Spoiled for Choice?*

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Abstract

The transition from a theory that turned out trivial to a consistent replacement need not proceed in terms of inconsistencies, which are negation gluts. Logics that tolerate gluts or gaps (or both) with respect to any logical symbol may serve as the lower limit for adaptive logics that assign a minimally abnormal consequence set to a given premise set. The same obtains for logics that tolerate a combination of kinds of gluts and gaps. This result runs counter to the obsession with inconsistency that classical logicians and paraconsistent logicians share.

All such basic logics will be systematically reviewed, some variants will be outlined, and the claim will be argued for. While those logics tolerate gluts and gaps with respect to logical symbols, ambiguity logic tolerates ambiguities in non-logical symbols. Moreover, forms of tolerance may be combined, with zero logic as an extreme.

In the baffling plethora of corrective adaptive logics (roads from trivial theories to consistent replacements), adaptive zero logic turns out theoretically interesting as well as practically useful. On the one hand all meaning becomes contingent, depending on the premise set. On the other hand, precisely adaptive zero logic provides one with an excellent analyzing instrument. For example it enables one to figure out which corrective adaptive logics lead, for a specific trivial theory, to a suitable and interesting minimally abnormal consequence set.

1 Introduction

Inconsistency-adaptive logics were devised for a specific purpose. Consider a theory $T = \langle \Gamma, \mathbf{CL} \rangle$, in which Γ is a set of non-logical axioms and **CL** is Classical Logic. From the fact that the second element of T is **CL** we know that the theory is or was meant and believed to be consistent. Suppose, however, that an inconsistency is derived from Γ, whence T is trivial. Suppose moreover that T is an actual historical theory, that it was considered respectable in view of its nice applications, and that the removal of the triviality is not obvious.

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In this case, one will want to reason from T in view of applications and also in order to find a consistent alternative to T. What one will want to reason from is T in its full richness, except for the pernicious consequences of its inconsistency—see already [9]. *This* entity will obviously be inconsistent, but it will be *as consistent as possible* and hence not trivial.[1] Inconsistency-adaptive logics offer a minimally inconsistent 'interpretation' of theories in that they *localize* and *isolate* the inconsistencies within their consequence set. From this 'interpretation', one will later try to remove the inconsistencies.

Already in [13] I complained that both classical logicians and paraconsistent logicians are obsessed by inconsistency. The former take inconsistency, on the one hand, as beyond the limit of coherent thinking and, on the other hand, as the criterion of all correct derivation ($\Gamma \vdash A$ iff $\Gamma \cup \{\neg A\}$ is inconsistent). Paraconsistent logicians see only one remedy for avoiding triviality: allowing for true inconsistencies. Whenever a theory turns out to have no **CL**-models, both kinds of logicians analyse the situation as follows: for some formula A, the theory requires that both A and $\neg A$ are true.

This analysis, which classical logicians and paraconsistent logicians agree upon, is mistaken because very different approaches are possible. This is a first reason to read this paper. I shall show in Section 3 that many premise sets that have no **CL**-models have not only paraconsistent models but other models as well. Paraconsistent models leave room for inconsistencies, which may be considered as negation gluts—a precise definition follows in Section 3. Other non-classical models leave room for negation gaps, or for other kinds of gluts or gaps.

Rephrased in terms of logics, there are logics that are exactly like **CL**, except in that they leave room for gluts or gaps (or both) with respect to a logical symbol. Moreover, every combination of (zero or more) kinds of gluts and (zero or more) kinds of gaps characterizes a similar logic. I mean this literally. Even the combination of all kinds of gluts and gaps defines a logic, which will be called **CLo**. It is a terrifyingly weak logic, but nevertheless a logic. Each logic that leaves room for gluts or gaps, however weak the logic be, may function as the lower limit logic of an adaptive logic. Each such adaptive logic offers a minimally abnormal interpretation for some premise sets. For people not familiar with adaptive logics: the lower limit logic leaves room for certain kinds of abnormalities and the adaptive logic (set theoretically) minimizes the number of abnormalities that are considered as true.

Apart from all this, each of these adaptive logics has a number of variants and some of them may be combined along different combination schemata.

The adaptive logic $\mathbf{CLo}^m$ is interesting for theoretical reasons and this is a second reason to read this paper. Although no logical symbol has any meaning in the lower limit logic **CLo**, $\mathbf{CLo}^m$ delivers a minimally abnormal interpretation of a premise set—of each premise set actually. So this means that meaning of all logical symbols is contingent in $\mathbf{CLo}^m$. To be more precise, the meaning of the logical symbols depends on the premise set, on the sentences that make it up—the precise meaning of the sentences becomes known by applying the adaptive logic.

[1] If even this entity were trivial, T cannot have been meant as consistent.

While **CLo** assigns no meaning to any logical symbol, it leaves the non-logical symbols unaffected. Not all logics, however, do so. An example is ambiguity logic, which will be presented in Section 5. According to ambiguity logic different occurrences of the same non-logical symbol may have different meanings. Again, adaptive ambiguity logic minimizes ambiguities.

This leads to the fascinating adaptive zero logic, $\mathbf{CL}\emptyset^{\mathsf{m}}$, which is a third reason to read this paper. The lower limit logic leaves room for all considered abnormalities: all kinds of gluts and gaps *and* non-logical symbols of which the different occurrences have different meanings. According to the lower limit logic, no formula is derivable from any premise set—not even the premises themselves.[2] Adaptive zero logic minimizes the abnormalities. The relation with a formal hermeneutics is striking.

For all adaptive logics mentioned so far, the following holds. If a premise set Γ has **CL**-models, which comes to saying that it is consistent, then the adaptive logics assign to Γ exactly the same consequence set as **CL**. The fascinating part, however, concerns the case where Γ has no **CL**-models. When that obtains, the considered adaptive logics fall into two categories with respect to a given Γ. Adaptive logics of the first (possibly empty) category assign the trivial consequence set to Γ, just as **CL** does. These logics are not suitable for application to Γ (in view of the aim described in the first paragraph of this section). The logics in the second category assign to Γ a non-trivial consequence set. These sets are minimally abnormal 'interpretations' of Γ and are, for nearly every Γ, different from each other.

When this situation first became known, it was felt to cause a puzzle: How could one possibly justify the choice for one or a few adaptive logics from this plethora? There may be extra-logical reasons to opt for one or more abnormalities. Also, adaptive logics that assign the trivial consequence set to Γ are not suitable for Γ. Yet, for most Γ a large number of adaptive logics will not be ruled out by these considerations. A related puzzle is that $\mathbf{CL}\emptyset^{\mathsf{m}}$ may be theoretically interesting but seems rather pointless from a practical point of view. Indeed, to most premise sets, $\mathbf{CL}\emptyset^{\mathsf{m}}$ assigns only a few adaptive consequences, most of them disjuncts, sometimes very long disjuncts. So $\mathbf{CL}\emptyset^{\mathsf{m}}$ assigns in a sense a minimally abnormal 'interpretation' to premise sets, but this interpretation is mostly a very weak one—the example proof from Γ_1 in Section 4 illustrates this.

These two puzzles were solved and that is a fourth reason to continue reading. We shall see that $\mathbf{CL}\emptyset^{\mathsf{m}}$ offers an analysis that provides an overview of the consequences that are delivered by the different adaptive logics considered in this paper. Such an overview is obviously extremely useful, were it only because it informs us what we are choosing from. Moreover, $\mathbf{CL}\emptyset^{\mathsf{m}}$ serves other purposes as well. (i) In a sense, which will be specified below, it informs one about the consequences of the different choices. Some choices require only one kind of abnormalities (ambiguities or a kind of gluts or a kind of gaps); on some choices the set of ambiguities is numerically or set theoretically smaller than on others; some choices involve a stronger consequence set than others;

[2]This is an extreme logic, but still a logic: it assigns a unique consequence set of every premise set. It is extreme in assigning the same consequence set, viz. $\emptyset$, to every premise set.

some choices cause certain key formulas (selected by extra-logical preferences) to belong to the consequence set; and so on. (ii) $\mathbf{CL}\emptyset^{\mathsf{m}}$ is also the ideal environment for *conjectures*. The advantage of conjectures is that they are introduced in a defeasible way—conjectures cannot cause triviality—and with a certain priority—some conjectures have precedence over others. In principle a conjecture concerns a single abnormality. However, one may introduce an infinity of conjectures of the same logical form, viz. the form of a specific abnormality. Proceeding thus one obtains the same effect as by opting for a richer adaptive logic, except that the latter option may result in triviality while conjectures don't, irrespective of their number.

A fifth reason for reading this paper concerns a side effect. The paper implicitly presents a method to turn indeterministic semantic systems into deterministic ones in such a way that the two semantic systems are strongly equivalent (validate the same inferences). The idea was first applied in [11], but the many applications in the present paper will readily reveal the underlying method.

Making this paper self-contained from a technical point of view would require too many pages. For this reason, I shall make the paper self-contained in informal terms. This will suit most readers because they will understand the paper without wading through all the technicalities. Those who want to understand the latter are referred to [17] or to the survey section of one of the recent papers on adaptive logics, for example [18, 19, 20, 21, 24, 25, 27]. An aim of adaptive logicians is to characterize all defeasible reasoning by an adaptive logic, possibly under a translation. All logics in the present paper are corrective—roughly: weaker then **CL**—while other adaptive logics are ampliative.

2 Some Preliminaries

Let $\mathcal{L}_s$ be the language schema of **CL**, with $\mathcal{S}$ the set of sentential letters, $\mathcal{P}^r$ the set of predicative letters or rank r for any $r \in \{1, 2, \ldots\}$, $\mathcal{C}$ the set of individual constants, and $\mathcal{V}$ the set of individual variables. Let $\mathcal{L}_S$ be obtained from $\mathcal{L}_s$ by adding, for every logical symbol, a 'checked' variant: $\check{\neg}, \check{\vee}, \ldots, \check{\exists}, \check{=}$. The checked symbols occur in all considered logics and the logics are defined in such a way that a checked symbol has always the meaning that the standard symbol has in **CL**. From now on the checked symbols will be called *classical symbols*. Premise sets and conclusions will always be formulas of $\mathcal{L}_s$, but the classical symbols will very often facilitate technicalities. A simplistic illustration is that, however defective the standard symbols that occur in A and B, a model that verifies $A \check{\wedge} \check{\neg} B$ verifies A and falsifies B.

When describing semantic systems (and only there) I shall use pseudo-languages $\mathcal{L}_{\mathcal{O}}$—giving them all the same name is harmless for present purposes. A model $M = \langle D, v \rangle$, in which D is a set (the domain), will be described in terms of the language $\mathcal{L}_{\mathcal{O}}$ which is just like $\mathcal{L}_S$ except that it also has a set of pseudo-constants which has the same cardinality as D. I write "pseudo-constants" and "pseudo-language" because their set of pseudo-constants may be uncountable whereas the symbols of a language are required to be denumerable.

Describing an uncountable model in terms of a pseudo-language is not worse than any other description; the description needs anyway to quantify over the members of the uncountable domain.

The sets of the open and closed formulas of the three languages will be called $\mathcal{F}_s$, $\mathcal{F}_S$, and $\mathcal{F}_{\mathcal{O}}$. The sets of the closed formulas of the three languages will be called $\mathcal{W}_s$, $\mathcal{W}_S$, and $\mathcal{W}_{\mathcal{O}}$.[3]

Where **L** is a logic and Γ a set of formulas, $Cn_{\mathbf{L}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{L}} A\}$. The relevant language will always be clear from the context.

For future reference I list an axiom system and a semantics for **CL** in a style that will be most useful in Section 3. The expressions $\Gamma \vdash_{\mathbf{CL}} A$ and $\vdash_{\mathbf{CL}} A$ are defined as usual for $\mathcal{L}_s$.

A⊃1	$A \supset (B \supset A)$
A⊃2	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A⊃3	$((A \supset B) \supset A) \supset A$
A∧1	$(A \wedge B) \supset A$
A∧2	$(A \wedge B) \supset B$
A∧3	$A \supset (B \supset (A \wedge B))$
A∨1	$A \supset (A \vee B)$
A∨2	$B \supset (A \vee B)$
A∨3	$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
A≡1	$(A \equiv B) \supset (A \supset B)$
A≡2	$(A \equiv B) \supset (B \supset A)$
A≡3	$(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$
A¬1	$(A \supset \neg A) \supset \neg A$
A¬2	$A \supset (\neg A \supset B)$
A∀	$\forall \alpha A(\alpha) \supset A(\beta)$
A∃	$A(\beta) \supset \exists \alpha A(\alpha)$
A=1	$\alpha = \alpha$
A=2	$\alpha = \beta \supset (A(\alpha) \supset A(\beta))$
MP	From A and $A \supset B$ to derive B
R∀	To derive $\vdash A \supset \forall \alpha B(\alpha)$ from $\vdash A \supset B(\beta)$, provided β does not occur in either A or $B(\alpha)$.
R∃	To derive $\vdash \exists \alpha A(\alpha) \supset B$ from $\vdash A(\beta) \supset B$, provided β does not occur in either $A(\alpha)$ or B.

The semantics proceeds in terms of $\mathcal{L}_{\mathcal{O}}$. The assignment maps the members of $\mathcal{W}_{\mathcal{O}}$ (rather than sentential letters only) on $\{0, 1\}$—this has only an effect for the semantic systems in Section 3. Where $M = \langle D, v \rangle$ is a model, the assignment function v is defined by:[4]

C1	$v\colon \mathcal{W}_{\mathcal{O}} \rightarrow \{0, 1\}$
C2	$v\colon \mathcal{C} \cup \mathcal{O} \rightarrow D$ (where $D = \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$)
C3	$v\colon \mathcal{P}^r \rightarrow \wp(D^r)$

[3] To avoid complications, well-formedness is defined in such a way that no classical symbol occurs within the scope of a standard symbol.

[4] By the restriction in C2, the couple $\langle D, v \rangle$ is not a **CL**-model if $D \neq \{v(\alpha) \mid \alpha \in \mathcal{C} \cup \mathcal{O}\}$. In C3, $\wp(D^r)$ is the power set of the r-th Cartesian product of D.

The valuation function $v_M\colon \mathcal{W}_\mathcal{O} \to \{0,1\}$ determined by M is defined by:

C$\mathcal{S}$ where $A \in \mathcal{S}$, $v_M(A) = 1$ iff $v(A) = 1$
C$\mathcal{P}^r$ $v_M(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$
C$=$ $v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$
C$\neg$ $v_M(\neg A) = 1$ iff $v_M(A) = 0$
C$\supset$ $v_M(A \supset B) = 1$ iff $v_M(A) = 0$ or $v_M(B) = 1$
C$\wedge$ $v_M(A \wedge B) = 1$ iff $v_M(A) = 1$ and $v_M(B) = 1$
C$\vee$ $v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$
C$\equiv$ $v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$
C$\forall$ $v_M(\forall\alpha A(\alpha)) = 1$ iff $\{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\} = \{1\}$
C$\exists$ $v_M(\exists\alpha A(\alpha)) = 1$ iff $1 \in \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$

$M \Vdash A$ (M verifies A) iff $v_M(A) = 1$. The expressions $\Gamma \vDash_{\mathbf{CL}} A$ and $\vDash_{\mathbf{CL}} A$ are defined as usual for $\mathcal{W}_s$.

