ADM Quasi-merging and Pure-arbitration*

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Abstract

The present paper introduces two new information merging protocols for the family of adaptive logics **ADM**, for which majority merging has been defined in [19]. The new adaptive operators reflect the negotiation processes of quasi-merging and pure arbitration, known from the Integrity Constraints framework introduced in [13]. The *Adaptive Variant Counting* selection provides a result equivalent to the *GMax* family of merging operators: it selects a collective model for a multi-set of belief bases based on the number of disagreements verified by the various models according to a leximax function. The *Adaptive Minimax Counting* selection is a quasi-merging operator which applies a minimax function and it obtains a larger spectrum of possibilities than the previous selection: it simulates the behaviour of the *Max* family of operators from the Integrity Constraints framework, avoiding some of its counterintuitive results.

Keywords: Information Fusion, Negotiation Protocols, Arbitration, Quasi-Merging, Adaptive Logics.

1 Introduction

In the last decade, the logical literature has provided an increasing number of systems that formalize rational processes where multiple epistemic agents are involved. The description of processes of collective deliberation is especially relevant for judgement aggregation strategies, or information fusion architectures in the case of non-human rational systems.

The standard analysis of decision processes focuses naturally on reachable agreements, in order to perform the most effective selection of common goals and judgements in the group. But such a process might not be entirely successful and the presence of a certain degree of internal dissatisfaction can not be completely ruled out by the negotiation protocol. Frameworks defining knowledge merging operators, known as information fusion operators in the artificial intelligence

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literature, aim at modelling the needed selection procedures, especially in view of possibly inconsistent sets of data.

The merging of contradictory sources, whose study goes back to [5], has applications in distributed databases and information systems. In [7], [6] and [4] the general properties of the logical approaches to merging procedures for inconsistent information bases have been studied; these are surveyed in [12] and in the more recent [10].

The definition of an operator called *arbitration* was given first in [20], and later considerably reworked in [14] and [15]. In the latter work, the idea of arbitration is presented as a modification of the more standard revision operator from the AGM-paradigm in [1]: merging two information bases does not depend on any order of priority (namely of the newer base over the older one), as it is the case for update or revision operators. The process might require that information be preserved from one base in some cases and from the other in other cases. This general principle has been modified by the use of weights on the bases, in order to express the relevance of the information contents, rather than a strict priority relation. Weights have been expressed as priority values (as in [9]), they have been assigned either to propositional terms (see e.g. [8]) or to the set of models of formulas (as in [20]), and finally they have been formulated as possibility values (see [22]).

A second major group of merging operators formulate *majority* protocols: these formal selection systems are the main object of study especially in social choice theory. The ground distinction between majority and arbitration operators can be reflected in the following terms (see e.g. [11]): whereas majority merging operators aim at minimizing collective dissatisfaction, arbitration operators aim at maximizing individual satisfaction.

These two sub-classes of merging operators are defined in the general and standard framework of information merging under intergrity constraints (IC) in [13]. This framework allows for defining a family of three distinct operators:

1. the $\triangle^{\Sigma}$ operator satisfies the postulates for majority and it corresponds to the merging operator from [16];

2. the $\triangle^{GMax}$ operator satisfies a pure arbitration procedure, removing the original restriction on the number of bases involved in the arbitration process;

3. the $\triangle^{Max}$ operator is called a quasi-merging operator and it represents a pseudo-arbitration operator corresponding to the one from [15].

In [19], a majority operator defined in view of the dynamic semantics of adaptive logics (see [2, 3]) is formulated for bases with partial support in terms of the logic $\mathbf{ADM}^c$ (*Adaptive Doxastic Merging by Counting*). The adaptive *Counting strategy* selects from the set of models of a modal language those that minimize overall disagreements among agents: it is a general protocol of majority merging equivalent to the generalization under Integrity Constraints represented by the $\triangle^{\Sigma}$ operator. The application of this majority protocol to the judgment aggregation paradox is considered in [17]: it provides a non-paradoxical (though somehow inefficient) solution. Effectivity is reached by modifying the agenda of interaction.[1] In the present paper, we extend the family of adaptive logics **ADM** by providing adaptive strategies whose results are equivalent to the operators $\triangle^{GMax}$ and $\triangle^{Max}$ from the IC framework.

The formulation of the arbitration protocol for the family of logics **ADM** is given by the formulation of the logic $\mathbf{ADM}^{c+}$ for *Adaptive Doxastic Merging by Variant Counting*.[2] The semantic selection defined by this adaptive logic is based on a pre-order of models obtained by calculating disagreements among the agents following the *leximax rule*: this has a strict correspondance to the lexicographic order of models as defined in the IC framework. From this new logic, a third selection procedure is defined, equivalent to the *minimax rule* for quasi-merging. This selection procedure shall be introduced as the logic $\mathbf{ADM}^{c-mm}$, for *Adaptive Doxastic Merging by Minimax Counting*. With this last result, the family of logics **ADM** is shown to be a general framework to define all the various negotiation processes modelled by the standard merging operators, in particular those of the general Integrity Constraints framework.

We shall proceed as follows. After some preliminaries in section 2, we will consider briefly the quasi-merging and arbitration Integrity Constraints operators in section 3, and mention an example from the literature to clarify the functioning of these protocols. The general formulation of the adaptive logic for merging **ADM** is introduced in section 4, followed by the definition of the semantic selection of *Variant Counting* and *Minimax Counting*. Section 6 uses the already introduced example to show how these adaptive strategies give equivalent results as in the IC framework. In the final section further steps for the research on the adaptive procedures of merging are surveyed.

2 Preliminaries

Let $\mathcal{L}$ be the standard language of classical propositional logic (henceforth **CL**) that is formed from a finite set of atoms $\mathcal{P}$ in the usual way. A *literal* is an atom or negation of an atom; $\mathcal{P}^{\pm}$ will be used to refer to the set of literals and $\mathcal{W}$ to refer to all well-formed formulas of $\mathcal{L}$. Capital roman letters from the beginning of the alphabet will be used as meta-variables for well-formed formulas. As is common, the abbreviation $\bigvee\Delta$ will stand for the disjunction of the members of Δ, where Δ is a set of formulas.

