Proof Theories for Superpositions of Adaptive Logics

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Abstract

The standard format for adaptive logics offers a generic and unifying formal framework for defeasible reasoning forms. One of its main distinguishing features is a dynamic proof theory by means of which it is able to explicate actual reasoning.

In many applications it has proven very useful to superpose sequences of adaptive logics, such that each logic treats the consequence set of its predecessor as premise set. Although attempts have been made to define dynamic proof theories for some of the resulting logics, no generic proof theory is available yet. Moreover, the existing proof theories for concrete superpositions are suboptimal in various respects: the derivability relations characterized by these proposals are often not adequate with respect to the consequence relation of the superposed adaptive logics and in some cases they even trivialize premise sets. An adequate and generic proof theory is needed in order to meet the requirement of explicating defeasible reasoning in terms of superpositions of adaptive logics.

This paper presents two generic proof theories for superpositions of adaptive logics in standard format. By means of simple examples, the basic ideas behind these proof theories are illustrated and it is shown how the older proposals are inadequate.

1 Introduction

Adaptive Logics (henceforth, ALs) have been suggested as a generic and unifying framework to formally explicate defeasible reasoning. For this purpose a standard format for ALs has been developed by Diderik Batens (see [5]) which comes with a rich meta-theory. One of the most distinguishing features compared to other formal frameworks for defeasible reasoning is the dynamic proof theory of ALs. In adaptive proofs some rules are applied in a defeasible way, i.e., they are applied conditionally. An inference is considered to be valid as long as there is no reason to suppose that its condition is not fulfilled. The standard format of ALs gives a formally precise account of this idea (see Section 2).[1]

There are two types of dynamics that come with defeasible reasoning. First, there is the *external dynamics* ([3], also known under the name *synchronic defeasibility* [15, 16]). It occurs whenever new premises necessitate the retraction of previous inferences. Most logics for defeasible reasoning forms take care of the external dynamic by means of the nonmonotonicity of their consequence set.

Another, more neglected form of dynamics is the *internal dynamics* ([3], also known under the name *diachronic defeasibility* [15, 16]). It occurs when new insights gained by a logical analysis of the premises necessitate the retraction of certain inferences. Note that here no new premises enter the picture.

The distinguishing feature of ALs is that by means of their dynamic proof theory they explicate the actual reasoning processes that give rise to the two types of dynamics. Due to the introduction of a new premise at a certain stage of an adaptive proof, some inferences on previous lines may be invalidated, others may be validated. Hence, adaptive proofs explicate the external dynamics of defeasible reasoning. Similarly, by analyzing premises in an adaptive proof we may derive formulas that cause that some previous conclusions are retracted, or others are added again. This way, adaptive proofs also explicate the internal dynamics of defeasible reasoning.

In order to model defeasible reasoning forms, their combinations and/or defeasible reasoning with preferences and priorities, it is often very useful to combine adaptive logics. Examples of such combinations can be found in [8, 6, 4, 14, 26, 24, 27, 2, 22, 12, 17]. The increasing plurality of these systems makes it very useful to have a unifying, generic meta-theoretic account of them. Hence, various recent publications offer results along these lines: in [21] the authors present a generalized canonical form for prioritized ALs, in [19] the author presents a new way of combining adaptive logics by means of merging their consequence sets, in [23] the authors compare various ways of combining ALs with the prioritized format from [21].

In this paper we focus on the most frequent way ALs are combined, i.e. on superpositions of ALs. Given a premise set Γ, an AL $\mathbf{AL_2}$ is superposed on another AL $\mathbf{AL_1}$ in case $\mathbf{AL_2}$ is applied to the consequence set $Cn_{\mathbf{AL_1}}(\Gamma)$ of $\mathbf{AL_1}$. Given a sequence of adaptive logics $\langle \mathbf{AL_1}, \mathbf{AL_2}, \ldots, \mathbf{AL_n} \rangle$, the consequence set of the associated superposition of ALs $\mathbf{SAL}$ is given by

$$Cn_{\mathbf{SAL}}(\Gamma) = Cn_{\mathbf{AL_n}}(Cn_{\mathbf{AL_{n-1}}}(\ldots Cn_{\mathbf{AL_2}}(Cn_{\mathbf{AL_1}}(\Gamma))\ldots))$$

In Section 3 we generalize this characterization to the infinite case.

