Strong, universal and provably non-trivial set theory by means of adaptive logic

PETER VERDEE[1], *Centre for Logic and Philosophy of Science, Ghent University, Blandijnberg 2, B-9000 Ghent, Belgium.*
E-mail: peter.verdee@ugent.be.

Abstract

In this paper I present a non-trivial but inconsistent set theory based on the axioms of naive set theory. The theory is provably non-trivial and strong enough for most of the applications of regular mathematics. This is realized by distinguishing between strong and weak set membership and allowing for the derivation of strong membership from weak membership whenever this is not problematic (it does not lead to paradoxes). This idea of applying rules whenever unproblematic is formalized by means of an adaptive logic.

Keywords: adaptive logic, set theory, axiom of foundation, non-triviality, Curry's paradox, ZFC

1 Introduction

In what follows I define a theory to be a couple $\langle \mathfrak{A}_1 + \ldots + \mathfrak{A}_n, \mathbf{L}\rangle$, where $\mathfrak{A}_i$ are axioms or sets of axioms and $\mathbf{L}$ is a logic. $A \in \langle \mathfrak{A}_1 + \ldots + \mathfrak{A}_n, \mathbf{L}\rangle$ iff $A \in Cn_{\mathbf{L}}(\Gamma)$ where $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_n$ and for all $i \leq n$, $\Gamma_i = \{\mathfrak{A}_i\}$, if $\mathfrak{A}_i$ is an axiom, and $\Gamma_i = \mathfrak{A}_i$, if $\mathfrak{A}_i$ is a set of axioms.

A set theory is a theory that determines how a binary predicate $\in$, denoting set membership, functions. It formalizes the common scientific and mathematical practice of collecting objects into a larger object.

Now, what do we ideally expect from such a theory? I assume that one would ideally prefer a set theory that has the following properties.

(PNT) Provable non-triviality
(UNI) Universality
(MS) With mathematical strength

(PNT) says there should exist a finitistic proof for the non-triviality of the set theory, otherwise one is never certain that working with the theory makes any sense at all. (UNI) stands for the existence of every set defined by means of an expression of the form $\{x \mid A(x)\}$. Finally, (MS) states that one should be able to translate most of mathematics into the set theory. This means for example that one should be able to express all interesting theorems of complex number theory within the set theory. This might sound unrealistic, but it is generally accepted that most of classical mathematics can be reduced to classical set theory **ZFC**.

Let us now go through some existing candidates for a proper set theory.

1.1 Naive set theory

Naive set theory is the theory $\langle \mathsf{COMP}_1 + \mathsf{EXT}, \mathbf{CL}\rangle$, where **CL** is classical logic and EXT and COMP_1 are defined as follows[2]:

EXT $\quad \forall x \forall y(\forall z(z \in x \equiv z \in y) \to x = y)$
COMP_1 $\quad \exists x \forall y(y \in x \equiv A(y))$

This theory is trivial. Many famous paradoxes can be derived in this theory. The most important paradoxes are the following.

We start with Russell's paradox.

$$\exists x \forall y(y \in x \equiv \neg y \in y)$$

Less famous variants of the Russell paradox are the following generelizations, which are discovered by Quine (cf. [15]): for every n,

$$\exists x \forall y(y \in x \equiv (\neg \exists z_1 \ldots \exists z_n(y \in z_1 \wedge z_1 \in z_2 \wedge \ldots \wedge z_n \in y)))\,.$$

One could interpret this axiom as follows. It states that the set of sets $\{x|x$ is a set and x is not part of a loop of n sets$\}$ exists, where *a loop of m sets* is a series of sets a_0, a_1, $\ldots a_m$ with the property $a_0 \in a_1$, $a_1 \in a_2, \ldots a_{m-1} \in a_m$. Curry's paradox (cf. [7]) shows that one can also express Russell's paradox without a negation:

$$\exists x \forall y(y \in x \equiv (y \in y \to A))\,.$$

Also Quine's variants of Russell's paradox are expressible without a negation:

$$\exists x \forall y(y \in x \equiv (\exists z_1 \ldots \exists z_n(y \in z_1 \wedge z_1 \in z_2 \wedge \ldots \wedge z_n \in y) \to A)))\,.$$

Let us call this last series of paradoxes the *Quine-Curry-paradoxes* (QCP). Remark that all of the former paradoxes can be considered as instances of (QPC). Let me prove (QPC) for the case where $n = 2$. The other cases are very similar.

In the proof below, every implication $A \to B$ is replaced by its disjunctive counterpart $\neg A \vee B$, which is equivalent to $A \to B$ in classical logic (and in all other logics discussed in this paper). In the third column of the proof below, the justification for the derivation is mentioned. I$\forall$ and I$\neg\exists$ abbreviate the universal instantiation rules (respectively $\forall \alpha A(\alpha) \vdash A(\beta)$ and $\neg \exists \alpha A(\alpha) \vdash \neg A(\beta)$), E$\equiv$1 and E$\equiv$2 stand for the elimination of an equivalence (respectively $A \equiv B \vdash \neg A \vee B$ and $A \equiv B \vdash \neg B \vee A$). E$A \vee A$ and E$A \wedge A$ denote the elimination of twice the same disjunct resp. conjunct (respectively $A \vee A \vdash A$ and $A \wedge A \vdash A$). DM abbreviates the De Morgan rules. E$\neg\neg$ stands for the elimination of double negation ($\neg\neg A \vdash A$). D$\wedge\vee$ denotes the distributivity of conjunction and disjunction ($(A \wedge B) \vee C \vdash (A \vee C) \wedge (B \vee C)$). SIM abbreviates the simplification rule ($A \wedge B \vdash A$ and $A \wedge B \vdash B$). EQ stands for the equivalence rule ($A \equiv B, C \vdash C'$ where A is a subformula of C and C' is the result of substituting A by B in the formula C). ASS refers to the associativity of the disjunction and ADD to the addition rule ($A \vdash A \vee B$). Finally, EM denotes the excluded middle rule ($A \vee B, \neg A \vee B \vdash B$).

[2]Every instantiation of an axiom (schema) in this paper should be closed under universal quantification, prior to introduction in an object language proof. The axioms therefore stand for closed, well formed formulas, even if they are not closed in their bare form.

Note that I do not use every rule literally for efficiency reasons. The rules are often applied on a formula within another formula and commutativity of the disjunction and conjunction is often used implicitly. The reader can verify that this is valid in the relevant cases. Also note that I use the rule E$\exists$. This refers to the rule 'from $\exists\alpha A(\alpha)$ derive $A(\beta)$ where β is a new constant'. Of couse, this rule is not generally valid in **CL**, but this is harmless in the proof below because the new constants o_C and o_D do not occur in the arbitrary formula A, which is what I actually want to prove, and because $\exists\alpha A(\alpha), A(\beta) \to B \vdash_{\mathbf{CL}} B$ holds whenever β does not occur in $A(\alpha)$ or B.

Proof.