A *belief base* is a finite subset of $\mathcal{W}$. We shall use $T, T_1, \ldots$ as meta-variables for belief bases.[3] A *belief set* is a *multiset* of belief bases. We say that a literal A is *fully supported* by some belief base T if $T \models A$. We say that a literal A is *partially supported* by a belief base T if there is a set of literals Δ such that $A \in \Delta$, $T \models \bigvee\Delta$, $\not\models \bigvee\Delta$, and there is no $\Delta' \subset \Delta$ such that $T \models \bigvee\Delta$.[4] The letter Ψ will be used as a meta-variable for belief sets and we shall use square brackets to denote multisets, as in $[T_1, \ldots, T_n]$.

[1]This corresponds, informally, to an iteration of voting processes and, more formally, to relaxing the specification on *individual judgements* being accepted as input in the *Universal Domain Condition*.

[2]This result was first presented in [18].

[3]In [13] a belief base is seen as the formula obtained by the conjunction of the formulae in the belief base T. This is justified in view of the fact that belief bases are finite.

[4]The notions of full and partial support may easily be generalized to complex formulas, but we do not need them in the context of the present paper.

Integrity constraints μ are a finite set of sentences, i.e. a belief base. The merging of a multi-set Ψ under constraints μ is a function from $\Psi \times \mu$ to a new belief base. The result of a merging procedure on a multi-set Ψ under constraints μ shall be denoted as $\triangle_\mu(\Psi)$.

A **CL**-*model* is a function from $\mathcal{P}$ to $\{0, 1\}$. We shall use $M, M_1, \ldots$ as metavariables for **CL**-models and $\mathcal{M}$ to denote the set of all **CL**-models. A *model* M is a model *of* a belief base T iff all the members of T are true in it. As usual, $M \models A$ will denote that M verifies A. We shall use $Mod(T)$ to denote the set of all models of T, and $Cn(T)$ to denote the semantic consequence set of T.

From Section 3 on, we shall define the notion of distance between models and a belief set, which gives a preorder on $\mathcal{M}$ (also called order of plausibility on models). A preorder over the set of **CL**-models is a reflexive and transitive relation on $\mathcal{M}$. Where $\leq$ is a preorder, $<$ is defined as: $M < M'$ iff $M \leq M'$ and $M' \not\leq M$. The merging protocols will define functions for the ordering on models, and the models of the resulting merged base will be the *minimal* ones in such orderings. Where M is a subset of $\mathcal{M}$, we shall say that a model M is *minimal* in M with respect to $\leq$ iff $M \in \mathsf{M}$ and there is no $M' \in \mathsf{M}$ such that $M' < M$. We shall use $min(\mathsf{M}, \leq)$ to denote the set of models that are minimal in M with respect to $\leq$.

From Section 4 on, we shall also need a (multi-)modal language that includes belief operators. Such language $\mathcal{L}^{\mathcal{B}}$ is obtained by $\mathcal{L}$ extending it with a belief operator b_i for any $i \in \mathcal{I}$, where $\mathcal{I}$ is a set of indexes $\{0, 1, \ldots\}$. Each different base is given an operator b_i with $i \in \mathcal{I} \setminus 0$. The operator b_0 is used exclusively for the beliefs selected for the merging state, or for the constraints holding in such state. Intuitively, $b_i A$ (for $i > 0$) will express that agent i believes or supports A; the formula $b_0 A$ means that all agents agree on A or that their decision is constrained by the holding of A. Given a belief base T_i, a *modal belief base* that corresponds to T_i will be obtained by preceding each member of T_i by b_i. A *modal belief set* will be a set of modal belief bases extended with the set $\{b_i A \vee b_i \neg A \mid A \in \mathcal{P}^{\pm}; i \in \mathcal{I}\}$. We shall use the letter Υ exclusively to denote the latter set. Thus, where $\Psi = [T_1, \ldots, T_n]$, the modal translation of Ψ, denoted by $\Psi^{\mathcal{B}}$, is the set $\{b_i A \mid A \in T_i; T_i \in \Psi\} \cup \Upsilon$.[5] We shall use $\mathcal{W}^{\mathcal{B}}$ to refer to all well-formed formulas of $\mathcal{L}^{\mathcal{B}}$. Where necessary, we shall explicitly indicate whether a formula belongs to $\mathcal{W}$ or to $\mathcal{W}^{\mathcal{B}}$.

$\mathcal{L}^{\mathcal{B}}$ enables one to represent a *multi-set* of belief bases by a single set of premises. It also enables one to consider (modal) models that validate all the premises, rather than having to consider models for each of the belief bases separately. To keep things as simple as possible, and in view of what is needed for the intended application context, the modal language will be restricted to first degree modalities. So, only modal formulas in which no nested belief operators occur will henceforth be considered as well-formed.

The language $\mathcal{L}^{\mathcal{B}}$ is used to define the logic **DM**, which is a multi-modal version of the modal logic **D**. The logic **DM** has been introduced in [19] as the Lower Limit Logic (**LLL**) of $\mathbf{ADM}^c$: this is the same for the two adaptive logics to be introduced in the present paper.[6] In addition to all **CL**-axioms, the

[5] Extending the modal belief base in this way is, for the application context at issue, harmless and greatly simplifies matters, both at the object-level and at the meta-level. See also [19].

[6] All of the three are Adaptive Logics in standard format, which is extensively discussed in [3]. This means that two other elements are needed for their definition along with the **LLL**:

logic **DM** validates for any $i \in \mathcal{I}$

- Necessitation Rule: if $\vdash_{\mathbf{CL}} A$ then $\vdash_{\mathbf{DM}} b_i A$;
- Distribution: $b_i(A \supset B) \supset (b_i A \supset b_i B)$;
- Consistency: $b_i A \supset \neg b_i \neg A$.

We shall use $M^{\mathcal{B}}, M_1^{\mathcal{B}}, \ldots$ as meta-variables for **DM**-models and $\mathcal{M}^{\mathcal{B}}$ to denote the set of all **DM**-models.

3 The Integrity Constraints Merging Framework

In this section we introduce the Integrity Constraints (IC) merging protocols from [13] that are going to be mimicked by different strategies defined within the **ADM** family of adaptive logics. The protocols from the IC Merging framework are respectively Pure-Arbitration and Quasi-Merging.

