate wording requires that one adds: in a proof from Γ_1 that extends the present stage 8. Indeed, the logic we are considering is non-monotonic. So extending the premise set may result in line 6 being marked.

[13] The reader might think that, as p is also a **CLuN**-consequence of Γ_1, $(p \wedge q) \wedge \neg(p \wedge q)$ is also a **CLuN**-consequence of Γ_1. This however is mistaken. $\neg q \nvdash_{\mathbf{CLuN}} \neg(p \wedge q)$.

5	$\neg t$	PREM	$\emptyset$	
6	$u \vee t$	PREM	$\emptyset$	
7	$p \vee q$	PREM	$\emptyset$	
8	r	1, 3; RC	$\{p \wedge \neg p\}$	$\surd$
9	s	2, 4; RC	$\{q \wedge \neg q\}$	$\surd$
10	u	5, 6; RC	$\{t \wedge \neg t\}$	
11	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 2, 7; RC	$\emptyset$	

At stage 10 of the proof—when the proof consists of lines 1–10 only—no line is marked. At stage 11, however, lines 8 and 9 are both marked. Why is that? Line 11 gives us the information that either p or q behaves inconsistently on Γ_2, but does not inform us which of both behaves inconsistently. So a natural reaction is to consider both $p \wedge \neg p$ and $q \wedge \neg q$ as unreliable. This is the reaction that agrees with the Reliability strategy—we shall come across other strategies later. According to the Reliability strategy a line is marked if one of the members of its condition is unreliable. At this point in the paper, consider the unreliable formulas as the disjuncts of the *minimal* disjunctions of contradictions. If the "minimal" was not there, Addition would cause every contradiction to be unreliable as soon as one contradiction is unreliable.

In both example proofs, some lines were unmarked at a stage and marked at a later stage. The converse move is also possible, as is illustrated by a proof from $\Gamma_3 = \{(p \wedge q) \wedge t, \neg p \vee r, \neg q \vee s, \neg p \vee \neg q, t \supset \neg p\}$.

1	$(p \wedge q) \wedge t$	PREM	$\emptyset$	
2	$\neg p \vee r$	PREM	$\emptyset$	
3	$\neg q \vee s$	PREM	$\emptyset$	
4	$\neg p \vee \neg q$	PREM	$\emptyset$	
5	$t \supset \neg p$	PREM	$\emptyset$	
6	r	1, 2; RC	$\{p \wedge \neg p\}$	$\surd$
7	s	1, 3; RC	$\{q \wedge \neg q\}$	$\surd$
8	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 4; RU	$\emptyset$	

Both lines 6 and 7 are marked at stage 8 because $(p \wedge \neg p) \vee (q \wedge \neg q)$ is a minimal disjunction of contradictions that is derived at the stage. However, look what happens if stage 9 looks as follows—I do not repeat 1–5.

6	r	1, 2; RC	$\{p \wedge \neg p\}$	$\surd$ ✓
7	s	1, 3; RC	$\{q \wedge \neg q\}$	
8	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 4; RU	$\emptyset$	
9	$p \wedge \neg p$	1, 5; RU	$\emptyset$	

At stage 9 of this proof, $(p \wedge \neg p) \vee (q \wedge \neg q)$ is not a minimal disjunction of abnormalities because (the 'one disjunct disjunction') $p \wedge \neg p$ was derived. We knew already that either $p \wedge \neg p$ or $q \wedge \neg q$ was unreliable and now obtain the more specific information that it is actually $p \wedge \neg p$ that is unreliable. So $q \wedge \neg q$ is off the hook, whence line 7 is unmarked. Stage 9 of this proof is stable: no mark will be removed or added to lines 1–9 if the stage is extended. Actually nothing interesting happens in any such extension.

It is time to make the marking more precise. Dynamic proofs need to explicate the dynamic reasoning. So, at the level of the proofs, the dynamics needs to be controlled. The central features for this control are the conditions and the marking definition. The way in which conditions are introduced should be clear by now—precise generic rules follow in Section 7. However, how does one precisely figure out which lines are marked?

Only some adaptive logics are inconsistency-adaptive. So allow me to use a slightly more general terminology. The formulas that occur in conditions of lines—in the previous examples these were contradictions—are called abnormalities and Ω is the usual name for the set of abnormalities.

A classical disjunction of abnormalities will be called a *Dab-formula*—needless to say, a disjunction of formulas is always a disjunction of finitely many formulas. I shall often write $Dab(\Delta)$ to refer to the classical disjunction of the members of a finite $\Delta \subset \Omega$. A *Dab*-formula that is derived in a proof stage by RU at a line with condition $\emptyset$ will be called a *inferred Dab-formula* of the proof stage. Note that a *Dab*-formula introduced by Prem is not an inferred *Dab*-formula in the sense of this definition. $Dab(\Delta)$ is a *minimal inferred Dab-formula* of a proof stage if it is an inferred *Dab*-formula of the proof stage and there is no $\Theta \subset \Delta$ such that $Dab(\Theta)$ is an inferred *Dab*-formula of the proof stage. Where $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ are the minimal inferred *Dab*-formulas of stage s, the set of *unreliable formulas of stage* s is $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$. Where Θ is the condition of line i, line i is *marked* iff $\Theta \cap U_s(\Gamma) \neq \emptyset$. This is the marking definition for the Reliability strategy—every strategy has its own marking definition.

Marks come and go. As they determine which formulas are considered as derived, derivability seems to be unstable; it changes from stage to stage. Let this unstable derivability be called *derivability at a stage*. Apart from it, we want a stable form of derivability, which is called *final derivability* and is noted as $\Gamma \vdash_{\mathbf{CLuN}^r} A$. There are several ways to define final derivability. At this point in my story, the following seems most handy. If A is derived at an unmarked line i of a stage of a proof from Γ and the stage is *stable* with respect to i—line i is not marked in any extension of the stage—then A is finally derived from Γ.

Just as we wanted the stable entity called final derivability, we also want to have some further entities that refer to what is **CLuN**-derivable from the premise set Γ rather than referring to a stage of a proof from Γ.

Definition 1 $Dab(\Delta)$ *is a* minimal *Dab*-consequence of Γ *iff* $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$ *and, for all* $\Delta' \subset \Delta$, $\Gamma \nvdash_{\mathbf{CLuN}} Dab(\Delta')$.

Definition 2 *Where* $Dab(\Delta_1), \ldots, Dab(\Delta_n)$ *are the minimal Dab-consequences of* Γ, $U(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$.

The set $U(\Gamma)$ is defined in view of the Reliability strategy. A very different set will be introduced later in view of Minimal Abnormality.

The reader may expect a section on semantics at this point, but I shall only deal with the semantics as defined by the standard format.

6 The Standard Format SF

There is a large diversity of adaptive logics. Every new adaptive logic requires that one delineates its syntax (proof theory), its semantics (models), and, what is the hard bit, its metatheory (study of properties of the system). This suggested the search for a common structure for a large set of adaptive logics, if possible for all of them. The idea was that the structure would take care of most of the work beforehand, that the proof theory and semantics would be defined in terms of the common structure and that the metatheoretic properties would be provable from the structure. The common structure would be a function of certain parameters and specifying these would result in a specific adaptive logic with all required features available. This common structure is called the *standard format*.

An adaptive logic **AL** in Standard Format is defined as a triple comprising:[14]

- a *lower limit logic* **LLL**: a logic that has static proofs and contains classical disjunction,
- a *set of abnormalities* Ω, a set of formulas that share a (possibly restricted) logical form or a union of such sets,
- a *strategy* (Reliability, Minimal Abnormality, ...).

That the lower limit logic contains a classical disjunction means that one of the logical symbols is implicitly or explicitly defined in such a way that it has the meaning of the **CL**-disjunction. Explaining the notion of static proofs goes beyond the scope of the present paper, but the reader may for all useful purposes replace the requirement by: a formal and compact Tarski logic.

"Abnormality" is a technical term, different adaptive logics require that different formulas are seen as abnormalities. Only the abnormalities of corrective adaptive logics—those with **LLL** weaker than **CL**—are **CL**-falsehoods. In nearly all inconsistency-adaptive logics, existentially closed contradictions are abnormalities. Also other formulas may belong to the Ω, for example Universally closed contradictions or formulas of the form $A \wedge \neg(A \vee B)$. Some examples of restricted and unrestricted logical forms will be presented below.

