A Rich Paraconsistent Extension of Full Positive Logic*

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Abstract

In the present paper we devise and study the most natural predicative extension of Schütte's maximally paraconsistent logic. With some of its large fragments, this logic, **CLuNs**, forms the most popular family of paraconsistent logics. Devising the system involves some entanglements, and the system itself raises several interesting questions. As the system and fragments were studied by other authors, we restrict our attention to results that we have not seen in press.

1 Aim of this Paper

In [33], Schütte presents a propositional logic $\mathbf{\Phi_v}$. The logic is paraconsistent ($A, {\sim}A \nvdash_{\mathbf{\Phi_v}} B$) and displays all usual negation properties that 'drive negations inwards': ${\sim}{\sim}A \equiv A$, ${\sim}(A \wedge B) \equiv ({\sim}A \vee {\sim}B)$, etc. Schütte devised $\mathbf{\Phi_v}$ for a special purpose, a purpose for which he does not need a predicative version of it. In the present paper we devise the most natural such extension, and call it **CLuNs** for reasons that become obvious later. Devising this system involves several entanglements and raises some interesting questions.

Actually, **CLuNs** and some of its fragments obtained by dropping certain logical symbols became the most popular paraconsistent logics. For some examples see [5], [6], [19], [20], [21], [22], [23], [24], [25] and [30], [34]—with thanks to João Marcos for some of these references. There are not many references in the paraconsistent literature, though, even after $\mathbf{\Phi_v}$ was (explicitly ascribed to Schütte and) studied, together with other propositional paraconsistent logics in [7].

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Some paraconsistent logicians object to a detachable material implication, but like the other properties of the logic. For example, Priest's preferred paraconsistent system, **LP**, is (at the propositional level) the $\sim$-$\wedge$-$\vee$-fragment of **CLuNs**.

$\mathbf{\Phi_v}$ contains a constant, now usually written as "$\perp$", that represents 'The Falsehood' (or the 'conjunction of all formulas') and is characterized by $\perp \supset A$. In this system, classical negation may be defined by $\neg A =_{df} A \supset \perp$. In [7], the $\perp$-less (and $\neg$-less) fragment of $\mathbf{\Phi_v}$ is studied (under the name $\mathbf{PI}^s$) and is shown to be maximally paraconsistent—i.e. propositional **CL** is the only non-trivial logic that extends $\mathbf{\Phi_v}$. In the present paper, we shall distinguish **CLuNs**, in which classical negation and bottom are primitive or definable, from pure paraconsistent **CLuNs**, in which classical negation is not definable.

It is not our aim, in the present paper, to offer a complete study of **CLuNs**, but rather to describe some properties that thus far went largely unnoticed. Three main topics are dealt with. First, we devise **CLuNs** as a natural predicative extension of $\mathbf{\Phi_v}$ and present a variety of semantics for it—the system turns out to be rather natural under a large class of very different descriptions. Next we offer some comments on definability in **CLuNs** and consider the (remarkable) relation between non-equivalent formulas containing a single propositional letter—we refer to [19] for an interesting study of definable propositional connectives in **CLuNs**. Finally we consider some further properties of the system.

A separate motivation for devising **CLuNs** is that we want to study, in a separate paper, the properties of the inconsistency-adaptive logics—see, *e.g.*, [10] or [13]—that are based upon it. Although our preferred inconsistency-adaptive logics for studying inconsistencies in empirical (scientific and everyday) theories have **CLuN**—see below—as their lower limit logic, most inconsistencies in mathematical theories seems to require inconsistency-adaptive logics that have **CLuNs** as their lower limit logic.

2 Syntax

Let $\mathcal{L}$ be the language of **CL** (with identity but without function symbols). We shall take "$\sim$" to be the standard negation of the language—the unqualified word "negation" will always refer to it. For future reference we shall say that $\mathcal{L}$ is *defined* (in the usual way) *from* $\langle \mathcal{S}, \mathcal{C}, \mathcal{V}, \mathcal{P}^1, \mathcal{P}^2, \ldots \rangle$, in which $\mathcal{S}$ is the set of sentential letters, $\mathcal{C}$ the set of (letters for) individual constants, $\mathcal{V}$ the set of variables, and $\mathcal{P}^r$ the set of predicates of rank r.

In agreement with the presentation in [33], we shall take $\mathcal{L}$ to contain bottom ($\perp$). It will have no meaning in pure paraconsistent **CLuNs**, but is implicitly defined by the axiom schema $\perp \supset A$ in full **CLuNs**.[1] The negation $\neg$, explicitly defined by $\neg A =_{df} A \supset \perp$, is coextensive with $\sim$ in **CL**, but not in **CLuNs**. So **CLuNs** may be seen as weaker than **CL**, but also as an extension of **CL** obtained by adding a (rich) paraconsistent negation $\sim$.

CLuNs is an extension of the basic paraconsistent logic **CLuN**,[2] which

[1]This greatly simplifies metatheoretic proofs whereas the properties of pure paraconsistent **CLuNs** are derivable by simple means.

[2]**CLuN** is basic in the following sense. Where $\neg$ is considered as the standard negation of **CL**, **CLuN** is the intersection of all $\sim$-complete extensions of **CL**. Without $\neg$, **CLuN** is the

consists of the full positive fragment of **CL** together with $A \vee {\sim}A$.[3]

It is worth pointing out that Replacement of Equivalents and Replacement of Identicals are not generally valid in **CLuN**. If $\vdash_{\mathbf{CLuN}} A \equiv B$ and D is obtained by replacing A by B in C, then $\vdash_{\mathbf{CLuN}} C \equiv D$ *provided* the replacement did not take place within the scope of a "$\sim$". The origin of the proviso is easily detected. The positive fragment of **CL** does not allow for the replacements within the scope of $\sim$, and adding $A \vee {\sim}A$ does not repair this. Similarly for Replacement of Identicals.

The propositional part of **CLuN** is axiomatized by:

MP	From A and $A \supset B$ to derive B
A⊃1	$A \supset (B \supset A)$
A⊃2	$((A \supset B) \supset A) \supset A$
A⊃3	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A⊥	$\bot \supset A$
A∧1	$(A \wedge B) \supset A$
A∧2	$(A \wedge B) \supset B$
A∧3	$A \supset (B \supset (A \wedge B))$
A∨1	$A \supset (A \vee B)$
A∨2	$B \supset (A \vee B)$
A∨3	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
A≡1	$(A \equiv B) \supset (A \supset B)$
A≡2	$(A \equiv B) \supset (B \supset A)$
A≡3	$(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$
A∼1	$(A \supset {\sim}A) \supset {\sim}A$

Full **CLuN** is obtained by adding:

R∀	To derive $\vdash A \supset (\forall\alpha)B(\alpha)$ from $\vdash A \supset B(\beta)$, provided β does not occur in either A or $B(\alpha)$.
A∀	$(\forall\alpha)A(\alpha) \supset A(\beta)$
R∃	To derive $\vdash (\exists\alpha)A(\alpha) \supset B$ from $\vdash A(\beta) \supset B$, provided β does not occur in either $A(\alpha)$ or B.
A∃	$A(\beta) \supset (\exists\alpha)A(\alpha)$
A=1	$\alpha = \alpha$
A=2	$\alpha = \beta \supset (A \supset B)$ where B is obtained by replacing in A an occurrence of α that occurs outside the scope of a negation by β

The propositional fragment of **CLuNs**, viz. $\mathbf{\Phi_v}$, is obtained by adding to that for **CLuN** a set of axiom schemas that 'drive negation inwards' in the expected way:

A∼∼	${\sim}{\sim}A \equiv A$
A∼⊃	${\sim}(A \supset B) \equiv (A \wedge {\sim}B)$
A∼∧	${\sim}(A \wedge B) \equiv ({\sim}A \vee {\sim}B)$
A∼∨	${\sim}(A \vee B) \equiv ({\sim}A \wedge {\sim}B)$
A∼≡	${\sim}(A \equiv B) \equiv ((A \vee B) \wedge ({\sim}A \vee {\sim}B))$

intersection of all $\sim$-complete extensions of full positive **CL**. We refer to [26] for a proof at the propositional level which is easily generalized to the full logic.

[3]Where negation in **CL** is characterized by the consistency and the completeness presupposition, **CLuN** just retains the latter, thus allowing for gl*u*ts with respect to *n*egation.

To obtain **CLuNs** without identity, add the pertinent axiom schemas and rules of **CLuN** together with:

A$\sim\forall$ $\quad \sim(\forall\alpha)A \equiv (\exists\alpha)\sim A$
A$\sim\exists$ $\quad \sim(\exists\alpha)A \equiv (\forall\alpha)\sim A$

It is worth pointing out two interesting facts at this point. First equivalence is not in general contraposable. Next, the contraposed versions of A$\sim\sim$, A$\sim\wedge$, A$\sim\vee$, A$\sim\forall$, and A$\sim\exists$ are derivable, but those of A$\sim\supset$ and A$\sim\equiv$ are not. (It follows at once that the rule of Replacement of (Provable) Equivalents is not derivable, but it is possible to define another equivalence that warrants replacement—see Section 6.)

How should identity behave in **CLuNs**? We may associate it with "$\equiv$", in which case it will, as in **CLuN**, lead to the Replacement of Identicals that do *not* occur within the scope of a negation. Alternatively, we may require that identity behaves fully classical in sanctioning Replacement of Identicals everywhere. There are three good reasons for the latter decision. The first is that the Replacement of Identicals is of the same type as other 'natural' rules, such as de Morgan properties—compare section 1. The second reason is this. As we shall see in Section 6, it is possible to define in **CLuNs** an equivalence that warrants replacement of formulas that are equivalent (in this sense). Given this, it would be odd not to have full Replacement of Identicals. The third reason is related to the relation between **CLuNs** and **CL**—we postpone its discussion to Section 7. So, while there is no *formal* objection against keeping A=2, we shall take identity in **CLuNs** to be defined by A=1 and A=2^s:

A=2^s $\quad \alpha = \beta \supset (A \supset B)$ where B is obtained by replacing in A an occurrence of α by β

Of course one may consider the variant defined by A=2—there is no formal objection to this.

The *pure paraconsistent* versions of **CLuN** and **CLuNs** are obtained by dropping the axiom A$\perp$. In pure paraconsistent **CLuN** no logical symbol can be eliminated by defining it from the others. In pure paraconsistent **CLuNs** some logical symbols can be eliminated by defining them from the others, as we shall see in Section 6.

3 Semantics and Some Metatheory

We begin with a semantics for **CLuNs** that is arrived by modifying and extending the **CLuN**-semantics—see [10] and especially [17].

According to the **CLuN**-semantics the assignment function v assigns a truth value to all closed formulas—henceforth wffs—of the form $\sim A$. In view of the clause

$$v_M(\sim A) = 1 \text{ iff } v_M(A) = 0 \text{ or } v(\sim A) = 1,$$

CLuN-models are negation-complete but possibly inconsistent. In **CLuNs** the value of the negation of a complex wff depends on the value of its subformulas and/or their negations. Moreover, we have to make sure that A=2^s comes out valid; if $v(a) = v(b)$, then, for example, it is required that $v(\sim Pa) = v(\sim Pb)$.