An adaptive logic, **AL**, in *standard format* is a triple:

1. A *lower limit logic* **LLL**: roughly a compact Tarski logic that has a characteristic semantics and contains **CL** (in terms of the classical logical symbols).

2. A *set of abnormalities* Ω: a set of **LLL**-contingent formulas, characterized by a (possibly restricted) logical form F; or a union of such sets.[5]

3. An *adaptive strategy*: Reliability or Minimal Abnormality.

The lower limit logic delineates the non-defeasible part of the adaptive logic; what follows from the premises by **LLL** will never be revoked. Abnormalities are supposed to be false, 'unless and until proven otherwise'. Strategies are ways to cope with derivable *Dab*-formulas (classical disjunctions of abnormalities). I shall mainly consider Minimal Abnormality in this paper, offering the definitions for Reliability without much discussion. The *upper limit logic* **ULL** (in the present paper always **CL**) is obtained by extending **LLL** with an axiom that declares all abnormalities false (i.e. engender triviality).

In expressions like $Dab(\Delta)$, Δ is a finite subset of Ω and $Dab(\Delta)$ is the *classical* disjunction of the members of Δ. If $Dab(\Delta)$ is a *Dab*-consequence of Γ, it is *minimal* iff $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Delta')$ for any $\Delta' \subset \Delta$. Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal *Dab*-consequences of Γ, $\Phi(\Gamma)$ is the set of minimal choice sets of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$—every choice set contains an element of each Δ_i.

The lines of an annotated **AL**-proof have four elements: a line number, a formula, a justification (at most referring to preceding lines) and a *condition*. Where

$$A \qquad \Delta$$

[5] A logical form may be identified with a metalinguistic expression. The set $\{A \wedge B \mid A, B \in \mathcal{W}_s\}$ is defined in terms of an unrestricted logical form, whereas $\{A \wedge B \mid A, B \in \mathcal{S}\}$ is defined in terms of a restricted logical form—A and B should be sentential letters. That Ω is characterized by a logical form warrants that the adaptive logic is a formal logic. Compare to the formula preferential systems from [6]. Incidentally, all formula preferential systems were characterized by adaptive logics under a translation, whereas the converse characterization seems impossible.

abbreviates that A occurs in the proof as the formula of a line that has Δ as its condition, the (generic) inference rules are:

PREM If $A \in \Gamma$:

$$\begin{array}{ll} \dots & \dots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \dots, A_n \vdash_{\mathbf{LLL}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \dots & \dots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \dots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} Dab(\Theta)$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \dots & \dots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

Every application of a rule brings a proof to its next *stage*. While the rules are determined by the lower limit logic and the set of abnormalities, the strategy determines which lines are *marked* at a stage. The formula of a line that is marked at stage s is considered as not derived at s.

Where $Dab(\Delta_1)$, …, $Dab(\Delta_n)$ are the minimal Dab-formulas that occur in stage s of a proof from Γ, $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \dots, \Delta_n\}$ and $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$. Where A is derived on the condition Δ at line l, line l is *unmarked for Minimal Abnormality* at stage s iff (i) there is a $\varphi \in \Phi_s(\Gamma)$ for which $\varphi \cap \Delta = \emptyset$ and (ii) for every $\varphi \in \Phi_s(\Gamma)$, there is a line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$. The marking definition for Reliability is much simpler. Where A is derived on the condition Δ at line l, line l is *marked for Reliability* iff $U_s(\Gamma) \cup \Delta \neq \emptyset$.

A formula A is *derived at stage s* of a proof from Γ iff it is the formula of a line that is unmarked at s. As marks may come and go, one also wants a stable notion of derivability, which is called final derivability. A is *finally derived* from Γ at line l of a stage s iff (i) A is the second element of l, (ii) l is unmarked at s, and (iii) every extension of s in which l is marked may be further extended in such a way that line l is unmarked.[6] $\Gamma \vdash_{\mathbf{AL}} A$ (A is *finally* **AL**-*derivable* from Γ) iff A is finally derived on a line of a proof from Γ.

The (adequate) semantics for an adaptive logic is obtained as follows. A **LLL**-model M of Γ is *minimally abnormal* iff no **LLL**-model of Γ verifies set-theoretically less abnormalities than M. Where the adaptive logic $\mathbf{AL}^m$ has Minimal Abnormality as its strategy, $\Gamma \vDash_{\mathbf{AL}^m} A$ iff A is verified by all minimally abnormal models of Γ. Let $U(\Gamma)$ be the set of all abnormalities that are verified by a minimal abnormal model of Γ.[7] A **LLL**-model M of Γ is *reliable* iff it verifies no other abnormalities than those in $U(\Gamma)$. Where the adaptive logic $\mathbf{AL}^r$ has Reliability as its strategy, $\Gamma \vDash_{\mathbf{AL}^r} A$ iff A is verified by all reliable models of Γ.

[6]See [19] for an attractive game-theoretic interpretation of this definition.

[7]$U(\Gamma)$ is usually defined without referring to minimal abnormal models, as may be seen for example from [17]. Here, however, I try to be as concise as possible.

3 Not Only Inconsistency-Adaptive Logics

The **CL**-clause for negation may be seen as consisting of the consistency requirement

$$\text{if } v_M(A) = 1 \text{ then } v_M(\neg A) = 0$$

which rules out negation gluts—for some A, both A and $\neg A$ are true—and the (negation-)completeness requirement

$$\text{if } v_M(A) = 0 \text{ then } v_M(\neg A) = 1$$

which rules out negation gaps—for some A, both A and $\neg A$ are false. Both classical logicians and paraconsistent logicians concentrate only on negation gluts. Classical logicians identify the triviality of a theory with the presence of negation gluts, whereas paraconsistent logicians stress that some theories display negation gluts without being trivial.

Consider the set $\{p, \neg\neg\neg p\}$. According to the paraconsistent logic[8] **CLuN**, this set has three kinds of models: (i) those in which p, $\neg p$, and $\neg\neg\neg p$ are true and $\neg\neg p$ is false, (ii) those in which p, $\neg\neg p$, and $\neg\neg\neg p$ are true and $\neg p$ is false, and (iii) those in which p, $\neg p$, $\neg\neg p$, and $\neg\neg\neg p$ are all true. If, however, negation gaps are logically possible, then models of $\{p, \neg\neg\neg p\}$ will verify p as well as $\neg\neg\neg p$ and falsify $\neg p$ as well as $\neg\neg p$. So the premise set has models with negation gluts (and without negation gaps) and it also has models with negation gaps and without negation gluts.

The logic which is a 'counterpart' of **CLuN** but leaves room for negation gaps rather than negation gluts will be called **CLaN**—it is just like **CL** except that it tolerates g*a*ps with respect to *n*egation. Its indeterministic semantics is obviously obtained by dropping the negation-completeness requirement from the **CL**-semantics. Its deterministic semantics and axiomatization will be spelled out below. Please check: all **CLaN**-models of $\{\neg p, q \supset p, \neg q \supset p, r \supset p\}$ verify $\check{\neg} p$, $\neg p$, $\check{\neg} q$, $\check{\neg}\neg q$, and $\check{\neg} r$; some verify $\neg r$ whereas others falsify it.

Consider a theory T that had **CL** as its underlying logic but turns out to be trivial. Suppose moreover that T has **CLaN**-models and hence that one may remove its triviality by replacing the underlying logic **CL** by **CLaN**. The result, call it T', is a negation-incomplete theory. By the same reasoning as for inconsistent theories, T' is too weak in comparison to what T was intended to be. Indeed, **CLaN** invalidates all *rules* that depend on negation-completeness, whereas a number of *applications* of those rules may very well be unproblematic in view of the premises. So what we need this time is 'T in its full richness, except for the pernicious consequences of its negation-incompleteness'—compare Section 1. In other words we want to interpret the negation-incomplete T' as negation-complete as possible; we want to minimize the negation gaps. To do so, we go adaptive.

Going adaptive requires, according to the standard format, a lower limit logic, a set of abnormalities, and a strategy. The lower limit logic is obviously

[8] **CLuN** is defined below in the text, axiomatically as well as semantically. For now, take the example in the text at face value: **CLuN** is a paraconsistent logic that validates neither direction of Double Negation.

CLaN and the strategy is Minimal Abnormality or Reliability. What is the set of abnormalities? Clearly we want $A \vee \neg A$ to be *true* unless the premises require it to be false. However, the set of abnormalities should comprise the formulas that will be considered as *false* unless the premises require them to be true. The presence of the classical logical symbols enables one to express this: the abnormalities will be the formulas of the form $\check{\neg}(A \vee \neg A)$.

If we need to use classical logical symbols anyway, there is a more transparent way to characterize the abnormalities. Consider a **CLaN**-model in which both A and $\neg A$ are false. Instead of saying that the model verifies $\check{\neg}(A \vee \neg A)$, we may just as well say that it verifies $\check{\neg}A \check{\wedge} \check{\neg}\neg A$. In **CLaN**, the standard conjunction has the same meaning as the classical conjunction. To use the classical conjunction in the present context will prove very handy in the sequel of this section.

The formula $\check{\neg}A \check{\wedge} \check{\neg}\neg A$ nicely expresses what we mean by an abnormality in the present context: A is *false* in the model and $\neg A$ is also *false* in it. And there is another suggestive reading: the model *verifies* $\check{\neg}A$ but *falsifies* $\neg A$. This clearly expresses a negation gap: the *classical* negation of A is verified but the *standard* negation of A is not. So the standard negation displays a gap. Of course, abnormalities have to be existentially closed for the predicative level. So we define $\Omega = \{\check{\exists}(\check{\neg}A \check{\wedge} \check{\neg}\neg A) \mid A \in \mathcal{F}_s\}$.

It is instructive to check what becomes of the **CLuN**-abnormalities if the same transformation is applied to them. In earlier papers, the set of **CLuN**-abnormalities was defined as $\Omega = \{\exists(A \wedge \neg A) \mid A \in \mathcal{F}_s\}$. It is just as good to define it as $\Omega = \{\check{\exists}(\check{\neg}\check{\neg}A \check{\wedge} \neg A) \mid A \in \mathcal{F}_s\}$. The form of these abnormalities clearly indicates a negation *glut*. Applied to models: the model falsifies the classical negation of A but nevertheless verifies the standard negation.

Let me reassure the suspicious reader that one obtains the same logics $\mathbf{CLuN}^r$ and $\mathbf{CLuN}^m$ if one defines $\Omega = \{\check{\exists}(\check{\neg}\check{\neg}A \check{\wedge} \neg A) \mid A \in \mathcal{F}_s\}$. For example whenever a model verifies $\check{\exists}(\check{\neg}\check{\neg}A \check{\wedge} \neg A)$ for some A, it verifies $\exists(A \wedge \neg A)$ for the same A; and *vice versa*. Whenever the first formula is derivable from a premise set for an A, so is the second formula for that A; and *vice versa*. The original formulation has the advantage that abnormalities are expressed in the standard language. What is attractive about the reformulation, however, is that it gives us a unified way to characterize negation gluts and negation gaps and that this characterization is transparent. Moreover, this approach may be generalized to all logical symbols.

Consider another example, the premise set $\{p, q, \neg(p \wedge q)\}$. At first sight, handling this set seems to require a logic that leaves room for inconsistencies (negation gluts). But consider a logic that rules out negation gaps but not conjunction gaps: if the classical conjunction of A and B is true, their standard conjunction may nevertheless be false. So the abnormalities will have the form $\check{\exists}((A \check{\wedge} B) \check{\wedge} \check{\neg}(A \wedge B))$. Some such models verify p and q, and hence also $p \check{\wedge} q$, but falsify $p \wedge q$, in which case they verify $\check{\neg}(p \wedge q)$ as well as (as there are no negation gaps) $\neg(p \wedge q)$. In other words, the premise set $\{p, q, \neg(p \wedge q)\}$ does not require paraconsistent models. It has just as well models in logics that leave room for conjunction gaps, even in those that forbid all other gluts and gaps.

Some premise sets are even more amusing, for example $\{p, r, \neg q \vee \neg r, (p \wedge r) \supset q\}$, which has no **CL**-models. It has models if one leaves room for negation gluts, but also if one leaves room for conjunction gaps, or for disjunction gluts, or for implication gluts. In general, for every gap or glut with respect to any logical symbol, there are premise sets that have no **CL**-models but have models in the logic that tolerates just such gluts or gaps.

I claimed that classical logicians and paraconsistent logicians are obsessed by negation gluts. There is an easy historical explanation for this: all gluts and gaps *surface* as inconsistencies if **CL** is applied to the premise set. Thus, if **CL** is applied to $\{p, \neg\neg\neg p\}$, one obtains the inconsistencies $p \wedge \neg p$ and $\neg p \wedge \neg\neg p$ (as well as all others of course). Similarly if **CL** is applied to $\{p, q, \neg(p \wedge q)\}$. The situation is the same for any other glut or gap: an inconsistency surfaces when one applies **CL**.

That all gluts and gaps surface as inconsistencies makes it understandable why there was and is ample interest in paraconsistent logics, but much less in logics that display other kinds of gluts or gaps. Nevertheless, it seems to me that it is a mistake to concentrate on consistency only. Remember that the plot behind inconsistency-adaptive logics was to localize and isolate the *problems* displayed by a theory or premise set and to do so in order to remove those problems. Inconsistency-adaptive logics always identify disjunctions of inconsistencies as the problems. Suppose one chooses a logic **L** that leaves room for other kinds of gluts or gaps and that one applies an adaptive logic that has **L** as its lower limit. Other formulas may then be identified as the problems and often there is some choice, as in the case of $\{p, r, \neg q \vee \neg r, (p \wedge q) \supset q\}$. Although *Dab*-formulas will be derivable for every choice, the *Dab*-formulas will be different. So different problems have to be resolved if one wants to regain consistency, whence different consistent alternatives are suggested. From a purely logical point of view, it is sensible to consider all possibilities. Some choices of gluts or gaps may cause less 'problems' than others or may cause problems that are easier to solve. Moreover, there may be extra-logical reasons to prefer certain consistent alternatives over others.

I shall now describe the basic logics that leave room for gluts or gaps in comparison to **CL**. Combinations of different kinds of gluts or gaps will be considered thereafter, but it is easier to mention the combination of gluts and gaps of the same kind from the very beginning.

Let us devise the basic logics in a systematic way. All clauses of the **CL**-semantics concern a 'basic form': schematic letters for sentences, primitive predicative expressions, and the forms characterized by a metalinguistic formula that contains precisely one logical symbol, identity included. Each of these clauses may be split into two implicative clauses. For formulas A of the considered basic form, one implicative clause states that $v_M(A) = 1$ if a certain condition obtains, the other that $v_M(A) = 0$ if another condition obtains.