The IC merging operators on a multiset Ψ, consisting of n belief bases $\Psi = [T_1, \ldots, T_n]$, define different ordering methods on the set of classical models: the resulting belief base $\triangle_\mu(\Psi)$ is "close" in a certain technical sense to the original multiset, being in general the one whose models are the minimal ones in the obtained ordering. The definition of distance between the involved belief bases and the set of interpretations is usually given as the Dalal distance from [8]:[7] its intuitive idea is to measure the distance between two models M and M', denoted as $dist(M, M')$, as the *number of atoms* whose valuation differs in the two models, and the distance between a **CL**-model M and a belief base T by the following definition:

$$dist(M, T) = min_{M' \in Mod(T)}(dist(M, M')) \tag{1}$$

in which "*min dist*" refers to the minimal distance. Where T is inconsistent, the value of $dist(M, T)$ is set to zero. Given a multiset of belief bases, a single model can be close to one of the bases in that set, and distant from another base in terms of the value of *dist*. The various merging protocols apply an ordering on the resulting values according to different functions to obtain the desired negotiation process.

Let us now consider an example introduced in [13], in order to show how the operators for arbitration and quasi merging perform their ordering and selections of models.

> *At a meeting of a block of flat co-owners, the chairman proposes for the coming year the construction of a swimming pool, of a tennis court and a private car park. But if two of these three items are built, the rent will increase significantly* ([13], p.787).

In the following the letters p, q, r stand respectively for the construction of the swimming pool, the tennis court and the private car park. The rent increase

the set of so-called abnormal formulas, which is also the same among the three logics; and the adaptive selections, which instead are the distinctive mark of each logic and provide different consequence sets for the same premise set.

[7] A variant definition is represented by the *Satoh distance* from [21].

will be denoted by s, which is implied by each conjunction of two out of the three items: this represents our constraint $\mu = ((p \wedge q) \vee (p \wedge r) \vee (q \wedge r)) \rightarrow s$. The set of beliefs for the co-owners is represented by $\Psi = [T_1, T_2, T_3, T_4]$:

$$\begin{array}{l} T_1 = \{p \wedge q \wedge r\} \\ T_2 = \{p \wedge q \wedge r\} \\ T_3 = \{\neg p \wedge \neg q \wedge \neg r \wedge \neg s\} \\ T_4 = \{q \wedge r \wedge \neg s\}. \end{array}$$

The first two co-owners want to build the three items and do not care about the rent; the third agent does not want the rent to increase nor anything built; the fourth one has a preference for the last two items, though he does not want the rent to increase. The result of merging shall select those models that are minimal with respect to a given pre-order and that satisfy the constraint.

In the following two subsections we shall introduce the definitions needed to define the IC merging protocols and show how these apply to the calculation of distances and selection on the set of models $\mathcal{M}$:

$$\begin{array}{l} M_1 = p, q, r, s \\ M_2 = p, q, r, \neg s \\ M_3 = p, q, \neg r, s \\ M_4 = p, q, \neg r, \neg s \\ M_5 = p, \neg q, r, s \\ M_6 = p, \neg q, r, \neg s \\ M_7 = p, \neg q, \neg r, s \\ M_8 = p, \neg q, \neg r, \neg s \\ M_9 = \neg p, q, r, s \\ M_{10} = \neg p, q, r, \neg s \\ M_{11} = \neg p, q, \neg r, s \\ M_{12} = \neg p, q, \neg r, \neg s \\ M_{13} = \neg p, \neg q, r, s \\ M_{14} = \neg p, \neg q, r, \neg s \\ M_{15} = \neg p, \neg q, \neg r, s \\ M_{16} = \neg p, \neg q, \neg r, \neg s \end{array}$$

3.1 IC Pure-Arbitration

The IC arbitration operator on a multiset Ψ is denoted by $\triangle_{\mu}^{GMax}(\Psi)$. An arbitration merging operator corresponds to a mapping from the belief set Ψ to a belief base whose models are minimal in a total preorder $\leq_{\Psi}$ obtained according to a *leximax* function.

Consider a multi-set $\Psi = [T_1, \ldots, T_n]$; for each classical model M, consider the list $D = (dist_1^M, \ldots, dist_n^M)$ of distances between M and the n belief bases in Ψ, i.e. the list of distances

$$dist_i^M = dist(M, T_i) \tag{2}$$

where $dist(M, T_i)$ is defined by 1 in the previous section. Let L_{Ψ}^M be the list obtained from D by sorting its members in descending order. Denote now by $\leq_{lex}$ the lexicographic order between sequences of integers of the same length, that is the lexicographic order between the L_{Ψ}^M lists. For any two models M_1

and M_2, $M_1 \leq_\Psi M_2$ holds if and only if $L_\Psi^{M_1} \leq_{lex} L_\Psi^{M_2}$. Where $\mathsf{M}_\mu \subseteq \mathcal{M}$ is the subset of models that satisfy the constraint μ, the selection by the $\triangle_\mu^{GMax}$ operator is then defined as follows:

$$Mod(\triangle_\mu^{GMax}(\Psi)) = min(\mathsf{M}_\mu, \leq_\Psi). \tag{3}$$

As concerns the example introduced in this section, the $\triangle_\mu^{GMax}(\Psi)$ operaotr presupposes that the distances of each $T_i \in \Psi$ with respect to the set of models $\mathcal{M}$ are calculated. From this set, the models contradicting the contraints μ are eliminated; this reduces the distances to the following list:

	T_1	T_2	T_3	T_4	L_Ψ^M
M_1	0	0	4	1	$4,1,0,0$
M_3	1	1	3	2	$3,2,1,1$
M_5	1	1	3	2	$3,2,1,1$
M_7	2	2	2	3	$3,2,2,2$
M_8	2	2	1	2	$2,2,2,1$
M_9	1	1	3	1	$3,1,1,1$
M_{11}	2	2	2	2	$2,2,2,2$
M_{12}	2	2	1	1	$2,2,1,1$
M_{13}	2	2	2	2	$2,2,2,2$
M_{14}	2	2	1	1	$2,2,2,1$
M_{15}	3	3	1	3	$3,3,3,1$
M_{16}	3	3	0	2	$3,3,2,0$

The lexicographic order between L_Ψ^M lists gives models M_{12}, M_{14} as the minimal ones, hence the result of merging by the $\triangle_\mu^{GMax}$ operator on Ψ is $(\neg p \wedge \neg q \wedge r \wedge \neg s) \vee (\neg p \wedge q \wedge \neg r \wedge \neg s)$. The preferred choice by the group of co-owners is therefore to build either the tennis court or the private car park without increasing the rent.