We shall meet this requirement by applying the (general) method suggested at the end of Section 8 of [10]: v does not assign a truth value to negations of wffs that contain constants, but rather assigns a set of n-tuples of members of the domain to some (specified) meta-linguistic formula of the same form. To simplify the notation, we write, where $\pi^r \in \mathcal{P}^r$, $\sim\pi^r$ instead of $\sim\pi^r\alpha_1\ldots\alpha_r$; similarly, we write $\sim=$ instead of $\sim\alpha=\beta$.

Let $\mathcal{O}$ be a set of *pseudo-constants*; $\mathcal{O}$ should have at least the cardinality of the domain of the largest models one wants to consider. Let the pseudo-language $\mathcal{L}+$ be defined from $\langle\mathcal{S},\mathcal{C}\cup\mathcal{O},\mathcal{V},\mathcal{P}^1,\mathcal{P}^2,\ldots\rangle$—see Section 2. Let $\mathcal{F}+$ and $\mathcal{W}+$ denote respectively the set of formulas and the set of wffs of $\mathcal{L}+$. Formulas that do not contain any logical symbols, except possibly for identity, will be called *primitive formulas*. Finally, let $^\sim\mathcal{S}=\{\sim A \mid A\in\mathcal{S}\}$, $^\sim\mathcal{P}^r=\{\sim\pi^r \mid \pi^r\in\mathcal{P}^r\}$ $(r>0)$, and extend $^\sim\mathcal{P}^2$ with $\sim=$.

A **CLuNs**-model is a couple $M=\langle D,v\rangle$ in which D is a non-empty set and v is an assignment function defined by:

C1.1 $v:\mathcal{S}\mapsto\{0,1\}$
C1.2 $v:\mathcal{C}\cup\mathcal{O}\mapsto D$ (where $D=\{v(\alpha)\mid\alpha\in\mathcal{C}\cup\mathcal{O}\}$)
C1.3 $v:\mathcal{P}^r\mapsto\wp(D^r)$ (the power set of the r-th Cartesian product of D)
C1.4 $v:{}^\sim\mathcal{S}\mapsto\{0,1\}$
C1.5 $v:{}^\sim\mathcal{P}^r\mapsto\wp(D^r)$

The valuation function v_M determined by M is defined as follows:

C2.1 $v_M:\mathcal{W}^+\mapsto\{0,1\}$
C2.2 where $A\in\mathcal{S}$, $v_M(A)=v(A)$; $v_M(\perp)=0$
C2.3 $v_M(\pi^r\alpha_1\ldots\alpha_r)=1$ iff $\langle v(\alpha_1),\ldots,v(\alpha_r)\rangle\in v(\pi^r)$
C2.4 $v_M(\alpha=\beta)=1$ iff $v(\alpha)=v(\beta)$
C2.5 $v_M(A\supset B)=1$ iff $v_M(A)=0$ or $v_M(B)=1$
C2.6 $v_M(A\wedge B)=1$ iff $v_M(A)=1$ and $v_M(B)=1$
C2.7 $v_M(A\vee B)=1$ iff $v_M(A)=1$ or $v_M(B)=1$
C2.8 $v_M(A\equiv B)=1$ iff $v_M(A)=v_M(B)$
C2.9 $v_M((\forall\alpha)A(\alpha))=1$ iff $v_M(A(\beta))=1$ for all $\beta\in\mathcal{C}\cup\mathcal{O}$
C2.10 $v_M((\exists\alpha)A(\alpha))=1$ iff $v_M(A(\beta))=1$ for at least one $\beta\in\mathcal{C}\cup\mathcal{O}$
C2.11 where $\sim A\in{}^\sim\mathcal{S}$, $v_M(\sim A)=1$ iff $v_M(A)=0$ or $v(\sim A)=1$
C2.12 where $r>0$, $v_M(\sim\pi^r\alpha_1\ldots\alpha_r)=1$ iff $v_M(\pi^r\alpha_1\ldots\alpha_r)=0$ or $\langle v(\alpha_1),\ldots,v(\alpha_r)\rangle\in v(\sim\pi^r)$
C2.13 $v_M(\sim\sim A)=v_M(A)$
C2.14 $v_M(\sim(A\supset B))=v_M(A\wedge\sim B)$
C2.15 $v_M(\sim(A\wedge B))=v_M(\sim A\vee\sim B)$
C2.16 $v_M(\sim(A\vee B))=v_M(\sim A\wedge\sim B)$
C2.17 $v_M(\sim(A\equiv B))=v_M((A\vee B)\wedge(\sim A\vee\sim B))$
C2.18 $v_M(\sim(\forall\alpha)A(\alpha))=v_M((\exists\alpha)\sim A(\alpha))$
C2.19 $v_M(\sim(\exists\alpha)A(\alpha))=v_M((\forall\alpha)\sim A(\alpha))$

Truth in a model, semantic consequence, and validity are defined as usual—we sometimes shall write $M\models A$ to express that M verifies A.

Any model is equivalent to (verifies the same wffs as) a $\mathcal{N}$-minimal model, viz. a model in which $v(\sim A)=0$ whenever $v_M(A)=0$. A model is consistent and $\mathcal{N}$-minimal if $v(\sim A)=0$ for all A; if the condition is not fulfilled the model may still be consistent, in which case it is not $\mathcal{N}$-minimal.

The Deduction Theorem is obviously provable. Similarly for Compactness with respect to derivability, semantic consequence, satisfiability, triviality, $\neg$-consistency, and $\sim$-consistency.

Theorem 1 **CLuNs** *is sound with respect to the semantics.*

Proof. The only non-trivial case concerns the truth of $A \vee \sim A$ in every model. To show this, we prove, by an induction on the complexity of A, that $v_M(\sim A) = 1$ if $v_M(A) = 0$. The base case follows immediately from C2.11 and C2.12. For the induction step we consider one clause as an example. Let A be of the form $B \wedge C$. Suppose $v_M(B \wedge C) = 0$. By C2.6 $v_M(B) = 0$ or $v_M(C) = 0$. Hence, by the induction hypothesis, $v_M(\sim B) = 1$ or $v_M(\sim C) = 1$. Consequently $v_M(\sim(B \wedge C)) = v_M(\sim B \vee \sim C) = 1$. ■

For the following theorem, consider a denumerable $\mathcal{O}^\circ \subseteq \mathcal{O}$ and let $\mathcal{L}^\circ$ be the defined from $\langle \mathcal{S}, \mathcal{C} \cup \mathcal{O}^\circ, \mathcal{V}, \mathcal{P}^1, \mathcal{P}^2, \ldots \rangle$.

Theorem 2 **CLuNs** *is strongly complete with respect to the semantics.*

Proof. Suppose that $\Gamma \nvdash_{\mathbf{CLuNs}} A$. Consider, as for the proof in **CL**, a sequence $B_1, B_2, \ldots$ that contains all wffs (of $\mathcal{L}^\circ$) and in which each wff of the form $(\exists\alpha)C$ is followed immediately by an instance with a constant that does not occur in Γ, in A, or in any previous member of the sequence. We then define

$$\begin{aligned}
\Delta_0 &= Cn_{\mathbf{CLuNs}}(\Gamma) \\
\Delta_{i+1} &= Cn_{\mathbf{CLuNs}}(\Delta_i \cup \{B_{i+1}\}) \text{ if } A \notin Cn_{\mathbf{CLuNs}}(\Delta_i \cup \{B_{i+1}\}), \text{ and} \\
\Delta_{i+1} &= \Delta_i \text{ otherwise} \\
\Delta &= \Delta_0 \cup \Delta_1 \cup \ldots
\end{aligned}$$

Each of the following is provable:

(i) $\Gamma \subseteq \Delta$ (by the construction).

(ii) $A \notin \Delta$ (by the construction).

(iii) Δ is deductively closed (by the definition of Δ).

(iv) Δ is maximally non-trivial. To see this, remark first that $A \supset C \in \Delta$ for all C. Indeed, if $A \supset C \notin \Delta$, then there is a Δ_i such that $\Delta_i \cup \{A \supset C\} \vdash A$; hence $\Delta_i \vdash (A \supset C) \supset A$ by the Deduction Theorem; hence, in view of A$\supset$2, $\Delta_i \vdash A$, which is impossible. If $E \notin \Delta$, then there is a Δ_i such that $\Delta_i \cup \{E\} \vdash A$ and hence $\Delta \cup \{E\} \vdash A$; as $A \supset C \in \Delta$ for all C, $\Delta \cup \{E\}$ is trivial.

(v) Δ is prime, i.e.: if $C \vee E \in \Delta$, then $C \in \Delta$ or $E \in \Delta$. Suppose that $C \vee E \in \Delta$, $C \notin \Delta$ and $E \notin \Delta$; hence, as in the proof of (iv), $\Delta \cup \{C\} \vdash A$ and $\Delta \cup \{D\} \vdash A$, and also $\Delta \vdash C \supset A$ and $\Delta \vdash D \supset A$ by the Deduction Theorem; but then $\Delta \vdash (C \vee D) \supset A$ and hence $\Delta \vdash A$, which is impossible.

(vi) Δ is ω-complete with respect to $\mathcal{L}^\circ$.[4] As for **CL**, the order of the sequence $B_1, B_2, \ldots$ and R$\exists$ warrant that, if $(\exists\alpha)C(\alpha) \in \Delta$, then $C(\beta) \in \Delta$ for some $\beta \in \mathcal{C} \cup \mathcal{O}^\circ$.

We now define a **CLuNs**-model M from Δ. Let $[\![\alpha]\!]$, the equivalence class of $\alpha \in \mathcal{C} \cup \mathcal{O}^\circ$, be such that $\beta \in [\![\alpha]\!]$ iff $\alpha = \beta \in \Delta$.

[4] Δ is ω-complete iff, if $(\exists\alpha)A(\alpha) \in \Delta$, then $A(\beta) \in \Delta$ for some $\beta \in \mathcal{C} \cup \mathcal{O}^\circ$.

1. $D = \{[\![\alpha]\!] \mid \alpha \in \mathcal{C} \cup \mathcal{O}^\circ\}$;
2. for all $C \in \mathcal{S}$, $v(C) = 1$ iff $C \in \Delta$;
3. for all $\alpha \in \mathcal{C} \cup \mathcal{O}^\circ$, $v(\alpha) = [\![\alpha]\!]$;
4. for all r, $v(\pi^r) = \{\langle [\![\alpha_1]\!], \ldots, [\![\alpha_r]\!]\rangle \mid \pi^r\alpha_1 \ldots \alpha_r \in \Delta\}$;
5. for all $\sim C \in {}^{\sim}\mathcal{S}$, $v(\sim C) = 1$ iff $\sim C \in \Delta$;
6. for all $\pi^r \in \mathcal{P}^r$, $v(\sim\pi^r) = \{\langle [\![\alpha_1]\!], \ldots, [\![\alpha_r]\!]\rangle \mid \sim\pi^r\alpha_1 \ldots \alpha_r \in \Delta\}$.

We finally show, by an induction on the complexity of the wffs of $\mathcal{L}^\circ$, that, for every wff C, $v_M(C) = 1$ iff $C \in \Delta$.