A logic **L** tolerates *gluts* with respect to a basic form A iff there are **L**-models M such that $v_M(A) = 1$ for a formula A of the form A while other properties of M are sufficient for $v_M(A) = 0$ according to the **CL**-semantics. A logic **L** tolerates *gaps* with respect to a basic form iff the same obtains with $v_M(A) = 1$ and $v_M(A) = 0$ exchanged.

Consider first gluts for a particular logical form A. Each of the logics described below leaves room for a single kind of gluts and for no gaps. The *indeterministic* semantics is obtained by removing from the **CL**-semantics the implicative clause that has $v_M(\mathsf{A}) = 0$ as its implicatum. In order to illustrate the naming scheme, I shall list all glut variants, including gluts for sentential letters and for primitive predicative expressions.[9] In view of what precedes, the names of the logics are self-explanatory, except perhaps the use of "M" for material implication—I need the "I" for identity—and the use of "X", the second letter of "existential"—I need the "E" for equivalence.

logic	*removed* implicative clause
CLuS	where $A \in \mathcal{S}$, if $v(A) = 0$ then $v_M(A) = 0$
CLuP	if $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \notin v(\pi^r)$ then $v_M(\pi^r \alpha_1 \ldots \alpha_r) = 0$
CLuI	if $v(\alpha) \neq v(\beta)$ then $v_M(\alpha = \beta) = 0$
CLuN	if $v_M(A) = 1$ then $v_M(\neg A) = 0$
CLuM	if $v_M(A) = 1$ and $v_M(B) = 0$, then $v_M(A \supset B) = 0$
CLuC	if $v_M(A) = 0$ or $v_M(B) = 0$, then $v_M(A \wedge B) = 0$
CLuD	if $v_M(A) = 0$ and $v_M(B) = 0$, then $v_M(A \vee B) = 0$
CLuE	if $v_M(A) \neq v_M(B)$, then $v_M(A \equiv B) = 0$
CLuU	if $\{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\} \neq \{1\}$, then $v_M(\forall \alpha A(\alpha)) = 0$
CLuX	if $1 \notin \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$, then $v_M(\exists \alpha A(\alpha)) = 0$

Each of these logics has a *deterministic* semantics. In it, the logical term tolerating gluts is characterized by a clause of the form "$v_M(\mathsf{A}) = 1$ iff [condition]". The condition is obtained from the **CL**-semantics by disjoining the condition of the standard clause with the correct reference to the assignment value: "$v(\mathsf{A}) = 1$" for the right A. I again list all the logics.

logic	replacing clause
CLuS	where $A \in \mathcal{S}$, $v_M(A) = 1$ iff $v(A) = 1$ or $v(A) = 1$
CLuP	$v_M(\pi^r \alpha_1 \ldots \alpha_r) = 1$ iff
	$\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$ or $v(\pi^r \alpha_1 \ldots \alpha_r) = 1$
CLuI	$v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$ or $v(\alpha = \beta) = 1$
CLuN	$v_M(\neg A) = 1$ iff $v_M(A) = 0$ or $v(\neg A) = 1$
CLuM	$v_M(A \supset B) = 1$ iff $(v_M(A) = 0$ or $v_M(B) = 1)$ or $v(A \supset B) = 1$
CLuC	$v_M(A \wedge B) = 1$ iff $(v_M(A) = 1$ and $v_M(B) = 1)$ or $v(A \wedge B) = 1$
CLuD	$v_M(A \vee B) = 1$ iff $v_M(A) = 1$ or $v_M(B) = 1$ or $v(A \vee B) = 1$
CLuE	$v_M(A \equiv B) = 1$ iff $v_M(A) = v_M(B)$ or $v(A \equiv B) = 1$
CLuU	$v_M(\forall \alpha A(\alpha)) = 1$ iff
	$\{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\} = \{1\}$ or $v(\forall \alpha A(\alpha)) = 1$
CLuX	$v_M(\exists \alpha A(\alpha)) = 1$ iff
	$1 \in \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$ or $v(\exists \alpha A(\alpha)) = 1$

All other clauses of the **CL**-semantics are obviously retained.

The indeterministic semantics defines the same logic as the deterministic semantics. I skip the long but transparent proof that, for every model M of one semantics, there is a model of the other semantics that verifies exactly the same formulas as M.

[9] These cause trouble on which I shall comment later in the text.

Nearly all glut-logics have nice adequate axiomatizations in $\mathcal{W}_s$. For **CLuC**, for example, it is sufficient to remove from the axiom system of **CL** the axioms A∧1 and A∧2, and to attach to A=2 the restriction that $A(\alpha)$ be a primitive formula of $\mathcal{W}_s$. However, as the reader will have seen, this way of proceeding gets us into trouble when we come to implication gluts,[10] whether separate or in combination with other gluts and gaps.

A different road is possible, and it is instructive. Consider the axiom system of **CL**, replace in every axiom and rule every standard symbol by the corresponding classical symbol, and attach to axiom schema A=2 the restriction that A does not contain standard logical symbols (but only classical symbols). Call this axiom system **CLC**.[11] Next add, for every logical symbol, the axiom that gives the standard symbol the same meaning as the classical symbol—example: $\neg A \check{\equiv} \check{\neg} A$. So all standard symbols have their **CL**-meaning in **CLC**. To obtain an axiomatic system that leaves room for gluts with respect to a specific logical form A, remove the relevant equivalence and replace it by a glut-tolerating implication. I do not list all of them as they are all similar. Gluts with respect to sentential letters and primitive predicative formulas will be commented upon below.

logic	axiom
CLuI	$\alpha \check{=} \beta \check{\supset} \alpha = \beta$
CLuN	$\check{\neg} A \check{\supset} \neg A$
⋮	⋮
CLuX	$\check{\exists} \alpha A(\alpha) \check{\supset} \exists \alpha A(\alpha)$

So the matter is utterly simple. As the standard symbol may display gluts, the formula containing the standard symbol is logically implied by the formula containing the corresponding classical symbol, but not *vice versa*.

Note the direct relation between the implicative glut-tolerating axiom and the relevant retained clause in the indeterministic semantics. Just as **CLuI** contains the axiom $\alpha \check{=} \beta \check{\supset} \alpha = \beta$ and not its converse, the indeterministic **CLuI**-semantics contains the clause "if $v(\alpha) = v(\beta)$ then $v_M(\alpha = \beta) = 1$". Note that the antecedent of the clause, $v(\alpha) = v(\beta)$, is the semantic definition of the antecedent of the axiom, $\alpha \check{=} \beta$.

As I promised, I now comment on the logics **CLuS** and **CLuP**. No axiomatic system for **CLuS** is mentioned in the previous paragraphs. There is no need to do so, as it is obvious from the deterministic semantics that **CLuS** is identical to **CL**. So I shall never refer to it again by the name **CLuS**.

For **CLuP** the matter is more complicated. Again, no axiomatic system for it is presented in the previous paragraphs. **CLuP** has decent axiomatizations, but its peculiarities are incompatible with **CLC**. To see this, it is sufficient to realize that $v_M(\pi^r \alpha_1 \ldots \alpha_r)$ may be 1 because $v(\pi^r \alpha_1 \ldots \alpha_r) = 1$, even if $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \notin v(\pi^r)$. So if $v(\alpha_1) = v(\beta)$, it is possible that

[10] If the implication is not classical, the resulting axiom system does not correctly define the other logical symbols.

[11] This is an axiom system for **CL**. The restriction on A=2 causes no weakening because one may derive the original version of A=2 for all formulas that do not contain standard logical symbols..

$v_M(\pi^r\beta\alpha_2\ldots\alpha_r) = 0$ whereas $v_M(\pi^r\alpha_1\alpha_2\ldots\alpha_r) = 1$. It follows that classical identity does not have the right meaning because Replacement of Identicals is invalid. In other words, **CLuP** is an odd logic that does not extend **CLC**. No adaptive logic in standard format can be built on **CLuP**.[12]

We are done with the basic logics for gluts and move on to logics in which one kind of gaps is logically possible. All these logics will have a lower case "a", referring to the possibility of gaps, where their glut-counterparts have a lower case "u". By now, I suppose that the reader understood the plot and skip most of the logics. Comments on gaps for sentential letters and for primitive predicative expressions follow below.

logic	*removed* implicative clause
CLaS	where $A \in \mathcal{S}$, if $v(A) = 1$ then $v_M(A) = 1$
CLaP	if $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$ then $v_M(\pi^r\alpha_1\ldots\alpha_r) = 1$
CLaI	if $v(\alpha) = v(\beta)$ then $v_M(\alpha = \beta) = 1$
CLaN	if $v_M(A) = 0$ then $v_M(\neg A) = 1$
$\vdots$	$\vdots$
CLaX	if $1 \in \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$, then $v_M(\exists\alpha A(\alpha)) = 1$

Each of these logics has a deterministic semantics, which requires a clause of the form "$v_M(\mathsf{A}) = 1$ iff [condition]". This clause is obtained from the **CL**-semantics by conjoining the condition of the standard clause with the correct reference to the assignment value: "$v(\mathsf{A}) = 1$".

logic	replacing clause
CLaS	where $A \in \mathcal{S}$, $v_M(A) = 1$ iff $v(A) = 1$ and $v(A) = 1$
CLaP	$v_M(\pi^r\alpha_1\ldots\alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$ and $v(\pi^r\alpha_1\ldots\alpha_r) = 1$
CLaI	$v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$ and $v(\alpha = \beta) = 1$
CLaN	$v_M(\neg A) = 1$ iff $v_M(A) = 0$ and $v(\neg A) = 1$
$\vdots$	$\vdots$
CLaX	$v_M(\exists\alpha A(\alpha)) = 1$ iff $1 \in \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$ and $v(\exists\alpha A(\alpha)) = 1$

As for the glut-variants, all other clauses of the **CL**-semantics are retained.

The way in which gluts and gaps are realized is fully transparent. In the case of gluts, the classical condition is sufficient but not necessary for $v_M(A) = 1$; in the case of gaps, the classical condition is necessary but not sufficient for $v_M(A) = 1$. So in both cases we may restore an equivalence by taking the 'arbitrary' missing element from the assignment. By doing so, the model (in the strict sense) determines the valuation function.

For the axiomatization, I shall again follow the road taken for the glut-allowing logics. Here are the axioms.

[12] The attentive reader may have remarked that variants for **CLuS** and **CLuP** may be devised in which one explicitly distinguishes between the classical meaning of sentential letters and predicates, denoted for example as $\check{p}$ and $\check{P}a$, and the standard meaning of such entities, denoted by p and Pa. On the semantics, $\check{p} \check{\equiv} p$ and $\check{P}a \check{\supset} Pa$ are valid, but not the converse of the latter. I shall not pursue this road here in view of the result from Section 5.

logic	axiom
CLaI	$\alpha = \beta \check{\supset} \alpha \check{=} \beta$
CLaN	$\neg A \check{\supset} \check{\neg} A$
$\vdots$	$\vdots$
CLaX	$\exists \alpha A(\alpha) \check{\supset} \check{\exists} \alpha A(\alpha)$

Again, the matter is utterly simple. As the standard symbol may display gaps (and no gluts), the formula containing the classical symbol is logically implied by the formula containing the corresponding standard symbol, but not *vice versa*. Again, all logical symbols for which no gaps are permitted are characterized by an axiom stating that a formula containing the standard symbol is classically equivalent to the corresponding expression containing the classical symbol.

Some will find the classical contraposition of the axioms more transparent, for example $\check{\neg} \alpha \check{=} \beta \check{\supset} \check{\neg} \alpha = \beta$ for **CLaI**. This also illustrates the direct connection between the axiom and the corresponding retained clause of the indeterministic semantics.

I still have to comment upon **CLaS** and **CLaP**. No axiomatic system for **CLaS** is provided above, and rightly so as it is obvious from the deterministic semantics that **CLaS** is identical to **CL**. So I shall no more use the name **CLaS**.

The logic **CLaP** is identical to **CLuP** and displays the same oddities. I shall not refer to it in the sequel because this logic cannot function as the lower limit of an adaptive logic in standard format.

Let us now move to the case where gluts and gaps for the same logical form are combined. The names of the logics contain a lower case "o" to indicate that *both* gluts and gaps are possible. For the indeterministic semantics, one removes both the clause preventing gluts and the clause preventing gaps. This means that one removes the **CL**-clause altogether.

logic	removed implicative clauses
CLoS	where $A \in \mathcal{S}$, if $v(A) = 0$ then $v_M(A) = 0$
	where $A \in \mathcal{S}$, if $v(A) = 1$ then $v_M(A) = 1$
CLoP	if $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \notin v(\pi^r)$ then $v_M(\pi^r \alpha_1 \ldots \alpha_r) = 0$
	if $\langle v(\alpha_1), \ldots, v(\alpha_r) \rangle \in v(\pi^r)$ then $v_M(\pi^r \alpha_1 \ldots \alpha_r) = 1$
CLoI	if $v(\alpha) \neq v(\beta)$ then $v_M(\alpha = \beta) = 0$
	if $v(\alpha) = v(\beta)$ then $v_M(\alpha = \beta) = 1$
CLoN	if $v_M(A) = 1$ then $v_M(\neg A) = 0$
	if $v_M(A) = 0$ then $v_M(\neg A) = 1$
$\vdots$	$\vdots$
CLoX	if $1 \notin \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$, then $v_M(\exists \alpha A(\alpha)) = 0$
	if $1 \in \{v_M(A(\beta)) \mid \beta \in \mathcal{C} \cup \mathcal{O}\}$, then $v_M(\exists \alpha A(\alpha)) = 1$

The deterministic semantics is also simple: the value of composing entities play no role whatsoever.

logic	replacing clause
CLoS	where $A \in \mathcal{S}$, $v_M(A) = v(A)$
CLoP	$v_M(\pi^r\alpha_1 \ldots \alpha_r) = v(\pi^r\alpha_1 \ldots \alpha_r)$
CLoI	$v_M(\alpha = \beta) = v(\alpha = \beta)$
CLoN	$v_M(\neg A) = v(\neg A)$
$\vdots$	$\vdots$
CLoX	$v_M(\exists\alpha A(\alpha)) = v(\exists\alpha A(\alpha))$

The way to obtain the axiomatic system corresponds closely to the indeterministic semantics: one removes the axiom concerning the symbol, for example $\alpha \check{=} \beta \check{\equiv} \alpha = \beta$ for **CLoI**. As a result, the standard identity does not occur in any axiom of **CLoI**, while all other standard symbols are identified with their classical counterparts. The logic **CLoS** is again identical to **CL**, whereas **CLoP** is the same logic as **CLuP** and **CLaP**.

Incidentally, many of the logics from this section and from the next are extensions of **CL**. In many of those logics, some standard symbols have the same meaning as the corresponding **CL**-symbols and other **CL**-symbols may be defined. This is fairly obvious for most of the logics. Slightly unexpected might be that $\sim A =_{df} A \supset \neg A$ defines classical negation within **CLaN**, **CLaNs**, and other logics in which implication gluts as well as negation gluts are logical falsehoods—in those logics $A \supset \neg A \vdash A \check{\supset} \check{\neg} A$.