1	$\exists x \forall y(y \in x \equiv (\neg\exists z(y \in z \wedge z \in y) \vee A))$	COMP_1
2	$\forall y(y \in o_C \equiv (\neg\exists z(y \in z \wedge z \in y) \vee A))$	E$\exists$;1
3	$o_C \in o_C \equiv (\neg\exists z(o_C \in z \wedge z \in o_C) \vee A)$	I$\forall$;2
4	$\neg o_C \in o_C \vee \neg\exists z(o_C \in z \wedge z \in o_C) \vee A$	E$\equiv$1;3
5	$\neg o_C \in o_C \vee \neg(o_C \in o_C \wedge o_C \in o_C) \vee A$	I$\neg\exists$;4
6	$\neg o_C \in o_C \vee \neg o_C \in o_C \vee A$	E$A\wedge A$;5
7	$\neg o_C \in o_C \vee A$	E$A\vee A$;6
8	$\neg(\neg\exists z(o_C \in z \wedge z \in o_C) \vee A) \vee o_C \in o_C$	E$\equiv$2;3
9	$(\neg\neg\exists z(o_C \in z \wedge z \in o_C) \wedge \neg A) \vee o_C \in o_C$	DM;8
10	$(\exists z(o_C \in z \wedge z \in o_C) \wedge \neg A) \vee o_C \in o_C$	E$\neg\neg$;9
11	$(\exists z(o_C \in z \wedge z \in o_C) \vee o_C \in o_C) \wedge (\neg A \vee o_C \in o_C)$	D$\wedge\vee$;10
12	$\exists z(o_C \in z \wedge z \in o_C) \vee o_C \in o_C$	SIM;11
13	$(o_C \in o_D \wedge o_D \in o_C) \vee o_C \in o_C$	I$\exists$;12
14	$(o_C \in o_D \vee o_C \in o_C) \wedge (o_D \in o_C \vee o_C \in o_C)$	D$\wedge\vee$;13
15	$o_D \in o_C \vee o_C \in o_C$	SIM;14
16	$o_D \in o_C \equiv (\neg\exists z(o_D \in z \wedge z \in o_D) \vee A)$	I$\forall$; 2
17	$(\neg\exists(o_D \in z \wedge z \in o_D) \vee A) \vee o_C \in o_C$	EQ;15,16
18	$(\neg(o_D \in o_C \wedge o_C \in o_D) \vee A) \vee o_C \in o_C$	EQ;17
19	$\neg(o_D \in o_C \wedge o_C \in o_D) \vee (A \vee o_C \in o_C)$	ASS;18
20	$(o_C \in o_D \wedge o_D \in o_C) \vee (A \vee o_C \in o_C)$	ADD;13
21	$A \vee o_C \in o_C$	EM;19,20
22	A	EM;7,21

■

1.2 ZFC-set theory

ZFC-set theory is the theory $\langle \mathsf{EXT} + \mathsf{SUB} + \mathsf{UNION} + \mathsf{POWER} + \mathsf{INF} + \mathsf{REPL} + \mathsf{FOUND} + \mathsf{CHOICE}, \mathbf{CL}\rangle$, where **CL** is classical logic and EXT, SUB, UNION, POWER, INF, REPL, FOUND, and CHOICE are defined as follows:

EXT	$(x \in y \equiv x \in z) \to y = z$
REPL	$\forall w \exists y \exists z (\forall y A(w,y) \to z = y) \to \exists y \forall z (z \in y \equiv \exists w (w \in x \wedge \forall y A(w,z))$
UNION	$\exists y \forall z (z \in y \equiv \exists u (u \in x \wedge z \in u))$
POWER	$\exists y \forall z (z \in y \equiv \forall w (w \in z \to w \in x))$
INF	$\exists y (x \in y \wedge \forall z (z \in y \to \exists w (z \in w \wedge w \in y)))$
FOUND	$\exists y (y \in x) \to \exists y (y \in x \wedge \forall z (z \in x \to \neg z \in y))$
CHOICE	$\exists y \forall z \forall w ((z \in w \wedge w \in x) \to \exists v \forall u (\exists t ((u \in w \wedge w \in t) \wedge (u \in t \wedge t \in y)) \equiv u = v))$

The theory **ZFC** is the most commonly used set theory (cf. [9]). It is an extension of Zermelo's set theory (cf. [22]).

It is commonly accepted that most of actual mathematics can be done in ZFC. In the project Metamath (cf. `http://us.metamath.org`), for example, many interesting mathematical thoerems of very different domains of mathematics are formally proved from the axioms of **ZFC**.

Due to Gödel's second incompleteness theorem (cf. [10]) and the reducibility of Peano arithmatic to **ZFC**, there cannot be a straight forward way to finitistically prove the non-triviality of **ZFC**.

The theory is not universal as many sets (among which the set of all sets and the set of all ordinals) do not exists according to the axioms.

1.3 The axioms of naive set theory with a monotonic non-classical logic

Many systems have been proposed to give a non-trivial version of the comprehension axiom. They use (monotonic) fragments of **CL** as their underlying logic. Most of the proposed set theories use a relevant logic that does not validate the rule of contraction to avoid Curry's paradox (cf. [5], [6], [19] and [20]). One can also use other logics without contraction (cf. [12], [8], [17] and [21]) or a weak paraconsistent logic like **LP** (cf. [14] and [16]).

I shall refer to the set theoretic system described in [16] by means of the abbreviation **LPST**. This is the theory $\langle \mathsf{COMP}_1 + \mathsf{EXT}, \mathbf{LP} \rangle$, where **LP** is a predicative version of Priest's Logic of Paradox (cf. [13]).

All set theories of this type are universal (they use full comprehension), but are often too weak for useful mathematical purposes or they are strong enough too formalize basic arithmatic, in which case they have the same problem as **ZFC** with respect to proving non-triviality.

1.4 Adaptive set theory

In this paper I shall construct a set theory based on adaptive logic. This universal set theory is able to accept only the unproblematic consequences of the comprehension axiom schema, and avoid the ones that lead to paradoxes. Adaptive logics are perfectly suitable for this mission as they localize the problems of theories and restrict this theory only wherever this is strictly necessary. The resulting set theory will be universal, evidently non-trivial, and, on the condition that **ZFC** is consistent, the theory will also have mathematical strength.

Now, there are of course several ways to construct such a set theory, but anyhow, one has to start from a so called lower limit logic. This is a usual monotonic, transitive and reflexive logic. The rules of this logic determine the entirely unproblematic consequences of the adaptive logic.

Next to this logic one also needs to define a set of axioms for the adaptive set theory. Together with the lower limit logic, this set of axioms already constitutes a weak universal set theory. This theory should be as strong as possible, but still be evidently paradox free. The adaptive logic will extend this theory with some more dangerous consequences: the classical logic consequences, except the ones that lead to paradoxes. In the next section, I define this weak monotonic set theory.

2 Weak universal set theory

2.1 The logic **LPC**

The logic I shall use is basically Priest's **LP** enriched with a classical negation $\neg$ and a stronger (defined) equivalence connective. Of course one could use another logic, and there may even be much stronger or more sensible logics available, but this logic is sufficiently strong and quite elegant. Let us call it **LPC**. I will define the logic semantically. But let us start with the language of the logic and some preliminaries.