In view of C2.2–4, 1–6 warrant that, where C is a primitive wff, $v_M(C) = 1$ iff $C \in \Delta$—the proof is completely standard. Also, if C is a primitive wff, then 5 and 6 warrant that $v_M(\sim C) = 1$ iff $\sim C \in \Delta$.[5]

With primitive wffs and their negations as the base case, we proceed by the usual induction. Let us consider one of the many cases, viz. $C = \sim(D \wedge E)$:

$\sim(D \wedge E) \in \Delta$	iff	$\sim D \vee \sim E \in \Delta$ (as Δ is deductively closed)
	iff	$\sim D \in \Delta$ or $\sim E \in \Delta$ (as Δ is prime)
	iff	$v_M(\sim D) = 1$ or $v_M(\sim E) = 1$ (by the induction hypothesis)
	iff	$v_M(\sim(D \wedge E)) = 1$ (by C2.7 and C2.15)

As $v_M(C) = 1$ iff $C \in \Delta$, (i) and (ii) give us: $v_M(B) = 1$ for all $B \in \Gamma$, and $v_M(A) = 0$. Hence $\Gamma \nvDash_{\mathbf{CLuNs}} A$. ∎

The semantics for the pure paraconsistent version of **CLuNs** is obtained by dropping the subclause on $\perp$ from C2.2. The proof of all aforementioned theorems for that version is easily derived from the above proofs. The situation is exactly the same for the semantic systems presented in subsequent sections, whence we shall not repeat it there.

4 Three-Valued Semantics

Several brands of semantic styles allow for more elegant characterizations of **CLuNs**. We shall mention four of them: a three-valued semantics (this Section), a plus-minus semantics, a Priest-style semantics, and an ambiguity semantics (next section). The elegance of the three-valued semantics resides especially in the fact that all logical constants are truth-functions in it—this was shown in [8] for the propositional version and is extended here for the predicative version.

Consider the values T, I, and F, corresponding to "consistently true", "inconsistent" and "consistently false" respectively. Where $M = \langle D, V\rangle$ (defined for the language $\mathcal{L}+$) is a three-valued **CLuNs**-model, the valuation function V_M maps $\mathcal{W}+$ on $\{T, I, F\}$. A is *true in* M iff $V_M(A) \in \{T, I\}$. Let us start with the propositional fragment. The behaviour of propositional letters is characterized by:

[5] This can still be proved by relying on C2.11 and C2.12 if one requires, in 5, that $C, \sim C \in \Delta$, and, in 6, that $\pi^r\alpha_1 \ldots \alpha_r, \sim\pi^r\alpha_1 \ldots \alpha_r \in \Delta$. In this case M is $\mathcal{N}$-minimal.

$V : \mathcal{S} \mapsto \{T, I, F\}$
where $A \in \mathcal{S}$, $V_M(A) = V(A)$; $V_M(\bot) = F$

The meaning of three connectives is defined by the following matrices:

	$\sim$
T	F
I	I
F	T

$\supset$	T	I	F
T	T	I	F
I	T	I	F
F	T	T	T

$\wedge$	T	I	F
T	T	I	F
I	I	I	F
F	F	F	F

whereas the two further connectives may be defined explicitly—we list the tables for the reader's ease:[6]

$A \vee B =_{df} \sim(\sim A \wedge \sim B)$
$A \equiv B =_{df} (A \supset B) \wedge (B \supset A)$

$\vee$	T	I	F
T	T	T	T
I	T	I	I
F	T	I	F

$\equiv$	T	I	F
T	T	I	F
I	I	I	F
F	F	F	T

In order to extend this to the predicative level, we let V assign elements of D to members of $\mathcal{C} \cup \mathcal{O}$ in such a way that $D = \{V(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$. Next, we let V assign a triple $\langle \Sigma_1, \Sigma_2, \Sigma_3 \rangle$ to members of $\mathcal{P}^r$ such that $\Sigma_1, \Sigma_2, \Sigma_3 \in \wp(D^r)$, $\Sigma_1 \cap \Sigma_2 = \Sigma_1 \cap \Sigma_3 = \Sigma_2 \cap \Sigma_3 = \emptyset$, and $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 = \wp(D^r)$. To simplify the notation, we consider V as composed in this case of the three functions V^T, V^I, and V^F, with $V^T(\pi^r) = \Sigma_1$, $V^I(\pi^r) = \Sigma_2$, and $V^F(\pi^r) = \Sigma_3$. The three functions determine for which r-tuples the predicate is true, inconsistent, and false respectively. The values of primitive predicative expressions are obviously determined by:

$V_M(\pi^r\alpha_1 \ldots \alpha_r) = T$ iff $\langle V(\alpha_1), \ldots, V(\alpha_r) \rangle \in V^T(\pi^r)$
$V_M(\pi^r\alpha_1 \ldots \alpha_r) = I$ iff $\langle V(\alpha_1), \ldots, V(\alpha_r) \rangle \in V^I(\pi^r)$
$V_M(\pi^r\alpha_1 \ldots \alpha_r) = F$ iff $\langle V(\alpha_1), \ldots, V(\alpha_r) \rangle \in V^F(\pi^r)$

Identity is considered as a binary predicate with the special characteristic that $V^T(=) \cup V^I(=) = \{\langle o, o \rangle \mid o \in D\}$. This obviously warrants that $V_M(\alpha = \alpha) \in \{T, I\}$ for all α and M.

Finally, the value of universally quantified wffs is determined by:

$V_M((\forall\alpha)A(\alpha)) = T$ iff $V_M(A(\beta)) = T$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$
$V_M((\forall\alpha)A(\alpha)) = F$ iff $V_M(A(\beta)) = F$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$
$V_M((\forall\alpha)A(\alpha)) = I$ iff $V_M(A(\beta)) \in \{T, I\}$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$ and $V_M(A(\beta)) = I$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$

whereas the existential quantifier can be explicitly defined by

$(\exists\alpha)A(\alpha) =_{df} \sim(\forall\alpha)\sim A(\alpha)$

Remark that the value of universally and existentially quantified formulas corresponds respectively to that of the infinite conjunctions and disjunctions of their instances in $\mathcal{L}^+$—compare to the instructive table on p. 140 of [19].

We shall say that two semantic systems are *equivalent* iff their semantic consequence relations coincide.

[6] It follows that the system $\mathbf{RM}_3^{\supset}$ from [5] is identical to the propositional fragment of pure paraconsistent **CLuNs**.

Theorem 3 *The two-valued* **CLuNs**-*semantics is equivalent to the three-valued* **CLuNs**-*semantics.*

Proof. It is obvious that any two-valued model $M = \langle D, v\rangle$ may be transformed to a three-valued model $M' = \langle D, V\rangle$, and that any three-valued model $M' = \langle D, V\rangle$ may be transformed to a two-valued model $M = \langle D, v\rangle$ such that

(i) Where $\alpha \in \mathcal{C} \cup \mathcal{O}$, $V(\alpha) = v(\alpha)$.
(ii) where $A \in \mathcal{S}$, $V(A) = T$ iff $v(A) = 1$ and $v({\sim}A) = 0$, $V(A) = I$ iff $v(A) = 1$ and $v({\sim}A) = 1$, $V(A) = F$ iff $v(A) = 0$ and $v({\sim}A) = 1$.
(iii) where $\pi^r \in \mathcal{P}^r$, $V^T(\pi^r) = v(\pi^r) - v({\sim}\pi^r)$, $V^I(\pi^r) = v(\pi^r) \cap v({\sim}\pi^r)$, $V^F(\pi^r) = v({\sim}\pi^r) - v(\pi^r)$.[7]

The method for obtaining the three-valued model from the two-valued one is immediate and elementary transformations provide the method for the converse.[8]

We leave it to the reader to check that, whenever A is an a primitive wff or its negation, the following equivalences hold:[9]

(1) $V_{M'}(A) = T$ iff $v_M(A) = 1$ and $v_M({\sim}A) = 0$
(2) $V_{M'}(A) = I$ iff $v_M(A) = 1$ and $v_M({\sim}A) = 1$
(3) $V_{M'}(A) = F$ iff $v_M(A) = 0$ and $v_M({\sim}A) = 1$

By the usual induction on the complexity of wffs, it is easily seen that (1)–(3) hold for all wffs. It follows that M and M' verify exactly the same wffs. ■

5 Some Further Semantic Characterizations

Rather elegant characterizations are obtained by a so-called plus-minus semantics.[10] One of the sources of paraconsistency is that, in some circumstances and for some A, one has good reasons to assert A and one also has good reasons to deny A. The idea is naturally rendered by a valuation function that assigns to each wff an assertion value as well as a denial value. Similarly, the assignment function will assign a couple of values to members of $\mathcal{S}$, $\mathcal{P}^1$, $\mathcal{P}^2$ (including identity), $\mathcal{P}^3$, ... Negation is then analysed by identifying the assertion value of ${\sim}A$ with the denial value of A. Where v is the assignment function, we shall refer to the elements of the couple separately by v^+ and v^-; similarly for the valuation function v_M.

A model is a couple $M = \langle D, \mathsf{v}\rangle$ in which D is a set and v is an assignment function defined by:

C1.1 $\mathsf{v}^+ : \mathcal{S} \mapsto \{0, 1\}$
$\mathsf{v}^- : \mathcal{S} \mapsto \{0, 1\}$
restriction: where $A \in \mathcal{S}$, $\mathsf{v}^+(A) + \mathsf{v}^-(A) \geq 1$

[7] Remember that this handles identity.

[8] For example (iii) is equivalent to "where $\pi^r \in \mathcal{P}^r$, $v(\pi^r) = V^T(\pi^r) \cup V^I(\pi^r)$ and $v({\sim}\pi^r) = V^F(\pi^r) \cup V^I(\pi^r)$".

[9] (1)–(3) are obviously equivalent to (1′) $v_M(A) = 1$ iff $V_{M'}(A) \in \{T, I\}$ and (2′) $v_M({\sim}A) = 1$ iff $V_{M'}(A) \in \{F, I\}$.

[10] To the best of our knowledge, this type of semantics was derived from Asenjo's semantics for the logic of antinomies (see for example [4]) in which two n-place relations are assigned to each predicate of rank n. It is not difficult to show that **CLuNs** coincides with the antinomic predicate calculus (if it is described in the standard metalanguage and if one disregards $\perp$).