3.2 IC Quasi-Merging

The pseudo-arbitration operator from [15] has the main property of being constrained to only two bases and to require consistency to be obtained without the average principle on bases to be preserved. This means that the negotiation procedure is performed among the belief bases rather than among the propositional letters having different truth values. As a result, if the operator is applied to two bases that are inconsistent with each other, it will provide their disjunction without taking into account any combination of consistent contents.

The $\triangle^{Max}$ operator from [13] is meant to model the very same procedure of arbitration, without the restriction imposed on the number of belief bases involved in the negotiation process. This operator is a less fine-grained one than the $\triangle^{GMax}$, because it provides a larger spectrum of possible results, and therefore it is called a *quasi-merging* operator.

The IC quasi-merging $\triangle^{Max}$ operator is defined in terms of the *minimax* function. Let $\Psi = [T_1, \ldots, T_n]$ be the usual belief set, M a classical model and *dist* the standard Dalal's distance value. A new Max-distance (d_{Max}) between a model M and a belief set Ψ is defined as follows:

$$d_{Max}(M, \Psi) = max_{T \in \Psi} dist(M, T); \tag{4}$$

that is the maximal distance of model M with respect to each belief base $T \in \Psi$. Then a preorder $M_1 \leq_{\Psi}^{Max} M_2$ holds if and only if the corresponding maximal distance for M_1 is less or equal than the maximal distance for M_2: $d_{Max}(M_1, \Psi) \leq d_{Max}(M_2, \Psi)$. Given integrity constraints μ, the $\triangle_{\mu}^{Max}(\Psi)$ operator selects in $\mathcal{M}$ the subset M_{μ} of models occupying the lower position (minimal value) in the obtained pre-order:

$$Mod(\triangle_{\mu}^{Max}(\Psi)) = min(\mathsf{M}_{\mu}, \leq_{\Psi}^{Max}). \tag{5}$$

As for the example, one first calculates the maximal distance between each belief base in Ψ and each of the models (considering those remaining from $\mathcal{M}$ after those contradicting the constraints μ have been eliminated):

	T_1	T_2	T_3	T_4	$d_{Max}(M, \Psi)$
M_1	0	0	4	1	4
M_3	1	1	3	2	3
M_5	1	1	3	2	3
M_7	2	2	2	3	3
M_8	2	2	1	2	2
M_9	1	1	3	1	3
M_{11}	2	2	2	2	2
M_{12}	2	2	1	1	2
M_{13}	2	2	2	2	2
M_{14}	2	2	1	1	2
M_{15}	3	3	1	3	3
M_{16}	3	3	0	2	3

The obtained pre-order gives as minimal models $M_{8,11,12,13,14}$, which means that the result of IC-merging by the $\triangle^{Max}$ operator is $((p \wedge \neg q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge q \wedge \neg r \wedge s) \vee (\neg p \wedge q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge \neg q \wedge r \wedge \neg s))$. The result of this merging procedure is therefore to build one of the three items and not to increase the rent, or to build either the car park or the tennis court increasing the rent.

4 The Adaptive Logic for Merging

In the following sections, the two Adaptive Logics $\mathbf{ADM}^{c+}$ and $\mathbf{ADM}^{c-mm}$ are introduced. These logics complete the definition of the family **ADM**, for which the *Counting Strategy for Majority* was defined in [19].

The adaptive strategies for the new merging protocols are the *Variant Counting* and the *Minimax Counting* strategy:[8] they perform selections on the set of **DM**-models of a given set of premises $\Psi^{\lfloor}$, providing results equivalent to those of the $\triangle^{GMax}$ and $\triangle^{Max}$ operators respectively. The operator $\triangle^{c+}$ (eventually $\triangle_{\mu}^{c+}$ when some set of constraints μ is given) is used for the result of the Variant Counting strategy and $Mod(\triangle^{c+}(\Psi^{\mathcal{B}}))$ to refer to the subset of $\mathcal{M}^{\mathcal{B}}$ correspondingly selected; the operator $\triangle^{c-mm}$ ($\triangle_{\mu}^{c-mm}$ respectively) is used for the result obtained by the Minimax Counting Strategy, $Mod(\triangle^{c-mm}(\Psi^{\mathcal{B}}))$ referring to the subset of $\mathcal{M}^{\mathcal{B}}$ selected by that strategy.

[8]As mentioned in a previous footnote, the strategy is the peculiar element for each of the logics of the family **ADM**.

Let us consider as an example a set of belief bases

$$T_1 = \{p \vee q\}$$
$$T_2 = \{\neg p\}$$
$$T_3 = \{\neg q\}.$$

These belief bases are given a modal translation in any of the logics belonging to the family **ADM** as the premise set $\Psi^{\mathcal{B}} = \{b_1(p \vee q), b_2\neg p, b_3\neg q\} \cup \Upsilon$. Each of the adaptive semantic selections on the set of **DM**-models establishes a related consequence set: the consequence set of each logic correspond to the result of a negotiation procedure performed according to a given merging protocol.

The selection is formulated in view of the disagreements derivable from the modal premise set according to the lower limit logic. Such disagreements are formalised in terms of a special class of formulas, called *abnormalities*. In the case of the previously introduced premise set $\Psi^{\mathcal{B}} = \{b_1(p \vee q), b_2\neg p, b_3\neg q\} \cup \Upsilon$, some of the **DM**-models of $\Psi^{\mathcal{B}}$ verify the formula $b_3\neg q \supset b_0\neg q$, whereas others falsify it; or, what comes to the same, verify $b_3\neg q \wedge \neg b_0\neg q$. An abnormality is precisely a formula of the form $b_iA \wedge \neg b_0A$, i.e. a formula expressing the support that some agent i gives to a literal A, which is not merged in view of some other agent's disagreement. In all **DM**-models of $\Psi^{\mathcal{B}}$, at least one instance of such an abnormality is verified. In our example, there will be two types of **DM**-models of $\Psi^{\mathcal{B}}$: those that verify $b_0\neg q$ and those that verify $\neg b_0\neg q$. Models that verify $\neg b_0\neg q$, necessarily verify the abnormality $b_3\neg q \wedge \neg b_0\neg q$; those that verify $b_0\neg q$ necessarily verify $(b_1q \wedge \neg b_0q) \vee (b_1p \wedge \neg b_0p) \vee (b_2\neg p \wedge \neg b_0\neg p)$.

The selection gives us a precise way to decide which models should be chosen.

5 The Semantics of ADM

In what follows, we present a formally precise formulation of the semantics of **DM** and of the adaptive selection procedures on **DM**-models.