It is obviously possible to formulate logics that tolerate a combination of gluts and gaps for different symbols. We may form names for such logics by combining the qualifications that appear in the already used names. Thus negation gluts, negation gaps, and implication gaps are logically possible in **CLoNaM**. To obtain, for example, the indeterministic semantics of **CLoNaM**, remove both implicative clauses on negation, as it was done for **CLoN**, and moreover remove the clause that prevents implication gaps. To obtain the deterministic semantics, one starts, for example, from the semantics for **CLoN** and replaces the implication clause by the implication clause from the **CLaM**-semantics. Similarly for the axiomatic systems.

There are logics that tolerate any combination of gluts and gaps. Among them, there is a logic that tolerates all kinds of gluts and gaps. Let us call it **CLo**. In this logic, no standard symbol is given a meaning. So if $\Gamma \subseteq \mathcal{W}_s$, then $Cn_{\mathbf{CLo}}(\Gamma) \cap \mathcal{W}_s = \Gamma$.[13] All this will seem of little interest, unless one remembers the reason to consider all these logics, which is to let them function as the lower limit of an adaptive logic. So let us have a look at the adaptive logics.

As announced, I shall disregard the logics that (attempt to) display gluts or gaps with respect to sentential letters or primitive predicative expressions. For the other logics, the matter is simple. I have already described the lower limits. To obtain adaptive logics in standard format, we need to combine those with either Reliability or Minimal Abnormality as well as with the right set of

[13]In **CL** infinitely (but denumerably) many logical symbols can be defined. These too have obviously no meaning in **CLo** as described above. Nothing unexpected and nothing interesting seems to happen if some or all of those symbols are added to $\mathcal{L}_s$. Where changes are required, they are as expected.

abnormalities. So all I have to describe are the sets of abnormalities and it was outlined before in which way these are obtained. Typically, every kind of gluts or gaps requires a specific set of abnormalities. Let us first look at gluts.

LLL	set of abnormalities Ω
CLuI	$\{\check{\exists}(\check{\neg}\alpha \mathrel{\check{=}} \beta \mathbin{\check{\wedge}} \alpha = \beta) \mid \alpha, \beta \in \mathcal{C} \cup \mathcal{V}\}$
CLuN	$\{\check{\exists}(\check{\neg}\check{\neg}A \mathbin{\check{\wedge}} \neg A) \mid A \in \mathcal{F}_s\}$
CLuM	$\{\check{\exists}(\check{\neg}(A \mathbin{\check{\supset}} B) \mathbin{\check{\wedge}} (A \supset B)) \mid A, B \in \mathcal{F}_s\}$
$\vdots$	$\vdots$
CLuX	$\{\check{\exists}(\check{\neg}\check{\exists}\alpha A(\alpha) \mathbin{\check{\wedge}} \exists\alpha A(\alpha)) \mid A \in \mathcal{F}_s\}$

And here are the adaptive logics allowing for one kind of gaps.

LLL	set of abnormalities Ω
CLaI	$\{\check{\exists}(\alpha \mathrel{\check{=}} \beta \mathbin{\check{\wedge}} \check{\neg}\alpha = \beta) \mid \alpha, \beta \in \mathcal{C} \cup \mathcal{V}\}$
CLaN	$\{\check{\exists}(\check{\neg}A \mathbin{\check{\wedge}} \check{\neg}\neg A) \mid A \in \mathcal{F}_s\}$
CLaM	$\{\check{\exists}((A \mathbin{\check{\supset}} B) \mathbin{\check{\wedge}} \check{\neg}(A \supset B)) \mid A, B \in \mathcal{F}_s\}$
$\vdots$	$\vdots$
CLaX	$\{\check{\exists}(\check{\exists}\alpha A(\alpha) \mathbin{\check{\wedge}} \check{\neg}\exists\alpha A(\alpha)) \mid A \in \mathcal{F}_s\}$

If the lower limit logic leaves room for gluts as well as gaps with respect to the same logical symbol, the appropriate set of abnormalities is the union of two sets of abnormalities: that of the corresponding logic tolerating gluts and that of the corresponding logic tolerating gaps. Thus the appropriate set of abnormalities for **CLoI** is $\Omega = \{\check{\exists}(\check{\neg}\alpha \mathrel{\check{=}} \beta \mathbin{\check{\wedge}} \alpha = \beta) \mid \alpha, \beta \in \mathcal{C} \cup \mathcal{V}\} \cup \{\check{\exists}(\alpha \mathrel{\check{=}} \beta \mathbin{\check{\wedge}} \check{\neg}\alpha = \beta) \mid \alpha, \beta \in \mathcal{C} \cup \mathcal{V}\}$ and the appropriate set of abnormalities for **CLoX** is $\{\check{\exists}(\check{\neg}\check{\exists}\alpha A(\alpha) \mathbin{\check{\wedge}} \exists\alpha A(\alpha)) \mid A \in \mathcal{F}_s\} \cup \{\check{\exists}(\check{\exists}\alpha A(\alpha) \mathbin{\check{\wedge}} \check{\neg}\exists\alpha A(\alpha)) \mid A \in \mathcal{F}_s\}$.

Handling logics that combine gluts or gaps for different logical symbols is just as easy. The appropriate set of abnormalities is the union of the sets that contain those gluts and gaps. Thus the appropriate set of abnormalities for **CLoNaM** is $\Omega = \{\check{\exists}(\check{\neg}\check{\neg}A \mathbin{\check{\wedge}} \neg A) \mid A \in \mathcal{F}_s\} \cup \{\check{\exists}(\check{\neg}A \mathbin{\check{\wedge}} \check{\neg}\neg A) \mid A \in \mathcal{F}_s\} \cup \{\check{\exists}((A \mathbin{\check{\supset}} B) \mathbin{\check{\wedge}} \check{\neg}(A \supset B)) \mid A, B \in \mathcal{F}_s\}$.

The appropriate set of abnormalities for **CLo** is obviously the union of all sets of abnormalities mentioned (explicitly or implicitly) in the two preceding tables. Incidentally, one may also use this union as the set of abnormalities of *all* corrective adaptive logics considered so far in this section. Some **CLo**-abnormalities are logically impossible for certain lower limit logics, but these have no effect on the adaptive logic anyway.[14]

Let me summarize. In this section, the basic logics for handling gluts and gaps with respect to one logical symbol were defined, together with all logics that combine those gluts and gaps. For each of these logics, there is an 'appropriate set' of abnormalities. Combining such a logic with the appropriate set of abnormalities and with the Reliability or Minimal Abnormality strategy results in an adaptive logic in standard format. Note that $\mathbf{CLuN}^m$ and $\mathbf{CLuN}^r$ are such adaptive logics. There are many more and in view of the obvious naming

[14] Consider the adaptive logic $\mathbf{CLoC}^m$ and let B be an implication glut. So $\Gamma \vdash_{\mathbf{CLoC}} \check{\neg}B$ for all Γ. It follows that $\Gamma \vdash_{\mathbf{CLoC}} A \mathbin{\check{\vee}} Dab(\Delta)$ iff $\Gamma \vdash_{\mathbf{CLoC}} A \mathbin{\check{\vee}} Dab(\Delta) \mathbin{\check{\vee}} B$.

schema, it is at once clear what is meant by $\mathbf{CLaI}^r$, $\mathbf{CLoNaM}^m$, or $\mathbf{CLo}^r$. These logics may be used as such, but may also serve other functions, as we shall see in the next section.

4 Variants and Combinations

This section contains further comments on the adaptive logics presented in the previous section. Three topics will be considered: (i) variants of the lower limit logics, (ii) choosing among the adaptive logics from the previous section for handling a given premise set, and (iii) combining the adaptive logics. Some of the comments remain sketchy because to describe them in more detail seems pointless. Either the matter is obvious, or the elaboration does not seem to engender any really new features.

The first topic concerns variants on the glut-logics and gap-logics. Three kinds of variants will be briefly considered. A first type concerns the rule of Replacement of Identicals. With the obvious exception of **CL**, no logic presented in the previous section validates this rule. However, all those logics have variants that validate Replacement of Identicals and leave the meaning of all other logical symbols unchanged. Several ways to do so are applied in [23] and in [48].[15]

A very different kind of enrichment is related to the reduction of complex expressions containing gappy or glutty symbols to simpler such expressions. The paraconsistent logic **CLuN**, for example, may be extended to **CLuNs** by adding double negation, de Morgan properties, and other axioms or rules that drive negations inside.[16] It can be shown that **CLuNs** is maximally paraconsistent (no paraconsistent logics are strictly stronger than **CLuNs** and strictly weaker than **CL**). Exactly the same axioms or rules cause a similar effect if one extends **CLaN** or **CLoN**. It is not difficult to find similar axiom schemas, and semantic clauses, for other logical symbols. Take implication. Among the obvious candidates, in which I use at once classical logical symbols for the sake of generality, are such equivalences as $((A \vee B) \supset C) \check{\equiv} ((A \supset C) \check{\wedge} (B \supset C))$, $((A \wedge B) \supset C) \check{\equiv} ((A \supset C) \check{\vee} (B \supset C))$, and so on. There is no need to spell all this out here.

The third kind of variants is analogous to the enrichment discussed in Section 5 of [21]. Where for example **CLuN** is the lower limit, the idea is to consider not only (i) $(p \vee q) \wedge \neg(p \vee q)$ as an abnormality, but also (ii) $p \wedge \neg(p \vee q)$ and (iii) $q \wedge \neg(p \vee q)$. A premise set may **CLuN**-entail (i) but neither (ii) nor (iii); and it may **CLuN**-entail (ii) but not (iii), or vice versa. The insight leads to a combined adaptive logic that drastically enriches $\mathbf{CLuN}^m$ consequence sets. Technically, we need a 'specifying part' relation (p and q are specifying parts of $p \vee q$). This can easily be adjusted for other gluts or gaps. Where,

[15] The central point of the last paper is that all those logics can be faithfully embedded in **CL**, a fact which has dramatic consequences for the application of partial decision methods.

[16] **CLuNs** is, under sundry names, the most popular paraconsistent logic. To the best of my knowledge, the propositional version appears first in [42]. Further useful references to studies of **CLuNs** and of its fragments are [1, 2, 3, 8, 23, 28, 31, 32, 33, 34, 35, 36, 39, 43].

for example, the lower limit logic is **CLuM**, $p \supset q$ will be a specifying part of $(p \wedge r) \supset q$.

Let us move to the second topic: choosing among the adaptive logics from the previous section for handling a given premise set. I have commented upon this choice in the previous section. Here, my main aim is to show that the **CLo**^{m}-proofs may help one to pick the right choice. Let us consider a simple example: $\Gamma_1 = \{p, r, \neg q \vee \neg r, (p \wedge r) \supset q, \neg p \vee s\}$. I introduce the classical symbols step by step in order to make the proof fully transparent.

1	p	Premise	$\emptyset$	
2	r	Premise	$\emptyset$	
3	$\neg q \vee \neg r$	Premise	$\emptyset$	
4	$(p \wedge r) \supset q$	Premise	$\emptyset$	
5	$\neg p \vee s$	Premise	$\emptyset$	
6	$\neg q \check{\vee} \neg r$	3; RC	$\{\check{\neg}(\neg q \check{\vee} \neg r) \check{\wedge} (\neg q \vee \neg r)\}$	$\checkmark^{16}$
7	$\check{\neg}\neg r$	2; RC	$\{\check{\neg}\check{\neg} r \check{\wedge} \neg r\}$	$\checkmark^{16}$
8	$\neg q$	6, 7; RU	$\{\check{\neg}(\neg q \check{\vee} \neg r) \check{\wedge} (\neg q \vee \neg r), \check{\neg}\check{\neg} r \check{\wedge} \neg r\}$	$\checkmark^{16}$
9	$\neg p \check{\vee} s$	5; RC	$\{\check{\neg}(\neg p \check{\vee} s) \check{\wedge} (\neg p \vee s)\}$	
10	$\check{\neg}\neg p$	1; RC	$\{\check{\neg}\check{\neg} p \check{\wedge} \neg p\}$	
11	s	9, 10; RU	$\{\check{\neg}(\neg p \check{\vee} s) \check{\wedge} (\neg p \vee s), \check{\neg}\check{\neg} p \check{\wedge} \neg p\}$	
12	$p \wedge r$	1, 2; RC	$\{(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)\}$	$\checkmark^{16}$
13	$(p \wedge r) \check{\supset} q$	4; RC	$\{\check{\neg}((p \wedge r) \check{\supset} q) \check{\wedge} ((p \wedge r) \supset q)\}$	$\checkmark^{16}$
14	q	12, 13; RU	$\{\check{\neg}((p \wedge r) \check{\supset} q) \check{\wedge} ((p \wedge r) \supset q),$ $(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)\}$	$\checkmark^{16}$
15	$\check{\neg} q$	8; RC	$\{\check{\neg}(\neg q \check{\vee} \neg r) \check{\wedge} (\neg q \vee \neg r), \check{\neg}\check{\neg} r \check{\wedge} \neg r,$ $\check{\neg}\check{\neg} q \check{\wedge} \neg q\}$	$\checkmark^{16}$
16	$(\check{\neg}((p \wedge r) \check{\supset} q) \check{\wedge} ((p \wedge r) \supset q)) \check{\vee} ((p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)) \check{\vee}$ $(\check{\neg}(\neg q \check{\vee} \neg r) \check{\wedge} (\neg q \vee \neg r)) \check{\vee} (\check{\neg}\check{\neg} r \check{\wedge} \neg r) \check{\vee} (\check{\neg}\check{\neg} q \check{\wedge} \neg q)$	14, 15; RD	$\emptyset$	

The proof is constructed in such a way that a single abnormality is added to the condition of every line at which RC is applied. These abnormalities are a disjunction glut at lines 6 and 9, a negation glut at lines 7, 10 and 15, a conjunction gap at line 12, and an implication glut at line 13. At line 16 I use the derived rule RD.

The example proof provides us with an analysis of the situation. It reveals which gluts and gaps have to be ruled out, globally or locally, in order to obtain certain consequences. By choosing a lower limit which is stronger than **CLo** together with the associated set of abnormalities one obtains a stronger final consequence set. The *Dab*-formula 16 is obtained because the occurrence of q and $\check{\neg} q$ triggers RD. The *Dab*-formula indicates which gluts and gaps are unavoidable. So it indicates which lower limit logics are not viable choices.

The above **CLo**^{m}-proof is easily transformed to a proof in terms of any of the stronger adaptive logics referred to in the previous paragraph. To illustrate this, and to illustrate at once the point from the previous paragraph, consider first the familiar adaptive logic **CLuN**^{m}. The difference between the **CLo**^{m}-proof and the **CLuN**^{m}-proof is simply that all abnormalities that are not **CLuN**^{m}-abnormalities are removed from the conditions of the lines and hence also from the only *Dab*-formula derived in the proof. Here is the so obtained proof.