C1.2 $\mathsf{v} : \mathcal{C} \cup \mathcal{O} \mapsto D$
C1.3 $\mathsf{v}^+ : \mathcal{P}^r \mapsto \wp(D^r)$ (the power set of the r-th Cartesian product of D)
$\mathsf{v}^- : \mathcal{P}^r \mapsto \wp(D^r)$
restriction: $\mathsf{v}^+(\pi^r) \cup \mathsf{v}^-(\pi^r) = D^r$
C1.4 $\mathsf{v}^+(=) = \{\langle o, o\rangle \mid o \in D\}$
$\mathsf{v}^-(=) \subseteq D^2$
restriction: $\mathsf{v}^+(=) \cup \mathsf{v}^-(=) = D^2$

The valuation function v_M determined by the model M is defined by

C2.1 $\mathsf{v}_M^+ : \mathcal{W}^+ \mapsto \{0,1\}$
$\mathsf{v}_M^- : \mathcal{W}^+ \mapsto \{0,1\}$
C2.2 where $A \in \mathcal{S}$, $\mathsf{v}_M^+(A) = \mathsf{v}^+(A)$; $\mathsf{v}_M^+(\bot) = 0$
where $A \in \mathcal{S}$, $\mathsf{v}_M^-(A) = \mathsf{v}^-(A)$; $\mathsf{v}_M^-(\bot) = 1$
C2.3 $\mathsf{v}_M^+(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle \mathsf{v}(\alpha_1), \ldots, \mathsf{v}(\alpha_r)\rangle \in \mathsf{v}^+(\pi^r)$
$\mathsf{v}_M^-(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle \mathsf{v}(\alpha_1), \ldots, \mathsf{v}(\alpha_r)\rangle \in \mathsf{v}^-(\pi^r)$
C2.4 $\mathsf{v}_M^+(\alpha = \beta) = 1$ iff $\langle \mathsf{v}(\alpha), \mathsf{v}(\beta)\rangle \in \mathsf{v}^+(=)$
$\mathsf{v}_M^-(\alpha = \beta) = 1$ iff $\langle \mathsf{v}(\alpha), \mathsf{v}(\beta)\rangle \in \mathsf{v}^-(=)$
C2.5 $\mathsf{v}_M^+(\sim A) = \mathsf{v}_M^-(A)$
$\mathsf{v}_M^-(\sim A) = \mathsf{v}_M^+(A)$
C2.6 $\mathsf{v}_M^+(A \supset B) = 1$ iff $\mathsf{v}_M^+(A) = 0$ or $\mathsf{v}_M^+(B) = 1$
$\mathsf{v}_M^-(A \supset B) = 1$ iff $\mathsf{v}_M^+(A) = 1$ and $\mathsf{v}_M^-(B) = 1$
C2.7 $\mathsf{v}_M^+(A \wedge B) = 1$ iff $\mathsf{v}_M^+(A) = 1$ and $\mathsf{v}_M^+(B) = 1$
$\mathsf{v}_M^-(A \wedge B) = 1$ iff $\mathsf{v}_M^-(A) = 1$ or $\mathsf{v}_M^-(B) = 1$
C2.8 $\mathsf{v}_M^+((\forall\alpha)A(\alpha)) = 1$ iff $\mathsf{v}_M^+(A(\beta)) = 1$ for all $\beta \in \mathcal{C} \cup \mathcal{O}$
$\mathsf{v}_M^-((\forall\alpha)A(\alpha)) = 1$ iff $\mathsf{v}_M^-(A(\beta)) = 1$ for at least one $\beta \in \mathcal{C} \cup \mathcal{O}$

$A \vee B$, $A \equiv B$, and $(\exists\alpha)A$ are defined as in Section 4. A is true in a model M iff $\mathsf{v}_M^+(A) = 1$. Semantic consequence and validity are defined as usual.

The reader may easily check that the clauses are quite intuitive. For example, one has a reason to deny $A \wedge B$ iff one has a reason for denying at least one of them; one has a reason to deny a universally quantified statement iff one has a reason for denying at least one instance of it (supposing that we had no trouble naming every object in the domain), etc.

Theorem 4 *The three-valued* **CLuNs**-*semantics is equivalent to the 'plus-minus'* **CLuNs**-*semantics.*

Proof. The proof is longwinded but obvious. A three-valued model M is turned into a 'plus-minus' model M', and *vice versa*, in view of the following equivalences:

(i) Where $\alpha \in \mathcal{C} \cup \mathcal{O}$, $V(\alpha) = \mathsf{v}(\alpha)$.
(ii) where $A \in \mathcal{S}$, $\mathsf{v}^+(A) = 1$ iff $V(A) \in \{T, I\}$, $\mathsf{v}^-(A) = 1$ iff $V(A) \in \{I, F\}$.
(iii) where $\pi^r \in \mathcal{P}^r$, $\mathsf{v}^+(\pi^r) = V^T(\pi^r) \cup V^I(\pi^r)$, $\mathsf{v}^-(\pi^r) = V^I(\pi^r) \cup V^F(\pi^r)$.

Next one establishes that the following equivalences hold for all primitive wffs of $\mathcal{L}^+$, and one applies an induction similar to that in the proof of Theorem 3 to generalize this result to all wffs of $\mathcal{L}^+$:

(1) $V_M(A) = T$ iff $\mathsf{v}_{M'}^+(A) = 1$ and $\mathsf{v}_{M'}^-(A) = 0$

(2) $V_M(A) = I$ iff $\mathsf{v}^+_{M'}(A) = 1$ and $\mathsf{v}^-_{M'}(A) = 1$
(3) $V_M(A) = F$ iff $\mathsf{v}^+_{M'}(A) = 0$ and $\mathsf{v}^-_{M'}(A) = 1$

■

It seems worthwhile to look at some variants of the present semantics. First, the requirements in the definition of the assignment may be dropped, provided one ensures the validity of $A \vee {\sim}A$ by the valuation functions. For example, C2.2 then needs to be modified by (leaving $\perp$ alone and) either changing the first part to

$$\text{where } A \in \mathcal{S},\ \mathsf{v}^+_M(A) = 1 \text{ iff } \mathsf{v}^+(A) = 1 \text{ or } \mathsf{v}^-(A) = 0$$

or by changing the second part to

$$\text{where } A \in \mathcal{S},\ \mathsf{v}^-_M(A) = 1 \text{ iff } \mathsf{v}^-(A) = 1 \text{ or } \mathsf{v}^+(A) = 0$$

In proceeding thus, the assignment itself is neutral with respect to properties of $\sim$-consistency and $\sim$-completeness, and the valuation determines whether the models are interpreted classically, paraconsistently, paracompletely, or both paraconsistently and paracompletely.

It may be more elegant to loosen C1.4 thus:

$$\mathsf{v}^+(=) \supseteq \{\langle o, o\rangle \mid o \in D\}$$
$$\mathsf{v}^-(=) \supseteq \{\langle o_1, o_2\rangle \mid o_1, o_2 \in D \text{ and } o_1 \neq o_2\}$$

Both identity and its negation then behave abnormally in a symmetric way. Technically, $a = b \vdash A(a) \equiv A(b)$ is warranted by defining equivalence classes of members of D such that $[o_1] = [o_2]$ iff $\langle o_1, o_2\rangle \in \mathsf{v}^+(=)$, and by letting v assign such equivalence classes to members of $\mathcal{C} \cup \mathcal{O}$ and r-tuples of such equivalence classes to members of $\mathcal{P}^r$.

The same idea may be realized in an even simpler way. Let S be a non-empty set, R an equivalence relation over S, and D the set of the equivalence classes obtained from R. $\mathsf{v}(a) \in D$ is then a set of members of S. Identity may be handled directly by the valuation thus:

$$\mathsf{v}^+_M(\alpha = \beta) = 1 \text{ iff } \mathsf{v}(\alpha) = \mathsf{v}(\beta)$$
$$\mathsf{v}^-_M(\alpha = \beta) = 1 \text{ iff } o_1 \neq o_2 \text{ for some } o_1 \in \mathsf{v}(\alpha) \text{ and an } o_2 \in \mathsf{v}(\beta)$$

The upshot is that $\mathsf{v}^+_M(\alpha = \beta) = 1$ and $\mathsf{v}^-_M(\alpha = \beta) = 0$ iff $\mathsf{v}(\alpha) = \mathsf{v}(\beta)$ and $\mathsf{v}(\alpha)$ is a singleton; $\mathsf{v}^+_M(\alpha = \beta) = 1 = \mathsf{v}^-_M(\alpha = \beta)$ iff $\mathsf{v}(\alpha) = \mathsf{v}(\beta)$ and $\mathsf{v}(\alpha)$ is not a singleton; $\mathsf{v}^+_M(\alpha = \beta) = 0$ and $\mathsf{v}^-_M(\alpha = \beta) = 1$ iff $\mathsf{v}(\alpha) \neq \mathsf{v}(\beta)$. In other words, inconsistencies with respect to identity arise just in case two terms refer to the same equivalence class, but refer inconsistently, viz. to a multiplicity of objects that are 'erroneously' identified. The idea is related to collapsed models in the sense of [31]. We shall see below that it may be generalized.

In Priest's preferred semantic style, the truth-values are not members but subsets of $\{0, 1\}$. This is combined with the plus-minus approach for predicative letters. In view of Theorems 3 and 4, we can be very brief. First, the three-valued values T, I, and F are translated as $\{1\}$, $\{1, 0\}$, and $\{0\}$ respectively. Next, primitive predicative expressions (including identities) are evaluated by

$$\mathsf{V}_M(\pi^r \alpha_1 \ldots \alpha_r) = \{1\} \text{ iff } \langle \mathsf{V}(\alpha_1), \ldots, \mathsf{V}(\alpha_r)\rangle \in \mathsf{V}^+(\pi^r) - \mathsf{V}^-(\pi^r)$$

$$\mathsf{V}_M(\pi^r\alpha_1\dots\alpha_r) = \{0,1\} \text{ iff } \langle \mathsf{V}(\alpha_1),\dots,\mathsf{V}(\alpha_r)\rangle \in \mathsf{V}^+(\pi^r)\cap\mathsf{V}^-(\pi^r)$$
$$\mathsf{V}_M(\pi^r\alpha_1\dots\alpha_r) = \{0\} \text{ iff } \langle \mathsf{V}(\alpha_1),\dots,\mathsf{V}(\alpha_r)\rangle \in \mathsf{V}^-(\pi^r) - \mathsf{V}^+(\pi^r)$$

That the resulting **CLuNs**-semantics is equivalent to the semantic systems listed before is immediate. It follows at once that Priest's **LP** —see *e.g.*, [30]—is the $\sim$-$\vee$-$\wedge$-$\forall$-$\exists$-fragment of **CLuNs**.[11]

This semantic style is attractive for dialetheists like Priest. They want their paraconsistent logic as the logic of the metalanguage, and want to say that some A is both true and false, rather than saying that both A and $\sim A$ are true. Indeed, the three values $\{0\}$, $\{1\}$, $\{0,1\}$ may be interpreted as "false only", "true only", and "both true and false". Much of the attractiveness vanishes if one realizes that the dialetheist seems unable to formulate this semantics in his preferred metalanguage.[12]

The assignment functions of all semantic systems mentioned up to this point seem to suggest that **CLuNs** presupposes that "the world" is in one way or other inconsistent. This, however, is not the case as may be seen from the semantics presented in the Appendix of [11]. We briefly outline a (simplified and) two-valued counterpart to that semantics, and shall call it here the *ambiguity semantics* for **CLuNs** to distinguish it from the two-valued semantics from Section 3.

Where the assignment function of the standard **CL**-model assigns an element of a set S to some non-logical symbol, the assignment function of an ambiguity model assigns to the symbol a non-empty subset of S. Intuitively, the symbol may have different meanings rather than one.[13]

A model is a couple $M = \langle D, \mathtt{v}\rangle$ in which D is a set and $\mathtt{v}$ is an assignment function defined by:

C1.1 $\mathtt{v} : \mathcal{C}\cup\mathcal{O} \mapsto (\wp(D) - \emptyset)$ (where $\wp(D) - \emptyset = \{\mathtt{v}(\alpha) \mid \alpha \in \mathcal{C}\cup\mathcal{O}\}$)
C1.2 $\mathtt{v} : \mathcal{S} \mapsto (\wp(\{0,1\}) - \emptyset)$
C1.3 $\mathtt{v} : \mathcal{P}^r \mapsto (\wp(\wp(D^r)) - \emptyset)$

Identity is not handled as a predicate of rank 2, but will be handled directly by the valuation function.