The semantics of the lower limit logic **DM** is a standard possible world semantics, with multiple accessibility relations. A **DM**-model is a quadruple $\langle W, w_o, \mathcal{R}, v\rangle$ where W is a set of possible worlds, $w_o \in W$ is the actual world, $\mathcal{R}$ is a set of serial accessibility relations R_i ($i \in \mathcal{I}$) over W, and $v : \mathcal{P} \times W \to \{0, 1\}$ is an assignment function.

The valuation function defined by a model $M^{\mathcal{B}}$ is characterized as follows:

C1 where $A \in \mathcal{P}$, $v_{M^{\mathcal{B}}}(A, w) = v(A, w)$
C2 $v_{M^{\mathcal{B}}}(\neg A, w) = 1$ iff $v_{M^{\mathcal{B}}}(A, w) = 0$
C3 $v_{M^{\mathcal{B}}}(A \vee B, w) = 1$ iff $v_{M^{\mathcal{B}}}(A, w) = 1$ or $v_{M^{\mathcal{B}}}(B, w) = 1$
C4 $v_{M^{\mathcal{B}}}(A \wedge B, w) = 1$ iff $v_{M^{\mathcal{B}}}(A, w) = 1$ and $v_{M^{\mathcal{B}}}(B, w) = 1$
C5 $v_{M^{\mathcal{B}}}(A \supset B, w) = 1$ iff $v_{M^{\mathcal{B}}}(A, w) = 0$ or $v_{M^{\mathcal{B}}}(B, w) = 1$
C6 $v_{M^{\mathcal{B}}}(b_iA, w) = 1$ iff $v_{M^{\mathcal{B}}}(A, w') = 1$ for all w' such that R_iww'

The standard semantic notions are defined as usual: a model $M^{\mathcal{B}}$ verifies A iff $v_{M^{\mathcal{B}}}(A, w_0) = 1$, $\Psi^{\mathcal{B}} \models_{\mathbf{DM}} A$ iff all **DM**-models of $\Psi^{\mathcal{B}}$ verify A, and $\models_{\mathbf{DM}} A$ iff all **DM**-models verify A.

Abnormalities express conflicts derivable from a premise set that contains the beliefs of the agents according to its lower limit logic models; the following is the formal definition:

Definition 1 (Set of Abnormalities Ω) $\Omega = \{b_i A \wedge \neg b_0 A \mid i \in \mathcal{I} \setminus 0, A \in \mathcal{P}^{\pm}\}$.

In each of the adaptive logics defined as a selection protocol on models of the lower limit logic **DM**, a disjunction of abnormalities may be **DM**-derivable without any of its disjuncts being **DM**-derivable. Consider again $\Psi^{\mathcal{B}} = \{b_1(p \vee q),\ b_2\neg p,\ b_3\neg q\} \cup \Upsilon$. From this, none of the following abnormalities is **DM**-derivable: neither $b_3\neg q \wedge \neg b_0 \neg q$ nor $b_1 q \wedge \neg b_0 q$ nor $b_1 p \wedge \neg b_0 p$ nor $b_2 \neg p \wedge \neg b_0 \neg p$; but their disjunction $(b_3\neg q \wedge \neg b_0\neg q) \vee (b_1 q \wedge \neg b_0 q) \vee (b_1 p \wedge \neg b_0 p) \vee (b_2\neg p \wedge \neg b_0 \neg p)$ is **DM**-derivable. Disjunctions of abnormalities will be called *Dab*-formulas, and the abbreviation $Dab(\Delta)$ will be used to refer to them:

Definition 2 (*Dab*-Formula) *$Dab(\Delta)$ stands for $\bigvee(\Delta)$ where $\Delta \subseteq \Omega$.*

If Δ is a singleton, $Dab(\Delta)$ is a single abnormality; if $\Delta = \emptyset$, any disjunction $A \vee Dab(\Delta)$ corresponds to A. A *Dab*-formula that is **DM**-derivable from $\Psi^{\mathcal{B}}$ will be called a *Dab*-consequence of $\Psi^{\mathcal{B}}$:

Definition 3 (*Dab*-Consequence) *$Dab(\Delta)$ is a Dab-consequence of a premise set $\Psi^{\mathcal{B}}$ iff $\Psi^{\mathcal{B}} \models_{\mathbf{DM}} Dab(\Delta)$.*

If $Dab(\Delta)$ is a *Dab*-consequence of a set $\Psi^{\mathcal{B}}$, then so is any $Dab(\Delta')$ such that $\Delta' \supset \Delta$. This is why a further definition is needed:

Definition 4 (Minimal *Dab*-Consequence) *A disjunction of abnormalities $Dab(\Delta)$ is a* minimal *Dab-consequence of $\Psi^{\mathcal{B}}$ iff $\Psi^{\mathcal{B}} \models_{\mathbf{DM}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ such that $\Psi^{\mathcal{B}} \models_{\mathbf{DM}} Dab(\Delta')$.*

It is in view of the derivability of *Dab*-formulas from a premise set that the *adaptive strategy* is needed. Intuitively, the adaptive strategy specifies what it means, in the case of disjunctions of abnormalities, that the abnormalities are false *unless and until proven otherwise*. Given the same lower limit logic and the same set of abnormalities, there are different ways to interpret a set of premises *as normally as possible*: the precise interpretation of this ambiguous phrase is determined by the adaptive strategy.

In the present case, we will distinguish between interpreting a premise set as normally as possible in view of the Variant Counting Strategy in $\mathbf{ADM}^{c+}$, and in view of the Minimax Counting Strategy in $\mathbf{ADM}^{c-mm}$.

5.1 Variant Counting for Arbitration

The selection by Variant Counting gives rise to the adaptive logic $\mathbf{ADM}^{c+}$. The selection is performed on the basis of the disagreements verified in each **DM**-model with respect to each agent: the number of disagreements provides the ratio by which models are ordered. The task is to select those models that – according to such ordering – satisfy the smallest combination of disagreements among agents.

Consider first all the formulas $A \in \Omega$ such that the b-operator indexed 1 occurs in A: typically, this will be the set of all abnormalities of the form $b_1 A \wedge \neg b_0 A$ or, in other words, all the formulas expressing a conflict involving agent 1. Call this set Ω^1. Then consider the set of all formulas of the same kind occurring with b-operator indexed 2 – that is the set of formulas expressing the conflicts involving agent 2 – and call this set Ω^2, and so on up to index n. Hence, for any given index $i \in \mathcal{I}$, a corresponding set of abnormalities can be formulated:

Definition 5 $\Omega^i = \{A \in \Omega \mid \; b_i$ *occurs in* $A \; (i \in \mathcal{I} \setminus 0)\}$.