This way we obtain the consequence set of a combined AL. Although the standard format equips each $\mathbf{AL_i}$ with a dynamic proof theory, we have no proof theory for $\mathbf{SAL}$. Proof theories have been developed for concrete superpositions of ALs. However, neither of the proposals is apt for the generic case where each $\mathbf{AL_i}$ is any adaptive logic in standard format and where no restrictions on the

[1]We only give an informal account of ALs and their proof theory in this introduction. The reader finds a formally precise formulation in Section 2.

premise sets are made. This is a severe shortcoming, since the fact that ALs explicate actual reasoning by means of their dynamic proof theory is, after all, one of their most salient features.

This paper fills this gap by presenting two generic proof theories for superpositions of ALs. As we will demonstrate, these proof theories are very similar to the proof theory of ALs in standard format. This has the advantage that users familiar with ALs will easily adjust to the superposed case, and, more importantly, that all the attractive design features of usual adaptive proofs carry over to the new formats.

In Section 2 we will introduce the reader to the standard format of ALs and their dynamic proof format while in Section 3 we present superpositions of ALs. In Sections 4 and 5 we explicate our generic proof theories for superpositions of ALs. We discuss a simplification of both formats that is apt for a subclass of superpositions of ALs in Section 6. Finally, in Section 7 we conclude by pointing to related future research. In the appendix we present the meta-theoretical proofs for the adequacy of our proof theories.

2 Adaptive Logics and Their Standard Format

ALs in standard format are characterized by means of three elements. We first state them and then explicate them.

1. A *lower limit logic*: a reflexive, transitive, and monotonic logic that has a characteristic semantics;[2]
2. A *set of abnormalities* Ω: a set of formulas that is characterized by a logical form F; or a union of such sets;
3. An *adaptive strategy*: the Reliability or the Minimal Abnormality strategy.

The strategy is usually indicated by a superscript: we write $\mathbf{AL}^{\mathbf{r}}$ to indicate the Reliability strategy, and $\mathbf{AL}^{\mathbf{m}}$ to indicate the Minimal Abnormality strategy.

The central idea behind ALs is to interpret a given set of premises "as normally as possible". The set of abnormalities and the adaptive strategy make precise what "normally" and "as possible" mean in this phrase. But before we elaborate more on this, we introduce some notation.

Let $\mathcal{L}$ be the language of the lower limit logic and $\mathcal{W}$ the set of all well-formed formulas in $\mathcal{L}$. In the remainder we presuppose that premise sets are formulated in $\mathcal{L}$. In ALs $\mathcal{L}$ is enhanced by "checked" classical connectives $\check{\neg}, \check{\vee}, \check{\wedge}, \ldots$ and in case of predicate logic also $\check{\exists}$ and $\check{\forall}$. These additional connectives are used in order to express statements concerning normality and may hence also be part of the logical form F that characterizes the set of abnormalities Ω. The set of well-formed formulas is obtained by closing $\mathcal{W}$ under the checked symbols.[3] Let

[2] A logic $\mathbf{L}$ is reflexive iff for all premise sets Γ, $\Gamma \subseteq Cn_{\mathbf{L}}(\Gamma)$; $\mathbf{L}$ is transitive iff for all premise sets Γ, whenever $\Gamma' \subseteq Cn_{\mathbf{L}}(\Gamma)$ then $Cn_{\mathbf{L}}(\Gamma') \subseteq Cn_{\mathbf{L}}(\Gamma)$; $\mathbf{L}$ is monotonic iff for all premise sets Γ, $Cn_{\mathbf{L}}(\Gamma) \subseteq Cn_{\mathbf{L}}(\Gamma \cup \Gamma')$.

[3] This is done in the usual way respecting the arity of the checked connectives: e.g. where $A, B \in \mathcal{W}$, both $A \check{\vee} B$ and $\check{\neg} A$ are in the obtained set of well-formed formulas. Note that checked connectives do not occur within the scope of non-checked connectives.

in the remainder **LLL** be the lower limit logic enriched by the classical checked connectives.[4] As we will explicate below, (classical) disjunctions of abnormalities play a central role in ALs. Thus, it is useful to introduce a notational convention: We call a checked disjunction of abnormalities, $\check{\bigvee}\Delta$ where $\Delta \subseteq \Omega$, a Dab-formula (**D**isjunction of **ab**normalities) and write $\mathsf{Dab}(\Delta)$.