1	p	Premise	$\emptyset$	
2	r	Premise	$\emptyset$	
3	$\neg q \vee \neg r$	Premise	$\emptyset$	
4	$(p \wedge r) \supset q$	Premise	$\emptyset$	
5	$\neg p \vee s$	Premise	$\emptyset$	
6	$\neg q \check{\vee} \neg r$	3; RU	$\emptyset$	
7	$\check{\neg}\neg r$	2; RC	$\{\check{\neg}\check{\neg} r \check{\wedge} \neg r\}$	$\checkmark^{16}$
8	$\neg q$	6, 7; RU	$\{\check{\neg}\check{\neg} r \check{\wedge} \neg r\}$	$\checkmark^{16}$
9	$\neg p \check{\vee} s$	5; RU	$\emptyset$	
10	$\check{\neg}\neg p$	1; RC	$\{\check{\neg}\check{\neg} p \check{\wedge} \neg p\}$	
11	s	9, 10; RU	$\{\check{\neg}\check{\neg} p \check{\wedge} \neg p\}$	
12	$p \wedge r$	1, 2; RU	$\emptyset$	
13	$(p \wedge r) \check{\supset} q$	4; RU	$\emptyset$	
14	q	12, 13; RU	$\emptyset$	
15	$\check{\neg} q$	8; RC	$\{\check{\neg}\check{\neg} r \check{\wedge} \neg r, \check{\neg}\check{\neg} q \check{\wedge} \neg q\}$	$\checkmark^{16}$
16	$(\check{\neg}\check{\neg} r \check{\wedge} \neg r) \check{\vee} (\check{\neg}\check{\neg} q \check{\wedge} \neg q)$	14, 15; RD	$\emptyset$	

It is useful to compare the present proof with the preceding one. To maximally retain the parallelism, I did not remove the lines at which classical disjunction and classical implication are introduced. These are useless but cause no harm. Apart from the announced deletion of certain formulas from the conditions and the *Dab*-formula, the only change is that RC is replaced by RU where the inference step does not depend on a $\mathbf{CLuN}^m$-abnormality. Note that the occurrence of a classical contradiction still leads to the *Dab*-formula 16.

There is a gain in the last example proof in comparison to the $\mathbf{CLo}^m$-proof: q is finally derivable. It is easy enough to choose an adaptive logic from the previous section that provides us with the opposite gain: that $\neg q$ as well as $\check{\neg} q$ are finally derivable. Moreover, the $\mathbf{CLo}^m$-proof shows us the way. One possibility is to allow only for conjunction gaps, in other words, to choose the adaptive logic $\mathbf{CLaC}^m$. The proof then goes as follows.

1	p	Premise	$\emptyset$	
2	r	Premise	$\emptyset$	
3	$\neg q \vee \neg r$	Premise	$\emptyset$	
4	$(p \wedge r) \supset q$	Premise	$\emptyset$	
5	$\neg p \vee s$	Premise	$\emptyset$	
6	$\neg q \check{\vee} \neg r$	3; RU	$\emptyset$	
7	$\check{\neg}\neg r$	2; RU	$\emptyset$	
8	$\neg q$	6, 7; RU	$\emptyset$	
9	$\neg p \check{\vee} s$	5; RU	$\emptyset$	
10	$\check{\neg}\neg p$	1; RU	$\emptyset$	
11	s	9, 10; RU	$\emptyset$	
12	$p \wedge r$	1, 2; RC	$\{(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)\}$	$\checkmark^{16}$
13	$(p \wedge r) \check{\supset} q$	4; RU	$\emptyset$	
14	q	12, 13; RU	$\{(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)\}$	$\checkmark^{16}$

15	$\check{\neg} q$	8; RU	$\emptyset$
16	$(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)$	14, 15; RD	$\emptyset$

Nearly the same effect is obtained by choosing $\mathbf{CLuM}^m$, which allows only for implication gluts. In that proof, $\check{\neg}((p \wedge r) \check{\supset} q) \check{\wedge} ((p \wedge r) \supset q)$ is the formula of line 16 and the singleton comprising this formula is the condition of lines 13 and 14, whence these lines are marked.

What happens if one chooses the adaptive logic $\mathbf{CLaN}^m$? All conditions become empty, so q and $\check{\neg} q$ are derived unconditionally. While transforming the $\mathbf{CLo}^m$-proof, the formula of line 16 is turned into the empty string. What this means is that we have to delete the line because RD cannot be applied. Moreover, as we derived a classical inconsistency, q and $\check{\neg} q$, and we derived it on the empty condition, we obtain triviality. In other words $\mathbf{CLaN}^m$ is not a suitable adaptive logic for the present premise set in view of the projected goal, which is to obtain a minimally abnormal 'interpretation' of Γ_1.

In the $\mathbf{CLuN}^m$-proof, q is unconditionally derived. This is also the case if one chooses the logic $\mathbf{CLuD}^m$, which tolerates disjunction gluts only. Moreover, the $\mathbf{CLo}^m$-proof reveals that this is a secure choice. Indeed, allowing for disjunction gluts causes $\check{\neg} q$ not to be a final consequence of the premise set. So this avoids triviality.

By now the reader should be convinced that $\mathbf{CLo}^m$-proofs offer an instrument for obtaining minimally abnormal interpretations of premise sets.[17] If no *Dab*-formulas are derived in the $\mathbf{CLo}^m$-proof, the premise set is apparently normal.[18] If that is so, its interpretation in terms of **CL** is normal. If *Dab*-formulas are derived, a minimally abnormal interpretation of the premises is obtained if the premise set is closed under an adaptive logic from the previous section that fulfils two properties: (i) no *Dab*-formula derived by RD is turned into the empty string, and (ii) every otherwise derived *Dab*-formula counts at least one disjunct that is an abnormality of the chosen logic.[19] Recall that some of the lower limit logics combine different gluts and gaps. The matter is completely straightforward. We can read off the minimally abnormal interpretations from the $\mathbf{CLo}^m$-proof. In sum, constructing proofs in $\mathbf{CLo}^m$ (or $\mathbf{CLo}^r$) offers an analysis that allows one to decide which adaptive logics from the previous section may be applied to handle a given premise set, and which may not because they assign a trivial consequence set to the premise set. The analysis also reveals which adaptive logics offer a richer consequence set than others.

The logic $\mathbf{CLo}^m$ is interesting in itself for a theoretical reason. Indeed, in this logic, the meaning of all standard logical symbols is *contingent*: the meaning of an occurrence of a standard symbol—no other symbol should occur in the premises or in the (main) conclusion—depends fully on the premise set.

[17] Not all of them, of course, because there are variants of the lower limit logics—they were briefly surveyed in the text of the present section. Yet, whatever the number of (kinds of) gluts and gaps they tolerate, the logics from Section 3 do not spread abnormalities but maximally isolate them—this property of **CLuN** was already discussed in [9].

[18] I write "apparently" because the judgement concerns only the present stage of the $\mathbf{CLo}^m$-proof.

[19] If these conditions are not fulfilled, the premise set is trivial according to the chosen adaptive logic.

To put it in a pompous way: $\mathbf{CLo}^m$ provides one with a formal hermeneutics—but see Section 6 for a more impressive result in this respect.

The story does not end here. Until now I have considered logics from the previous section and have illustrated the way in which they lead to different non-trivial but inconsistent 'interpretations' of an inconsistent theory. However, the logics from the previous section may, in a specific sense, also be combined. I shall illustrate that this leads to further non-trivial but inconsistent 'interpretations' of an inconsistent theory. This approach requires some clarification before we start.

Consider the premise set $\Gamma_2 = \{p, r, (p \vee q) \supset s, (p \vee t) \supset \neg r, (p \wedge r) \supset \neg s, (p \wedge s) \supset t\}$. I shall not write out the $\mathbf{CLo}^m$-proof, but if one writes it out, one readily sees that Γ_2 can be interpreted non-trivially by allowing for disjunction gaps as well as for conjunction gaps. The $\mathbf{CLo}^m$-proof moreover reveals that it may be interesting to first eliminate the disjunction gaps and next the conjunction gaps, something which typically may be realized by a combined adaptive logic. The question is what this combined logic precisely looks like.

The simplest combination, $Cn_{\mathbf{CLaC}^m}(Cn_{\mathbf{CLaD}^m}(\Gamma_2))$, does not have the desired effect—I skip a technical detail on superpositions because it is irrelevant for the point I want to make. One the one hand, every conjunction of members of $Cn_{\mathbf{CLaD}^m}(\Gamma_2)$ is itself a member of that set because the standard conjunction behaves like the classical conjunction in **CLaD**. So closing $Cn_{\mathbf{CLaD}^m}(\Gamma_2)$ under $\mathbf{CLaC}^m$ does not add any conjunctions. On the other hand, the standard *disjunction* behaves like the classical disjunction in **CLaC**. This means that if $A \in \Gamma$ and hence $A \in Cn_{\mathbf{CLaD}^m}(\Gamma_2)$, then $A \vee B \in Cn_{\mathbf{CLaC}^m}(Cn_{\mathbf{CLaD}^m}(\Gamma_2))$ for every B. This may very well cause triviality. The reader may easily verify this by reinterpreting the subsequent proof from Γ_2 as a proof for $Cn_{\mathbf{CLaC}^m}(Cn_{\mathbf{CLaD}^m}(\Gamma_2))$.[20]

What we need is rather obvious. We want to superimpose two simple adaptive logics that leave room for disjunction gaps as well as for conjunction gaps, but we want first to minimize the set of disjunction gaps and only thereafter the set of conjunction gaps. So, following the naming scheme from the previous section, we first need an adaptive logic composed of the lower limit logic **CLaDaC**, the set of abnormalities $\Omega = \{\check{\exists}((A \check{\vee} B) \check{\wedge} \check{\neg}(A \vee B)) \mid A, B \in \mathcal{F}_s\})$, comprising the disjunction gaps, and say Minimal Abnormality. One might call this logic $\mathbf{CLaDaC}^m_{aD}$—the subscript refers to the kind of abnormalities that is minimized (here disjunction gaps). Next, we want to close the consequence set of this logic by an adaptive logic composed of the lower limit logic **CLaDaC**, the set of abnormalities $\Omega = \{\check{\exists}((A \check{\wedge} B) \check{\wedge} \check{\neg}(A \wedge B)) \mid A, B \in \mathcal{F}_s\}$, comprising the conjunction gaps, and Minimal Abnormality. One might call this logic $\mathbf{CLaDaC}^m_{aC}$.

Let us move to a proof from Γ_2 in this combined logic. All logical symbols have their classical meaning with the exception of disjunction and conjunction. The reader should be informed that, in this specific combined logic, the first

[20]The disjunction $p \vee t$ is **CLaC**-derivable from p and hence is derivable on the empty condition in the so reinterpreted proof. But then so are both r and $\neg r$, whence triviality results.

round of marking proceeds in terms of the minimal *Dab*-formulas that have disjunction gaps as their disjuncts and are derived on the empty condition, whereas the second round proceeds in terms of the minimal *Dab*-formulas that have conjunction gaps as their disjuncts and are derived at an unmarked line the condition of which may contain disjunction gaps but no conjunction gaps. I try to make the proof more transparent for the reader by first deriving the required disjunctions, applying $\mathbf{CLaDaC}^m_{aD}$, and only thereafter deriving the required conjunctions by applying $\mathbf{CLaDaC}^m_{aC}$. The distinction between the two conditional rules is self-explanatory.

1	p	Premise	$\emptyset$	
2	r	Premise	$\emptyset$	
3	$(p \vee q) \supset s$	Premise	$\emptyset$	
4	$(p \vee t) \supset \neg r$	Premise	$\emptyset$	
5	$(p \wedge r) \supset \neg s$	Premise	$\emptyset$	
6	$(p \wedge s) \supset t$	Premise	$\emptyset$	
7	$p \vee q$	1; RC1	$\{(p \check{\vee} q) \check{\wedge} \check{\neg}(p \vee q)\}$	
8	s	3, 7; RU	$\{(p \check{\vee} q) \check{\wedge} \check{\neg}(p \vee q)\}$	
9	$p \vee t$	1; RC1	$\{(p \check{\vee} t) \check{\wedge} \check{\neg}(p \vee t)\}$	$\checkmark^{11}$
10	$\neg r$	4, 9; RU	$\{(p \check{\vee} t) \check{\wedge} \check{\neg}(p \vee t)\}$	$\checkmark^{11}$
11	$(p \check{\vee} t) \check{\wedge} \check{\neg}(p \vee t)$	2, 10; RD	$\emptyset$	
12	$p \wedge r$	1, 2; RC2	$\{(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)\}$	$\checkmark^{14}$
13	$\neg s$	5, 12; RU	$\{(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)\}$	$\checkmark^{14}$
14	$(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)$	8, 13; RD	$\{(p \check{\vee} q) \check{\wedge} \check{\neg}(p \vee q)\}$	
15	$p \wedge s$	1, 8; RC2	$\{(p \check{\vee} q) \check{\wedge} \check{\neg}(p \vee q), (p \check{\wedge} s) \check{\wedge} \check{\neg}(p \wedge s)\}$	
16	t	6, 15; RU	$\{(p \check{\vee} q) \check{\wedge} \check{\neg}(p \vee q), (p \check{\wedge} s) \check{\wedge} \check{\neg}(p \wedge s)\}$	

On line 14, the general form of rule RD is applied. The set of consequences of the combined logic can be 'summarized' as $\{p, r, s, t, \neg(p \wedge r), \neg(p \vee t)\}$. Note that I write classical negation in the abnormalities in the proof to be coherent with the rest of this paper, but that the standard negation has the same meaning. The same result cannot be obtained by any of the logics described in the previous section.[21]

There may be specific logical or extra-logical reasons to prefer a combined adaptive logic or another of the aforementioned logics to obtain a minimally abnormal 'interpretation' of Γ_2; or there may be reasons to consider the 'interpretation' as a sensible alternative. As mentioned before, such reasons may become apparent in view of a $\mathbf{CLo}^m$-proof from Γ_2. The choices considered for Γ_1 were extended with the choice of an order in which the abnormalities are minimized.

The upper limit logic of all simple adaptive logics presented in this paper is **CL**. So these logics, and all the combined adaptive logics built from them, assign the same consequence set as **CL** to all premise sets that have **CL**-models. While this is an interesting feature in itself, the interest of the diversity of the logics lies with premise sets that have no **CL**-models.

[21] Line 14 witnesses that $((p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)) \check{\vee} ((p \check{\vee} q) \check{\wedge} \check{\neg}(p \vee q))$ is derivable on the empty condition from Γ_2. With respect to the superposition, it shows that $(p \check{\wedge} r) \check{\wedge} \check{\neg}(p \wedge r)$ is a a final $\mathbf{CLaDaC}^m_{aD}$-consequence of Γ.

5 Ambiguity-Adaptive Logics

In [46], Guido Vanackere presented the first ambiguity-adaptive logic. The underlying idea is simple but ingenious. The inconsistency of a text may derive from the ambiguity of its non-logical symbols. To take these possible ambiguities into account, one *indexes* all occurrences of non-logical symbols. This roughly means that every occurrence receives a different superscript and that symbols with a different index are considered as different. An ambiguity-adaptive logic interprets a set of premises as unambiguous as possible. It presupposes that two non-logical symbols that differ only in their index have the same meaning unless and until proven otherwise.