The set of abnormalities with a given index is a proper subset of Ω and in turn the set Ω is the union of all the various Ω^i sets:

Definition 6 (The set of indexed abnormalities)

$$\Omega = \bigcup_{i=1}^{n} \Omega^i.$$

For each model $M^{\mathcal{B}}$, consider now the set of abnormal formulas with index i verified by that model. We shall call this set the abnormal part with index i of the model $M^{\mathcal{B}}$:

Definition 7 (The abnormal part with index i of a model) $Ab^i(M^{\mathcal{B}}) = \{A \mid A \in \Omega^i$ *and* $M^{\mathcal{B}} \models_{\mathbf{DM}} A\}$.

For any model $M^{\mathcal{B}}$, let us denote by $\mathcal{C}^i_{M^{\mathcal{B}}} = |Ab^i(M^{\mathcal{B}})|$ the cardinality of its abnormal part of index i:

Definition 8 (Cardinality of the abnormal part with index i of a model) *Given the abnormal part* $Ab^i(M^{\mathcal{B}})$ *with index* i *of a* **DM**-*model, we call its cardinality* $\mathcal{C}^i_{M^{\mathcal{B}}}$ *the number of abnormal formulas* $A \in \Omega^i$ *such that* $M^{\mathcal{B}} \models_{\mathbf{DM}} A$.

The cardinality $\mathcal{C}^i_{M^{\mathcal{B}}}$ expresses the number of disagreements involving agent i verified by the model $M^{\mathcal{B}}$. For each model $M^{\mathcal{B}}$, we construct the list $(\mathcal{C}^1_{M^{\mathcal{B}}}, \ldots, \mathcal{C}^n_{M^{\mathcal{B}}})$, where n is the number of elements of $\mathcal{I}$ occurring in $\Psi^{\mathcal{B}}$. For each $M^{\mathcal{B}}$, let $L^{M^{\mathcal{B}}}$ be the list obtained by $(\mathcal{C}^1_{M^{\mathcal{B}}}, \ldots, \mathcal{C}^n_{M^{\mathcal{B}}})$ by sorting its elements in descending order. Let now $\leq_{lex}$ be the lexicographic order between the various lists $L^{M^{\mathcal{B}}}$. On the basis of the ordering $\leq_{lex}$, a total preorder $\leq^{\mathcal{C}}_{\Psi^{\mathcal{B}}}$ holds among the models $M^{\mathcal{B}}_1, \ldots, M^{\mathcal{B}}_n$ in the following way:

Definition 9 (Preorder by Minimal Cardinality) *A total preorder* $\leq^{\mathcal{C}}_{\Psi^{\mathcal{B}}}$ *holds between* **DM**-*models according to the following definition:*

$$M^{\mathcal{B}}_i \leq^{\mathcal{C}}_{\Psi^{\mathcal{B}}} M^{\mathcal{B}}_j \text{ iff } L^{M^{\mathcal{B}}_i} \leq_{lex} L^{M^{\mathcal{B}}_j}. \tag{6}$$

According to this definition, a pre-order on $\mathcal{M}^{\mathcal{B}}$ is obtained by ordering lexicographically the descending lists of cardinalities for each $M^{\mathcal{B}} \in \mathcal{M}^{\mathcal{B}}$ with respect to their abnormal parts. Where $\mathsf{M}_{\Psi^{\mathcal{B}}}$ stands for the set of **DM**-models for the literals contained in $\Psi^{\mathcal{B}}$, the Variant Counting strategy $\triangle^{c+}(\Psi^{\mathcal{B}})$ will select among those models the minimal ones with respect to the ordering obtained by $\leq^{\mathcal{C}}_{\Psi^{\mathcal{B}}}$:

Definition 10 (Selection of Models by ADM^{c+})

$$Mod(\triangle^{c+}(\Psi^{\mathcal{B}})) = Min(\mathsf{M}_{\Psi^{\mathcal{B}}}, \leq^{\mathcal{C}}_{\Psi^{\mathcal{B}}}). \tag{7}$$

The merging operator $\triangle^{c+}$ reflects an arbitration selection on the **DM**-models: it selects the models of the belief bases that support contents corresponding to the median choices among each $T_i \in \Psi^{\mathcal{B}}$. In the case of the adaptive selection, this means that the selected models are those with lower position in the pre-order given by cardinalities of their abnormal parts.

5.2 Minimax Counting for Quasi-Merging

In the present section the *Minimax Counting* selection of the adaptive logic $\mathbf{ADM}^{c-mm}$ is introduced. It aims at providing the same kind of negotiation process obtained by the $\triangle^{Max}$ operator. The resulting $\triangle^{c-mm}$ operator for the Minimax Adaptive Counting applies the minimax rule to select among the **DM**-models of a premise set in view of the cardinality of their abnormal parts with a given index. This latter notion is the same as in Definition 8.

A new maximal distance Max is defined by the selection of the first element in each list $L^{M}_{\Psi^{\mathcal{B}}}$ for each model $M^{\mathcal{B}}$:

Definition 11 (Maximal Abnormal Distance) $Max(M^{\mathcal{B}}, \Psi^{\mathcal{B}}) = \mathcal{C}^{i}_{M^{\mathcal{B}}}$ *such that there is no index k for which $|Ab^k(M^{\mathcal{B}})| > |Ab^i(M^{\mathcal{B}})|$ holds.*

i.e. $Max(M, \Psi^{\mathcal{B}})$ picks the highest value $\mathcal{C}^i$ for each model $M^{\mathcal{B}}$. Informally, the Maximal Abnormal Distance expresses – for each model – the highest number of disagreements in which a given agent is involved. On its basis, one derives a new total pre-order for the abnormal models in the following way:

Definition 12 (Preorder by Maximal Abnormal Distance) *A total preorder $\leq^{Max}_{\Psi^{\mathcal{B}}}$ on* **DM***-models holds according to the following definition*

$$M_i^{\mathcal{B}} \leq^{Max}_{\Psi^{\mathcal{B}}} M_j^{\mathcal{B}} \text{ iff } Max(M_i^{\mathcal{B}}, \Psi^{\mathcal{B}}) \leq Max(M_j^{\mathcal{B}}, \Psi^{\mathcal{B}}). \tag{8}$$