Let $\mathcal{L}$ be the language with the logical symbols $\neg$, $\check{\neg}$, $\wedge$, $\vee$, $\rightarrow$, $\mathrel{\equiv\!\!\!\!\!\neg}$, $\equiv$, $\forall$, $\exists$, and $=$ (but without function symbols). $\mathcal{L}$ is defined in the usual way *from* $\langle \mathcal{C}, \mathcal{V}, \mathcal{P}^0, \mathcal{P}^1, \ldots \rangle$, in which $\mathcal{C}$ is the set of individual constants, $\mathcal{V}$ the set of individual variables, and $\mathcal{P}^r$ the set of predicates of rank $r \geq 0$—predicates of rank 0 will function as sentential letters, with the restriction that $\mathrel{\equiv\!\!\!\!\!\neg}$ cannot occur within the scope of another $\mathrel{\equiv\!\!\!\!\!\neg}$. Let $\mathcal{F}$ and $\mathcal{W}$ denote respectively the set of formulas and the set of closed formulas of $\mathcal{L}$. Let $\mathbb{P} \subset \mathcal{F}$ denote the set of primitive formulas and let $\mathbb{P}^{\neg} = \{\neg A \mid A \in \mathbb{P}\}$.

There are two negations. A paraconsistent negation $\sim\! A$ and a classical negation $\neg A$. Unlike many other paraconsistent logics, the classical negation can occur within the scope of a paraconsistent negation.

Formulas will have four truth values. T stands for pure truth, F for pure falsity, B for both A and $\sim\! A$ true and D for A false, $\sim\! A$ true, but $\sim\!\neg A$ also true. The fourth value is necessary because the classical negation $\neg A$ can occur within the scope of the paraconsistent negation $\sim$. The semantics is recursive and every inconsistency boils down to inconsistencies on the level of primitive formulas or on the level of the classical negation of primitive formulas. The designated truth values are T and B.

Because I only need the paraconsistent negation $\sim$ for the definition of a special equivalence symbol $\mathrel{\equiv\!\!\!\!\!\neg}$ (which I need for a non-trivial version of the comprehension axiom), I take the $\mathrel{\equiv\!\!\!\!\!\neg}$-symbol as a primitive symbol and define the $\sim$-symbol.

In order to simplify the characterization of the semantics, I introduce a pseudo-language. Let $\mathcal{O}$ be a set of *pseudo-constants*; $\mathcal{O}$ should have at least the cardinality of your largest set—the domain of a model is a set and a member of $\mathcal{O}$ should be mapped by the assignment v on every element of $\mathcal{O}$. The pseudo-language ${}^{+}\mathcal{L}$ is defined from $\langle \mathcal{C} \cup \mathcal{O}, \mathcal{V}, \mathcal{P}^0, \mathcal{P}^1, \ldots \rangle$. Let ${}^{+}\mathcal{F}$ and ${}^{+}\mathcal{W}$ denote respectively the set of formulas and the set of closed formulas of ${}^{+}\mathcal{L}$.

Let, for every $r > 0$, $D^{(r)}$ denote the r-th Cartesian product of D and let $D^{(0)} = \{\emptyset\}$, i.e. a 0-tuple will be identified with $\emptyset$.

	$\mathcal{V}_M(A \Rrightarrow B)$				$\mathcal{V}_M(A \vee B)$				$\mathcal{V}_M(\neg A)$
$\mathcal{V}_M(A)$ \ $\mathcal{V}_M(B)$	B	T	F	D	B	T	F	D	
B	T	T	F	F	B	T	B	B	D
T	T	T	F	F	T	T	T	T	F
F	F	F	T	T	B	T	F	D	T
D	T	T	T	T	B	T	D	D	B

Table 1. Matrices for the propositional symbols.

An **LPC**-model M (for the language $\mathcal{L}$) is a couple $\langle D, v\rangle$ in which D is a non-empty set and the assignment v is as follows:

C1.1 $v\colon \mathcal{C} \cup \mathcal{O} \to D$ (where $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$)
C1.2 $v_{\mathfrak{V}}\colon \mathcal{P}^r \to \wp(D^{(r)})$ (for every $r \in \mathbb{N}$ and every $\mathfrak{V} \in \{B, T, F, D\}$)
C1.3 $\bigcap\{v_{\mathfrak{V}}(\pi) | \mathfrak{V} \in \{B, T, F, D\}\} = \emptyset$
C1.4 $\bigcup\{v_{\mathfrak{V}}(\pi) | \mathfrak{V} \in \{B, T, F, D\}\} = D^{(r)}$

The following clauses define how a model M determines the truth values $\mathcal{V}_M$ formulas receive in that model.

C2.1 $\mathcal{V}_M(\pi^r\alpha_1 \dots \alpha_r) = \mathfrak{V}$ iff $\langle v(\alpha_1), \dots, v(\alpha_r)\rangle \in v_{\mathfrak{V}}(\pi^r)$ where $\mathfrak{V} \in \{B, T, F, D\}$[3].
C2.2 $\mathcal{V}_M(\neg A)$, $\mathcal{V}_M(A \Rrightarrow B)$ and $\mathcal{V}_M(A \wedge B)$ are determined according to the truth values in table 1.
C2.3a $\mathcal{V}_M(\forall\xi A(\xi)) = T$ iff $\mathcal{V}_M(A(\alpha)) = T$ for all $\alpha \in \mathcal{C} \cup \mathcal{O}$
C2.3b $\mathcal{V}_M(\forall\xi A(\xi)) \in \{B, T\}$ iff $\mathcal{V}_M(A(\alpha)) \in \{B, T\}$ for all $\alpha \in \mathcal{C} \cup \mathcal{O}$
C2.3c $\mathcal{V}_M(\forall\xi A(\xi)) = F$ iff $\mathcal{V}_M(A(\alpha)) = F$ for at least one $\alpha \in \mathcal{C} \cup \mathcal{O}$
C2.3d $\mathcal{V}_M(\forall\xi A(\xi)) \in \{F, D\}$ iff $\mathcal{V}_M(A(\alpha)) \in \{F, D\}$ for at least one $\alpha \in \mathcal{C} \cup \mathcal{O}$
C2.4a $\mathcal{V}_M(\alpha = \beta) \in \{T, F\}$
C2.4b $\mathcal{V}_M(\alpha = \beta) = T$ iff $v(\alpha) = v(\beta)$

Some symbols are defined from the other symbols. The defined symbols function as mere abbreviations of more complex formulas.

D1 $A \to B =_{\text{def}} \neg A \vee B$
D2 $A \wedge B =_{\text{def}} \neg(\neg A \vee \neg B)$
D3 $A \equiv B =_{\text{def}} (A \to B) \wedge (B \to A)$
D4 $\exists\alpha A(\alpha) =_{\text{def}} \neg\forall\alpha\neg A(\alpha)$
D5 $\sim A =_{\text{def}} \neg A \Rrightarrow (A \vee \neg A)$

Definition 2.1
LPC-satisfaction. Where $A \in \mathcal{L}$, $\Gamma \subseteq \mathcal{L}$ and $M = \langle v, D\rangle$ is an **LPC**-model, $M \models A$ iff $\mathcal{V}_M(A) \in \{B, T\}$ and $M \models \Gamma$ iff $M \models A$ for every $A \in \Gamma$.