Adaptive strategies will be discussed at some length later in this section.

If the lower limit logic **LLL** is extended with a set of rules or axioms that trivialize abnormalities (and no other formulas), then one obtains a logic called the *upper limit logic* **ULL**. Examples follow but it should be clear by now that, for all $A \in \Omega$ and for all $B \in \mathcal{W}$, A/B should be a derivable rule in **ULL**. As Ω is characterized by a logical form, it is in possible to obtain **ULL** by extending **LLL** with a set of rules.

I shall suppose that a characteristic semantics of **LLL** is known. This will enable me to define the semantics of **AL** in terms of the standard format. The **LLL**-models that verify no member of Ω form a semantics for **ULL**.[15] A premise set that has **ULL**-models is often called a *normal premise set*; it does not require that any abnormality is true.

It is instructive to have a closer look at the difference between **ULL** and **AL**. **ULL** extends **LLL** by validating some further rules of inference. **AL** extends **LLL** by validating certain *applications* of **ULL**-rules. The point is easily illustrated in connection to Disjunctive Syllogism. **CL** validates this rule, while in the (not yet precise) toy examples of proofs from Section 5, some but not all applications of Disjunctive Syllogism were sanctioned as correct. As those examples clarify, it depends on the premises—or should one say on the content

[14]Names like **LLL**, **AL**, $\mathbf{AL}^r$, and **ULL** are used as generic names to define the standard format and to study its features. The names refer to arbitrary logics that stand in a certain relation to each other.

[15]Similarly for those models together with the trivial model—the model that verifies all formulas.

of the premises—which applications turn out valid. In other words, adaptive logics display a form of *content-guidance*.[16] A different way of phrasing the matter is that $Cn_{\mathbf{AL}}(\Gamma)$ comes to $Cn_{\mathbf{LLL}}(\Gamma)$ extended with what is derivable if *as many* abnormalities are false *as* the premises permit. This phrase is obviously ambiguous, but strategies disambiguate it, as we shall see.

An important supposition on the language $\mathcal{L}$ of **AL** is that it contains a classical disjunction. It may of course contain several disjunctions, but one of them should be classical. In the sequel of this paper, the symbol $\hat{\vee}$ will always refer to this disjunction.[17] Similarly, $\sim$ will always refer to a classical negation. This is *not* supposed to occur in every considered language schema.

As we already have seen in Section 5, we need $\hat{\vee}$ for *Dab*-formulas—but see Section 11 for an alternative. In Section 5, I also introduced inferred *Dab*-formulas and minimal inferred *Dab*-formulas of a proof stage as well as the notation $Dab(\Delta)$.

Let us consider some examples of adaptive logics. Expressions $\exists A$ will denote the existential closure of A, viz. A preceded by an existential quantifier over every variable free in A.

The adaptive logic $\mathbf{CLuN}^m$ is defined by the following triple:

· lower limit logic: **CLuN**,

· set of abnormalities $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$

· strategy: Minimal Abnormality.

The upper limit logic is **CL**, obtained by extending **CLuN** with, for example, the axiom schema $(A \wedge \neg A) \supset B$.[18] It is not difficult to prove that the **CLuN**-models that verify no abnormality form a semantics of **CL**.

The logic $\mathbf{CLuNs}^m$ is defined by:

· lower limit logic: **CLuNs**,

· set of abnormalities $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s^a\}$

· strategy: Minimal Abnormality,

in which $\mathcal{F}_s^a$ is the set of atomic (open and closed) formulas of $\mathcal{L}_s$—atomic formulas are those in which no logical symbols occur except possibly for identity $=$. The upper limit logic is **CL**, obtained by extending **CLuNs** with, for example, the axiom schema $(A \wedge \neg A) \supset B$.[19] Semantically: the **CLuNs**-models that verify no abnormality form a **CL**-semantics.

Some further examples are easy variants. $\mathbf{CLuN}^r$ is like $\mathbf{CLuN}^m$, except that Minimal Abnormality is replaced by Reliability. $\mathbf{LP}^m$ is like $\mathbf{CLuNs}^m$

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[16]The notion played a rather central role in discussions on scientific heuristics. A very clear and argued position was for example proposed by Dudley Shapere [59].

[17]This obviously does not mean that $\hat{\vee}$ is a symbol of the language. It is a conventional name to refer to a symbol of the language that has the meaning of classical disjunction. It may even refer ambiguously: if there are several classical disjunctions, $\hat{\vee}$ need not always refer to the same one.

[18]Axioms are suppose to be closed formulas. So $A \in \mathcal{W}_s$. The idea is that **CLuN**-valid rules are fully retained in the extension. One of these rules is: from $\vdash A(a) \supset B$ to derive $\vdash \exists x A(x) \supset B$ provided a does not occur in B.

[19]The axiom schema may be restricted to $A \in \mathcal{W}_s^a$, but there is no need to do so.

except that **CLuNs** is replaced by Priest's **LP**—see Section 3 for the relation between **CLuNs** and **LP**.

In these examples **LLL** or the strategy are varied. What about the difference between the set of abnormalities of $\mathbf{CLuN}^m$ as opposed to $\mathbf{CLuNs}^m$? In a sense this is just a variation. Yet, if the Ωs are exchanged, the resulting variant of $\mathbf{CLuN}^m$ is still an inconsistency-adaptive logic, but its **ULL** is weaker than **CL**—a feature that is difficult to justify with respect to applications. If the Ω are exchanged, the resulting variant of $\mathbf{CLuNs}^m$ is also still an inconsistency-adaptive logic, but it is a flip-flop logic—see Section 12, where also more variation will be considered.

If an adaptive logic is in standard format, this fact (not specific properties of the logic) provides it with:

- its proof theory,
- its semantics (models),
- most of its metatheory (*including* soundness and completeness).

So the standard format provides guidance in devising new adaptive logics. Moreover, once a new adaptive logic is phrased in standard format, most of the hard work is over.

7 SF: Proof Theory

As we already know, every adaptive logic requires a set of rules of inference and a marking definition. The rules of inference are determined by **LLL** and Ω; the marking definition is determined by Ω and by the strategy. We also know that the dynamics of the proofs is controlled by attaching conditions (finite subsets of Ω) to derived formulas, or, if you prefer, to lines at which formulas are derived. We also have seen what is special about annotated dynamic proofs: their lines consist of four rather than three elements: a number, a formula, a justification, and a condition. The rules govern the addition of lines, the marking definition determines for every line i at every stage s of a proof whether i is unmarked or marked— this means that it is respectively IN or OUT—in view of (i) the condition of i and (ii) the minimal inferred *Dab*-formulas of stage s.

The rules of inference can be presented as three generic rules. Let Γ be the premise set and let

$$A \qquad \Delta$$

abbreviate that A occurs in the proof on the condition Δ.

Prem If $A \in \Gamma$:

$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \mathbin{\hat{\vee}} Dab(\Theta)$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

Only RC *introduces* new non-empty conditions (adds a non-empty set to the conditions of the local premises). Prem introduces empty conditions and RU merely carries conditions over and adds them up in a union.

Easy illustrations: RU may be applied in view of $p, p \supset q \vdash_{\mathbf{CLuN}} q$; RC may be applied in view of $p, \neg p \vee q \vdash_{\mathbf{CLuN}} q \mathbin{\hat{\vee}} (p \wedge \neg p)$. In view of the formulation of the antecedent of RU and RC, all rules are *finitary*—have a finite number of local premises. This formulation does not in any way affect the adaptive logic **AL** because **LLL** is a compact logic anyway. Incidentally, it is instructive to review the toy examples in terms of the precise formulation of the rules.

Marking definitions proceed in terms of the minimal inferred *Dab*-formulas at the proof stage. Where $Dab(\Delta_1)$, $\ldots$, $Dab(\Delta_n)$ are the minimal inferred *Dab*-formulas at stage s, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$.

Definition 3 *Marking for Reliability: where Δ is the condition of line i, line i is marked at stage s iff $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

The idea behind the definition consists of two steps. First, the minimal inferred *Dab*-formulas of stage s of a proof from Γ provide, at stage s, the best available estimate of the minimal *Dab*-consequences of Γ. So their disjuncts, which are abnormalities, cannot be safely considered as false. Next, the formula of a line can only be considered as derived (by present insights) if the abnormalities in the condition of the line can be considered as false. If they cannot, the line is marked.