We shall use R, R_1, etc. as variables for relations over D (sets of r-tuples of members of D). Where π is a predicate of rank r, $\mathtt{v}(\pi)$ is a set of relations of adicity r. This explains phrases as the following: $\langle o_1,\dots,o_r\rangle \in R$ for some $R \in \mathtt{v}(\pi)$. Remark that, where $A \in \mathcal{S}$, $\mathtt{v}(A) \in \{\{0\},\{1\},\{0,1\}\}$.

The valuation function $\mathtt{v}_M : \mathcal{W} \mapsto \{0,1\}$ is defined as follows for primitive wffs and their negations:

C2.1 where $A \in \mathcal{S}$,
$\mathtt{v}_M(A) = 1$ iff $1 \in \mathtt{v}(A)$
$\mathtt{v}_M(\sim A) = 1$ iff $0 \in \mathtt{v}(A)$
$\mathtt{v}_M(\bot) = 0$
$\mathtt{v}_M(\sim\bot) = 1$

[11] Where we use a classical metalanguage, Priest uses a metalanguage that has **LP** as its underlying logic. However, as was shown in [9], the statement in the text holds true under both metalinguistic descriptions. See, however, the following paragraph in the text.

[12] Some arguments to this effect are presented in [9]. A more extensive and updated discussion, including arguments from for example [2, pp. 496–497] and [32], is presented in [14].

[13] The symbol has an unambiguous meaning iff it is assigned a singleton.

C2.2 where $\pi \in \mathcal{P}^r$ and $\alpha_1, \ldots, \alpha_r \in \mathcal{C} \cup \mathcal{O}$,
$\mathtt{v}_M(\pi\alpha_1 \ldots \alpha_r) = 1$ iff $\langle o_1, \ldots, o_r \rangle \in R$ for some $o_1 \in \mathtt{v}(\alpha_1)$, …, for some $o_r \in \mathtt{v}(\alpha_r)$ and for some $R \in \mathtt{v}(\pi)$,
$\mathtt{v}_M({\sim}\pi\alpha_1 \ldots \alpha_r) = 1$ iff $\langle o_1, \ldots, o_r \rangle \notin R$ for some $o_1 \in \mathtt{v}(\alpha_1)$, …, for some $o_r \in \mathtt{v}(\alpha_r)$ and for some $R \in \mathtt{v}(\pi)$

C2.3 where $\alpha, \beta \in \mathcal{C} \cup \mathcal{O}$,
$\mathtt{v}_M(\alpha = \beta) = 1$ iff $\mathtt{v}(\alpha) = \mathtt{v}(\beta)$
$\mathtt{v}_M({\sim}\alpha = \beta) = 1$ iff $o_1 \neq o_2$ for some $o_1 \in \mathtt{v}(\alpha)$ and $o_2 \in \mathtt{v}(\beta)$

All other wffs are handled by clauses C2.5–10 and C2.13–19 of the two-valued semantics from the Section 3 (replacing v_M by $\mathtt{v}_M$).

In order to clarify the second half of the proof of the following theorem, we mention that a **CLuNs**-model verifies $a = b \wedge {\sim}a = b$ iff it verifies both $a = b$ and $a = a \wedge {\sim}a = a$ and the latter holds just in case, in the two-valued semantics, $v({\sim}a = a) = 1$.

Theorem 5 *The ambiguity semantics is equivalent to the two-valued semantics.*

Proof. We outline the proof that, from each ambiguity model M, an equivalent two-valued model M' may be defined, and *vice versa.* To simplify the notation, D will be the domain of the ambiguity model M and o, o', o_1 etc. will refer to members of D; D' will be the domain of the two-valued model M' and x, x', x_1 etc. will refer to members of D'.

From an ambiguity model $M = \langle D, \mathtt{v} \rangle$ we define a two-valued model $M' = \langle D', v \rangle$ as follows.

(1) $D' = \wp(D) - \emptyset$.
(2) Where $A \in \mathcal{S}$, $v(A) = 1$ iff $1 \in \mathtt{v}(A)$, and $v({\sim}A) = 1$ iff $0 \in \mathtt{v}({\sim}A)$.
(3) Where $\alpha \in \mathcal{C} \cup \mathcal{O}$, $v(\alpha) = \mathtt{v}(\alpha)$—remark that $v(\alpha) \in D'$ as required.
(4) Where $\pi \in \mathcal{P}^r$, $v(\pi)$ is the set of $\langle \mathsf{x}_1, \ldots, \mathsf{x}_r \rangle$ such that $\langle \mathsf{o}_1, \ldots, \mathsf{o}_r \rangle \in R$ for some $\mathsf{o}_1 \in \mathsf{x}_1$, …, for some $\mathsf{o}_r \in \mathsf{x}_r$ and for some $R \in \mathtt{v}(\pi)$.
(5) $v({\sim}{=})$ is the set of $\langle \mathsf{x}_1, \mathsf{x}_2 \rangle$ such that $\mathsf{o}_1 \neq \mathsf{o}_2$ for some $\mathsf{o}_1 \in \mathsf{x}_1$ and $\mathsf{o}_2 \in \mathsf{x}_2$.
(6) Where $\pi \in \mathcal{P}^r$ is different from $=$, $v({\sim}\pi)$ is the set of $\langle \mathsf{x}_1, \ldots, \mathsf{x}_r \rangle$ such that $\langle \mathsf{o}_1, \ldots, \mathsf{o}_r \rangle \notin R$ for some $\mathsf{o}_1 \in \mathsf{x}_1$, …, for some $\mathsf{o}_r \in \mathsf{x}_r$ and for some $R \in \mathtt{v}(\pi)$.

We leave to the reader the (by now obvious) task to show that $v_{M'}(A) = \mathtt{v}_M(A)$, first for all primitive wffs A, and next, by the standard induction on the complexity of wffs, for all wffs A.

From a two-valued model $M' = \langle D', v \rangle$ we define an ambiguity model $M = \langle D, \mathtt{v} \rangle$ as follows. Let f be a function such that, for all $\mathsf{x} \in D'$, $f(\mathsf{x}) = \{\mathsf{x}\}$ if $\langle \mathsf{x}, \mathsf{x} \rangle \notin v({\sim}{=})$, and $f(\mathsf{x}) = \{\mathsf{x}, \{\mathsf{x}\}\}$ if $\langle \mathsf{x}, \mathsf{x} \rangle \in v({\sim}{=})$.

(1) $D = \bigcup \{f(\mathsf{x}) \mid \mathsf{x} \in D'\}$
(2) Where $A \in \mathcal{S}$,
   - $1 \in \mathtt{v}(A)$ iff $v(A) = 1$ and
   - $0 \in \mathtt{v}(A)$ iff $v(A) = 0$ or $v({\sim}A) = 1$.
(3) Where $\alpha \in \mathcal{C} \cup \mathcal{O}$, $\mathtt{v}(\alpha) = f(v(\alpha))$.
(4) Where $\pi \in \mathcal{P}^r$, $\mathtt{v}(\pi) = \{R_\pi, R'_\pi\}$ in which
   - $R_\pi = \{\langle \mathsf{o}_1, \ldots, \mathsf{o}_r \rangle \mid \mathsf{o}_1 \in f(\mathsf{x}_1), \ldots, \mathsf{o}_r \in f(\mathsf{x}_r),$ for some $\langle \mathsf{x}_1, \ldots, \mathsf{x}_r \rangle \in v(\pi) - v({\sim}\pi)\}$ and

- $R'_\pi = \{\langle \mathsf{o}_1, \ldots, \mathsf{o}_r \rangle \mid \mathsf{o}_1 \in f(\mathsf{x}_1), \ldots, \mathsf{o}_r \in f(\mathsf{x}_r),$ for some $\langle \mathsf{x}_1, \ldots, \mathsf{x}_r \rangle \in v(\pi)\}$.

We now show that $\mathtt{v}_M(A) = v_{M'}(A)$ for all primitive formulas A.

Consider some $A \in \mathcal{S}$. We have (with some notational abuse):

(i) $v_{M'}(A) = 1$ iff $v(A) = 1$ iff $1 \in \mathtt{v}(A)$ iff $\mathtt{v}_M(A) = 1$, and
(ii) $v_{M'}({\sim}A) = 1$ iff ($v(A) = 0$ or $v({\sim}A) = 1$) iff $0 \in \mathtt{v}(A)$ iff $\mathtt{v}_M({\sim}A) = 1$.

Where $\alpha, \beta \in \mathcal{C} \cup \mathcal{O}$, we have for identity:

(i) $v_{M'}(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$ iff $\mathtt{v}(\alpha) = f(v(\alpha)) = f(v(\beta)) = \mathtt{v}(\beta)$ iff $\mathtt{v}_M(\alpha = \beta) = 1$,
(ii) $v_{M'}({\sim}\alpha = \beta) = 1$ iff ($v(\alpha) \neq v(\beta)$, or $\langle v(\alpha), v(\beta) \rangle \in v({\sim}{=})$) iff ($\mathtt{v}(\alpha) \neq \mathtt{v}(\beta)$, or $\mathtt{v}(\alpha) = \mathtt{v}(\beta) = \{v(\alpha), \{v(\alpha)\}\}$) iff $\mathsf{o}_1 \neq \mathsf{o}_2$ for some $\mathsf{o}_1 \in \mathtt{v}(\alpha)$ and $\mathsf{o}_2 \in \mathtt{v}(\beta)$ iff $\mathtt{v}_{M'}({\sim}\alpha = \beta) = 1$.

Consider some $\pi \in \mathcal{P}^r$ that is different from identity.