While the idea is simple and attractive, elaborating the technical details requires hard work. Most published papers on ambiguity-adaptive logics evade some unsolved problems. There is a reason why the matter is confusing. The languages underlying ambiguity-adaptive logics may serve diverse, unexpected, and attractive purposes. All purposes require a monotonic logic that is close to **CL**, but many purposes demand that the logic deviate from **CL** in one or other detail, and each purpose requires a different deviation. I now spell out a systematic and sensible variant of ambiguity logic.

In the language $\mathcal{L}_s$, the sets of schematic letters[22] for non-logical symbols are $\mathcal{S}$, $\mathcal{C}$, $\mathcal{V}$, and $\mathcal{P}^r$ (for each rank $r \in \mathbb{N}$). Let us replace each of these sets with a set of indexed letters: $\mathcal{S}^I = \mathcal{S} \cup \{\lambda^i \mid \lambda \in \mathcal{S}; i \in \mathbb{N}\}$, and similarly for $\mathcal{C}^I$, $\mathcal{V}^I$, and $\mathcal{P}^{rI}$. The resulting sets are obviously denumerable. From these sets we define a language $\mathcal{L}_s^I$, with $\mathcal{F}_s^I$ as its set of formulas and $\mathcal{W}_s^I$ as its set of closed formulas. The language $\mathcal{L}_s^I$ is exactly as one expects, *except that the quantifiers still range over the variables of* $\mathcal{L}_s$. The reason for this convention will be explained later on.

Next, we define a logic **CLI** over this language. The logic is almost identical to **CL**, except for the way in which quantified formulas are handled. To phrase the semantics, we need to add an indexed set $\mathcal{O}^I$ of pseudo-constants, which is defined from $\mathcal{O}$ in the same way as $\mathcal{C}^I$ is defined from $\mathcal{C}$. The resulting pseudo-language $\mathcal{L}_{\mathcal{O}}^I$ has $\mathcal{W}_{\mathcal{O}}^I$ as its set of closed formulas. A **CLI**-model $M = \langle D, v\rangle$, in which D is a set and v is an assignment function. The function v is like for **CL**, except that it now interprets the indexed sets.

C1 $v\colon \mathcal{W}_{\mathcal{O}}^I \to \{0,1\}$
C2 $v\colon \mathcal{C}^I \cup \mathcal{O}^I \to D$ (where $D = \{v(\alpha) \mid \alpha \in \mathcal{C}^I \cup \mathcal{O}^I\}$)
C3 $v\colon \mathcal{P}^{rI} \to \wp(D^r)$

The valuation function $v_M\colon \mathcal{W}_{\mathcal{O}}^I \to \{0,1\}$ determined by M is defined by the following clauses: C$\neg$, C$\supset$, C$\wedge$, C$\vee$, and C$\equiv$ from the **CL**-semantics plus:

C$\mathcal{S}^I$ where $A \in \mathcal{S}^I$, $v_M(A) = 1$ iff $v(A) = 1$
C$\mathcal{P}^{rI}$ where $\pi^r \in \mathcal{P}^{rI}$ and $\alpha_1 \ldots \alpha_r \in \mathcal{C}^I \cup \mathcal{O}^I$,
$v_M(\pi^r\alpha_1 \ldots \alpha_r) = 1$ iff $\langle v(\alpha_1), \ldots, v(\alpha_r)\rangle \in v(\pi^r)$
C= where $\alpha, \beta \in \mathcal{C}^I \cup \mathcal{O}^I$, $v_M(\alpha = \beta) = 1$ iff $v(\alpha) = v(\beta)$

[22] The name "letter" is slightly misleading. Most schematic letters are actually strings composed from a finite sequence of symbols.

C$\forall^I$ $\quad v_M(\forall\alpha A(\alpha^{i_1},\ldots,\alpha^{i_n})) = 1$ iff $\{v_M(A(\beta^{i_1},\ldots,\beta^{i_n})) \mid \beta \in \mathcal{C}\cup\mathcal{O}\} = \{1\}$

C$\exists^I$ $\quad v_M(\exists\alpha A(\alpha^{i_1},\ldots,\alpha^{i_n})) = 1$ iff $1 \in \{v_M(A(\beta^{i_1},\ldots,\beta^{i_n})) \mid \beta \in \mathcal{C}\cup\mathcal{O}\}$

$M \Vdash A$ iff $v_M(A) = 1$, which defines $\vDash_{\mathbf{CLI}} A$ as well as $\Gamma \vDash_{\mathbf{CLI}} A$.

The clauses C$\forall$ and C$\exists$ deserve some clarification. Note that the quantifiers range over a variable α and that the α^{i_j} are indexed occurrences of this variable in A. Thus $M \Vdash \forall x(P^1x^1 \supset Q^1x^2)$ holds iff $M \Vdash P^1\alpha^1 \supset Q^1\alpha^2$ holds for all $\alpha \in \mathcal{C}\cup\mathcal{O}$. Similarly, $M \Vdash \exists x(P^1x^1 \wedge Q^1x^2)$ holds iff $M \Vdash P^1\alpha^1 \wedge Q^1\alpha^2$ holds for some $\alpha \in \mathcal{C}\cup\mathcal{O}$.

The behaviour of the quantifiers causes a connection between variables that differ only from each other in their index, because the same quantifiers bind them all. The quantifiers also connect indexed variables to the constants with the same indices. Thus, among the semantic consequences of $\forall x(P^1x^1 \supset Q^1x^2)$ are $P^1a^1 \supset Q^1a^2$ as well as $P^1b^1 \supset Q^1b^2$, but not, for example, $P^1a^1 \supset Q^1b^2$ or $P^1a^1 \supset Q^1a^3$. It will become clear later that this peculiar logic is tailored in order to suit the ambiguity-adaptive logic of which it is the lower limit.

To spell out an axiomatic system for **CLI** is left as an easy exercise for the reader. Hint: take the **CL**-axiomatization from Section 2, letting the metavariables range over indexed entities; next adjust A$\forall$ to $\forall\alpha A(\alpha^{i_1},\ldots,\alpha^{i_n})) \supset A(\beta^{i_1},\ldots,\beta^{i_n})$, and adjust A$\exists$, R$\forall$, and R$\exists$ similarly.

The idea of (non-adaptive) ambiguity logics is that, where $\Gamma \subseteq \mathcal{W}_s$ and $A \in \mathcal{W}_s$, $\Gamma \vdash A$ iff a certain translation of A is a **CLI**-consequence of a certain translation of Γ. The presumably unexpected handling of the quantifiers will be easier understood after I presented the translation. Let $\Gamma^\dagger$ be obtained from Γ by adding superscripted indices from an $I \subset \mathbb{N}$ to all non-logical symbols in Γ in such a way that every index occurs at most once. Next, let $A^\ddagger$ be obtained from A by adding superscripted indices from $\mathbb{N} - I$ to all non-logical symbols in A in such a way that every index occurs at most once.[23] The ambiguity logic **CLA**, defined over the language $\mathcal{L}_s$, is defined by

$$\Gamma \vdash_{\mathbf{CLA}} A \text{ iff } \Gamma^\dagger \vdash_{\mathbf{CLI}} A^\ddagger\,.$$

In order to define **CLA**, we need only a certain fragment of **CLI**. For every premise set Γ and conclusion A, $\Gamma^\dagger \cup \{A^\ddagger\}$ is a set of members of $\mathcal{W}_s^I$ that has a very specific property: all non-logical symbols are indexed and no two occurrences of the same non-logical symbol have the same index. One of the effects of this is that there are no $\Gamma \subseteq \mathcal{W}_s$ and $A \in \mathcal{W}_s$ for which $\Gamma^\dagger \vdash_{\mathbf{CLI}} A^\ddagger$, whereas there obviously are $\Gamma \subseteq \mathcal{W}_s^I$ and $A \in \mathcal{W}_s^I$ for which $\Gamma \vdash_{\mathbf{CLI}} A$, for example $p^1 \wedge q^2 \vdash_{\mathbf{CLI}} p^1$.

At this point, the handling of the quantifiers should be more transparent. If no quantifiers occur in Γ or A, we have $\Gamma \nvdash_{\mathbf{CLA}} A$. For example, $p \wedge q \nvdash_{\mathbf{CLA}} p$ because $p^1 \wedge q^2 \nvdash_{\mathbf{CLI}} p^3$—in some **CLI**-models $v(p^1) = v(q^2) = 1$ and $v(p^3) = 0$. Also $p \nvdash_{\mathbf{CLA}} p$ because $p^1 \nvdash_{\mathbf{CLI}} p^3$. However, if the quantifiers ranged over the indexed variables, we would have $\forall x\, x = x \wedge q \vdash_{\mathbf{CLA}} \forall x\, x = x$ as well as

[23]Other ways of indexing are equally adequate. As explained below in the text, every two occurrences of the same symbol in $\Gamma \cup \{A\}$ should have different indices and no individual variable should have the same index as an individual constant.

$\forall x\, x = x \vdash_{\mathbf{CLA}} \forall x\, x = x$ because $\forall x^1\, x^1 = x^1 \wedge q^2 \vDash_{\mathbf{CLI}} \forall x^3\, x^3 = x^3$ as well as $\forall x^1\, x^1 = x^1 \vDash_{\mathbf{CLI}} \forall x^3\, x^3 = x^3$.[24] But then quantified statements would behave oddly, because they would form classical exceptions in the ambiguity logic.

Let us take a closer look at this. The point is actually related to theorems of logic. Thus $\nvdash_{\mathbf{CLA}} p \vee \neg p$ because $\nvdash_{\mathbf{CLI}} p^1 \vee \neg p^2$. In general, **CLA** does not have any theorems at the propositional level. Note that the absence of theorems derives from the *translation*, not from **CLI**, which obviously has all the right theorems, for example $\vdash_{\mathbf{CLI}} p^1 \vee \neg p^1$. When one moves to the predicative level, **CL**-theorems turn out to be non-theorems of **CLA**. For example $\nvdash_{\mathbf{CLA}} a = a$ because $\nvDash_{\mathbf{CLI}} a^1 = a^2$ and $\nvdash_{\mathbf{CLA}} \forall x Px \supset Pa$ because $\nvdash_{\mathbf{CLI}} \forall x P^1 x^2 \supset P^3 a^4$—even if the quantifiers ranged over the indexed variables, we would still have $\nvdash_{\mathbf{CLI}} \forall x^2 P^1 x^2 \supset P^3 a^4$ because P^1 and P^3 are different schematic letters for predicates. However, if the quantifiers ranged over indexed variables, we would have $\vdash_{\mathbf{CLA}} \forall x\, x = x$ because $\vdash_{\mathbf{CLI}} \forall x^1\, x^1 = x^1$—note that $\forall x^1\, x^1 = x^2$ is not a closed formula. So this would reintroduce logical theorems at a unique specific point, which would be an anomaly.

The reader may think that another approach is equally sensible: to let the quantifiers range over indexed variables while multiplying the quantifiers where this is necessary to obtain closed formulas. Thus the translation of $\forall x\, x = x$ would be, for example, $\forall x^1 \forall x^2\, x^1 = x^2$. This, however, would not work. Indeed, from this formula, one might first obtain $\forall x^2\, a^1 = x^2$ and next $a^1 = b^2$, which would blur the difference between two very different formulas, $\forall x\, x = x$ and $\forall x \forall y\, x = y$. Moreover, even this kind of translation would introduce some theorems. Thus $\vdash_{\mathbf{CLA}} \forall x \forall y\, x = y \supset \forall x\, x = x$ would be translated by the true statement $\vdash_{\mathbf{CLI}} \forall x^1 \forall y^2\, x^1 = y^2 \supset \forall x^3 \forall x^4\, x^3 = x^4$.

Actually, letting the quantifiers range over the original variables causes no trouble, as the **CLI**-semantics reveals. Indeed, there are **CLI**-models that verify $\forall x\, x^1 = x^2$, and there are that do not, just as we want it. So $\nvdash_{\mathbf{CLA}} \forall x\, x = x$. Similarly $\forall x\, x = x \nvdash_{\mathbf{CLA}} \forall x\, x = x$ because $\forall x\, x^1 = x^2 \nvdash_{\mathbf{CLI}} \forall x\, x^3 = x^4$. Moreover, that the quantifiers range over the non-indexed variables in **CLI** guarantees that all indexed occurrences of the same variable are instantiated at the same time. This will turn out essential for the adaptive logic that will have **CLA** as its lower limit.

The logic **CLA** is intriguing. Nothing is valid in it, nothing is derivable from any premise set. Post-modernists should be pleased. Sensible people, however, will regard **CLA** as a lower limit logic, and will try to minimize abnormalities. They will admit that some texts (or premise sets) force one to consider non-logical terms as ambiguous,[25] but they will also stress that each non-logical term has to be considered as unambiguous "unless and until proven otherwise". In other words, they will go adaptive.

It is not difficult to see what going adaptive comes to. The lower limit logic will be **CLI** and the strategy either Reliability or Minimal Abnormality. We need a set of abnormalities containing three kinds of formulas: ambiguities pertaining respectively to sentential letters, to individual constants and variables,

[24] Do not think something is wrong here before you read the next two paragraphs.

[25] The texts force one to do so if one supposes that the *logical* symbols have their usual meaning and this is unique and stable.

and to predicative letters. In order to save some space in the example proofs, I shall introduce abbreviations for each of these kinds of abnormalities. Ambiguities for sentential letters have the form $\neg(A^i \equiv A^j)$, with $A \in \mathcal{S}$ and $i, j \in \mathbb{N}$.[26] These will be abbreviated as $A^{i \cdot j}$, for example $p^{5 \cdot 8}$ abbreviates $\neg(p^5 \equiv p^8)$. Ambiguities for individual constants and variables will have the form $\exists \neg\, \alpha^i = \alpha^j$, with $\alpha \in \mathcal{C} \cup \mathcal{V}$ and $i, j \in \mathbb{N}$. These will be abbreviated as $\alpha^{i \cdot j}$, for example $a^{6 \cdot 7}$ abbreviates $\neg\, a^6 = a^7$ and $x^{4 \cdot 8}$ abbreviates $\exists x \neg\, x^4 = x^8$. Finally, ambiguities for predicative letters have the form $\exists \neg(\pi^i \alpha_1 \ldots \alpha_r \equiv \pi^j \alpha_1 \ldots \alpha_r)$, with $\pi \in \mathcal{P}^r$, $i, j \in \mathbb{N}$, and $\alpha_1 \ldots \alpha_r \in \mathcal{V}^I$. These will be abbreviated as $\pi^{i \cdot j} \alpha_1 \ldots \alpha_r$, for example, where $P \in \mathcal{P}^1$, $P^{3 \cdot 5} x^1$ abbreviates $\exists x \neg(P^3 x^1 \equiv P^5 x^1)$ and, where $R \in \mathcal{P}^3$, $R^{2 \cdot 8} a^1 x^1 b^2$ abbreviates $\exists x \neg(R^2 a^1 x^1 b^2 \equiv R^8 a^1 x^1 b^2)$.[27]

The meaning of the abnormalities requires hardly any clarification: different occurrences of a symbol have different meanings. The matter is straightforward for sentential letters, individual constants and individual variables. There is a difference, however. Occurrences of the same constant may have different denotations. So it is possible that $\neg\, a^1 = a^2$, $\neg\, a^1 = a^3$, and $\neg\, a^2 = a^3$, and so on for any number of occurrences of the same constant. The matter is different for propositional letters. As there are (on the present approach) only two truth-values, 0 and 1, the occurrence of p^1, p^2 and p^3 necessarily leads to $p^1 \equiv p^2$, to $p^3 \equiv p^1$, or to $p^3 \equiv p^2$. The case of predicative letters is slightly more sophisticated. If both $P^1 a^2$ and $\neg P^3 a^2$ hold true, the object denoted by a^2 belongs to the extension of P^1 but not to that of P^3. In other words, P^1 and P^3 differ in extension with respect to the object denoted by a^2. If moreover both $P^1 a^4$ and $\neg P^3 a^4$ hold true, there is a further ambiguity: P^1 and P^3 also differ in extension with respect to the object denoted by a^4. This is the reason why abnormalities pertaining to predicates require a more complex abbreviation than the other abnormalities.