However sensible this may sound, Minimal Abnormality offers a more refined approach. A *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ is a set that contains one element out of each member of Σ. A *minimal choice set* of Σ is a choice set of Σ of which no proper subset is a choice set of Σ. Where $Dab(\Delta_1)$, $\ldots$, $Dab(\Delta_n)$ are the minimal inferred *Dab*-formulas of stage s, $\Phi_s(\Gamma)$ is the set of the minimal choice sets of $\{\Delta_1, \ldots, \Delta_n\}$.

Definition 4 *Marking for Minimal Abnormality: where A is the formula and Δ is the condition of line i, line i is marked at stage s iff (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$,* or *(ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.*

The set $\Phi_s(\Gamma)$ is the best estimate, at stage s, of $\Phi(\Gamma)$, which is the set of minimal choice sets of the minimal *Dab*-consequences of Γ. The $\varphi \in \Phi(\Gamma)$ are the minimal sets of abnormalities that are true if Γ is true. On the Minimal

Abnormality strategy, a formula A is an adaptive consequence of Γ iff A is a consequence for every $\varphi \in \Phi(\Gamma)$. So, for every $\varphi \in \Phi(\Gamma)$, there should be a Θ such that $A \hat{\vee} Dab(\Theta)$ is a **LLL**-consequence of Γ and all members of Θ can be false, viz. none of them is a member of φ.

The difference between Minimal Abnormality and Reliability can be nicely illustrated by means of a toy proof. Considering again $\Gamma_2 = \{\neg p, \neg q, p \vee r, q \vee s, \neg t, u \vee t, p \vee q\}$, let us continue the second proof from Section 5. The premise lines 1–7 are not repeated.

8	r	1, 3; RC	$\{p \wedge \neg p\}$	$\checkmark$
9	s	1, 4; RC	$\{q \wedge \neg q\}$	$\checkmark$
10	u	5, 6; RC	$\{t \wedge \neg t\}$	
11	$(p \wedge \neg p) \vee (q \wedge \neg q)$	1, 2, 7; RC	$\emptyset$	
12	$r \vee s$	8; RC	$\{p \wedge \neg p\}$	
13	$r \vee s$	9; RC	$\{q \wedge \neg q\}$	

Obviously $\Phi_{13}(\Gamma) = \Phi_{11}(\Gamma) = \{\{p \wedge \neg p\}, \{q \wedge \neg q\}\}$. So, on the Minimal Abnormality strategy, lines 12 and 13 are unmarked. Indeed, if $p \wedge \neg p$ is the case and $q \wedge \neg q$ is not, then $r \vee s$ is in view of line 13. If $q \wedge \neg q$ is the case and $p \wedge \neg p$ is not, then $r \vee s$ is in view of line 12. It follows that, on the Minimal Abnormality strategy, $r \vee s$ is an adaptive consequence of Γ_2. The matter is very different for Reliability. Indeed, $U_{13}(\Gamma) = \{p \wedge \neg p, q \wedge \neg q\}$, whence lines 12 and 13 are marked. As the displayed proof stage is stable for both strategies and $r \vee s$ is not **CLuN**-derivable from Γ_2 on any other condition, $\Gamma_2 \vdash_{\mathbf{CLuN}^m} r \vee s$ but $\Gamma \nvdash_{\mathbf{CLuN}^r} r \vee s$.

In Section 5, I delineated final derivability in terms of a stable proof stage. This is not very handy as a general definition. Indeed, for some adaptive logics **AL**, premise sets Γ, and formulas A, only infinite **AL**-proofs of A from Γ are stable [11, §7]. But one obviously cannot write down infinite proofs. For this reason, the official definition of final derivability goes as follows.

Definition 5 *A is* finally derived *from Γ at line i of a finite proof stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

Definition 6 $\Gamma \vdash_{\mathbf{AL}} A$ *(A is* finally **AL**-derivable *from Γ) iff A is finally derived at a line of a proof stage from Γ.*

Establishing final derivability requires (i) a finite proof stage and (ii) a metatheoretic reasoning about extensions of the stage and extensions of these. Some comments on these definitions follow in Section 10.

8 SF: Semantics

The syntactic definition of minimal *Dab*-consequences of Γ was presented in Definition 1. As this proceeds in terms of **LLL** and an adequate semantics of this logic is supposed to be known, $Dab(\Delta)$ is a minimal *Dab*-consequence of Γ iff $\Gamma \vDash_{\mathbf{LLL}} Dab(\Delta)$ and, for all $\Delta' \subset \Delta$, $\Gamma \nvDash_{\mathbf{LLL}} Dab(\Delta')$.

Definition 7 *Where M is a* **LLL**-*model,* $Ab(M) = \{A \in \Omega \mid M \Vdash A\}$.

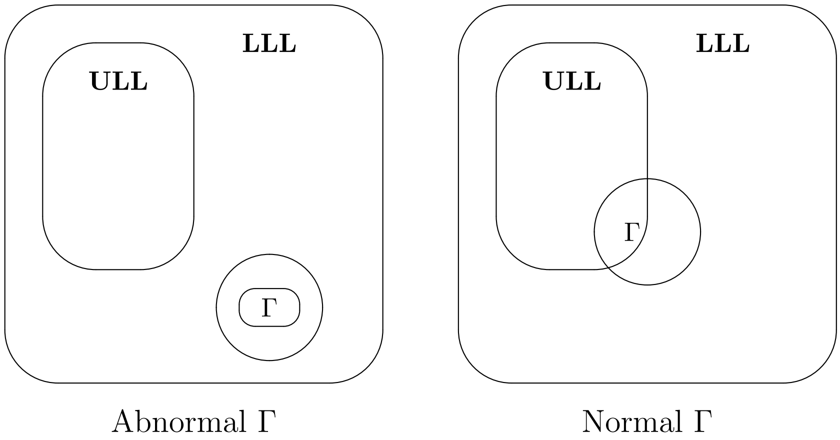

Figure 1: Comparison of Models

Consider first adaptive logics $\mathbf{AL}^r$ that follow the Reliability strategy. Let $\mathcal{M}_\Gamma^{\mathbf{LLL}}$ be the set of **LLL**-models of Γ.

Definition 8 *$M \in \mathcal{M}_\Gamma^r$ (M is a* reliable *model of* Γ*) iff* $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *and* $Ab(M) \subseteq U(\Gamma)$*.*

So the reliable models of Γ are the models of Γ that verify at most reliable abnormalities. Note that there are no reliable models, but only reliable models *of* a set of formulas Γ. The same holds for adaptive models in general.

Definition 9 $\Gamma \vDash_{\mathbf{AL}^r} A$ *(A is an* $\mathbf{AL}^r$*-consequence of* Γ*) iff* $M \Vdash A$ *for all* $M \in \mathcal{M}_\Gamma^r$*.*

So the $\mathbf{AL}^r$-semantics selects some **LLL**-models of Γ as $\mathbf{AL}^r$-models *of* Γ. The selection depends on Ω and on the strategy.

For adaptive logics $\mathbf{AL}^m$ that follow the Minimal Abnormality strategy, one may proceed in a very different way.

Definition 10 *$M \in \mathcal{M}_\Gamma^m$ (M is a* minimally abnormal *model of* Γ*) iff* $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *and no* $M' \in \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *is such that* $Ab(M') \subset Ab(M)$*.*

Definition 11 $\Gamma \vDash_{\mathbf{AL}^m} A$ *(A is an* $\mathbf{AL}^m$*-consequence of* Γ*) iff* $M \Vdash A$ *for all* $M \in \mathcal{M}_\Gamma^m$*.*

Lemma 14 below greatly clarifies the relation between the minimal abnormal models and the marking definition for Minimal Abnormality.

Have a look at Figure 1. For a normal premise set Γ, an adaptive logic simply selects the upper limit models of Γ, and hence delivers the same consequence set as the upper limit logic. Abnormal Γ have no **ULL**-models. Still, some exceptions aside,[20] adaptive logics select a proper subset of the set of **LLL**-models and hence deliver a larger consequence set than **LLL**.

[20]The exception may be caused by the logic, which is then called a flip-flop, or by the premise set—for example if the premise set comprises the formulas verified by a **LLL**-model.