(i) Suppose that $v_{M'}(\pi\alpha_1 \ldots \alpha_r) = 1$. It follows that $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \in v(\pi)$ and hence $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \in R'_\pi$. Hence $\langle \mathsf{o}_1, \ldots, \mathsf{o}_r \rangle \in R'_\pi$ for some $\mathsf{o}_1 \in \mathtt{v}(\alpha_1)$, …, for some $\mathsf{o}_r \in \mathtt{v}(\alpha_r)$. Hence $\mathtt{v}_M(\pi\alpha_1 \ldots \alpha_r) = 1$.
(ii) Suppose that $\mathtt{v}_M(\pi\alpha_1 \ldots \alpha_r) = 1$. Hence $\langle \mathsf{o}_1, \ldots, \mathsf{o}_r \rangle \in R_\pi \cup R'_\pi = R'_\pi$ for some $\mathsf{o}_1 \in \mathtt{v}(\alpha_1) = f(v(\alpha_1))$, …, for some $\mathsf{o}_r \in \mathtt{v}(\alpha_r) = f(v(\alpha_r))$. By the definitions of $\mathtt{v}(\alpha_i)$ and R'_π, if $\langle \mathsf{o}_1, \ldots, \{v(\alpha_i)\}, \ldots, \mathsf{o}_r \rangle \in R'_\pi$ then $\langle \mathsf{o}_1, \ldots, v(\alpha_i), \ldots, \mathsf{o}_r \rangle \in R'_\pi$ $(1 \leq i \leq r)$. But then, $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \in v(\pi)$, and hence $v_{M'}(\pi\alpha_1 \ldots \alpha_r) = 1$.
(iii) Suppose that $v_{M'}({\sim}\pi\alpha_1 \ldots \alpha_r) = 1$. Hence $v_{M'}(\pi\alpha_1 \ldots \alpha_r) = 0$ or $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \in v(\pi) \cap v({\sim}\pi)$. If $v_{M'}(\pi\alpha_1 \ldots \alpha_r) = 0$, then $\mathtt{v}_M(\pi\alpha_1 \ldots \alpha_r) = 0$ in view of (ii). If $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \in v(\pi) \cap v({\sim}\pi)$, then $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \notin R_\pi$. In both cases $\mathtt{v}_M({\sim}\pi\alpha_1 \ldots \alpha_r) = 1$.
(iv) Suppose that $v_{M'}({\sim}\pi\alpha_1 \ldots \alpha_r) = 0$. It follows that $v_{M'}(\pi\alpha_1 \ldots \alpha_r) = 1$ and $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \notin v({\sim}\pi)$. As $v_{M'}(\pi\alpha_1 \ldots \alpha_r) = 1$, $\mathtt{v}_M(\pi\alpha_1 \ldots \alpha_r) = 1$ in view of (i) and $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \in v(\pi)$. But then, in view of the definition of M, $\langle \mathsf{o}_1, \ldots, \mathsf{o}_r \rangle \in R_\pi \cap R'_\pi$ for all $\mathsf{o}_1 \in \mathtt{v}(\alpha_1) = f(v(\alpha_1))$, …, and $\mathsf{o}_r \in \mathtt{v}(\alpha_r) = f(v(\alpha_r))$. Hence $\mathtt{v}_M({\sim}\pi\alpha_1 \ldots \alpha_r) = 0$.

We leave to the reader the obvious task to show, by the standard induction on the complexity of wffs, that $v_{M'}(A) = \mathtt{v}_M(A)$ for all wffs A. ■

It follows immediately from the proof that any ambiguity model is equivalent to an ambiguity model in which $v(\ldots)$ comprises at most two members.

6 On Defining in CLuNs

In **CLuNs**, $\supset$ cannot be defined in terms of $\sim$ and $\wedge$ or in terms of $\sim$ and $\vee$. Similarly, $\vee$ (and $\wedge$) cannot be defined in terms of $\sim$ and $\supset$.[14] So pure paraconsistent **CLuNs** is not functionally complete—for example, classical negation cannot be defined in it.

The following definition is well-known from the literature:

[14] This was checked (indirectly) in terms the three-valued semantics by a computer program (82 different binary truth-functions may be defined in terms of "$\sim$" and "$\wedge$"; 896 different binary truth-functions may be defined in terms of "$\sim$" and "$\supset$"). Obviously proofs may be given (and are standard).

D$\sqsupset$ $\quad A \sqsupset B =_{df} {\sim}A \vee B$

This 'implication' is not detachable, but it is transposable: $A \sqsupset B$ and ${\sim}B \sqsupset {\sim}A$ are true in the same models (similarly for $A \sqsupset {\sim}B$ and $B \sqsupset {\sim}A$, etc.). Many relevant (and some other paraconsistent) logicians—see *e.g.*, [1] and [30]—have argued or claimed that "$\sqsupset$" *is* material implication, but 'they are mistaken'.[15]

Material implication, "$\supset$", is detachable but not transposable in **CLuNs**. It is, however, not difficult to define a *strong implication* that is both detachable and transposable:

D$\rightarrow$ $\quad A \rightarrow B =_{df} (A \supset B) \wedge ({\sim}B \supset {\sim}A)$

This implication has many relevant properties, such as: $A \nvdash_{\mathbf{CLuNs}} B \rightarrow A$; ${\sim}A \nvdash_{\mathbf{CLuNs}} A \rightarrow B$; ... Obviously, "$\rightarrow$" is not a relevant implication because it is a truth-function in the three-valued semantics, because $\vdash_{\mathbf{CLuNs}} (A \rightarrow B) \vee (B \rightarrow A)$ (and $A \wedge B \vdash_{\mathbf{CLuNs}} {\sim}A \rightarrow B$), and because $\vdash_{\mathbf{CLuNs}} A \rightarrow B$ does not warrant that A and B share a letter (*e.g.*, $\vdash_{\mathbf{CLuNs}} {\sim}(p \vee {\sim}p) \rightarrow (q \vee {\sim}q)$).

The Rule of Replacement of Equivalents is *not* derivable in **CLuNs**. Indeed, $\vdash_{\mathbf{CLuNs}} (p \vee {\sim}p) \equiv (q \vee {\sim}q)$, but $\nvdash_{\mathbf{CLuNs}} {\sim}(p \vee {\sim}p) \equiv {\sim}(q \vee {\sim}q)$—the latter is false in a model in which $V(p) = I$ and $V(q) = T$. However, the Rule of Replacement of Equivalents holds if the replacement takes place outside the scope of a negation sign—the proof proceeds by properties of positive logic and is standard. Moreover, it is possible to define a *strong equivalence* for which the Rule of Replacement of Strong Equivalents holds generally:

D$\leftrightarrow$ $\quad A \leftrightarrow B =_{df} (A \equiv B) \wedge ({\sim}A \equiv {\sim}B)$

The same connective is defined by $(A \rightarrow B) \wedge (B \rightarrow A)$.[16] In terms of the three-valued semantics: A and B have *the same value* in a model that verifies $A \leftrightarrow B$, and hence have the same value in all models iff $A \leftrightarrow B$ is valid; whence A and B can be replaced by each other, *salva veritate*, even *within* the scope of a negation (${\sim}$).

There is a different definable equivalence that warrants replacement of equivalents. One of its possible definitions is:

D$\Leftrightarrow$ $\quad A \Leftrightarrow B =_{df} (\neg\neg A \equiv \neg\neg B) \wedge (\neg\neg{\sim}A \equiv \neg\neg{\sim}B)$

We shall stick to $\leftrightarrow$ in the sequel. As appears from the following matrices, $\vdash_{\mathbf{CLuNs}} A \leftrightarrow B$ iff $\vdash_{\mathbf{CLuNs}} A \Leftrightarrow B$.

$\leftrightarrow$	T	I	F
T	T	F	F
I	F	I	F
F	F	F	T

$\Leftrightarrow$	T	I	F
T	T	F	F
I	F	T	F
F	F	F	T

This equivalence enables us to clarify the behaviour of negation in front of complex formulas in **CLuNs**. All of the following are valid:

[15] Remark also that the **CLuNs**-material implication ($\supset$) is a truth-functional connective in the strict sense of the term (in all semantic systems presented above).

[16] This connective is called "$\equiv^{\circ}$" in [4].

$\sim\sim A \leftrightarrow A$
$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
$\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
$\sim(\forall\alpha)A \leftrightarrow (\exists\alpha)\sim A$
$\sim(\exists\alpha)A \leftrightarrow (\forall\alpha)\sim A$

In general, $\vdash_{\mathbf{CLuNs}} A \equiv B$ is sufficient to warrant $\vdash_{\mathbf{CLuNs}} A \leftrightarrow B$, provided neither A nor B contains $\supset$ or $\equiv$, and neither A nor B are **CLuNs**-theorems.[17]

Also principles as the following hold:

$$A \leftrightarrow ((B \vee \sim B) \supset A)\,.$$

The reason is that $V_M(B \vee \sim B) \in \{T, I\}$ for all M, and that $V_M(C \supset D) = V_M(D)$ whenever $V_M(C) \in \{T, I\}$.

However, neither of the following is valid:

$$\sim(A \supset B) \leftrightarrow (A \wedge \sim B)$$
$$\sim(A \equiv B) \leftrightarrow ((A \vee B) \wedge (\sim A \vee \sim B))\,.$$

The corresponding material equivalences ($\equiv$) are valid. The failure of the strong equivalences derives from the difference between the values T and I.[18]

This seems the right place to warn the reader for a possible confusion. The classical negation of a formula that has a designated value has the value false, the classical negation of a formula that has a non-designated value has the value true.[19] If no paraconsistent or paracomplete negation is present in the system, this results in a two-valued semantics in which all logical constants are truth-values—thus the $\sim$-less fragment of **CLuNs** is simply **CL**. There is a single designated value in this semantics, and a single non-designated value. If A has the one, then $\neg A$ has the other, and hence $\neg\neg A$ has the same value as A.

The presence of a paraconsistent negation (or a paracomplete negation or both) changes the picture drastically. **CLuNs** clearly illustrates this. The negation $\sim$ is not a truth-function in the two-valued semantics (and is not a truth-function in any two-valued semantics), which comes to saying that consistent truth is distinguished from inconsistent truth. Given that three values have to be distinguished in a semantics in which all connectives are truth functions—the quantifiers being border cases—material equivalence ($\equiv$) fails to warrant replacement of equivalents. Only strong equivalence ($\leftrightarrow$) warrants this. As a result,

$$\vdash_{\mathbf{CLuNs}} \sim\sim A \leftrightarrow A$$

but

$$\nvdash_{\mathbf{CLuNs}} \neg\neg A \leftrightarrow A\,,$$

precisely because, in the presence of three distinct values, $\neg$ conflates them to two whereas $\sim$ does not:

[17] Given that $\vdash_{\mathbf{CLuNs}} A \equiv B$, A is a **CLuNs**-theorem iff B is. That **CLuNs**-theorems have to be ruled out is easily seen from the following example (out of many): $\vdash_{\mathbf{CLuNs}} (p \vee \sim p) \equiv (q \vee \sim q)$ whereas $\nvdash_{\mathbf{CLuNs}} (p \vee \sim p) \leftrightarrow (q \vee \sim q)$.

[18] If $V_M(A) = I$ and $V_M(B) = F$, then $V_M(\sim\neg A) = V_M(\neg\neg A) = T$, $V_M(\sim\neg A \wedge \sim B) = T$ and $V_M(A \wedge \sim B) = I$. Hence, $\sim(A \supset B)$ is only materially equivalent to $A \wedge \sim B$. Remark, however, that $\vdash_{\mathbf{CLuNs}} \sim(A \supset B) \leftrightarrow (\sim\neg A \wedge \sim B)$.

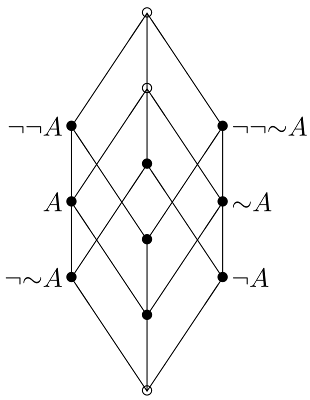
[19] The latter holds also if, *e.g.*, a fourth value is introduced to label negation-incompleteness, viz. that neither A nor $\sim A$ is true. Relevant logicians use to call this value *N(either)*.

A	$\sim A$	$\sim\sim A$	$\neg A$	$\neg\neg A$
T	F	T	F	T
I	I	I	F	T
F	T	F	T	F

The formula $\neg\neg A \leftrightarrow A$ is not **CLuNs**-valid because $\neg\neg A$ and A have a different truth value in the three-valued semantics, in which $\leftrightarrow$ is a truth-function. In the two-valued semantics $\neg\neg A$ and A have the same truth value, but $\leftrightarrow$ is not a truth-function; the value of $\neg\neg A \leftrightarrow A$ also depends on the values of $\sim A$ and $\sim\neg\neg A$—the latter has the same value as $\neg A$—and these need not be identical.