Let us proceed with the adaptive proof theory. The idea is to take all the inference rules of the lower limit logic for granted, but to additionally allow for defeasible applications of some rules. Defeasible inferences in adaptive proofs are conditional. Hence, the usual way lines in proofs are presented –by a line number, a formula, and a justification– is enriched by a fourth element: a condition. A condition in turn is a set of abnormalities.

Suppose some formula A is derived on the condition $\{B_1, B_2, \ldots, B_n\} \subseteq \Omega$. The intended reading is that A is derived under the assumption that all the abnormalities $B_1, \ldots, B_n$ are false.

Adaptive proofs are characterized by three generic rules and marking conditions. Let us first discuss the generic rules. In what follows we skip the line numbers and justification of lines.

PREM If $A \in \Gamma$:
$$\begin{array}{cc} \vdots & \vdots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:
$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \,\check{\vee}\, \mathsf{Dab}(\Theta)$:
$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

By means of PREM any premise may be introduced on the empty condition. Of course, we do not need any defeasible assumptions in order to state premises. The unconditional rule RU makes it possible to apply any inference rule of **LLL** in an adaptive proof. Note that these rules may also be applied to lines that were derived under defeasible assumptions, i.e. where $\Delta_i \neq \emptyset$. The idea is that all the defeasible conditions under which the A_i's were derived carry forward to the line at which B is derived. By means of PREM and RU, ALs inherit all the derivative power of **LLL**: any **LLL**-proof can be rephrased as an **AL**-proof just by adding the empty condition in the fourth column and by replacing the respective **LLL**-rule by RU.

[4] In some papers this logic is denoted by $\mathbf{LLL}^+$, in others **LLL** ambiguously denotes both, the lower limit logic and the enriched lower limit logic. Since in this paper we will not make further references to the lower limit logic without the checked symbols, we drop the $^+$ from the notation.

What makes adaptive proofs distinctive is the third rule. It allows for defeasible inferences. Suppose we can derive $B \check{\vee} \mathsf{Dab}(\Theta)$ by means of **LLL**, i.e. that either B is the case or some of the abnormalities in Θ. Then the conditional rule RC allows us to derive B under the assumption that none of the abnormalities in Θ is true. This is formally realized by stating Θ in the fourth column for conditions. Similarly as for RU, in case some of the lines that are used for the inference step are conditional inferences, we carry forward their conditions as well.

Obviously it is not enough to just be able to derive formulas conditionally. In order to model defeasible reasoning we need to formally explicate the retraction of inferences as well. In adaptive proofs this is realized by means of markings. A line is marked in case the assumption under which it was derived is considered as "unsafe". This idea is made precise by two marking definitions: one for the Reliability strategy and one for the Minimal Abnormality strategy.

The marking is determined by the Dab-formulas that have been derived at a given stage of the proof. A *stage* of a proof is a list of consecutive lines such that for each line in the list the lines called upon in its justification are also contained in the list. A proof at stage s is *extended* to some stage s' by inserting or appending lines.[5]

$\mathsf{Dab}(\Delta)$ is a *minimal* Dab-formula at stage s of the proof in case (a) it is derived on the empty condition at stage s and (b) there is no $\Delta' \subset \Delta$ such that $\mathsf{Dab}(\Delta')$ is derived on the empty condition at stage s. Where $\mathsf{Dab}(\Delta_1), \mathsf{Dab}(\Delta_2), \ldots$ are all the minimal Dab-formulas derived at stage s, $\Sigma_s(\Gamma) =_{\text{df}} \{\Delta_1, \Delta_2, \ldots\}$. The set of *unreliable formulas* at stage s, $U_s(\Gamma) =_{\text{df}} \bigcup \Sigma_s(\Gamma)$, contains all the members of the minimal Dab-formulas.

In case $\mathsf{Dab}(\Delta)$ is a minimal Dab-formula at stage s of the proof, we know according to the best insights available at stage s that some member of Δ has to be true. Since we don't know which member, all the members of Δ are labeled unreliable. The marking according to the Reliability strategy makes sure that any line whose condition contains an unreliable member is marked:

Definition 1 (Marking for Reliability)**.** A line l with condition Δ is marked at stage s, iff $\Delta \cap U_s(\Gamma) \neq \emptyset$.