9 SF: Metatheory

What follows is a selection of theorems. They are selected in view of their importance or in view of the insights they reveal in the context of the present introduction. They are all provable from the standard format [19, 23]. This means that they are provable from the common structure of all adaptive logics in standard format, independent of further specific properties.

Theorem 12 *$\Gamma \vDash_{\mathbf{AL}^r} A$ iff $\Gamma \vDash_{\mathbf{LLL}} A \,\check{\vee}\, Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$ for a finite $\Delta \subset \Omega$.*

Corollary 13 *$\Gamma \vdash_{\mathbf{AL}^r} A$ iff $\Gamma \vDash_{\mathbf{AL}^r} A$. (Soundness and Completeness for Reliability)*

Lemma 14 *$M \in \mathcal{M}^m_\Gamma$ iff $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ and $Ab(M) \in \Phi(\Gamma)$.*

Theorem 15 *$\Gamma \vdash_{\mathbf{AL}^m} A$ iff $\Gamma \vDash_{\mathbf{AL}^m} A$. (Soundness and Completeness for Minimal Abnormality)*

Strong Reassurance, also called Stopperedness or Smoothness, refers to the following property: if a model of the premises is not selected, this is justified by the fact that a selected model of the premises is less abnormal. If Strong Reassurance is absent, there are infinite sequences of models of a certain Γ in which each member of the sequence is less abnormal than its predecessor. This absence sometimes results in very odd consequence sets [12].

Theorem 16 *If $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma - \mathcal{M}^m_\Gamma$, then there is a $M' \in \mathcal{M}^m_\Gamma$ such that $Ab(M') \subset Ab(M)$. (Strong Reassurance for Minimal Abnormality.)*

Theorem 17 *If $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma - \mathcal{M}^r_\Gamma$, then there is a $M' \in \mathcal{M}^r_\Gamma$ such that $Ab(M') \subset Ab(M)$. (Strong Reassurance for Reliability.)*

All of the following theorems highlight important features of adaptive logics. The reader may find some more fascinating than others. This will depend on the reader's familiarity with certain aspects of non-monotonic reasoning and of defeasible reasoning in general.

Theorem 18 *Each of the following obtains:*

1. *$\mathcal{M}^m_\Gamma \subseteq \mathcal{M}^r_\Gamma$. Hence $Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma)$.*
2. *If $A \in \Omega - U(\Gamma)$, then $M \nVdash A$ for all $M \in \mathcal{M}^r_\Gamma$, whence $\sim A \in Cn_{\mathbf{AL}^r}(\Gamma)$ if $\sim$ is in $\mathcal{L}$.*
3. *If $Dab(\Delta)$ is a minimal Dab-consequence of Γ and $A \in \Delta$, then some $M \in \mathcal{M}^m_\Gamma$ verifies A and falsifies all members (if any) of $\Delta - \{A\}$.*
4. *$\mathcal{M}^m_\Gamma = \mathcal{M}^m_{Cn_{\mathbf{AL}^m}(\Gamma)}$ whence $Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{AL}^m}(Cn_{\mathbf{AL}^m}(\Gamma))$. (Fixed Point for Minimal Abnormality.)*
5. *$\mathcal{M}^r_\Gamma = \mathcal{M}^r_{Cn_{\mathbf{AL}^r}(\Gamma)}$ whence $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^r}(Cn_{\mathbf{AL}^r}(\Gamma))$. (Fixed Point for Reliability.)*
6. *For all $\Delta \subseteq \Omega$, $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$ iff $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$. (Immunity.)*

7. *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ *then* $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}}(\Gamma)$. *(Cautious Cut.)*

8. *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, *and* $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$. *(Cautious Monotonicity.)*

Theorem 19 *Each of the following obtains:*

1. *If* Γ *is normal, then* $\mathcal{M}_\Gamma^{\mathbf{ULL}} = \mathcal{M}_\Gamma^m = \mathcal{M}_\Gamma^r$ *whence* $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.

2. *If* Γ *is abnormal and* $\mathcal{M}_\Gamma^{\mathbf{LLL}} \neq \emptyset$, *then* $\mathcal{M}_\Gamma^{\mathbf{ULL}} \subset \mathcal{M}_\Gamma^m$ *and hence* $Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma) \subset Cn_{\mathbf{ULL}}(\Gamma)$.

3. $\mathcal{M}_\Gamma^{\mathbf{ULL}} \subseteq \mathcal{M}_\Gamma^m \subseteq \mathcal{M}_\Gamma^r \subseteq \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *whence* $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.

4. $\mathcal{M}_\Gamma^r \subset \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *iff* $\Gamma \cup \{A\}$ *is* **LLL**-*satisfiable for some* $A \in \Omega - U(\Gamma)$.

5. $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^r}(\Gamma)$ *iff* $\mathcal{M}_\Gamma^r \subset \mathcal{M}_\Gamma^{\mathbf{LLL}}$.

6. $\mathcal{M}_\Gamma^m \subset \mathcal{M}_\Gamma^{\mathbf{LLL}}$ *iff there is a (possibly infinite)* $\Delta \subseteq \Omega$ *such that* $\Gamma \cup \Delta$ *is* **LLL**-*satisfiable and there is no* $\varphi \in \Phi_\Gamma$ *for which* $\Delta \subseteq \varphi$.

7. *If there are* $A_1, \ldots, A_n \in \Omega$ $(n \geq 1)$ *such that* $\Gamma \cup \{A_1, \ldots, A_n\}$ *is* **LLL**-*satisfiable and, for every* $\varphi \in \Phi_\Gamma$, $\{A_1, \ldots, A_n\} \nsubseteq \varphi$, *then* $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^m}(\Gamma)$.

8. $Cn_{\mathbf{AL}^m}(\Gamma)$ *and* $Cn_{\mathbf{AL}^r}(\Gamma)$ *are non-trivial iff* $Cn_{\mathbf{LLL}}(\Gamma)$ *is non-trivial. (Reassurance)*

Theorem 20 *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, *then* $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') = Cn_{\mathbf{AL}}(\Gamma)$. *(Cumulative Indifference.)*

Theorem 21 *If* $\Gamma \vdash_{\mathbf{AL}} A$, *then every* **AL**-*proof from* Γ *can be extended in such a way that* A *is finally derived in it. (Proof Invariance)*

Theorem 22 *If* $\Gamma' \in Cn_{\mathbf{AL}}(\Gamma)$ *and* $\Gamma \in Cn_{\mathbf{AL}}(\Gamma')$, *then* $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma')$. *(Equivalent Premise Sets)*

10 SF: Decidability Matters And A Philosophical Comment

We have seen in Section 7 that final derivability is established by a finite proof stage and a metatheoretic reasoning about extensions of the stage and extensions of these. It is provable that, if $\Gamma \vdash_{\mathbf{AL}} A$, then A is derived on an unmarked line i of an **AL**-proof stage from Γ that is stable with respect to line i. The inconvenience is that the stage may be infinite,[21] whence Definition 5 is superior

The need for a metatheoretic argument reveals an ambiguity in the notion of a proof. On the one hand, there are proofs in the sense of constructions obtained by correct applications of the rules of inference. On the other hand, a proof in the strong sense establishes *by itself* that a certain formula is derivable

[21]Infinite stages can be extended by inserting lines in the sequence.

from a certain premise set. For compact Tarski logics, there are metatheoretic arguments that show that the existence of a proof in the weak sense warrants the existence of a proof in the strong sense—or that a proof in the weak sense constitutes a proof in the strong sense. For adaptive logics that matter is more sophisticated, as we shall see.

Definition 5 has a nice game-theoretic interpretation, actually several related such interpretations. As one might expect, the Proponent's task is to establish the proof, the Opponent's task to defeat it. In the simplest variant, the first move is for the Proponent who should produce a finite proof stage in which A is derived from Γ, say at line i. The next move is for the Opponent, who should extend the proof stage from Γ in such a way that i is marked. In the third move, the Proponent has to further extend the result in such a way that line i is unmarked. The Proponent has a winning strategy if, whatever the second move of the Opponent, the Proponent is able to carry out the third move successfully. Please check that this literally follows Definition 5.