It is instructive to consider the expressions that may be built from some wff A by the logical constants of **CLuNs**. Twelve distinct truth-functions are distinguished in **CLuNs**. They are represented by A, $\sim A$, $\neg A$, $\neg\sim A$, $\neg\neg A$, $\neg\neg\sim A$, $A \wedge \sim A$, $\neg\neg A \wedge \neg\neg\sim A$, $\neg\sim A \vee \neg A$, $A \vee \sim A$, $\neg\sim A \wedge \neg A$ (which is strictly equivalent to $\perp$), and $\neg\neg A \vee \neg\neg\sim A$ (which is strongly equivalent to $\neg\perp$). In Figure 1, we show the relations between these wffs: a line going up indicates derivability. The wffs not named in the Figure may be easily completed in terms of conjunctions and disjunctions. The two top nodes (on the middle row) are **CLuNs**-valid. Only the bottom node has no **CLuNs**-models.

Figure 1: Not strongly equivalent expressions built from A.



The figure may be seen as composed of three superposed 'squares'. Each of these may be related to a notion of truth. The middle square is related to truth *simpliciter*, characterized by a predicate that is definable as follows:

$$T(A) =_{df} A$$

If a notion of falsehood is connected to it, as is done in [30] and in [1], viz. by $F(A) =_{df} \sim A$, $T(A)$ and $F(A)$ do not exclude each other but one of them is bound to obtain. A is true simpliciter iff $\sim A$ is false simpliciter, and *vice versa.*

The lower square is related to *strong* or *consistent* truth, which may be defined by

$$T^s(A) =_{df} \neg\sim A$$

The corresponding *strong* or *consistent* falsehood is defined by $F^s(A) =_{df} \neg A$. Strong truth and strong falsehood exclude each other (have no common model), but both may fail to obtain (because both A and $\sim A$ may obtain). A is strongly true iff $\sim A$ is strongly false, and *vice versa*.

Finally, *weak* truth may be defined by

$$T^w(A) =_{df} \neg\neg A$$

and the corresponding weak falsehood by $F^w(A) =_{df} \neg\neg{\sim}A$. $T^w(A)$ and $F^w(A)$ do not exclude each other but one of them is bound to obtain. A is weakly true iff $\sim A$ is weakly false, and *vice versa*.

The difference between the three notions of truth is obviously related to the value I, which represents inconsistent truth. In the following table, we use $B(A)$ (both) to abbreviate $T(A) \wedge F(A)$, and $E(A)$ (either) to abbreviate $T(A) \vee F(A)$; similarly for $B^s(A)$, etc. To save space, we write T instead of $T(A)$, etc. The table lists the twelve wffs mentioned in Figure 1, in an order that we think to reveal the differences most clearly.[20]

A	B^s	T^s	F^s	E^s	B	T	F	E	B^w	T^w	F^w	E^w
T	F	T	F	T	F	T	F	T	F	T	F	T
I	F	F	F	F	I	I	I	I	T	T	T	T
F	F	F	T	T	F	F	T	T	F	F	T	T

Neither of these three notions of truth corresponds to the classical one (or to the notion of truth in a model, which actually is the classical one). Classical logicians collapse the three squares (by recognizing only $\neg A$ as the negation of A). Relevant logicians introduce four truth values (identifying I with the designated "Both" and introducing the undesignated "Neither" as well). Rejecting the **CLuNs**-connective "$\supset$" as a sensible logical connective, they end up with just the middle square of which the top node is valid but the bottom node (which is the $\sim$-negation of the top node) is not trivial. Dialetheists like Graham Priest stick to the three values of **CLuNs**, reject the **CLuNs**-connective "$\supset$" as a sensible logical constant, but recognize bottom ($\perp$) as a sensible non-logical constant. As a result, they end up with the middle diamond extended by the top and bottom node of Figure 1.[21]

Let us return to strong equivalence in **CLuNs**. $(A \vee A) \leftrightarrow A$, $(A \wedge {\sim}A) \leftrightarrow (A \equiv {\sim}A)$, $(A \supset A) \leftrightarrow ({\sim}A \vee A)$ and $A \leftrightarrow ((A \supset A) \supset A)$ are all valid; but $(A \supset B) \leftrightarrow ({\sim}A \vee B)$ and $A \leftrightarrow ((B \supset B) \supset A)$ are not.

Neither the disjunction defined by ${\sim}A \supset B$ nor that defined by $(A \supset B) \supset B$ are commutative in the strong sense (that is, with respect to $\leftrightarrow$). The disjunction defined by ${\sim}A \to B$ is commutative in this sense, but Addition does not hold for it. And yet, the latter disjunction is an important one.

Relevant logicians have capitalized on the distinction between extensional connectives, such as disjunction and conjunction, and intensional connectives such as relevant implication, to define fusion and fission—a kind of 'strong' conjunction and 'strong' disjunction. In **CLuNs**, there is a somewhat similar distinction between disjunction and conjunction on the one hand, and implication on the other hand. However, as mentioned in the previous paragraph, the

[20] We do not pursue the study of the properties of the structure in Figure 1. The interested reader might start by considering the behaviour of the functions $\neg$ and $\sim$.

[21] Bottom does not occur in Priest's original **LP**. In [30], however, Priest introduces a modal implication, and next combines it with bottom.

'strong' disjunction defined from this implication is not commutative. This is circumvented by defining fusion and fission from $A \to B$, rather than from $A \supset B$. The resulting definitions are $A \oplus B =_{df} {\sim}A \to B$, and $A \otimes B =_{df} {\sim}({\sim}A \oplus {\sim}B)$. This line of approach was followed, as was shown afterwards in [12], by Joke Meheus in [27] and [28], where the logic **AN**$\emptyset$ is defined by the $\sim$–$\to$–$\otimes$–$\oplus$–$\forall$–$\exists$–fragment of **CLuNs**.[22] It is instructive to list the matrices for the propositional connectives:

	$\sim$
T	F
I	I
F	T

$\to$	T	I	F
T	T	F	F
I	T	I	F
F	T	T	T

$\oplus$	T	I	F
T	T	T	T
I	T	I	F
F	T	F	F

$\otimes$	T	I	F
T	T	T	F
I	T	I	F
F	F	F	F

These define a paraconsistent logic that validates Modus Ponens, Modus Tollens, Disjunctive Syllogism, and similar 'analysing' rules, but not Addition, Irrelevance, and similar 'constructive' rules.

7 Some Further Metatheory

For the Interpolation Theorem and a set of Embedding Theorems, we refer to [16] and [15]. From the proofs of the Embedding theorems, it follows that the fragments that are known to be decidable in **CL** are decidable in **CLuNs**. Hence, all effective proof-search procedures for fragments of **CL** are effective for the corresponding fragments of **CLuNs**.

Theorem 6 **CLuNs** *and* **CL** *have the same valid wffs in the* $\sim$-$\vee$-$\wedge$-$\forall$-$\exists$-*fragment of* $\mathcal{L}$.

Proof. As **CL** extends **CLuNs**, A is **CL**-valid if it is **CLuNs**-valid. For the converse, remark first that any wff A of the intended fragment is **CL**-equivalent to a wff B that is in prenex conjunctive normal form. As all the required equivalences are valid strong equivalences in **CLuNs**, $\vdash_{\mathbf{CLuNs}} A \leftrightarrow B$. If B is **CL**-valid, each of its conjuncts has the form $\ldots \vee C \vee \ldots \vee {\sim}C \vee \ldots$ (in which each occurrence of "$\ldots$" may be empty). But then B, and hence A, is also **CLuNs**-valid. ■

Let us now turn to an interesting property of models.

Theorem 7 *If* M *is a non-trivial* **CLuNs**-*model, then* $\{A \mid A \in \mathcal{W}; M \models A\}$ *is deductively closed and maximally non-trivial.*

Proof. The set is obviously deductively closed. That it is maximally non-trivial is immediate from the semantics: if $M \not\models A$, then $M \models (A \supset B)$ for all B. ■

The theorem also holds for the pure paraconsistent fragment of **CLuNs**. The theorem does not hold for Priest's **LP**, viz. the $\sim$-$\wedge$-$\vee$-fragment of **CLuNs**—no set of formulas verified by the model warrants that A is false in the model.

The proof of Theorem 2 is easily transformed into a proof of each of the following:

[22] Meheus writes $\supset$ where we write $\to$, etc. The logic **AN** is obtained by reducing formulas to a specific prenex conjunctive normal form, and next by evaluating the latter in terms of **AN**$\emptyset$.

1. Every deductively closed, maximally non-trivial set $\Gamma \subseteq \mathcal{W}$ has a $\mathcal{N}$-minimal **CLuNs**-model.

2. All **CLuNs**-models of a deductively closed, maximally non-trivial set $\Gamma \subseteq \mathcal{W}$ are equivalent.

Clearly **CLuNs** is *not* Post complete: some **CL**-theorems are not **CLuNs**-theorems and **CL** is not trivial. A logic **L** is said to be Lindenbaum complete if the following holds in it: if no substitution instance of A is a theorem of **L**, then $\sim A$ is a theorem of **L**.

Theorem 8 **CLuNs** *is not Lindenbaum complete.*

Proof. $\sim\sim((p \supset (q \wedge \sim q)) \supset \sim p)$ is not a **CLuNs**-theorem. Indeed, it is invalid, viz. false in a model that verifies p, q, and $\sim q$ and falsifies $\sim p$. However, no wff of the form $\sim((A \supset (B \wedge \sim B)) \supset \sim A)$ is a **CLuNs**-theorem, which is easily seen from the fact that all **CLuNs**-theorems are **CL**-theorems (because all **CL**-models are **CLuNs**-models). ■

In [1, p. 121], Anderson and Belnap write: "We offer [Lindenbaum completeness] as a plausible syntactical condition which ought to be satisfied by a semantically complete system." This statement is clearly confusing. In many senses of the term, **CLuNs** is as semantically complete as any system could be. Needless to say, **CLuNs** is Lindenbaum complete with respect to the defined classical negation $\neg$.

A logic $\mathcal{L}$ is *strictly paraconsistent* iff $\mathfrak{A}, \dagger\mathfrak{A} \vdash_{\mathbf{L}} A$ is not a valid schema for any unary connective † and for any metalinguistic formula $\mathfrak{A}$ in which the metavariable A does not occur. That the propositional fragment of the pure paraconsistent **CLuNs** is strictly paraconsistent was shown in [7] (and is a corollary of Theorem 9). The propositional fragment of **CLuNs** is obviously not strictly paraconsistent. Yet, it is possible to show a related property of this logic.