Definition 2.2
LPC-consequence. Where $\Gamma \cup \{A\} \subseteq \mathcal{L}$, A is an **LPC**-consequence of Γ, in symbols $\Gamma \vDash_{\mathbf{LPC}} A$, iff $M \models A$, for every **LPC**-model M such that $M \models \Gamma$. Let $A \Dashv\vDash_{\mathbf{LPC}} B$ abbreviate $A \vDash_{\mathbf{LPC}} B$ and $B \vDash_{\mathbf{LPC}} A$.

[3] Remember that $\langle v(\alpha_1), \dots, v(\alpha_r)\rangle = \emptyset$ if $r = 0$. So $\mathcal{V}_M(\pi^0) = T$ iff $v_T(\pi^0) = \{\emptyset\}$.

$\mathcal{V}_M(A)$ \ $\mathcal{V}_M(B)$	$\mathcal{V}_M(A \wedge B)$ B	T	F	D	$\mathcal{V}_M(\sim A)$	$\mathcal{V}_M(\sim\neg A)$	$\mathcal{V}_M(A \Rrightarrow \neg A)$
B	B	B	F	D	T	T	F
T	B	T	F	D	F	T	F
F	F	F	F	F	T	F	F
D	D	D	F	D	T	T	T

TABLE 2. Truth functionality of some important formulas

One can easily devise a sound and complete proof theory for this logic (very similar to the proof theory **LP**), but for my current purpose, this is not needed. The reader can check the following theorem by means of truth tables.

THEOREM 2.3
Important properties of this logic are the following:

F1 $A \Rrightarrow B \dashv\vDash_{\mathbf{LPC}} (A \to B) \wedge (B \to \sim\neg A)$
F2 $\sim\neg A \nvDash_{\mathbf{PCL}} A$
F3 $A \wedge \sim A \dashv\vDash_{\mathbf{LPC}} \neg A \Rrightarrow A$
F3 $\neg A \wedge \sim\neg A \dashv\vDash_{\mathbf{LPC}} A \Rrightarrow \neg A$
F4 $\sim(A \wedge B) \dashv\vDash_{\mathbf{LPC}} \sim A \vee \sim B$
F5 $\sim(A \wedge B) \dashv\vDash_{\mathbf{LPC}} \sim A \wedge \sim B$
F6 if $A \dashv\vDash_{\mathbf{LPC}} B$ then $\sim A \dashv\vDash_{\mathbf{LPC}} \sim B$
F7 $A \Rrightarrow B \dashv\vDash_{\mathbf{LPC}} (A \equiv B) \vee (A \Rrightarrow \neg A)$
F8 if $\mathcal{V}_M(A) = D$ then $M \models A \Rrightarrow B$
F9 the truth functionality matrices in table 2 are correct
F10 $\neg A, \sim\neg A \vdash A \Rrightarrow B$
F11 $M \models \neg A$ iff $M \not\models A$
F12 $M \models A \wedge B$ iff $M \models A$ and $M \models B$
F13 $M \models A \wedge B$ iff $M \models A$ or $M \models B$
F14 $\Gamma \cup \{\neg(B \Rrightarrow \neg B) \mid B \in \mathbb{P} \cup \mathbb{P}^{\neg}\} \vDash_{\mathbf{PLC}} A$ iff $\Gamma \vDash_{\mathbf{CL}} A$

This logic is not paraconsistent because it is explosive (for all $A, B \in \mathcal{W}$, $A, \neg A \vDash_{\mathbf{LPC}} B$ holds), but $\sim$ is a fully paraconsistent negation, i.e. (for every $A \in \mathcal{W}$ there is a $B \in \mathcal{W}$ such that $A, \sim A \nvDash_{\mathbf{LPC}} B$).

2.2 A weak but universal and non-trivial set theory based on **LPC**

DEFINITION 2.4
The universal set theory **WUST** is defined as $\langle \mathsf{EXT} + \mathsf{COMP}_2, \mathbf{LPC}\rangle$, where EXT and COMP_2 are defined as follows

EXT $\forall x \forall y(\forall z(z \in x \equiv z \in y) \to x = y)$
COMP_2 $\forall x(x \in \llcorner\{\mathbf{y} \mid \mathbf{A}(\mathbf{y})\}\lrcorner \Rrightarrow A(x))$

Remark that COMP_2 does not have its more common existential form. We need the version in which the created sets have a name, because we need to distinguish between different individuals of the theory in function of the adaptive systems presented in the next section. However, the existential version is of course a consequences of this version, so I can do this without loss of generality.

Remark moreover that COMP_2 does not allow us to derive $\alpha \in \lfloor\{y \mid A(y)\}\rfloor$ from $A(\alpha)$. However one is able to derive $\sim\neg\alpha \in \lfloor\{y \mid A(y)\}\rfloor$ from $A(\alpha)$. So it makes sense to define what I shall call *weak membership*, denoted by the symbol $\tilde{\in}$: let $\alpha\tilde{\in}\beta =_{\text{def}} \sim\neg\alpha \in \beta$. I shall use the word *strong membership* for the usual membership predicate $\in$.

The following theorem states that the presence of a rule that allows for the derivation of strong membership from weak membership would make the set theory identical to naive set theory. It is an immediate consequence of F14.

Theorem 2.5
If one adds the rule $\sim\neg A \vdash A$ or the axiom $\neg(A \equiv \neg A)$, where $A \in \mathbb{P} \cup \mathbb{P}^{\neg}$, to **WUST**, one obtains naive set theory.

The following theorem is true in quite a strong sense, because for every formula $A(\alpha)$, (1) the set $a = \lfloor\{x \mid A(x)\}\rfloor$ for which holds that $(y \in a \equiv A(y)) \in$ **WUST** exists and (2) $A(\alpha)$ is **WUST**-equivalent to $\alpha\tilde{\in}\lfloor\{x \mid A(x)\}\rfloor$. Many other universal alternative set theories only have property (1) and not property (2), because they do not enable the possibility of defining a weak set membership predicate like $\tilde{\in}$.

Theorem 2.6
WUST is universal.

Theorem 2.7
The theory **WUST** is non-trivial, i.e. it has a model that does not make all formulas true.

Proof. Let D be the singleton $\{s\}$, let $v(\alpha) = s$ for every $\alpha \in \mathcal{C} \cup \mathcal{O}$ and let $\langle s, s\rangle \in v_D(\in)$ but $v_B(\in) \cup v_T(\in) \cup v_F(\in) = \emptyset$. Suppose $M = \langle D, v\rangle$. The truth functionality of $\neg$ and $\sim$ and the definition of **LPC**-satisfaction ensure that for every $\alpha, \beta \in \mathcal{C} \cup \mathcal{O}$, $M \models \neg\beta \in \alpha$ and $M \models \sim\neg\beta \in \alpha$. F10 warrants that $M \models \beta \in \alpha \equiv A$. Using the definition of $\forall$, one obtains $M \models \mathsf{COMP}_2$. Because $M \models \forall x \forall y x = y$ (D is a singleton), also $M \models \mathsf{EXT}$.

So **WUST** has at least one model. Moreover, M does not satisfy every formula, as e.g. $M \not\models \forall x x \in x$. ∎

WUST is a rather weak theory that is satisfied by (among more interesting models) a rather simplistic model consisting of only one individual. We can, however, literally add most of the **ZFC**-axioms (with their full classical meaning) to this theory, without losing this simplistic model (which warrants the non-triviality).