For the propositional fragment (and for other decidable fragments of **LLL**), final derivability from finite premise sets is *decidable*. For the full predicative logics, however, there is not even a positive test. Nevertheless, even at the predicative level, there are *criteria* for final derivability. Such criteria were developed by several means, for example a 'block analysis' of proofs [10], specific tableau methods [27, 28], and a specific prospective dynamics [15, 17, 70]. Some of these need some reworking in view of the present standard format. The third approach results in the formulation of proof procedures that provide a criterion. If the procedure stops, the state of the proof reveals whether a certain formula is or is not finally derivable from the premise set; however, it is also possible that the procedure does not stop.

What if no criterion applies? All one can do is act on present insights as revealed by a proof at a stage. This leads to two questions. The first is whether the dynamics of the proofs goes anywhere. In view of the block analysis of proofs (and of the connected block semantics), the following can be established. A stage of a proof provides an insight in the premises and every step of the proof can be either informative or non-informative—this is defined in a precise way. If the step is informative, more insight in the premises is gained; if the step is non-informative, no insight is gained but no insight is lost either.

Sensible proofs contain only informative steps and it is not difficult to avoid uninformative steps. There is, however, no guarantee on convergence because the computational complexity of some adaptive consequence sets, viz. where the logic follows the Minimal Abnormality strategy, is Π^1_1.[22] Let me be more explicit on convergence. There is convergence with respect to the set of *Dab*-consequences of the premise set. There is also convergence with respect to the set of minimal *Dab*-consequences of the premise set Γ. Both sets are recursively enumerable. However, there is no convergence with respect to final derivability from Γ. Suppose that A is derived on a condition, respectively a set of conditions, that warrants its final derivability with respect to $U(\Gamma)$, respectively $\Phi(\Gamma)$. As long as not all minimal *Dab*-consequences of Γ are derived, it is possible that the derivation of a non-minimal *Dab*-consequence of Γ causes A not

[22] It is ironic that the study of the computational complexity of adaptive logics started with a paper arguing that they are too complex [40]. The philosophical complaints and misunderstandings in that paper were answered in [26]; a mistaken theorem was corrected in [67]. Extremely interesting and more detailed studies followed [53, 54].

to be derived at the stage. Needless to say, there is convergence with respect to final derivability whenever the set of minimal *Dab*-consequences of Γ is finite.

If no criterion applies, there is, as announced, a second question: Does the application context require final derivability? Not always. Reconsider the role of inconsistency-adaptive logics with respect to (what I called) the original problem. After certain abnormalities are located and perhaps some abnormalities are narrowed down in view of personal constraints and the like—see Section 12—one may have a clear idea for replacement and this may be sufficient to launch a hypothesis for a replacement of the inconsistent theory. Several people may launch several hypotheses, but the located problems will usually be common. Even if these are far from complete, some of the launched hypotheses may be successful, for a while or forever. A good example is Frege's set theory. The Russell paradox was known and led to proposals for replacements. Several of these were not shown to be inconsistent until now. So, as far as we can tell, they are worthwhile proposals for consistent set theories. Only after most of these proposals were formulated, the Curry paradox was discovered. So the proposals were made without a full analysis of the inconsistencies in Frege's theory. A similar story may be told, although perhaps less convincingly, about Clausius' removal of an inconsistency from thermodynamics. The aim of applications with respect to creative processes is to arrive at sensible hypothetical proposals for consistent replacements. The means to reach this end is the analysis provided by the inconsistency-adaptive logic(s). In that respect $Cn_{\mathbf{AL}}(\Gamma)$ is merely an ideal. This ideal is studied in order to show that the applied mechanism is coherent and conceptually sound. To the extent that our estimate of $Cn_{\mathbf{AL}}(\Gamma)$ is better, we may arrive at better proposals. We know that, for some **AL** and Γ, the set $Cn_{\mathbf{AL}}(\Gamma)$ is beyond our reach. All we can do is go by present insights and hope that they are not too bad an estimate of the final consequence set. That's life. The only alternatives are dogmatic belief and gardening.

11 Variants To The Standard Format

The first versions of the standard format were published in [14] and [16]. It soon became clear that especially a universal formulation of the proof theory required the presence of a classical disjunction. Other classical logical symbols also proved very useful. If the abnormalities are contradictions or existentially closed contradictions, one better has a classical conjunction around. Having classical negation around also turned out attractive.

Let me illustrate the attractiveness of classical negation in terms of $\mathbf{CLuN}^r$—the subsequent illustration may be adjusted to any inconsistency-adaptive logic mentioned so far. If $p \wedge \neg p \notin U(\Gamma)$, then each of the following obtain: (i) if $\neg p, p \vee q \in Cn_{\mathbf{CLuN}^r}(\Gamma)$, then $q \in Cn_{\mathbf{CLuN}^r}(\Gamma)$, (ii) if $\neg p, q \supset p \in Cn_{\mathbf{CLuN}^r}(\Gamma)$, then $\neg q \in Cn_{\mathbf{CLuN}^r}(\Gamma)$, (iii) if $\neg p \in Cn_{\mathbf{CLuN}^r}(\Gamma)$, then $\neg(p \wedge q) \in Cn_{\mathbf{CLuN}^r}(\Gamma)$, and so forth and so on. Suppose, however, that **CLuN** is extended with the classical negation $\sim$.[23] As $p \wedge \neg p \notin U(\Gamma)$, we now obtain: if $\neg p \in Cn_{\mathbf{CLuN}^r}(\Gamma)$, then $\sim p \in Cn_{\mathbf{CLuN}^r}(\Gamma)$. Note, however, that this is a very basic step. Once we have derived $\sim p$ by the rule RC, all other steps follow by the rule RU. Indeed, in

[23]Stepwise: the language $\mathcal{L}_s$ of **CLuN** is extended with the symbol $\sim$ and **CLuN** is extended with axioms or rules that give $\sim$ its classical meaning—for example the schemas $A \supset (\sim A \supset B)$ and $(A \supset \sim A) \supset \sim A$.

the version of **CLuN** that contains a classical negation, (i) $\sim p, p \vee q \vdash_{\mathbf{CLuN}} q$, (ii) $\sim p, q \supset p \vdash_{\mathbf{CLuN}} \neg q$, (iii) $\sim p \vdash_{\mathbf{CLuN}} \neg(p \wedge q)$, and so forth and so on. So once the classical negation of p is derived, there is no further need to apply RC. This made classical negation quite interesting.

The situation became even more attractive when it turned out that, in certain combinations of adaptive logics—like in $Cn_{\mathbf{AL2}}(Cn_{\mathbf{AL1}}(\Gamma))$—not all information is carried over to the second logic unless $Cn_{\mathbf{AL1}}(\Gamma)$ contains a classical negation. Moreover, the formulation of the standard format turned out more elegant if classical connectives were around. I tried to avoid $\sim$ in Section 9—actually, $\sim$ only occurs in Item 2 of Theorem 18. However, many transparent and clarifying statements may be phrased as soon as classical negation is around. Just to mention one example: $Cn_{\mathbf{AL^r}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma \cup \{\sim A \mid A \in \Omega - U(\Gamma)\})$. Note that, thanks to the presence of $\sim$, this defines the $\mathbf{AL}^r$-consequences of Γ in terms of its **LLL**-consequences—even $U(\Gamma)$ is so defined. All this, and actually more, suggested the usefulness of classical symbols in general and of classical negation in particular. Moreover, adding the classical logical symbols (in a specific way) turned out to be easy and seemed philosophically unobjectionable. Over the years, this led to the view that, given a premise set $\Gamma \subseteq \mathcal{W}$, it is advisable to formulate adaptive logics handling Γ in terms of the extension of the native $\mathcal{L}$ with the classical symbols that do not belong to $\mathcal{L}$. In the interest of the elegance of the standard format, this was modified to: add classical symbols, even when they duplicate symbols of $\mathcal{L}$, and refer to them by specific 'checked' logical symbols $\check{\neg}$, $\check{\vee}$, etc.[24]

It later turned out that it was important to distinguish, with respect to proofs, between (what is now called) *Dab*-formulas and inferred *Dab*-formulas.[25] As the added symbols were around anyway, the distinction was originally introduced in terms of the checked disjunction $\check{\vee}$.