We shall say that a *unary connective* "†" is *strictly paraconsistent* in a logic **L** iff $\mathfrak{A}, \dagger\mathfrak{A} \nvdash_{\mathbf{L}} A$ whenever $\mathfrak{A}$ is a metatheoretic formula that does not contain $\perp$ and A does not occur in $\mathfrak{A}$.[23]

Theorem 9 *In* **CLuNs**, $\sim$ *is strictly paraconsistent.*

Proof. Consider a $B \in \mathcal{W}$ and a sentential letter A that does not occur in B. It is easily seen that there is a model M of the three-valued **CLuNs**-semantics such that $V(A) = F$ whereas $V_M(C) = I$ for all primitive formulas that occur in B. It follows that $V_M(B) = V_M(\sim B) = I$ and that $V_M(A) = F$. By Theorems 2 and 3, $B, \sim B \nvdash_{\mathbf{CLuNs}} A$. ■

A propositional logic $\mathcal{L}$ is maximally paraconsistent iff it has no 'extension' that is paraconsistent—we mean *only* extensions that are Compact and Monotonic, and the set of theorems of which is closed under Uniform Substitution. It was shown in [7] that the propositional fragment of pure paraconsistent

[23]We mean that $\mathfrak{A}$ does not contain $\perp$ and does not contain a logical symbol from which $\perp$ can be defined. The sense of the definition is that, for some paraconsistent negations, for example the one from [3], the Ex Falso Quodlibet does not hold generally, but $A \wedge B, \sim(A \wedge B) \vdash C$ does.

CLuNs is maximally paraconsistent. A related property may be proved for the propositional fragment of **CLuNs** (including $\bot$ and $\neg$), viz. that this fragment is maximally $\sim$-paraconsistent. Where $\mathcal{L}$ is restricted to its propositional part, a logic is *maximally $\sim$-paraconsistent* iff (i) it is $\sim$-paraconsistent (for some A, $A, \sim A \nvdash B$ and (ii) its only 'extensions' are not $\sim$-paraconsistent (viz. either (propositional) **CL** or the trivial logic). To interpret this claim, recall that $\sim$ is taken to be the standard negation of both **CL** and **CLuNs**, whereas $\neg$ is a defined negation (that is co-extensive with $\sim$ in **CL**). The set of extensions should obviously be restricted as above.

First we define the Conjunctive Normal Form, CNF, for **CLuNs**-formulas. Where A is a sentential letter, A, $\sim A$, $\neg A$, $\neg\sim A$, $\neg\neg A$, and $\neg\neg\sim A$ will be *atoms*.[24] Moreover, $\bot$ and $\neg\bot$ (to which $\sim\bot$ is strongly equivalent) will also be called atoms.

Definition 1 *A wff A is in CNF iff it has the form $(B_1 \wedge \ldots \wedge B_n)$ $(n \geq 1)$, each of these B_i is a disjunction of (one or more) atoms, no B_i is strongly implied by another B_j, and no atom that occurs in a B_i is strongly implied by another atom that occurs in the same B_i.*[25]

Remark that $\bot$ and $\neg\bot$ cannot both occur in the same B_i, and that, if one of them occurs in it, then it forms the only *conjunct* of the wff. We leave it to the reader to show, by nearly standard means, that any wff A is strongly equivalent to some wff B that is in CNF.

Theorem 10 *If A is a propositional formula and $\nvdash_{\mathbf{CLuNs}} A$, then any extension of* **CLuNs** *in which A is a theorem is not $\sim$-paraconsistent.*

Proof. Where $\nvdash_{\mathbf{CLuNs}} A$, let $\mathbf{CLuNs}^+$ be an extension of **CLuNs** in which A is a theorem. Let B be strongly equivalent to A and in CNF. At least one conjunct of B is a theorem of $\mathbf{CLuNs}^+$ and not a theorem of **CLuNs**. Let the following wff be such a conjunct

$$\begin{aligned}&\neg\sim C_1 \vee \ldots \vee \neg\sim C_{n_1} \vee D_1 \vee \ldots \vee D_{n_2} \vee \neg\neg E_1 \vee \ldots \vee \neg\neg E_{n_3} \vee \\ &\qquad \neg F_1 \vee \ldots \vee \neg F_{n_4} \vee \sim G_1 \vee \ldots \vee \sim G_{n_5} \vee \neg\neg\sim H_1 \vee \ldots \vee \neg\neg\sim H_{n_6} \quad (1)\end{aligned}$$

with $n_1 \geq 0, \ldots, n_6 \geq 0$ and $n_1 + \ldots + n_6 > 0$.

In view of the definition of CNF and the fact that (1) is not a **CLuNs**-theorem:

Fact 1 All C_i, E_i , F_i , G_i, and H_i are propositional letters and all D_i are propositional letters or some D_i is $\bot$, in which case it is the only disjunct of (1).

Fact 2 At most some C_i are identical to some F_i.

Indeed, by the definition of CNF, all C_i, D_i and E_i are different from one another, and all F_i, G_i and H_i are different from one another. As (1) is not a

[24]Compare Figure 1 and the subsequent table.

[25]See Figure 1 for strong implication between atoms.

CLuNs-theorem, all C_i, D_i and E_i are different from all G_i and H_i, and all F_i are different from all D_i and E_i.

Case 1. $n_1 > 0$ and $n_4 > 0$. Let I be a propositional letter that does not occur in (1). In view of Facts 1 and 2, one obtains a theorem of **CLuNs**$^+$ if one substitutes I for all C_i and F_i, $\sim\sim\perp$ for all D_i and E_i, and $\sim\perp$ for all G_i and H_i. Deleting disjuncts that occur twice, we obtain the formula:

$$\neg{\sim}I \vee \neg I \vee {\sim}{\sim}\perp \vee \neg\neg{\sim}{\sim}\perp$$

in which the last or next to last disjunct (or both) may be empty. This is **CLuNs**-equivalent to

$$(I \wedge {\sim}I) \supset \perp \tag{2}$$

As I is a propositional letter, **CLuNs**$^+$ is not paraconsistent (and is identical to **CL**).

Case 2. $n_1 = 0$. In view of Facts 1 and 2, one obtains a theorem of **CLuNs**$^+$ if one substitutes $\sim\sim\perp$ for all D_i and E_i, and substitutes $\sim\perp$ for all F_i, G_i and H_i. Deleting disjuncts that occur twice, we obtain:

$${\sim}{\sim}\perp \vee \neg{\sim}\perp \vee \neg\neg{\sim}{\sim}\perp$$

or one or two disjuncts of this formula. As this is **CLuNs**-equivalent to $\perp$, **CLuNs**$^+$ is the trivial system.

Case 3. $n_4 = 0$. In view of Facts 1 and 2, one obtains a theorem of **CLuNs**$^+$ if one substitutes $\sim\sim\perp$ for all C_i, D_i and E_i, and substitutes $\sim\perp$ for all G_i and H_i. Deleting disjuncts that occur twice, we obtain:

$$\neg{\sim}{\sim}{\sim}\perp \vee {\sim}{\sim}\perp \vee \neg\neg{\sim}{\sim}\perp$$

or one or two disjuncts of this formula. As this is **CLuNs**-equivalent to $\perp$, **CLuNs**$^+$ is the trivial system. ■

Corollary 1 *The propositional fragment of* **CLuNs** *is maximally* $\sim$*-paraconsistent.*

What about maximal paraconsistency in the predicative case? All we can offer here is, apart from complications, an open problem with a tentative answer.

First there is the complication related to a suitable substitution rule, studied very carefully in [29]. Next, a central difference with the propositional case is that there are many logics between (predicative) **CL** and the trivial logic. For example, one might add to **CL** an axiom schema that restricts the cardinality of the domain, $(\exists\alpha)(\exists\beta){\sim}\alpha = \beta$, or an axiom schema that requires all binary relations to be transitive, even $(\forall\alpha)(\forall\beta)(\forall\gamma)(A(\alpha\beta) \supset (A(\beta\gamma) \supset A(\alpha\gamma)))$, and so on. Third, it is quite obvious that **CLuNs** can be extended with axiom schemas that introduce Ex Falso Quodlibet for some logical form without introducing it for all of them. Thus adding the schema $(\alpha = \beta \wedge {\sim}\alpha = \beta) \supset A$ to **CLuNs** does not make $A, {\sim}A \vdash B$ hold in general.

The semantics suggests that $\sim$ is not strictly paraconsistent in any logic between **CLuNs** and **CL**, more precisely that the negation $\sim$ is not strictly paraconsistent in any logic **CLuNs**$^+$ obtained by extending **CLuNs** with an axioma schema that holds in **CL**.

This impression is further confirmed by attempts to falsify it. Extensions of **CLuNs** seem all to introduce Ex Falso Quodlibet for at least a specific form and under some condition, whence they all seem to be equivalent to an axiom schema of the form $C \supset (\mathsf{Q}(A \wedge {\sim}A) \supset B)$. If some metalinguistic formula has a more specific form (but also the above one) in which C is a **CLuNs**-theorem, $\mathsf{Q}(A \wedge {\sim}A) \supset B$ is derivable, and the **CLuNs**-extension is not strictly paraconsistent. So let us consider an extension of **CLuNs** obtained by adding the following axiom schema, of which the antecedent cannot be turned into a **CLuNs**-theorem:

$$(\exists\alpha)(\forall\beta)\alpha = \beta \supset (\exists\alpha)({\sim}\alpha = \alpha \supset B)\,, \tag{3}$$

which expresses that $x = x$ behaves consistently for at least one x in models with a singleton domain.

As (3) is a theorem of the extension, so is

$$(\exists\alpha)(\forall\beta)\alpha = \beta \supset (\forall\alpha)({\sim}\alpha = \alpha \supset B) \tag{4}$$

and as $(\forall\alpha)(\forall\beta)(\alpha = \beta \supset ({\sim}\alpha = \beta \supset {\sim}\alpha = \alpha))$ holds, it follows that

$$(\exists\alpha)(\forall\beta)\alpha = \beta \supset (\forall\alpha)(\forall\beta)(\alpha = \beta \supset ({\sim}\alpha = \beta \supset B)) \tag{5}$$

and from this easily follows

$$(\exists\alpha)(\forall\beta)\alpha = \beta \supset ({\sim}(\exists\alpha)(\forall\beta)\alpha = \beta \supset B)\,, \tag{6}$$

whence the extension is not paraconsistent. There is nothing puzzling here obviously. If identity behaves consistently in models with singleton domains, no model verifies both *implicantia* of (6).

Not finding a proof that $\sim$ is not strictly paraconsistent in any logic between **CLuNs** and **CL**, we tried a host of possible counterexamples, but without success. So we have to leave this an open problem (both for **CLuNs** and for pure paraconsistent **CLuNs**).

8 In Conclusion

The main interest of **CLuNs** seems to reside in the fact that it combines the theorems and rules of the full positive fragment of **CL** and the usual rules for driving negations inwards. As a side-effect, it also contains all theorems of the $\sim$-$\vee$-$\wedge$-$\forall$-$\exists$-$=$-fragment of **CL**. It follows that **CLuNs** contains all theorems of **CL** in that the aforementioned fragment is functionally complete.

Among the possible applications, both inconsistent empirical theories and inconsistent arithmetic seem attractive domains, except of course if there are reasons to prefer an inconsistency-adaptive logic. Remark that inconsistent arithmetic is often studied in terms of the $\sim$-$\vee$-$\wedge$-$\forall$-$\exists$-$=$-fragment of **CL**. The presence, in **CLuNs**, of a detachable implication for which the deduction theorem holds, makes it attractive for the aforementioned application contexts. Indeed, the presence of the implication warrants that the models are maximally non-trivial (see Theorem 7), and, combined with bottom, enables one to express falsehood (in the sense of the two-valued semantics) within the object language.

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