Definition 2.8
The universal set theory **WUST**+ is defined as $\langle \mathsf{EXT} + \mathsf{COMP}_2 + \mathsf{FOUND} + \mathsf{FAF} + \mathsf{REPL} + \mathsf{UNION} + \mathsf{CHOICE}, \mathbf{LPC}\rangle$, where FAF is defined as follows:

$$\mathsf{FAF} \qquad \bigvee\{\bigwedge\{\neg x_i \in x_j \mid i \leq n\} \mid j \leq n\} \qquad \text{where } n \in \mathbb{N}$$

FAF is a finitistic version of the axiom of Foundation (FOUND). It is simply the result of instantiating the universal quantifier $\forall x$ in FOUND with every possible finite set, i.e. with the sets $\{x_0\}$, $\{x_0, x_1\}$, $\{x_0, x_1, x_2\}$ and so on, for every possible set x_0, x_1, x_2, and so on (and afterwards simplifying the obtained expression). Hence, in combination with the POWER-axiom and the UNION-axiom of **ZFC**, FAF is a **CL**-consequence of FOUND, but this is not the case in **WUST+**, because **WUST+** does not prove simple **ZFC**-truths like $a \in \{a\}$, $a \in \{a, b\}$, or $a \in \{a, b, c\}$, and so on. Although FOUND is not a **CL**-consequence of FAF, FAF seems to be equivalent to FOUND for every sensible set theoretic model. Let me explain this. In fact FOUND states that for every set x of sets it holds that here is no loop of sets among the elements of x. If x is infinite and there would be a loop of sets among the members of x, then there must be a finite subset y of x, such that there is a loop of sets among the members of y. Now, remark that FAF exactly states that there is no loop of sets among the members of a finite set. The reason why FAF and FOUND are nevertheless not equivalent in a classical **ZFC** context is related to omega incompleteness, but elaborating on this particularity falls out of the scope of this paper.

After analyzing the axioms FOUND, FAF, REPL, UNION and CHOICE, one immediately discovers that they are **PLC**-consequences of the formula $\forall x \forall y \neg x \in y$, whence they are satisfied by the model described in proof 2.2. Hence, the latter proof is also sufficient to show the following theorem.

THEOREM 2.9
The theory **WUST+** is non-trivial.

As **WUST** $\subset$ **WUST+**, the following theorem is evident.

THEOREM 2.10
The theory **WUST+** is universal.

For the following theorem, let the weak equivalence $\equiv$ from **LP** correspond to the $\Rrightarrow$-equivalence in **LPC**. The proof of the theorem is rather straight forward considering the fact that **LPC** is conceived as an extension of **LP** and the fact that **LP** has a weaker equivalence-connective and that it does not have a classical negation nor an implication which can be used in a modus ponens rule.

THEOREM 2.11
LPST $\subset$ **WUST**, i.e. **WUST** is stronger than **LPST**.

Although **WUST+** is already stronger than **WUST**, also **WUST+** is terribly weak. Simple classical derivations like $\sim\neg A \vdash A$ are not allowed in **LPC**, which disables the derivation of evident truths like $\forall x(x \in \llcorner\{\mathsf{y} \mid \mathsf{y} = \mathsf{x}\}\lrcorner)$, although $\sim\neg\forall x(x \in \llcorner\{\mathsf{y} \mid \mathsf{y} = \mathsf{x}\}\lrcorner) \in$ **WUST**. Without the capacity of proving membership of even simple sets like this singleton, none of the interesting mathematical theorems are provable. Consequently, the following theorem holds.

THEOREM 2.12
The theories **WUST** and **WUST+** do not have mathematical strength.

Remark that this is unavoidable, given the fact that **WUST** is provably non-trivial and the fact that there is a positive test for **WUST**-theoremhood. Suppose a provably non-trivial theory like **WUST** would have mathematical strength, then it would be a system stronger than Peano arithmetic and it would be able to prove its own non-triviality. Gödels second theorem proves that this is impossible.

It is perfectly possible to make **WUST** stronger without losing its provable non-triviality. **WUST+** is a first step towards a stronger version of **WUST**. Without any doubt, one is able to make this even stronger in a sensible way. A sensibly stronger version of **WUST+** is definitely preferable for our present purpose. Nevertheless, one should be very careful with the addition of more **ZFC**-theorems to **WUST+** or **CL**-rules to **LPC**. All evident enrichments have the consequence that there are no finite models that satisfy the theory (i.e. only models in which there are infinitely many sets). This makes it very probable that the non-triviality proof for the theory (if there is any) will no longer be evidently or finitistically acceptable.

If one would add the **ZFC**-axioms to **WUST**, the resulting theory would be universal and would have mathematical strength, but one would fail to finitistically prove this theory's non-triviality.

3 Full adaptive set theory

We have observed that the set theory **WUST** is interesting because of its universality and its non-triviality, but at the same time it is so terribly weak that it becomes totally useless (although it is already stronger than Priest and Restall's **LPST**). Adding the rule $\sim\neg A \vdash A$ would make the theory trivial. Nevertheless, a lot of cases (formulas A) exist in which the application of $\sim\neg A \vdash A$ is probably harmless. How can we formally distinguish between the safe cases, and the cases that lead to paradoxes. Well, adaptive logic is the perfect tool to guide this process of applying rules only when this does not cause problems.

3.1 The adaptive logic $\mathbf{LPC^r}$

A reasoning form with rules of the type 'conclude A unless and until A delivers problems' is a typical form of *defeasible reasoning.* In that sense, the intuitive picture of the logic for the more subtle set theory we are looking for definitely refers to a form of defeasible reasoning.

Adaptive logics are excellent tools to formalize defeasible reasoning (cf. [2], [4], [3] and [1] for some general formal and philosophical introductions to **AL**). A large amount of very different types of defeasible reasoning have been characterized by means of an adaptive logic: abductive reasoning, inductive reasoning, inconsistency corrective reasoning, reasoning with vagueness, reasoning with ambiguity, reasoning about compatibility, question raising, coping with theories where statements are only plausibly true, diagnosis, causal discovery, belief merging and default reasoning. The semantics of adaptive logics defines the set of stable consequences of dynamic defeasible reasoning. The dynamic proofs of adaptive logics are intuitive explications for actual defeasible reasoning processes.

There is an elegant formal format for adaptive logics, called the *standard format of adaptive logic.* The dynamic proof theory, the semantics and the meta-theory of adaptive logics in standard format are generic and intuitive. An adaptive logic **AL** is defined by a triple:

1. A *lower limit logic* **LLL**: a reflexive, transitive, monotonic, and compact logic that has a characteristic semantics and contains **CL** (Classical Logic).
2. A *set of abnormalities* Ω: a set of **LLL**-contingent formulas, characterized by a (possibly restricted) logical form which contains at least one logical symbol.

3. An *adaptive strategy*: Reliability (r) or Minimal Abnormality (m).

The lower limit logic is the stable part of the adaptive logic; anything that follows from the premises by **LLL** will never be revoked. The lower limit logic is an extension of **CL** because it contains all the classical symbols next to its standard symbols. Abnormalities are supposed to be false, 'unless and until proven otherwise'. Strategies are ways to cope with derivable disjunctions of abnormalities: an adaptive strategy picks one specific way to interpret the premises as normally as possible.