There are mainly three reasons why I described a standard format without 'checked' logical symbols. First, the introduction of those symbols is rather tiresome. It requires a motivation and a lengthy and careful formulation. A standard format with checked symbols is definitely more complicated than one without, and one wonders whether the advantages of extending the language outweighs the complication. Next, the addition of classical negation will definitely raise suspicion from the side of dialetheists. So, as the addition is avoidable, it better is avoided—the formulation of a logic should refrain from taking a philosophical stance. Finally, the checked symbols led to confusion, for example to the mistaken claim that adaptive logics are in a sense incomplete because not all semantic consequences would be derivable from premise sets in which occur checked symbols [62, 63].[26]

All that we really need in the standard format is a classical disjunction, to which I *refer* by $\hat{\vee}$. The classical disjunction will occur in *Dab*-formulas and in disjunctions like $B \mathbin{\hat{\vee}} Dab(\Theta)$ in applications of RC. And even the requirement that a classical disjunction should occur in $\mathcal{L}$ may be dropped, as we shall see

[24]The classical symbols were actually *superimposed* on $\mathcal{L}$: in the extended language, they never occur within the scope of the original logical symbols of $\mathcal{L}$.

[25]The distinction warrants that the reference to a *finite* proof stage in Definition 5 is all right.

[26]The mistake is caused by a confusion between symbols and concepts. If $\check{\vee}$ occurs in a premise, and so in $\mathcal{L}$, then $\check{\vee}$ is not a new symbol of the extended language. So one needs to extend the language with another symbol, say $\tilde{\vee}$, and call that the checked disjunction.

after the next paragraph.

Do all adaptive logics that fit in the version of the standard format with added classical symbols also fit in the version without such added symbols? Not quite. However, the adaptive logics that do not belong to the standard format in the present (actually restored original)[27] version can be integrated by a single and simple strike. We shall see so in Section 13.

The requirement that classical disjunction should be a symbol of $\mathcal{L}$ may be dropped by moving to a *multiple-conclusion standard format*. This fact was first seen and used by Sergei Odintsov and Stanislav Speranski [54]; they formulated this version of the standard format for propositional logics, but the generalization to predicative logics is straightforward.

Where **L** is a logic, I shall write $\Gamma \vdash^{mc}_{\mathbf{L}} \Delta$ to express that, according to **L**, one of the members of Δ is true if all members of Γ are true. **LLL** should be specified to be left compact as well as right compact; so if $\Gamma \vdash^{mc}_{\mathbf{L}} \Delta$, then there is a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta' \subseteq \Delta$ such that $\Gamma' \vdash^{mc}_{\mathbf{L}} \Delta'$. Next, the condition of the rule RC can now be phrased as "If $A_1, \ldots, A_n \vdash^{mc}_{\mathbf{LLL}} \{B\} \cup \Theta$", in which Θ is a finite set as in the original RC. The multiple-conclusion standard format is also handy and interesting from a metatheoretic point of view. Remember the characterization of $\mathbf{AL}^r$ in terms of **LLL** phrased with the help of $\sim$: $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma \cup \{\sim A \mid A \in \Omega - U(\Gamma)\})$. This can be phrased without classical negation in multiple-conclusion terms: $\Gamma \vdash^{mc}_{\mathbf{AL}^r} \Delta$ iff $\Gamma \vdash^{mc}_{\mathbf{LLL}} \Delta \cup (\Omega - U(\Gamma))$. The multiple conclusion version of Theorem 12 follows from this by right compactness.

12 Variation

As adaptive logics are not deductive logics but formal characterizations of methods, a multiplicity of adaptive logics is required for every purpose. It is not up to the logician to decree which methods a scientist should use. This choice is up to the user, viz. the scientist, and perhaps to some extent to philosophers of science. The choice cannot be justified in terms of logical features. It depends on what one learned about how to learn (Shapere), and more precisely about learning within a specific domain. So the logician should provide a multiplicity of adaptive logics. Variation may have two sources. On the one hand, the logician should look at the facts, historical facts most of the time. As the saying justly goes, the facts often outdo our phantasy. On the other hand, the logician is well placed to devise a set of variations in terms of features of the formal machinery.

Let us first have a look at **LLL**-variation. In principle, the lower limit logic can be every formal paraconsistent logic that is reflexive, transitive, monotonic, and compact, for which there is a positive test, and that contains a classical disjunction—the latter is not even required in view of the multiple-conclusion standard format. So a multitude of potential lower limit logics is available. Logics between **CLuN** and **CL** (**CLuNs**, da Costa's $\mathbf{C}_n, \ldots$), fragments of the former, such as **LP**, all **LFI** that have a classical disjunction, Jaśkowski's **D2**,[28] practically all relevant logics, etc. Each of these can be combined with several Ω

[27] All that is new in the restored version is the notion of an inferred *Dab*-formula.

[28] Adaptive versions of **D2** and other Jaśkowski logics were extensively studied [47, 48, 49].

and with several strategies. Some **LLL** behave in an unexpected way if they are combined with an unsuitable Ω. However, a suitable Ω is usually easily located.

The set of abnormalities Ω may also be varied. We have already seen $\{\exists(A \wedge \neg A) \mid A \in \mathcal{W}_s\}$ as well as a restricted version $\{\exists(A \wedge \neg A) \mid A \in \mathcal{W}_s^a\}$, which is adequate for **CLuNs**, **LP**, and similar logics. At first sight, not much room seems to be left as the lower limit logic **CLuN** combined with $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{W}_s^a\}$ results in adaptive logics of which **CL** is not the upper limit, whereas the lower limit logic **CLuNs** combined with $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{W}_s\}$ results in a flip-flop logic—see below.

And yet, some variation is known. One example is that the set of abnormalities is extended as follows: $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\} \cup \{\forall(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$. The effect is rather transparent. Although $\forall(A \wedge \neg A) \vdash_{\mathbf{CLuN}} \exists(A \wedge \neg A)$, it makes a difference whether, next to minimizing $\exists(A \wedge \neg A)$ one also minimizes $\forall(A \wedge \neg A)$. Again, this Ω is suitable for **CLuN**; for **CLuNs** one needs to replace $\mathcal{F}_s$ by $\mathcal{F}_s^a$. Other variations require symbols not in $\mathcal{L}_s$—but **CL**-definable in $\mathcal{L}_s$. A nice example is the consistency operator from logics of formal inconsistency [31]. If **LLL** is a compact such logic (and $\hat{\vee}$ is present in its language schema), it may be combined with $\{\neg \circ A \mid A \in \mathcal{W}\}$, possibly restricted to, for example, $\{\neg \circ A \mid A \in \mathcal{W}^a\}$. A few more suitable sets of abnormalities for inconsistency-adaptive logics are known, but it seems wiser to postpone their introduction for a few paragraphs.

So let us turn to variations to the strategy. Reliability and Minimal Abnormality are the oldest and still central strategies. A few others are worth being mentioned. The first strategy that comes to the mind of people new in the domain is the Simple strategy.

Definition 23 *Marking for Simple: where Δ is the condition of line i, line i is marked at stage s iff some $A \in \Delta$ is an inferred Dab-formula of s.*

This strategy is suitable iff, in view of properties of **LLL** or of the specific premise set Γ, every minimal *Dab*-consequence of Γ has only one disjunct and so is just an abnormality. It is easily seen that, if this is the case, Reliability, Minimal Abnormality, and Simple define the same adaptive logic. Where Simple is suitable, its semantics is like that of Reliability or Minimal Abnormality—the semantics for those coincide whenever Simple is suitable.

The Normal Selections Strategy was mainly developed in order to characterize some non-monotonic logics known from the literature in terms of an adaptive logic—see Section 13. The relation with Minimal Abnormality is obvious in view of Section 8.

Definition 24 *Marking for Normal Selections: where Δ is the condition of line i, line i is marked at stage s iff $\varphi \cap \Delta = \emptyset$ for all $\varphi \in \Phi_s(\Gamma)$.*

The following theorem shows that the computational complexity of adaptive logics that follow the Normal Selections strategy is less complex than the definition suggests.

Theorem 25 *Where $\mathbf{AL}^n$ is an adaptive logic following the Normal Selections strategy, $\mathbf{AL}^n$-final consequence sets are identical to the final consequence sets assigned by an adaptive logic* **AL1** *that is exactly like $\mathbf{AL}^n$ except that marking is defined as follows:*

where Δ is the condition of line i, line i is marked at stage s iff, for a $\Theta \subseteq \Delta$, $Dab(\Theta)$ is an inferred Dab-formula of stage s.