Example 1. We take as a lower limit logic the modal logic **K**.[6] Moreover, we enrich **K** by the "checked" connectives as discussed above.[7] Read $\lozenge A$ as "A is plausible". The idea is to defeasibly infer from "A is plausible" that A is the case. This can be achieved by means of taking $\lozenge A \wedge \neg A$ as the logical form for

[5] In case a line is inserted the following lines have to be renumbered accordingly. The reason why we do not restrict the notion of extending proofs to the appending of lines is that in order to define the derivability relation of ALs we need to be able to talk about extensions of infinite proofs.

[6] Recall that **K** is axiomatized by all the axiom schemes of classical propositional logic, the axiom scheme (K), $\vdash \Box(A \supset B) \supset (\Box A \supset \Box B)$, the rule (NEC), if $\vdash A$ then $\vdash \Box A$, and Modus Ponens.

[7] For the sake of simplicity we will denote this logic also by **K**.

our set of abnormalities where A is a literal.[8] Note that $\Diamond A \vdash_{\mathbf{K}} A \check{\vee} (\Diamond A \wedge \neg A)$. Hence, by the generic rule RC we may derive A on the condition $\{\Diamond A \wedge \neg A\}$ from $\Diamond A$.

Now, let us have a look at the following proof from the premises $\{\Diamond p, \Diamond q, \Diamond r, p \supset o, q \supset o, \neg p \vee \neg q\}$ where p, q, r and o are sentential letters.

	1	$\Diamond p$	PREM	$\emptyset$
	2	$\Diamond q$	PREM	$\emptyset$
	3	$\Diamond r$	PREM	$\emptyset$
	4	$p \supset o$	PREM	$\emptyset$
	5	$q \supset o$	PREM	$\emptyset$
✓	6	r	3; RC	$\{\Diamond r \wedge \neg r\}$
✓	7	p	1; RC	$\{\Diamond p \wedge \neg p\}$
✓	8	q	2; RC	$\{\Diamond q \wedge \neg q\}$
✓	9	o	7; RU	$\{\Diamond p \wedge \neg p\}$
✓	10	o	8; RU	$\{\Diamond q \wedge \neg q\}$
	11	$\neg p \vee \neg q$	PREM	$\emptyset$
	12	$(\Diamond p \wedge \neg p) \check{\vee} (\Diamond q \wedge \neg q)$	1,2,11; RU	$\emptyset$

According to the Reliability strategy, line 6 is unmarked since the abnormality in its condition is not part of any minimal Dab-formula and hence not unreliable: $\Diamond r \wedge \neg r \notin U_{12}(\Gamma)$. However, lines 7–10 are marked since each of these lines contains one of the two unreliable formulas, either $\Diamond p \wedge \neg p$ or $\Diamond q \wedge \neg q$. After all, by means of line 12 we know that either $\Diamond p \wedge \neg p$ or $\Diamond q \wedge \neg q$ is the case. Interestingly, o is derived on both conditions (at lines 9 and 10).

The idea behind the Minimal Abnormality strategy is to give a more rigorous account of "interpreting the premises as normally as possible" compared to the Reliability strategy. As a consequence, the marking for Minimal Abnormality is less skeptical in nature. Let us demonstrate this by means of Example 1. Line 12 indeed indicates that one of the two abnormalities is the case and we don't know which one. However, we may still defeasibly assume that one of the two abnormalities is false. This is in contrast to the Reliability strategy which treats both abnormalities as unreliable. As a consequence, if a formula is derivable on both conditions, then the inference is considered safe. Hence, according to the Minimal Abnormality strategy, lines 9 and 10 are not marked while lines 7 and 8 are marked. Take for instance p: it is only derivable on one of the two abnormalities, namely $\Diamond p \wedge \neg p$. This abnormality may very well be the one

[8] A literal is a sentential letter or a negated sentential letter. The reader may wonder why we restrict the logical form that characterizes our abnormalities to literals. The reason is that otherwise we would obtain a so-called *flip-flop AL*. In flip-flop ALs any abnormality can be involved in a minimal Dab-formula whenever some Dab-formula is derivable by means of the lower limit logic: a rather unwanted property in most applications. Take for instance the premise set $\Gamma_{\mathsf{ff}} = \{\Diamond p, \Diamond q, \neg p\}$. Obviously the abnormality $\Diamond p \wedge \neg p$ is $\mathbf{K}$-derivable. Intuitively we would derive q on the condition $\{\Diamond q \wedge \neg q\}$ from $\Diamond q$ at some line l and expect this line not to get marked in any extension of the proof. However, we can also derive $A = \Diamond q \wedge \neg q \check{\vee} (\Diamond(p \vee \neg q) \wedge \neg(p \vee \neg q))$ from Γ_{ff} by means of $\mathbf{K}$. Hence, would we not restrict the logical form of our abnormalities to literals, we would have to mark l as soon as we derived A and there would be no way to remove the marking in a further extension of the proof.

which is true. Hence, line 7 is marked.