Where $\mathsf{M}_{\Psi^{\mathcal{B}}}$ stands for the set of **DM**-models for the literals contained in $\Psi^{\mathcal{B}}$, the Minimax Counting strategy of $\mathbf{ADM}^{c-mm}$ will select the minimal models with respect to the ordering obtained by $\leq^{Max}_{\Psi^{\mathcal{B}}}$:

Definition 13 (Selection of Models by $\mathbf{ADM}^{c-mm}$)

$$Mod(\triangle^{c-mm}(\Psi^{\mathcal{B}})) = Min(\mathsf{M}_{\Psi^{\mathcal{B}}}, \leq^{Max}_{\Psi^{\mathcal{B}}}). \tag{9}$$

This selection first considers the highest cardinality of the indexed abnormal parts fro each **DM**-models; then it orders these values and it considers only those models that have the minimal among these maximal values. The result of this selection expresses a negotiation procedure that accounts for all the possible consistent combinations of contents.

6 An Example

We shall now refer to the example introduced in section 3, in order to show how the operators $\triangle^{c+}$ and $\triangle^{c-mm}$ work, and that the same result is obtained as by the corresponding IC-operators. The corresponding premise set in **DM** is of the form $\Psi^{\mathcal{B}} = \{b_1(p \wedge q \wedge r), b_2(p \wedge q \wedge r), b_3(\neg p \wedge \neg q \wedge \neg r \wedge \neg s), b_4(q \wedge r \wedge \neg s)\} \cup \Upsilon$.

Let us now consider the **DM**-models for which cardinalities of their abnormal parts shall be calculated:[9]

$$
\begin{aligned}
M_1^{\mathcal{B}} &= b_0p, b_0q, b_0r, b_0s\\
M_2^{\mathcal{B}} &= b_0p, b_0q, b_0r, b_0\neg s\\
M_3^{\mathcal{B}} &= b_0p, b_0q, b_0\neg r, b_0s\\
M_4^{\mathcal{B}} &= b_0p, b_0q, b_0\neg r, b_0\neg s\\
M_5^{\mathcal{B}} &= b_0p, b_0\neg q, b_0r, b_0s\\
M_6^{\mathcal{B}} &= b_0p, b_0\neg q, b_0r, b_0\neg s\\
M_7^{\mathcal{B}} &= b_0p, b_0\neg q, b_0\neg r, b_0s\\
M_8^{\mathcal{B}} &= b_0p, b_0\neg q, b_0\neg r, b_0\neg s\\
M_9^{\mathcal{B}} &= b_0\neg p, b_0q, b_0r, b_0s\\
M_{10}^{\mathcal{B}} &= b_0\neg p, b_0q, b_0r, b_0\neg s\\
M_{11}^{\mathcal{B}} &= b_0\neg p, b_0q, b_0\neg r, b_0s\\
M_{12}^{\mathcal{B}} &= b_0\neg p, b_0q, b_0\neg r, b_0\neg s\\
M_{13}^{\mathcal{B}} &= b_0\neg p, b_0\neg q, b_0r, b_0s\\
M_{14}^{\mathcal{B}} &= b_0\neg p, b_0\neg q, b_0r, b_0\neg s\\
M_{15}^{\mathcal{B}} &= b_0\neg p, b_0\neg q, b_0\neg r, b_0s\\
M_{16}^{\mathcal{B}} &= b_0\neg p, b_0\neg q, b_0\neg r, b_0\neg s
\end{aligned}
$$

In view of our constraint μ, the models $M_2^{\mathcal{B}}, M_4^{\mathcal{B}}, M_6^{\mathcal{B}}, M_{10}^{\mathcal{B}}$ are rejected, i.e. any model satisfying $b_0((p \wedge q) \vee (p \wedge r) \vee (q \wedge r)) \wedge \neg s$ is ignored. The following models are left:

$$
\begin{aligned}
M_1^{\mathcal{B}} &= b_0p, b_0q, b_0r, b_0s\\
M_1^{\mathcal{B}} &= b_0p, b_0q, b_0r, b_0s\\
M_5^{\mathcal{B}} &= b_0p, b_0\neg q, b_0r, b_0s\\
M_7^{\mathcal{B}} &= b_0p, b_0\neg q, b_0\neg r, b_0s\\
M_8^{\mathcal{B}} &= b_0p, b_0\neg q, b_0\neg r, b_0\neg s\\
M_9^{\mathcal{B}} &= b_0\neg p, b_0q, b_0r, b_0s\\
M_{11}^{\mathcal{B}} &= b_0\neg p, b_0q, b_0\neg r, b_0s\\
M_{12}^{\mathcal{B}} &= b_0\neg p, b_0q, b_0\neg r, b_0\neg s\\
M_{13}^{\mathcal{B}} &= b_0\neg p, b_0\neg q, b_0r, b_0s\\
M_{14}^{\mathcal{B}} &= b_0\neg p, b_0\neg q, b_0r, b_0\neg s\\
M_{15}^{\mathcal{B}} &= b_0\neg p, b_0\neg q, b_0\neg r, b_0s\\
M_{16}^{\mathcal{B}} &= b_0\neg p, b_0\neg q, b_0\neg r, b_0\neg s
\end{aligned}
$$

6.1 Arbitration

For each of the remaining models we calculate the cardinality with respect to the sets of abnormalities $\Omega^i = \{b_iA \wedge \neg b_0A \mid A \in \mathcal{P}^{\pm}; i \in \mathcal{I}\}$. For each model $M^{\mathcal{B}}$ and any indexed set of abnormalities Ω^i, there will be a value to $\mathcal{C}^i_{M^{\mathcal{B}}}$. These values are listed in the following table, where at the intersection of each $M_j^{\mathcal{B}}$ and $\mathcal{C}^i$ one has the value of $|Ab^i(M^{\mathcal{B}})|$; in the last column $L^{M^{\mathcal{B}}}$ is the list of the obtained values of cardinalities of the abnormal parts of each model in descending order:

[9] In the following we restricts the representation of models only to formulas prefixed by the b_0-operator.