Definition 26 $\Gamma \vDash_{\mathbf{AL}^n} A$ *iff, for some $\varphi \in \Phi(\Gamma)$, $M \Vdash A$ for all $M \in \mathcal{M}^m_\Gamma$ with $Ab(M) = \varphi$.*

Some adaptive logics **AL** are called flip-flops. For normal premise sets Γ, $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$, which is as desired and holds for all adaptive logics. For abnormal Γ—those that have no **ULL**-models—$Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma)$, which is usually not what one wants. As was explained in Section 4, a central aim of adaptive logics is to isolate the abnormalities in abnormal Γ and to validate applications of **ULL**-rules whenever no abnormality is involved. Flip-flops do this only in the crudest possible way. In the case of inconsistency-adaptive logics, for example, flip-flops deliver the full **CL**-consequence set of normal Γ and nevertheless avoid triviality in the case of abnormal Γ. Unlikely as it may appear, there are application contexts in which a flip-flop is precisely what one wants. For such cases, it is useful to have a strategy around to define flip-flops.

Definition 27 *Marking for Flip-Flops: where Δ is the condition of line i, line i is marked at stage s iff $\Delta \neq \emptyset$ and there is at least one inferred Dab-formula of s.*

The Blindness strategy handles abnormal premise sets as if they were normal. Replacing the strategy of any of the aforementioned inconsistency-adaptive logics by Blindness results in **CL**.

Definition 28 *Marking for Blindness: mark no lines.*

By varying the strategy, one may also define some logic-like entities. A first example is the Single Selection Strategy. It consists in *choosing* a $\varphi \in \Phi_s(\Gamma)$ and in marking lines with condition Δ iff $\varphi \cap \Delta = \emptyset$. The result is not a logic because there is an element of choice that is not specified in the premise set. There are several ways in which the consequence set may be characterized in terms of an adaptive logic. I mention the most obvious one. Let $\mathbf{AL}^s$ have the same lower limit and set of abnormalities as the logic-like object but the Simple strategy instead. The intended consequence set is provably identical to $Cn_{\mathbf{AL}^s}(\Gamma \cup \varphi)$.[29]

Another logic-like entity is defined by the All Selections Strategy. The entity is at best logic-like because it maps premise sets to sets of consequence sets, rather than to consequence sets: $\wp(\mathcal{W}) \rightarrow \wp(\wp(\mathcal{W}))$. Each of the consequence sets is associated with a $\varphi \in \Phi(\Gamma)$. One also needs to associate a mark to each $\varphi \in \Phi(\Gamma)$. A line with condition Δ is φ-marked iff $\Delta \cap \varphi \neq \emptyset$.[30]

Leaving strategy variations, let us have a look at some more drastic 'variants'. A first variant comes in a sense to digging deeper in abnormalities. The point is that an inconsistency like $(p \vee q) \wedge \neg(p \vee q)$ may have several 'causes' and that the causes themselves may be considered as abnormalities. The inconsistency $(p \vee q) \wedge \neg(p \vee q)$ may be derivable from the premises because $p \wedge \neg(p \vee q)$

[29] The low computational complexity of the consequence set is rather artificial. We suppose that at least one $\varphi \cap \Delta = \emptyset$ is given, but precisely locating a φ may be a very complex task.

[30] The logic-like entity has a rather limited application field. For some Γ, $\Phi(\Gamma)$ is not only infinite but also uncountable.

$\mathbf{CLuN}^m$	$\mathbf{CLuN}_c^m$	$\mathbf{CLuNs}^m$	$\mathbf{LP}^m$
p	p	p	p
		$\neg p$	$\neg p$
$\neg\neg p$	$\neg\neg p$	$\neg\neg p$	$\neg\neg p$
	$\neg r$	$\neg r$	$\neg r$
q	q		
s	s	s	

Table 1: Comparison for $\Gamma = \{p, \neg p \vee q, \neg(p \vee r), \neg\neg p \supset s\}$

is derivable, or because $q \wedge \neg(p \vee q)$ is derivable. It is also possible that neither of the two is derivable, but that $(p \vee q) \wedge \neg(p \vee q)$ still is. So this leaves us with three different sorts of (non-independent) abnormalities rather than one. What is fascinating in this approach? Let me explain in terms of Reliability. Even if $(p \vee q) \wedge \neg(p \vee q) \in U(\Gamma)$, it is possible that r is derivable on the condition $\{p \wedge \neg(p \vee q)\}$ and that $p \wedge \neg(p \vee q) \notin U(\Gamma)$. On the one hand this approach forms an Ω-variant. On the other hand, a net gain is obtained if one applies this approach to, for example, $Cn_{\mathbf{CLuN}^m}(\Gamma)$ rather than to Γ itself. I refer to a published paper [21] for the precise (but rather lengthy) definition of the new set of abnormalities. It is instructive to compare the new combined logic—call it $\mathbf{CLuN}_c^m$—with the well studied $\mathbf{CLuN}^m$, $\mathbf{CLuNs}^m$, and $\mathbf{LP}^m$. I present one example of a premise set in Table 1. The consequence set of the combined logic is rather fascinating. On the one hand, it extends the $\mathbf{CLuN}^m$-consequence set. On the other hand, where a member of the $\mathbf{CLuNs}^m$-consequence set is absent ($\neg p$ in the example), this results in a more interesting consequence (q in the example); an inconsistency is avoided in order to obtain a different consequence.

A very different variant concerns the reduction of abnormalities in terms of plausibilities or preferences. Suppose that $A_1, \ldots, A_n \in \Omega$ and that $A_1 \hat{\vee} \ldots \hat{\vee} A_n$ is a minimal inferred *Dab*-formula at stage s of a proof from Γ. One may have reasons not to consider the n abnormalities A_i as equally affected, but to opt for or against a specific abnormality A_i. Of course, the choice should be made defeasibly to avoid triviality on the one hand and superfluous inconsistency on the other. So one will add the premise $\Diamond A_i$ or $\Diamond\neg A_i$, in which $\Diamond$ functions as a plausibility operator. Abnormalities may be the formulas of the form $\Diamond A \wedge {\sim}A$ and those of the form $\Diamond{\sim}A \wedge A$. So, for example, from $\Diamond{\sim}A$ one may derive ${\sim}A$ on the condition $\{\Diamond{\sim}A \wedge A\}$. The upshot will be that plausible statements will be defeasibly turned into full premises and that *Dab*-formulas from the inconsistency-adaptive logic will be reduced. If $A_1 \hat{\vee} \ldots \hat{\vee} A_n$ is a minimal inferred *Dab*-formula at stage s and A_1 came out of the plausibility logic, then $A_2, \ldots, A_n$ are off the hook. If, to the contrary, ${\sim}A_1$ came out of the plausibility logic, then $A_2 \hat{\vee} \ldots \hat{\vee} A_n$ is **LLL**-derivable and hence is a minimal inferred *Dab*-formula. It is often more appropriate to have different degrees of plausibility available: $\Diamond A$ for very plausible, $\Diamond\Diamond B$ for a bit less plausible, and so on. Technically speaking, one first adds the layers of plausibility statements—as much as possible of the most plausible statements, next as much as possible of the second-most plausible statements, and so on, and finally one applies the inconsistency-adaptive logic. This approach to weeding out abnormalities was studied along with several variants for expressing and handling plausibilities or preferences [18].

And now to a third type of variant, and again a completely different one: other gluts, gaps, and ambiguities. Remember that, in the original problem, the aim was to obtain minimally inconsistent theories that may serve as a starting point to devise a consistent theory. Until now, I have followed the official line of thought: as the theory under consideration is inconsistent, one has to replace **CL** by a paraconsistent logic. This, however, is not the only way out. Inconsistencies may be seen as negation gluts: the classical condition for $\neg A$ to be false is present (in that A is true), but nevertheless $\neg A$ is true. Negation gaps may be understood in a similar way. Moreover, gluts as well as gaps with respect to other logical symbols may also be understood along the same line. We are for example confronted with an existential gap if $\exists x Px$ is false although Pa is true. Furthermore, non-logical symbols may be ambiguous in that different occurrences of the same symbol may have a different meaning, whence different occurrences of the same formula may have different truth values. Sundry gluts or gaps may be allowed, possibly along with ambiguities, in order to avoid triviality; next, the gluts and gaps and ambiguities may be minimized in order to interpret the premise set as much as possible in the way **CL** interprets it—the first ambiguity-adaptive logics were devised by Guido Vanackere [64, 65, 66].