It is time to identify the set of related abnormalities. I shall do this in terms of the introduced abbreviations: $\Omega = \{A^{i \cdot j} \mid A \in \mathcal{S}; i, j \in \mathbb{N}; i \neq j\} \cup \{\alpha^{i \cdot j} \mid \alpha \in \mathcal{C} \cup \mathcal{V}; i, j \in \mathbb{N}; i \neq j\} \cup \{\pi^{i \cdot j} \alpha_1 \ldots \alpha_r \mid \pi \in \mathcal{P}^r; i, j \in \mathbb{N}; \alpha_1 \ldots \alpha_r \in \mathcal{C}^I \cup \mathcal{V}^I; i \neq j\}$. When reading this, remember that all logical symbols have their classical meaning. The adaptive logics $\mathbf{CLI}^m$ and $\mathbf{CLI}^r$ are now fully defined.

In terms of $\mathbf{CLI}^m$, we define the logic $\mathbf{CLA}^{\mathsf{m}}$:

$$\Gamma \vdash_{\mathbf{CLA}^{\mathsf{m}}} A \text{ iff } \Gamma^\dagger \vdash_{\mathbf{CLI}^m} A^\ddagger\,,$$

and similarly for $\mathbf{CLA}^{\mathsf{r}}$. I write the superscripts of $\mathbf{CLA}^{\mathsf{m}}$ and $\mathbf{CLA}^{\mathsf{r}}$ in a different type to indicate that these logics are not themselves adaptive logics in standard format, but are characterized in terms of such logics.

Let us consider some example proofs. The set $\Gamma_3 = \{\forall x(Px \supset Qx), Pa\}$ is normal. So the $\mathbf{CLA}^{\mathsf{m}}$-consequence set (and $\mathbf{CLA}^{\mathsf{r}}$-consequence set) of Γ_3 is

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[26] If the intention is to combine ambiguity logics with logics from Sections 3 or 4, the abnormalities are better phrased with the help of classical logical symbols.

[27] The use of ambiguities in the variables is illustrated by $\exists x \neg(P^1 x^2 \equiv P^3 x^4) \vdash_{\mathbf{CLI}} \exists x \neg(x^2 = x^4) \vee \exists x \neg(P^1 x^2 \equiv P^3 x^2)$. Incidentally, $\neg(p^1 \equiv p^2)$ and $\neg(p^2 \equiv p^1)$ are officially considered as different (but equivalent) abnormalities. Similarly $p^{1 \cdot 2}$ and $p^{2 \cdot 1}$ are officially seen as abbreviations of different formulas. Both decisions are obviously purely conventional.

identical to its **CL**-consequence set, as the reader expected. Here is an example proof for $\Gamma_3 \vdash_{\mathbf{CLA}^r} Qa$. This is translated for example by $\forall x(P^1x^2 \supset Q^3x^4)$, $P^5a^6 \vdash_{\mathbf{CLI}^m} Q^7a^8$.

1	$\forall x(P^1x^2 \supset Q^3x^4)$	Prem	$\emptyset$
2	P^5a^6	Prem	$\emptyset$
3	$P^1a^2 \supset Q^3a^4$	1; RU	$\emptyset$
4	P^1a^2	2; RC	$\{P^{1\cdot 5}a^6, a^{2\cdot 6}\}$
5	Q^3a^4	3, 4; RU	$\{P^{1\cdot 5}a^6, a^{2\cdot 6}\}$
6	Q^7a^8	5; RC	$\{P^{1\cdot 5}a^6, a^{2\cdot 6}, Q^{3\cdot 7}a^4, a^{4\cdot 8}\}$

As $\{\forall x(P^1x^2 \supset Q^3x^4), P^5a^6\}$ is normal with respect to $\mathbf{CLI}^m$, no *Dab*-formula is derivable from it, whence no line is marked in any extension of the proof.

Some readers may find the proof a bit fast. Here is the trick, applied to the transition from 2 to 4. In the slower proof fragment displayed below, the condition of line 2.1 is the negation of the formula of that line. So the line results from the **CLI**-theorem $(P^5a^6 \equiv P^1a^6) \vee \neg(P^5a^6 \equiv P^1a^6)$. Similarly for line 2.3, which results from the **CLI**-theorem $a^2 = a^6 \vee \neg a^2 = a^6$.

2	P^5a^6	Prem	$\emptyset$
2.1	$P^5a^6 \equiv P^1a^6$	RC	$\{P^{1\cdot 5}a^6\}$
2.2	P^1a^6	2, 2.1; RU	$\{P^{1\cdot 5}a^6\}$
2.3	$a^2 = a^6$	RC	$\{a^{2\cdot 6}\}$
4	P^1a^2	2.2, 2.3; RU	$\{P^{1\cdot 5}a^6, a^{2\cdot 6}\}$

If predicative expressions are ambiguous, the ambiguity can lie with a predicate, an individual constant, or a variable. This often leads to a disjunction of such abnormalities. For example $P^1a^2, \neg P^3a^4 \vdash_{\mathbf{CLI}} \neg a^2 = a^4 \vee \neg(P^1a^4 \equiv P^3a^4)$. This will be illustrated in the next example proof.

It is instructive to consider a further example: $\Gamma_4 = \{\forall x(Px \supset Qx), Pa, \neg Qa, Pb\}$. Its translation is, for example, $\{\forall x(P^1x^2 \supset Q^3x^4), P^5a^6, \neg Q^7a^8, P^9b^{10}\}$. Let us check wether $\Gamma_4 \vdash_{\mathbf{CLA}^m} Qa$ and $\Gamma_4 \vdash_{\mathbf{CLA}^m} Qb$. As the indices 1–10 occur in the translation of Γ_4, the indexed conclusions will be, for example, $Q^{11}a^{12}$ and $Q^{11}b^{12}$ respectively.

1	$\forall x(P^1x^2 \supset Q^3x^4)$	Prem	$\emptyset$	
2	P^5a^6	Prem	$\emptyset$	
3	$\neg Q^7a^8$	Prem	$\emptyset$	
4	P^9b^{10}	Prem	$\emptyset$	
5	$P^1a^2 \supset Q^3a^4$	1; RU	$\emptyset$	
6	P^1a^2	2; RC	$\{P^{1\cdot 5}a^6, a^{2\cdot 6}\}$	$\checkmark^{10}$
7	Q^3a^4	5, 6; RU	$\{P^{1\cdot 5}a^6, a^{2\cdot 6}\}$	$\checkmark^{10}$
8	$Q^{11}a^{12}$	7; RC	$\{P^{1\cdot 5}a^6, a^{2\cdot 6}, Q^{3\cdot 11}a^4, a^{4\cdot 12}\}$	$\checkmark^{10}$
9	$\neg Q^3a^4$	3; RC	$\{Q^{7\cdot 3}a^8, a^{8\cdot 4}\}$	$\checkmark^{10}$
10	$P^{1\cdot 5}a^6 \vee a^{2\cdot 6} \vee Q^{7\cdot 3}a^8 \vee a^{8\cdot 4}$	7, 9; RD	$\emptyset$	

Besides 10, many other *Dab*-formulas are **CLI**-derivable from the premises. For any suitable i and j, Q^ia^j is derivable from Q^3a^4 on the condition $\{P^{1\cdot 5}a^6, a^{2\cdot 6}, Q^{3\cdot i}a^4, a^{4\cdot j}\}$ and $\neg Q^ia^j$ is derivable from $\neg Q^7a^8$ on the condition $\{Q^{7\cdot i}a^8, a^{8\cdot j}\}$.

So the disjunction of members of both conditions is **CLI**-derivable on the empty condition. This entails that the line at which $Q^{11}a^{12}$ is derived will ultimately be and remain marked in any proof from Γ_4; $Q^{11}a^{12}$ is not a final $\mathbf{CLI}^m$-consequence of $\{P^{1\cdot 5}a^6, a^{2\cdot 6}, Q^{3\cdot i}a^4, a^{4\cdot j}\}$ and $\Gamma_4 \nvdash_{\mathbf{CLA}^{\mathsf{m}}} Qa$.

The situation is obviously very different for $Q^{11}b^{12}$. Let us have a look at the continuation of the previous proof.

11	$P^1b^2 \supset Q^3b^4$	1; RU	$\emptyset$
12	P^1b^2	4; RC	$\{P^{1\cdot 9}b^{10}, b^{2\cdot 10}\}$
13	Q^3b^4	11, 12; RU	$\{P^{1\cdot 9}b^{10}, b^{2\cdot 10}\}$
14	$Q^{11}b^{12}$	13; RC	$\{P^{1\cdot 9}b^{10}, b^{2\cdot 10}, Q^{3\cdot 11}b^4, b^{4\cdot 12}\}$

None of these lines will be marked in any extension of the proof. The reason is that the conditions of the lines contain only abnormalities that explicitly mention b, whereas no such abnormality is **CLI**-derivable from $\{\forall x(P^1x^2 \supset Q^3x^4), P^5a^6, \neg Q^7a^8, P^9b^{10}\}$. So $Q^{11}b^{12}$ is a final **CLI**-consequence of the translated premise set and $\Gamma_4 \vdash_{\mathbf{CLA}^{\mathsf{m}}} Qb$.

Some readers may wonder why the proofs contain no examples of abnormalities that pertain to variables. This is partly a matter of style. For example, the lines 11–14 of the last proof may just as well be replaced by the following lines in which I also proceed a bit faster.

11	$\forall x(P^9x^{10} \supset Q^{11}x^{12})$	1; RC	$\{P^{1\cdot 9}x^2, x^{2\cdot 10}, Q^{3\cdot 11}x^4, x^{4\cdot 12}\}$
12	$P^9b^{10} \supset Q^{11}b^{12}$	11; RU	$\{P^{1\cdot 9}x^2, x^{2\cdot 10}, Q^{3\cdot 11}x^4, x^{4\cdot 12}\}$
13	$Q^{11}b^{12}$	4, 12; RU	$\{P^{1\cdot 9}x^2, x^{2\cdot 10}, Q^{3\cdot 11}x^4, x^{4\cdot 12}\}$

In other cases, for example in order to establish $\forall x(Px \supset Qx), \forall x(Qx \supset Rx) \vdash_{\mathbf{CLA}^{\mathsf{m}}} \forall x(Px \supset Rx)$, abnormalities pertaining to variables are unavoidable, unless of course when dummy constants would be introduced.

Before leaving the matter, two points are worth some attention. The first concerns my promise to clarify the translation, the second concerns variants for the present ambiguity-adaptive logics.

The translation is actually a simple matter. When describing it, I required (in footnote 23) that no two occurrences of the same symbol receive the same index *and* that no individual constant receives the same index as an individual variable. The first requirement is obvious. That two occurrences of the same symbol receive the same index amounts to declaring them to have the same meaning. If ambiguities may be around, there is no logical justification for doing so. The second requirement may be easily explained. Consider the premise set $\{\forall x\, Px, \neg Pa\}$ and note that Pa is derivable from the first premise. If, for example, the first premise is translated as $\forall x\, P^1x^2$, then P^1a^2 is a **CLI**-consequence of it. So there either is an ambiguity in P or there is an ambiguity in a. But suppose that the premise set were translated as $\{\forall x\, P^1x^1, \neg P^2a^1\}$—this translation fulfils the first requirement but not the second. As P^1a^1 is a **CLI**-consequence of this, so is the abnormality $P^{1\cdot 2}a^1$. But this is obviously mistaken because it locates the ambiguity definitely in P, neglecting the possible ambiguity in a.

It is instructive to return for a moment to Γ_4 and to describe the abnormality of the premise set in non-technical terms with reference to $\mathbf{CLA}^{\mathsf{m}}$. The

Dab-formula derived at line 10 of the proof teaches us that there is an ambiguity in P, in Q, or in a. It is also possible to derive in the proof the Dab-formula $P^{1\cdot 5}a^2 \vee x^{2\cdot 6} \vee Q^{3\cdot 7}x^4 \vee x^{4\cdot 12}$, which teaches us that there is an ambiguity in P, in Q, or in x. An important insight is that both statements are rather rudimentary. They locate a connected ambiguity, but tell one nothing about the effects of the ambiguities. A more fine-grained analysis goes in terms of derivable and non-derivable formulas. On the one hand, the connected ambiguity prevents Qa from being derivable. The ambiguity resides either in the P that occurs in the formulas from which Qa would be derivable, or in the Q that occurs in those formulas, viz. in $\forall x(Px \supset Qx)$ and in Qa itself, or the ambiguity resides in the a that occurs in Qa and in the formulas from which it would follow, viz. Pa.[28] On the other hand, the connected ambiguity does not prevent the derivability of Qb. So even if the ambiguity resides in P or in Q, this does not prevent the derivability of Qb from Pb and $\forall x(Px \supset Qx)$; these occurrences of P and Q are taken to be unambiguous. Note also that the connected ambiguities have no effect on the derivability of $Pa \vee Ra$ from Pa. Even if the ambiguity resides in P, the occurrences of P in Pa and in $Pa \vee Ra$ have the same meaning. Note also that, for a similar reason, $\neg Qa \wedge Qb$ is a final consequence of Γ_4. So the Dab-formulas that are **CLI**-derivable from $\Gamma^\dagger$ indicate connected ambiguities in non-logical symbols. However, which couples of occurrences of those symbols have a different meaning is only revealed by a careful study of the final derivability relation.