In order to spell out the marking definition for Minimal Abnormality we introduce some more notions. A *choice set* of a set of sets $\{\Delta_1, \Delta_2, \ldots\}$ is a set that contains a member out of each Δ_i. A choice set φ is *minimal* in case there is no choice set φ' of $\{\Delta_1, \Delta_2, \ldots\}$ for which $\varphi' \subset \varphi$. We denote the set of minimal choice sets of $\Sigma_s(\Gamma)$ by $\Phi_s(\Gamma)$. In our example $\Phi_{12}(\Gamma) = \{\{\lozenge p \wedge \neg p\}, \{\lozenge q \wedge \neg q\}\}$. The marking for Minimal Abnormality is defined as follows:

Definition 2 (Marking for Minimal Abnormality)**.** A line l with formula A and condition Δ is marked at stage s iff

(i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\Delta \cap \varphi \neq \emptyset$, or
(ii) for a $\varphi \in \Phi_s(\Gamma)$: there is no line l' at stage s with formula A and condition Θ such that $\Theta \cap \varphi = \emptyset$.

One way to interpret the marking definition is in terms of an argumentation game. Suppose the proponent derives a formula A on a line with condition Δ at stage s. Each minimal choice sets $\varphi \in \Phi_s(\Gamma)$ represents a minimally abnormal interpretation of the Dab-formulas derived at stage s: each $B \in \varphi$ is true in this interpretation while each $B \in \Omega \setminus \varphi$ is false. Each minimal choice set φ thus represents a potential counter-argument against the defeasible assumption used by our proponent in order to derive A (namely that all members of Δ are false). φ is a counter-argument in case the defeasible assumption, i.e. the condition of line l, contains elements of φ. In this case the assumption of line l is not valid in the interpretation offered by φ.

In case there is no minimally abnormal interpretation φ in which the assumption holds (see point (i)), the proponent cannot defend herself and her inference to A is retracted in terms of being marked. But suppose there is a φ such that $\Delta \cap \varphi = \emptyset$. In this case there is at least one minimally abnormal interpretation in which the assumption of our proponent holds. But what about minimally abnormal interpretations in which the assumption does not hold, i.e. φ's for which $\varphi \cap \Delta \neq \emptyset$? In this case the proponent has to offer for each such φ another argument whose assumption is valid in φ (see point (ii)). If she is able to do so, i.e. if she is able to defend herself against all counter-arguments, then her argument is valid and hence line l is not marked at stage s.

Marking is a dynamic enterprise. Suppose for instance that the premise set of Example 1 is enriched by $\lozenge q \wedge \neg q$ and that at line 13 of the proof above $\lozenge q \wedge \neg q$ is derived on the empty condition. In that case the Dab-formula at line 12 would not be anymore minimal at that stage. Also, the only minimal choice set would be $\{\lozenge q \wedge \neg q\}$. Hence, according to both, the Reliability and the Minimal Abnormality strategy, line 7 would be unmarked, while line 8 remains marked.

Altogether, markings come and go: previously marked lines may be unmarked at a later stage and previously unmarked lines may be marked. In order to define the consequence set of ALs we hence need a more stable notion than derivability at a stage.

Definition 3. A is *finally derived* at a line l of a finite stage s in an **AL**-proof, iff (i) line l is unmarked at stage s, and (ii) every extension of the proof in which l is marked can be further extended in such a way that l is unmarked.

A is *finally derivable* from Γ in **AL** iff there is a proof from Γ in which A is finally derived.

We write $\Gamma \vdash_{\mathbf{AL}} A$ in case A is finally derivable from Γ and define the consequence set of **AL** by: $Cn_{\mathbf{AL}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{AL}} A\}$.

One way of looking at final derivability is by means of a two-person-game (see [7]). The proponent conditionally derives A on a line l by means of a finite argument. Now the opponent has the opportunity to extend the proof of the proponent in such a way that line l is marked. If the proponent can extend the proof further in such a way that l is unmarked again, she wins. A is finally derived at l iff there is a winning strategy for the proponent, i.e., whatever the opponent replies, she always has a way of winning the game.[9]

Note also that ALs are equipped with a semantics that is based on the semantics of the lower limit logic. The idea for Minimal Abnormality is to select all so-called minimally abnormal models from the **LLL**-models of a premise set Γ, i.e., models that validate a minimal set of abnormalities. For Reliability **LLL**-models are selected that verify only abnormalities that are also verified by some of the minimally abnormal models (see [5, 25, 20]).