The premise set $\Gamma_4 = \{p, r, (p \vee q) \supset s, (p \vee t) \supset \neg r, (p \wedge r) \supset \neg s, (p \wedge s) \supset t\}$ may serve as an illustration. Γ_4 has models (i) of logics that allow for negation gluts, (ii) of logics that allow for negation gaps, (iii) of logics that allow for conjunction gaps as well as disjunction gaps, (iv) of logics that allow for implication gluts, (v) of logics that allow for ambiguities in the non-logical symbols, and of course of logics that allow for several of the mentioned gluts and gaps and ambiguities. Each of these possibilities defines a different adaptive theory. Each of these theories is a sensible solution of the original problem. So, again, a multiplicity of approaches is available and this is as it should be. All those abnormalities *surface* as inconsistencies when one applies **CL** to premise sets, but this does not mean that paraconsistency is the only possible answer. The combinations lead up to adaptive zero logic $\mathbf{CL}\emptyset^m$. In this logic, all meaning is contextual. According to $\mathbf{CL}\emptyset$ nothing is derivable from any premise set, not even the premises. Nevertheless, the adaptive $\mathbf{CL}\emptyset^m$ assigns to normal premise sets the same consequence set as **CL**. Apart from its own interest, $\mathbf{CL}\emptyset^m$ was shown to have an important heuristic value for determining which combinations of gaps or gluts or ambiguities lead to maximally normal interpretations of a given premise set. A detailed study is available [24].

13 Integration

Once the standard format was described, it was not difficult to devise many new logics and this pragmatic attitude led to useful work. However, it is also important to unify the domain of 'defeasible logics'. It is important to find out whether all defeasible logics can be subsumed under the same schema or, if that turns out impossible, whether the number of schemas can be reduced. Needless to say, it cannot be settled today which schemes have most unifying power. However, studying the unifying power of adaptive logics seems sensible because there is a clear underlying concept. This is why a lot of attention was given to integrating existing mechanisms into adaptive logics. There is a book

[62] that contains many relevant results and a list of papers that I shall not add to the references.

As I see it, the aim should be to integrate the realistic and potentially realistic defeasible reasoning forms. It goes without saying that truckloads of defeasible mechanisms may be defined, especially in semantic terms. It goes equally without saying that many of them cannot be integrated in any finite set of unifying schemas. This is as unimportant as it is obvious. Among the possible sources for potentially realistic reasoning forms are (i) defeasible reasoning forms described by different approaches, (ii) old and 'unusual' adaptive logics that are not in standard format, (iii) new defeasible reasoning forms that are useful in view of the philosophy of science, the philosophy of mathematics, and everyday reasoning.

Two examples of integration follow, one 'external' and one 'internal'. The external one concerns the Strong Consequence Relation devised by Nicholas Rescher [57]. Consider a version of **CLuN** with classical negation $\sim$—the variant will not be given a different name. Let Γ' comprise the members of Γ with $\neg$ replaced by $\sim$ and let $\Gamma^{\neg\sim} = \{\neg{\sim}A \mid A \in \Gamma'\}$. It was proven [13] that $\Gamma \vdash_{\text{Strong}} A$ iff $\Gamma^{\neg\sim} \models_{\mathbf{CLuN}^m} A$. So the corrective consequence relation Strong is characterized by (the variant of) the adaptive logic $\mathbf{CLuN}^m$ under a translation. The characterization in adaptive terms reveals at once a whole set of properties of the Strong consequence relation. It also enables one to devise so-called direct proofs: adequate dynamic proofs that proceed in the original language (with one negation symbol) [29].

By internal integration I mean that adaptive logics that are not in standard format are characterized in terms of an adaptive logic in standard format. It may be shown, for example, that adaptive logics following the Normal Selections strategy can be characterized in terms of adaptive logics that follow the Minimal Abnormality strategy. The example I shall use as an illustration here is the one promised in Section 11: adaptive logics that fall under the standard format with checked logical symbols but not under the standard format without, may (all and in one sweep) be characterized in terms of adaptive logics that fall under the new standard format.

Let **AL1** be the adaptive logic that requires integration because it requires the presence of checked symbols whereas some (or even all) classical symbols are absent from its native language. One simply proceeds as follows. First, the native language $\mathcal{L}$ of **AL1** is extended to $\mathcal{L}^+$ by superimposing $\hat{\vee}$[31] as well as all other classical symbols. Next, define **AL2** like **AL1** except that **AL2** is defined over $\mathcal{L}^+$. So, whatever classical symbols were required for defining **AL1** are available in the native language of **AL2**, which is in the present standard format. Finally, define $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}^+}(\Gamma) \cap \mathcal{W}$—obviously no translation function is required, or rather, the translation function is such that $\text{tr}(A) = A$. The reader should not be misled by this example. Here integration is nearly obvious. In other cases, however, integration may require quite some ingenuity.

14 In Conclusion: Applications

From the very first ideas on, my motivation for developing adaptive logics was always guided by the aim to handle sensible applications in a sensible way.

[31]That is (i) $\mathcal{W} \subseteq \mathcal{W}^+$ and (ii) if $A, B \in \mathcal{W}^+$, then $(A \hat{\vee} B) \in \mathcal{W}^+$.

Moreover, this aim was to understand and explicate the actual defeasible reasoning. Attention for models and for formal properties came only afterwards, as a means rather than as an end.

We have seen that the 'original problem' was to construct minimally abnormal interpretations of mathematical or empirical theories that were intended as consistent but turned out to be inconsistent. This was the central application context for inconsistency-adaptive logics as well as for combinations of inconsistency-adaptive logics with other adaptive logics.

In the previous paragraph, "theory" should not be taken too literally. There are many cases in which one deals with inadvertently inconsistent premise sets the content of which is much more disparate than are the theorems of a theory. A nice example is that inconsistency-adaptive logics allow one to incorporate the inconsistent case in belief revision [33]. This broadens an existing approach, making room for inconsistency. A similar move may be made with respect to many other approaches, for example question evocation [44]. A different matter is that existing mechanisms that are able to handle inconsistency have more attractive adaptive versions [45].

Graham Priest, who edited my oldest paper on the topic, was fascinated by the application of adaptive logics to a very different problem. Inconsistency-adaptive logics offer the possibility to understand most of classical reasoning and actually to understand it as correct. Not as correct by logical standards, but as correct by logical standards extended with the presumption that inconsistencies are false. For dialetheists the presumption is justified by the low frequency of true inconsistencies. That a person with so different a view on logic saw a use in inconsistency-adaptive logics has been a great source of encouragement.

Recently a very different type of application turned out to be fascinating. In view of the limitative theorems in mathematics, (i) the axiomatic method is known to have a rather limited scope and (ii) some of our present mathematical theories may very well turn out to be inconsistent and hence, as their underlying logic is **CL**, trivial. In view of each of these facts, it became attractive to phrase theories that have an adaptive logic as their underlying logic. These theories, viz. their set of theorems, are obviously not semi-recursive. That is precisely one of the advantages. Notwithstanding their finitary rules and notwithstanding the simplicity of dynamic proofs-at-a-stage, adaptive logics enable one to axiomatize Π^1_1-complex theories. So although it is too complex, for either humans or Turing machines, to figure out whether some formula is or is not a theorem of the theory, the theory at least defines correctly a certain complex consequence set.[32]

With respect to the possible triviality of classical mathematical theories, the advantage of adaptive theories is similar. Well-wrought inconsistency-adaptive theories display the following feature: if the classical theory is consistent, then the adaptive theory defines exactly the same set of theorems; if the classical theory is inconsistent, it is trivial and so pointless, but the adaptive theory, which we may phrase today, will still define a non-trivial consequence set that is 'as close to' the intended consequence set 'as is possible'.

Until now only a few adaptive theories have been formulated and studied

[32]Classical theories, which have **CL** as underlying logic, fail to define such a theory. Their consequence relation is much less complex. If A is not a theorem of a classical theory, humans or Turing machines may never find this out. However, if A is a theorem of the classical theory, humans or Turing machines will find that out at a finite point in time. As this point may be two million years from here, the point is slightly theoretical.

[22, 68, 69], but the results seem fascinating.

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