If the lower limit logic is extended with an axiom that declares all abnormalities logically false, one obtains the *upper limit logic* **ULL**. If a premise set Γ does not require that any abnormalities are true, the **AL**-consequences of Γ are identical to its **ULL**-consequences.

For the logic $\mathbf{LPC^r}$ the **LLL** is **LPC**, $\Omega = \{A \Rrightarrow \neg A \mid A \in \mathbb{P}\} \cup \{\neg A \Rrightarrow A \mid A \in \mathbb{P}\}$ and the strategy is Reliability. The **ULL** of this logic is full classical logic.

In the expression $Dab(\Delta)$, Δ is a finite subset of Ω and $Dab(\Delta)$ denotes the *classical* disjunction of the members of Δ. $Dab(\Delta)$ is called a *Dab*-formula. $Dab(\Delta)$ is a *minimal Dab-consequence of* Γ iff $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ whereas $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Delta')$ for any $\Delta' \subset \Delta$. Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal *Dab*-consequences of Γ, $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$ The set $U(\Gamma)$ comprises the abnormalities that are *unreliable* with respect to Γ. Where M is a **LLL**-model, $\mathrm{Ab}(M)$ is the set of abnormalities verified by M.

Definition 3.1
A **LPC**-model M of Γ is *reliable* iff $\mathrm{Ab}(M) \subseteq U(\Gamma)$.

Definition 3.2
$\Gamma \vDash_{\mathbf{LPC^r}} A$ iff A is verified by all reliable models of Γ.

There exists an interesting dynamic proof theory for adaptive logics, but I do not need it for the present purpose. For an introduction, cf. [4]. The proof theory is sound and complete with respect to the semantics. Hence, *syntactic adaptive consequence* $\Gamma \vdash_{\mathbf{AL}^m} A$ or $\Gamma \vdash_{\mathbf{AL}^r} A$ is equivalent to semantic adaptive consequence $\Gamma \vDash_{\mathbf{AL}^m} A$ or $\Gamma \vDash_{\mathbf{AL}^r} A$, respectively.

The fact that $\mathbf{LPC^r}$ is in standard format also immediately shows the following lemma (cf. [2]).

Lemma 3.3
If there are **LPC**-models for Γ, there are also reliable **LPC**-models for Γ.

3.2 First attempt

Definition 3.4
The universal set theory $\mathbf{WUST^r}$ is defined as $\langle \mathsf{EXT} + \mathsf{COMP}_2, \mathbf{LPC^r} \rangle$

Theorem 3.5
$\mathbf{WUST^r}$ is universal and non-trivial.

Proof. The universality is an immediate consequence of the fact that **WUST** is non-trivial and the fact that $\mathbf{WUST^r}$ is an extension of **WUST**. The non-triviality is a consequence of lemma 3.3. ∎

Intuitively this results in a theory which is much richer than **WUST**. Every time one can only derive $\sim\neg A$ in **WUST**, the adaptive logic allows for the conditional derivation of A from $\sim\neg A$. The idea is that whenever it turns out that problematic instances of COMP_2 are used, one can also (unconditionally) derive $\neg A$. Whenever this happens, the derivation of A from $\sim\neg A$ is not any more adaptively valid. If no problems occur, the derivation of A should be adaptively accepted.

This is very attractive. However, the Curry-Quine-paradoxes make this adaptive set theory useless, i.e. the formulas true in all **PCL**-models of the axioms are the same as the formulas true in the reliable models of the axioms. Suppose for arbitrary α and β that $\sim\neg\alpha \in \beta \in \mathbf{WUST}$ and $\alpha \in \beta$ is a harmless **ZFC**-theorem (for example $a \in \{a\}$). Hence also $\alpha \in \beta \vee (\alpha \in \beta \Rrightarrow \neg\alpha \in \beta) \in \mathbf{WUST}$ (1). Now, by COMP_2, we know that $\forall x(x \in \llcorner\{\mathbf{y} \mid \neg\mathbf{y} \in \mathbf{y} \vee \neg\alpha \in \beta\}\lrcorner \Rrightarrow (\neg x \in x \vee \neg\alpha \in \beta))$. Hence, by F7 and a reasoning similar to the one used in proof 1.1, $\neg\alpha \in \beta \vee (s_c \in s_c \equiv \neg s_c \in s_c) \in \mathbf{WUST}$ (2), where s_c abbreviates $\llcorner\{\mathbf{y} \mid \neg\mathbf{y} \in \mathbf{y} \vee \neg\alpha \in \beta\}\lrcorner$. From (1) and (2) we can conclude that $(\alpha \in \beta \Rrightarrow \neg\alpha \in \beta) \vee (s_c \in s_c \equiv \neg s_c \in s_c) \in \mathbf{WUST}$. Because $\alpha \in \beta$ is supposed to be a harmless consequence, $\alpha \in \beta \Rrightarrow \neg\alpha \in \beta \notin \mathbf{WUST}$. Observe moreover that $s_c \in s_c \equiv \neg s_c \in s_c \notin \mathbf{WUST}$ (otherwise $\neg s_c \in s_c \in \mathbf{WUST}$ would be true, but there is a **WUST**-model M such that $M \models s_c \in s_c$ and $M \not\models \alpha \in \beta$). Consequently, $(\alpha \in \beta \Rrightarrow \neg\alpha \in \beta) \vee (s_c \in s_c \equiv \neg s_c \in s_c)$ is a minimal *Dab*-consequence of the axioms and both abnormalities are unreliable, which means that there are reliable models M in which $M \models \alpha \in \beta \Rrightarrow \neg\alpha \in \beta$ whence $M \models \neg\alpha \in \beta$. Those reliable models will not satisfy the unproblematic consequence $\alpha \in \beta$ whence $\alpha \in \beta \notin \mathbf{WUST}^r$.

Generalizing this problem, we obtain the following theorem.

THEOREM 3.6
With respect to the set theory **WUST**, going adaptive does not result in a stronger theory, i.e. $\mathbf{WUST}^r = \mathbf{WUST}$.

This theorem holds for universal set theory with underlying logic **PLC**, but as far as I have checked, exactly the same problem occurs in about every sensible candidate for a lower limit logic.

3.3 Further localization of the inconsistencies

Fortunately, there is an easy way out of this problem. The solution is achieved by further localization of the inconsistencies. The basic problem of the previous subsection is related to the fact that no explicit inconsistency is derivable from the Curry-Quine-paradoxes. Such an explicit inconsistency is nevertheless derivable in the presence of the FAF-axioms. This is stated in the following lemma.

LEMMA 3.7
$(\neg s_c \in s_c) \in \mathbf{WUST}+$ where $s_c = \llcorner\{\mathbf{x} \mid \neg\exists \mathbf{y_1} \ldots \exists \mathbf{y_n}(\mathbf{x} \in \mathbf{y_1} \wedge \mathbf{y_1} \in \mathbf{y_2} \wedge \ldots \wedge \mathbf{y_n} \in \mathbf{x}) \vee \mathbf{A}\}\lrcorner$.