	$\mathcal{C}^1_{M^{\mathcal{B}}}$	$\mathcal{C}^2_{M^{\mathcal{B}}}$	$\mathcal{C}^3_{M^{\mathcal{B}}}$	$\mathcal{C}^4_{M^{\mathcal{B}}}$	$L^{M^{\mathcal{B}}}$
$M^{\mathcal{B}}_1$	0	0	4	1	$4, 1, 0, 0$
$M^{\mathcal{B}}_3$	1	1	3	2	$3, 2, 1, 1$
$M^{\mathcal{B}}_5$	1	1	3	2	$3, 2, 1, 1$
$M^{\mathcal{B}}_7$	2	2	2	3	$3, 2, 2, 2$
$M^{\mathcal{B}}_8$	2	2	1	2	$2, 2, 2, 1$
$M^{\mathcal{B}}_9$	1	1	3	1	$3, 1, 1, 1$
$M^{\mathcal{B}}_{11}$	2	2	2	2	$2, 2, 2, 2$
$M^{\mathcal{B}}_{12}$	2	2	1	1	$2, 2, 1, 1$
$M^{\mathcal{B}}_{13}$	2	2	2	2	$2, 2, 2, 2$
$M^{\mathcal{B}}_{14}$	2	2	1	1	$2, 2, 2, 1$
$M^{\mathcal{B}}_{15}$	3	3	1	3	$3, 3, 3, 1$
$M^{\mathcal{B}}_{16}$	3	3	0	2	$3, 3, 2, 0$

The lexicographic order $\leq^{\mathcal{C}}_{\Psi^{\mathcal{B}}}$ among the sequences of $L^{M^{\mathcal{B}}}$ gives the total preorder among the various models:[10]

$$\begin{gathered} M^{\mathcal{B}}_{12,14} \leq M^{\mathcal{B}}_8 \leq M^{\mathcal{B}}_{11,13} \leq M^{\mathcal{B}}_9 \leq M^{\mathcal{B}}_{3,5} \leq \\ M^{\mathcal{B}}_7 \leq M^{\mathcal{B}}_{16} \leq M^{\mathcal{B}}_{15} \leq M^{\mathcal{B}}_1. \end{gathered} \tag{10}$$

The result of merging according to $min(\mathsf{M}, \leq^{\mathcal{C}}_{\Psi^{\mathcal{B}}})$ is given by the disjunction of the two minimal models $(12, 14)$ in the preorder:

$$\triangle^{c+}_{\mu}(\Psi^{\mathcal{B}}) = b_0((\neg p \wedge q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge \neg q \wedge r \wedge \neg s)). \tag{11}$$

This is also the result of the pure arbitration $\triangle^{GMax}$ operator from [13].

6.2 Quasi-merging

By means of the same example, we show now the selection performed by the $\triangle^{c-mm}$ operator for Minimax Adaptive Counting. From the very same premise set $\Psi^{\mathcal{B}} = \{b_1(p \wedge q \wedge r), b_2(p \wedge q \wedge r), b_3(\neg p \wedge \neg q \wedge \neg r \wedge \neg s), b_4(q \wedge r \wedge \neg s)\} \cup \Upsilon$, the same list of $\mathcal{C}^i_{M^{\mathcal{B}}}$ values are formulated for the cardinalities of the indexed abnormal part of each model. The models that allow the combination $b_0((p \wedge q) \vee (p \wedge r) \vee (q \wedge r) \wedge \neg s)$ are obviously still rejected in view of the constraint μ.

According to Definition 11, one selects the Maximal Abnormal Distance for each of the remaining models:

[10] To make it easier for the reader, super- and subscripts on the ordering symbol $\leq^{\mathcal{C}}_{\Psi^{\mathcal{B}}}$ have been removed.

	$\mathcal{C}^1_{M^{\mathcal{B}}}$	$\mathcal{C}^2_{M^{\mathcal{B}}}$	$\mathcal{C}^3_{M^{\mathcal{B}}}$	$\mathcal{C}^4_{M^{\mathcal{B}}}$	$Max(M_j, \Psi^{\mathcal{B}})$
$M_1^{\mathcal{B}}$	0	0	4	1	4
$M_3^{\mathcal{B}}$	1	1	3	2	3
$M_5^{\mathcal{B}}$	1	1	3	2	3
$M_7^{\mathcal{B}}$	2	2	2	3	3
$M_8^{\mathcal{B}}$	2	2	1	2	2
$M_9^{\mathcal{B}}$	1	1	3	1	3
$M_{11}^{\mathcal{B}}$	2	2	2	2	2
$M_{12}^{\mathcal{B}}$	2	2	1	1	2
$M_{13}^{\mathcal{B}}$	2	2	2	2	2
$M_{14}^{\mathcal{B}}$	2	2	1	1	2
$M_{15}^{\mathcal{B}}$	3	3	1	3	3
$M_{16}^{\mathcal{B}}$	3	3	0	2	3

from which the following preorder based on $\leq^{Max}_{\Psi^{\mathcal{B}}}$ is obtained:

$$M^{\mathcal{B}}_{8,11,12,13,14} \leq^{Max}_{\Psi^{\mathcal{B}}} M^{\mathcal{B}}_{3,5,7,9,15,16} \leq^{Max}_{\Psi^{\mathcal{B}}} M^{\mathcal{B}}_1. \tag{12}$$

From these values the selection of models with the minimal values provides the following alternatives:

$$\begin{aligned} \triangle^{c-mm}_{\mu}(\Psi^{\mathcal{B}}) = b_0((p \wedge \neg q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge q \wedge \neg r \wedge s) \vee \\ (\neg p \wedge q \wedge \neg r \wedge \neg s) \vee (\neg p \wedge \neg q \wedge r \wedge s) \vee \\ (\neg p \wedge \neg q \wedge r \wedge \neg s)). \end{aligned} \tag{13}$$

Also in this case, the adaptive selection by $\triangle^{c-mm}$ provides the same result as the $\triangle^{Max}$ operator from [13].

7 Conclusion

In this paper, adaptive selection procedures corresponding to pure-arbitration and quasi-merging fusion protocols have been provided. The obtained adaptive logics complete the formal presentation of the family **ADM**, together with the adaptive logic for majority $\mathbf{ADM}^c$ from [19]. The formulation of an adaptive proof theory for $\mathbf{ADM}^{c+}$ and $\mathbf{ADM}^{c-mm}$ shall easily follow along the line of the derivability and marking relations defined for the majority protocol. We also expect to be able to define a selection procedure for **ADM** that reflects the $\triangle^n$ operators from [11], which define protocols belonging simultaneously to majority and arbitration. The formulation of positive and negative results for the application of the adaptive protocols, as for example in relation to judgment aggregation procedures (see [17]), is foreseen.

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