Let us now move to variants. Actually, **CLI** and similar logics contain a very rich potential—see for example [14, 22] for applications that have nothing to do with ambiguity-adaptive logic. However, also the ambiguity-adaptive logics deserve further attention. A striking point concerns ambiguities in sentential letters. As we have seen before, if there are three occurrences of the same sentential letter, at least two of them 'have the same meaning'. This is so because having the same meaning is expressed by equivalence, equivalence is truth-functional, and there are only two truth values. However, it is obvious that the same sentential letter (or the same sentence in a natural language) may be used with more than two different meanings. This suggests that one tries to dig deeper into meaning. The meaning of a linguistic entity may be seen as composed from different elements. Some bunches of such elements may actually be realistic, in that they occur in statements made in terms of the language, whereas other bunches do not. Moreover, it is well-known that speakers often want to express something close to, but slightly different from, a given realistic bunch and still use the same word or phrase. An approach that may enable one to dig deeper into meaning is available along these lines. Some work has been done on it. I cannot report on it here, but address the reader to some relevant papers: [29, 30, 44].

Before leaving the matter, an important proviso should be mentioned. Much so-called ambiguity arises from the fact that many predicates are *vague*. Vague-

[28] A similar comment applies to the second aforementioned Dab-formula. Actually, the ambiguity in x cannot be separated from the one in a. If the joint truth of Pa, $Pa \supset Qa$, and $\neg Qa$ is not caused by an ambiguity in P or in Q, then it is caused by an ambiguity in a. If that is so, there also is bound to be an ambiguity in x because $Pa \supset Qa$ is derived from $\forall x(Px \supset Qx)$.

ness obviously cannot be adequately handled by means of **CLA**, *pace* [45]. See [49] for a decent proposal to upgrade fuzzy *logics* adaptively.

6 Adaptive Zero Logic

In the previous sections, we met two extremely weak logics. The first was **CLo**, in which no standard logical symbol has any specific meaning. We have seen that $A \in \mathcal{W}_s$ is **CLo**-derivable from a premise set $\Gamma \subseteq \mathcal{W}_s$ iff $A \in \Gamma$. The second, even weaker logic, was **CLA**, in which different occurrences of a non-logical symbol may have different meanings. Recall that no $A \in \mathcal{W}_s$ is **CLA**-derivable from any premise set $\Gamma \subseteq \mathcal{W}_s$. It is not difficult to combine the weaknesses of both logics. I shall call the result **CL∅**, in words *zero logic*. In zero logic, logical symbols have no meaning whereas the meaning of non-logical symbols may vary by the occurrence. While zero logic in itself is utterly useless, it may function as the lower limit of a very useful adaptive logic. The idea of zero logic was first presented in [12]. The paper is clumsy at several points and uses terminology that has now been replaced.

Defining **CL∅** is easy. For the semantics, replace all standard logical symbols in the **CLI**-semantics by their classical counterparts and do not add anything for the standard logical symbols. Let this logic be called **CL∅I**. For its axiomatization, replace the standard logical symbols in the axiom system of **CLI** by their classical counterparts (and do not add anything for the standard logical symbols). From **CL∅I**, define **CL∅** by

$$\Gamma \vdash_{\mathbf{CL\emptyset}} A \text{ iff } \Gamma^{\dagger} \vdash_{\mathbf{CL\emptyset I}} A^{\ddagger}\,.$$

in which † and ‡ are as in Section 5. The logic **CL∅** is useless in itself. It is also odd. Even the difference between logical and non-logical symbols is blurred. To be more precise, the difference is obviously neat in the metalanguage, but nothing *within* the logic reveals it. This is really the logic that suits the post-modernist. It also shows that post-modernism, in its extreme form, is not viable. If, in a text, any occurrence of any symbol can have whatever meaning, then nothing sensible can be said about the text. I consider it plausible that **CL∅** is the logic present in our brains before we start to learn our mother tongue. Only as this learning proceeds, we start connecting words to entities in the world (things, actions, processes) or to representations of such entities, and we start connecting logical symbols to operations. In doing so, we are forced to turn the connection into a probabilistic and contextual one.

The most straightforward adaptive logics that have **CL∅I** as their lower limit logic combine it with Reliability or Minimal abnormality and with a specific set of abnormalities. This set is the union of two subsets: (i) the set containing all formulas that express gluts and gaps (as mentioned in the table at the end of Section 3), and (ii) the abnormalities of $\mathbf{CLI}^m$, duly phrased in terms of classical logical symbols. This gives us $\mathbf{CL\emptyset I}^m$ and $\mathbf{CL\emptyset I}^r$. From these we define

$$\Gamma \vdash_{\mathbf{CL\emptyset^m}} A \text{ iff } \Gamma^{\dagger} \vdash_{\mathbf{CL\emptyset I}^m} A^{\ddagger}\,.$$

and similarly for $\mathbf{CL\emptyset^r}$.

Pushed by a referee, I include a (terribly simple) example proof, viz. one for $p^1 \wedge q^2, p^3 \supset r^4 \vdash_{\mathbf{CL\emptyset I}^m} r^5$. Let α abbreviate $(p^1 \wedge q^2) \check{\wedge} \check{\neg}(p^1 \check{\wedge} q^2)$ and let β abbreviate $(p^3 \supset r^4) \check{\wedge} \check{\neg}(p^3 \check{\supset} r^4)$.

1	$p^1 \wedge q^2$	Premise	$\emptyset$
2	$p^1 \check{\wedge} q^2$	1; RC	$\{\alpha\}$
3	p^1	2; RU	$\{\alpha\}$
4	$p^3 \supset r^4$	Premise	$\emptyset$
5	$p^3 \check{\supset} r^4$	4; RC	$\{\beta\}$
6	$p^1 \check{\supset} r^5$	4; RC	$\{\beta, p^{1\cdot 3}, q^{4\cdot 5}\}$
7	r^5	3, 6; RU	$\{\alpha, \beta, p^{1\cdot 3}, q^{4\cdot 5}\}$

As the premise set is normal, no *Dab*-formula is derivable. So all formulas in the proof are finally derived and hence $p \wedge q, p \supset r \vdash_{\mathbf{CL}\emptyset^{\mathsf{m}}} r$. It is more important to comment on the use of adaptive zero logic.

Every symbol, logical or non-logical, has a contingent meaning in $\mathbf{CL\emptyset I}^m$. This means that the meaning of a specific occurrence of a symbol will depend on the premises. Of course, there are presuppositions, laid down by the abnormalities.[29] Thus logical symbols are supposed to have their classical meaning, unless and until proven otherwise. Different occurrences of non-logical terms are supposed to have the *same* meaning, unless and until proven otherwise—the fact that our logic is defined within a language *schema* causes these meanings to be left unspecified.

If applied to abnormal premise sets, $\mathbf{CL\emptyset I}^m$ is a marvellous instrument of analysis. It locates each and every possible explanation of the abnormality—but see the next to last paragraph of this section. The idea here is as explained in Section 4, except that the present analysis is richer: ambiguities in the non-logical terms are also considered. The analysis will give rise to different abnormal but non-trivial theories, obtained by blaming one kind of abnormality rather than another, or by blaming the abnormalities in a certain order (combined adaptive logics).

If applied to a normal premise set, $\mathbf{CL\emptyset I}^m$ delivers the **CL**-consequence set. This is fully the merit of the adaptivity of the logic, because the lower limit logic does not assign any meaning to any symbol. The lower limit logic prescribes literally nothing about any symbol. In $\mathbf{CL}\emptyset^{\mathsf{m}}$, the meaning of symbols is in a sense an empirical matter.

The last statements from the previous paragraph should be qualified. It obviously makes a difference which precise set of abnormalities is selected, because this defines the normal interpretation of the symbols. A first choice that underlies $\mathbf{CL\emptyset I}^m$ is that the upper limit logic is **CL**. Some will want to replace this by a different 'standard of deduction'. Next, the selected abnormalities are the plain ones, bare gluts and bare gaps for the logical symbols and plain ambiguity for the non-logical symbols. For the logical symbols, this may be modified into many variants, including those from combined logics.

By all means, the present results suggest a skeleton for a formal approach to the interpretation of texts. What should be added to the skeleton $\mathbf{CL\emptyset I}^m$

[29]The abnormalities are presumed to be false. By delineating their set, we specify which formulas are considered to be false unless and until proven otherwise.

is basically a set of suitable suppositions about the meaning of non-logical symbols. Next, contextual features should be taken into account. This is not the place to expand upon the topic, but it seemed worth pointing out this possible line of research. The reader will also note the connection with argumentation. Most contributions to that domain are on the non-formal side and close to natural language. $\mathbf{CL\emptyset I}^m$ provides an approach on the formal side and close to formal languages. It seems to me that both approaches may work towards each other—see (the old) [10] for some first ideas on the matter.

7 Conjectures

When (a variant of) zero logic was first discovered, the fascinating properties of this logic were at once noted. Yet, there was a puzzle. For many premise sets, formulas are only derivable at a stage on a rather complex condition. Moreover, often the set of derivable *Dab*-formulas is very large. An effect of this is that many premise sets have only rather complex disjunctions as final consequences. So while zero logic is interesting from a theoretical point of view, it seemed not very suitable for practical purposes. There was a similar puzzle in connection with the manifold of adaptive logics described in the previous sections. How might one justify a choice for one of the logics in connection with a specific premise set?

Those puzzles have meanwhile been resolved. Although zero logic may not be very interesting in itself, it forms an excellent instrument of analysis and thus contributes to the justification of a choice from the manifold of stronger corrective adaptive logics.[30] I stated this explicitly for the logics from Section 3. The logics from Section 4 may be easily involved in the comparison. At the worst they require that a separate column of conditions and marks is added to the proofs. Aside from its relevance to justifying choices of logics, however, zero logic has a further practical use.

Logics like $\mathbf{CL\emptyset}^{\mathsf{m}}$ provide an outstanding environment for applying conjectures of the kind considered in [16, 18]. In one possible approach the $\mathbf{T}$-modality $\Diamond$ is interpreted as plausible. Thus $\Diamond A$ states that A is plausible, $\Diamond\Diamond B$ states that A is plausibly plausible, which is weaker than plausible simpliciter, etc. Let $\Diamond^i$ abbreviate a sequence of i diamonds, whence, for each i, $\Diamond^i C$ states that C has a certain degree of plausibility, which is lower as i is larger. The adaptive logic handling plausibilities will have $\mathbf{T}$ as its lower limit; its set of abnormalities comprises the formulas of the form $\Diamond^i A \check{\wedge} \check{\neg} A$ in which $A \in \mathcal{W}_s$ is a primitive (or atomic) formula. Note that $\Diamond^i A \vdash_{\mathbf{T}} A \check{\vee} (\Diamond^i A \check{\wedge} \check{\neg} A)$. So in the adaptive proofs A is derivable from $\Diamond^i A$ on the condition $\{\Diamond^i A \check{\wedge} \check{\neg} A\}$. This adaptive logic takes care that the more plausible formulas are (provisionally and defeasibly) turned into truths before the less plausible ones.

One may have definite views on the plausibility of certain formulas from the outset. One may also form (or modify) such views as one studies the premise set (without the formulas expressing plausibility). Given a premise set, one first writes out a $\mathbf{CL\emptyset I}^m$-proof from it. In doing so, the attention

[30] Justification does not require uniqueness; several alternatives may be equally justified.

should be focussed on presumably interesting potential consequences. The $\mathbf{CL\emptyset I}^m$-proof will reveal the connection between potential conclusions and sets of abnormalities. In view of one's preferences for deriving certain conclusions, the connected abnormalities may be studied further. The relevant questions will be which conclusions depend on the abnormalities (have the abnormalities in their condition) and in which minimal *Dab*-formulas the abnormalities occur. The procedures described in [15, 47] will prove useful in this respect. Once sufficient insight is gained, some abnormalities (or their negations) may be stated to have a certain plausibility. These new premises will have defeasible effects.

The choices may be organized around an enrichment of the lower limit logic. This effect is obtained when the logical form that characterizes a certain set of abnormalities, say disjunction gaps, is declared logically impossible. This was described in previous sections. The choices may also be made in a piecemeal way, as is typical for the approach in terms of conjectures. Both ways of proceeding may also be combined.

Actually, the combination itself may be looked upon in two different ways. One may strengthen the logic by ruling out several kinds of gluts and gaps or by ruling out ambiguities altogether, and next introduce conjectures in a defeasible way. However, one may also proceed in a fully defeasible way. For example, instead of ruling out implication gluts by introducing the axiom schema $(A \supset B) \check{\supset} (A \check{\supset} B)$ in the lower limit logic, one may extend the premises with the set $\{\Diamond^i\check{\forall}((A \supset B) \check{\supset} (A \check{\supset} B)) \mid A, B \in \mathcal{F}_s\}$ for a chosen i—the members of the set will function as defeasible new premises.

Obviously, what one obtains at best in the end is a minimally abnormal interpretation of the premise set. The usual next step is the transition to a normal premise set. Often a *partial* execution of the first step will provide sufficient insights to move on to the second step. In the documented cases from the history of the sciences, whether mathematical or empirical, replacements for inconsistent theories were most often obtained by a few well-directed changes. Perhaps an ambiguity was resolved and a non-logical axiom was restricted.

8 Strength of Paraconsistency and Ambiguity

I have argued that each of the logics considered in this paper leads, with respect to some premise sets, to a different minimally abnormal 'interpretation'. Obviously, most of the logics trivialize some premise sets that have no **CL**-models. Consider all logics from Sections 3 and 4. Whether the logic is adaptive or not, the consequence set of $\{p, \neg p\}$ is trivial unless negation is paraconsistent. In this sense paraconsistency has a special status: strictly paraconsistent logics—those for which there is no A such that $A, \neg A \vdash B$ for all B—have models for *all* subsets of $\mathcal{W}_s$.

Ambiguity logics share the strength of paraconsistent logics. Every $\Gamma \subseteq \mathcal{W}_s$, even if it has no **CL**-models, has **CLA**-models.[31] Some paraconsistent logics may even be defined in terms of ambiguity logics—I have shown in [14] that this

[31] Obviously not every $\Gamma \subseteq \mathcal{W}^I_s$ has **CLI**-models.

holds for **LP** from [38, 40] and it is not impossible that a similar result holds for all paraconsistent logics. Note that this is a technical point. A philosophical point is that, even if all paraconsistent logics can be characterized in terms of ambiguity logics, the interpretation of both types of logics is nevertheless different. The question as to the precise meaning of negation should not be confused with the question whether ambiguities occur in non-logical symbols. In this respect, the philosophical tenet of David Lewis in [37] is mistaken. That a given text (or premise set) may be interpreted both ways is altogether a different matter.

What should be concluded from the strength of paraconsistency and ambiguity? Not much as I see it. These approaches offer a road to a maximally non-trivial interpretation of every premise set. However, if another logic provides also such a road for a given premise set, the latter road may be just as sensible. All the logic needs to do is offer a way for handling a theory T once it turned out to be **CL**-trivial. Which maximally non-trivial interpretation of T will turn out most interesting will always depend on non-logical considerations. As early as 1964, Nicholas Rescher remarked in [41, p. 37]: "And while the *recognition* of ambiguity does fall within the province of logic, its *resolution* is inevitably an extralogical matter." This holds for every cause of triviality.

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