This fact warrants that for every Curry-Quine-paradoxical set s_c a formula $(s_c \in s_c \Rrightarrow \neg s_c \in s_c) \vee (\alpha \in \beta \Rrightarrow \neg\alpha \in \beta)$ is never a minimal *Dab*-consequence in **WUST**+, whence the problem for $\mathbf{WUST^r}$ is solved in $\mathbf{WUST+^r}$.

DEFINITION 3.8
The universal set theory $\mathbf{WUST+^r}$ is defined as $\langle \mathsf{EXT} + \mathsf{COMP_2} + \mathsf{FOUND} + \mathsf{FAF} + \mathsf{REPL} + \mathsf{UNION} + \mathsf{CHOICE}, \mathbf{LPC^r}\rangle$

THEOREM 3.9
The adaptive set theory $\mathbf{WUST+^r}$ is universal and non-trivial.

THEOREM 3.10
If $A \in \mathbf{ZFC}$, then there is a $\Delta \subset \Omega$ such that $A \vee Dab(\Delta) \in \mathbf{WUST+^r}$, i.e. the conditional consequences of the adaptive set theory $\mathbf{WUST+^r}$ are a superset of the **ZFC**-theorems.

Although it seems unlikely that harmless **ZFC**-consequences would not be verified by all reliable **WUST**+-models, it seems to be rather difficult to prove such a statement, as it seems to require a full overview over the **ZFC**-consequences and the other conditional **WUST**+-consequences (otherwise one is unable to assure that there are no harmful minimal *Dab*-consequences).

CONJECTURE 3.11
If **ZFC** is non-trivial, the adaptive set theory $\mathbf{WUST+^r}$ has mathematical strength.

Why is it possible for adaptive set theory to be very strong and provably non-trivial while this is impossible for all usual classical, non-classical set theories? Well the answer is very simple: the Gödel proof relies heavily on the fact that the notion proof is recursive for the systems he talks about. A proof for finally demonstrating some adaptive consequence of a set of formulas, is definitely not recursive as it is based on the question whether some formulas are not **LLL**-consequences and this question is definitely not recursive.

3.4 Localizing the inconsistencies even further

Another, new (although arguably somewhat ad hoc) adaptive strategy can be conceived, in such a way that conjecture 3.11 can be proved for the resulting set theory.

Let $\Omega_{\mathbf{ZFC}} = \{A|A \in \Omega \text{ and } A \text{ only contains constants that refer to sets that exist according to } \mathbf{ZFC}\}$. Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal *Dab*-consequences of Γ,

$$\Delta'_i = \begin{cases} \Delta_i & \text{if } \Delta_i \subset \Omega_{\mathbf{ZFC}} \\ \Delta_i - \Omega_{\mathbf{ZFC}} & \text{otherwise,} \end{cases}$$

Let $U^{\mathbf{ZFC}}(\Gamma) = \Delta'_1 \cup \Delta'_2 \cup \ldots$ The set $U^{\mathbf{ZFC}}(\Gamma)$ comprises the abnormalities that are **ZFC**-*unreliable* with respect to Γ. Where M is a **LLL**-model, $\text{Ab}(M)$ is the set of abnormalities verified by M.

DEFINITION 3.12
A **PCL**-model M of Γ is **ZFC**-*reliable* iff $\text{Ab}(M) \subseteq U^{\mathbf{ZFC}}(\Gamma)$.

DEFINITION 3.13
$\Gamma \vDash_{\mathbf{PCL}^{\mathbf{ZFC}}} A$ iff A is verified by all **ZFC**-reliable models of Γ.

LEMMA 3.14
If there are **LPC**-models for Γ, there are also **ZFC**-reliable **LPC**-models for Γ.

LEMMA 3.15
If $A \in \mathbf{ZFC}$, then there is a $\Delta \subset \Omega_{\mathbf{ZFC}}$ such that $A \vee Dab(\Delta) \in \mathbf{WUST+}$, i.e. the conditional consequences of the adaptive set theory $\mathbf{WUST+^{ZFC}}$ are a superset of the **ZFC**-theorems.

THEOREM 3.16
If **ZFC** is non-trivial, the adaptive set theory $\mathbf{WUST+^{ZFC}} = \langle \mathsf{EXT} + \mathsf{COMP}_2 + \mathsf{FOUND} + \mathsf{FAF} + \mathsf{REPL} + \mathsf{UNION} + \mathsf{CHOICE}, \mathbf{LPC^{ZFC}} \rangle$ is universal, non-trivial and has mathematical strength.

PROOF. Non-triviality is a consequence of lemma 3.14. Universality is evident.

Let me prove the (conditional) mathematical strength of $\mathbf{WUST+^{ZFC}}$. Suppose that **ZFC** is non-trivial. In that case, there are no inconsistencies in **ZFC**, which means that every time $(\neg\alpha \in \beta) \in \mathbf{ZFC}$, also $(\alpha \in \beta) \notin \mathbf{ZFC}$, whence $(\alpha \in \beta \equiv \neg\alpha \in \beta) \notin \mathbf{WUST+}$ and $(\neg\alpha \in \beta \equiv \alpha \in \beta) \notin \mathbf{WUST+}$. Now suppose some **ZFC**-theorem A is not derivable in $\mathbf{WUST+^{ZFC}}$. In view of lemma 3.15, $A \vee Dab(\Delta) \in \mathbf{WUST+}$ for some $\Delta \subset \Omega_{\mathbf{ZFC}}$. It follows that $A \in \mathbf{WUST+^{ZFC}}$ if there is no $B \in \Delta$ such that also $B \in U^{\mathbf{ZFC}}(\Gamma)$.

So it suffices to prove that $B \notin U^{\mathbf{ZFC}}(\Gamma)$ for every possible $B \in \Delta$. If B would be in $U^{\mathbf{ZFC}}(\Gamma)$, then B would be in one of the sets $\Delta'_1, \Delta'_2, \ldots$. Suppose it would be in Δ'_i. In that case B would also be in Δ_i (because $B \in \Omega_{\mathbf{ZFC}}$), Δ_i would be a subset of $\Omega_{\mathbf{ZFC}}$ and Δ_i would be a minimal Dab-consequence, which conflicts with our assumption that **ZFC** would be non-trivial and hence consistent. ■

4 Discussion

I have presented several set theories. The first two set theories **WUST** and **WUST+** are monotonic extensions of Priest's **LPST**. Based on those two rather weak set theories, I have constructed the adaptive set theories $\mathbf{WUST^r}$, $\mathbf{WUST+^r}$ and $\mathbf{WUST+^{ZFC}}$. $\mathbf{WUST^r}$ turned out to be too weak, but the logics $\mathbf{WUST+^r}$ and $\mathbf{WUST+^{ZFC}}$ succeed in adding some unproblematic richer consequences to the monotonic consequences. By means of these systems, I have provided a set theory which is universal, non-trivial and sufficiently strong. This is exactly what is ideally required from a set theory.

Universality is an important property. It means that collecting objects by means of the criterion that they share some property is possible for every property. In other words, for every property there exists a set. Although one needs to be very cautious with the things one does with these sets (otherwise one runs into paradoxes), the intuitive idea of collecting objects by their common properties turns out to be possible without giving up to much of the mathematical strength of a usual theory of sets. Where, for example, most mathematicians do not consider the collection of all sets (formerly called the universal set) as a set itself (they would call it a class or an object of another type), it seems to be possible to state the existence of a universal set without risk of triviality. In the theory $\mathbf{WUST^r}$ the universal set exists (e.g. the set named $\llcorner\{\mathsf{x} \mid \mathsf{x} \in \mathsf{x} \vee \neg\mathsf{x} \in \mathsf{x}\}\lrcorner$) and it is a (weak) member of itself, as the following holds:

$$(\llcorner\{\mathsf{x} \mid \mathsf{x} \in \mathsf{x} \vee \neg\mathsf{x} \in \mathsf{x}\}\lrcorner \tilde{\in} \llcorner\{\mathsf{x} \mid \mathsf{x} \in \mathsf{x} \vee \neg\mathsf{x} \in \mathsf{x}\}\lrcorner) \in \mathbf{WUST^r}\,.$$

However, it is impossible to have all these wonderful advantages without disadvantages. The most important disadvantage is a substantial increase in complexity. Unlike in usual theories, there is no positive test nor a negative test for theoremhood. Of course, this can be considered as a major drawback, but compare this to the alternatives. All other universal set theories either use very weak logics or their non-triviality cannot be proved finitistically. This means that the former theories give some formal explication of a concept set, but these interesting formal tools cannot be considered as explications for the concept set used in actual modern mathematics (**LPST**, for example, does not even have the property that if $\alpha \in \{x \mid A(x)\}$, then also $A(\alpha)$, a very basic principle of set theory which cannot really be considered as a paradox). The latter set theories on the other hand use sufficiently strong logics, but have exactly the same problem as **ZFC**: one cannot convincingly prove their non-triviality.

Most mathematicians nowadays do not really care about non-triviality. They find it very unlikely that their systems would turn out to be inconsistent after so many decades of unproblematic work with these systems. Their triviality may indeed be very unlikely, but if someone would one day find a contradiction in some exotic part of **ZFC** (and there is no absolute guarantee that this will not happen), the whole framework of formal mathematics entirely collapses into triviality. Interesting results that are proved in many decades of work with **ZFC** become as valuable as the sentence $\forall x \forall y(x \in y \wedge \neg x \in y)$. Without tools that are as complex as adaptive logics, one would not be able to safe the parts of set theory that seem unproblematic, even after the inconsistency is found.

Now, how problematic is this increase in complexity, given that it is unavoidable? I argue that it is not problematic at all. One is still able to conditionally prove theorems in adaptive set theory (by means of the **ULL**, by supposing that some abnormalities are false), in the same way and with the same ease as one would do this in **ZFC**. The only problem is that one is never really certain that these conditional theorems are also reliable consequences. But the same is true in **ZFC**: one is never really certain that one's theorems have any more value than nonsensical sentences. The difference with the adaptive way of working is that the adaptive set theorist is constantly aware of the possibly problematic assumptions he has based his proof on. If one day he would find a contradiction, he only has to give up on some limited part of his system (and he knows which part this is!).

The adaptive set theories $\mathbf{WUST^r}$ and $\mathbf{WUST^{ZFC}}$ are Σ^0_3-complete (cf. [11]). For abstract purposes like my present purpose, the Minimal Abnormality strategy is a far more elegant and intuitive adaptive strategy, and it allows for more consequences, so one could claim that it would have been more sensible if I used this strategy, but because this strategy is even more complex (such a set theory would even be Π^1_1-complete, cf. [18]) and because I could not come up with any sensible extra consequences it delivers, I opted for the simpler Reliability strategy.

Apart from the fact that they are interesting theory in their own right, the presented adaptive set theories give a formal explication of how inconsistencies are localized and neutralized in inconsistent theories. Adaptive logics capture the methodology that can be used to resolve problems in mathematical theories. It is quite interesting to see the process of moving from the weak theory **WUST** to **WUST**+, to $\mathbf{WUST^r}$, to $\mathbf{WUST+^r}$ and to $\mathbf{WUST^{ZFC}}$. With every step of this metalogical process, the inconsistencies are further localized. This results in more consequences.

References

[1] Diderik Batens. The need for adaptive logics in epistemology. In Dov Gabbay, S. Rahman, J. Symons, and J. P. Van Bendegem, editors, *Logic, Epistemology and the Unity of Science*, pages 459–485. Kluwer Academic Publishers, Dordrecht, 2004.

[2] Diderik Batens. A universal logic approach to adaptive logics. *Logica Universalis*, 1:221–242, 2007.

[3] Diderik Batens. *Adaptive Logics and Dynamic Proofs. A Study in the Dynamics of Reasoning.* 200x. Forthcoming.

[4] Diderik Batens, Kristof De Clercq, Peter Verdée, and Joke Meheus. Yes fellows, most human reasoning is complex. *Synthese*, 166:113–131, 2009.

[5] Ross T. Brady. The simple consistency of a set theory based on the logic csq. *Notre Dame Journal of Formal Logic*, 24:431–449, 1983.

[6] Ross T. Brady and Richard Routley. The non-triviality of extensional dialectical set theory. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic: Essays on the Inconsistent*, pages 415–436. Philosophia Verlag, 1989.

[7] Haskell Curry. The inconsistency of certain formal logics. *Journal of Symbolic Logic*, 7:115–117, 1942.

[8] Hartry Field. *Saving Truth from Paradox.* Oxford University Press, Oxford, 2008.

[9] Abraham Adolf Fraenkel, Yehoshua Bar-Hillel, and Azriel Lvy. *Foundations of set theory.* North Holland, Amsterdam, 1973.

[10] Kurt Gödel. Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme. I. *Monatshefte für Mathematik und Physik*, 38:173–198, 1931.

[11] Leon Horsten and Philip Welch. The undecidability of propositional adaptive logic. *Synthese*, 158:41–60, 2007.

[12] Uwe Petersen. A contraction free approach to logic based on inclusion and unrestricted abstraction. first draft.

[13] Graham Priest. The logic of paradox. *Journal of Philosophical Logic*, 8:219–241, 1979.

[14] Graham Priest. *In Contradiction: A Study of the Transconsistent.* Martinus Nijhoff, The Hague, 1987.

[15] Willard V. Quine. On the theory of types. *Journal of Symbolic Logic*, 3(4):125–139, 1938.

[16] Greg Restall. A note on naïve set theory in *LP*. *Notre Dame Journal of Formal Logic*, 33(3):422–432, 1992.

[17] Greg Restall. How to be *Really* contraction free. *Studia Logica*, 52:381–391, 1993.

[18] Peter Verdée. Adaptive logics using the minimal abnormality strategy are Π^1_1-complex. *Synthese*, 167:93–104, 2009.

[19] Zach Weber. Extensionality and Restriction in Naive Set Theory. *Studia Logica*, 94(1), 2010.

[20] Zach Weber. Transfinite Numbers in Paraconsistent Set Theory. *Review of Symbolic Logic*, 3(1):71–92, 2010.

[21] Richard White. The consistency of the axiom of comprehension in the infinite valued predicate logic of łukasiewicz. *Journal of Philosophical Logic*, 8(1):503–534, 1979.

[22] Ernst Zermelo. Untersuchungen ber die grundlagen der mengenlehre. I. *Mathematische Annalen*, 65(2), 1908.

Received October 15, 2010