The standard format of ALs comes with a rich meta-theory. For a more detailed overview see e.g. [5]. Here we only list some important properties.

Theorem 1. $\Gamma \subseteq Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL^r}}(\Gamma) \subseteq Cn_{\mathbf{AL^m}}(\Gamma)$.

Moreover, the consequence set of ALs in standard format is a fixed point ($Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(Cn_{\mathbf{AL}}(\Gamma))$) and satisfies the cumulativity property introduced in [13] ($Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$ for all $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$).

3 Superpositions of Adaptive Logics

3.1 The consequence relation characterized by SAL

In this section we characterize superpositions of ALs in a generic way. Let $\langle \mathbf{AL_i^{x_i}} \rangle_I$ be an sequence of adaptive logics $\mathbf{AL_i^{x_i}} = \langle \mathbf{LLL}, \Omega_i, \mathbf{x_i} \rangle$ that all share the same lower limit logic **LLL**. The index set I is either a initial set of natural numbers $\{1, \ldots, n\}$ or $I = \mathbb{N}$. Each $\mathbf{AL_i^{x_i}}$ comes with a strategy, either Minimal Abnormality ($\mathbf{x_i} = \mathbf{m}$) or Reliability ($\mathbf{x_i} = \mathbf{r}$). For each $i \in I$ we can define the consequence set that corresponds to the superposition of all logics up to $\mathbf{AL_i^{x_i}}$:

$$Cn_{\mathbf{SAL_i}}(\Gamma) =_{\mathrm{df}} Cn_{\mathbf{AL_i^{x_i}}}(\ldots(Cn_{\mathbf{AL_2^{x_2}}}(Cn_{\mathbf{AL_1^{x_1}}}(\Gamma))))$$

[9] One of the subtleties of the adaptive proof theory is that both extensions of the proof, the one of the opponent and the one of the proponent, may be infinite (see [5] for a more detailed discussion). For Reliability the definition can be restricted to finite extensions.

In case I is finite we define $Cn_{\mathbf{SAL}}(\Gamma) =_{\rm df} Cn_{\mathbf{SAL_n}}(\Gamma)$, otherwise:

$$Cn_{\mathbf{SAL}}(\Gamma) =_{\rm df} \liminf_{i\to\infty} Cn_{\mathbf{SAL_i}}(\Gamma) = \limsup_{i\to\infty} Cn_{\mathbf{SAL_i}}(\Gamma)$$

Note that the sequence $\langle Cn_{\mathbf{SAL_i}}(\Gamma)\rangle_{i\in\mathbb{N}}$ converges to its limes inferior resp. to its limes superior due to the fact that the sequence is monotonic (see Theorem 1).

Example 2. In what follows we generalize the idea presented in Example 1 in such a way that we allow for different degrees of plausibility. We will use the resulting class of logics for demonstrative purposes in the remainder of the paper.

Recall that $\Diamond A$ was read as "A is plausible". The idea is now to indicate the degree of plausibility of a formula by means of sequences of $\Diamond$'s, i.e., the degree of plausibility of A is inversely proportional to the number of $\Diamond$'s that proceed it. We write $\Diamond^k$ for a sequence of k $\Diamond$'s.

Each degree i of plausibility is associated with an AL $\mathbf{Ki^x}$ where $\mathbf{x} \in \{\mathbf{r}, \mathbf{m}\}$ that has $\mathbf{K}$ as its lower limit logic and $\Omega_i^{\mathbf{K}} = \{\Diamond^i A \wedge \neg A \mid A \text{ is a literal}\}$ as its set of abnormalities. Given an index set I we can define logics by means of sequences $\langle \mathbf{Ki^{x_i}}\rangle_{i\in I}$ where $\mathbf{x_i} \in \{\mathbf{r}, \mathbf{m}\}$.

Let us take a look at the simple example $\mathbf{SK2^r}$ which is the superposition of $\mathbf{K2^r}$ on $\mathbf{K1^r}$. Our premise set is $\Gamma_{\mathsf{s}} = \{\Diamond p, \Diamond^2 q, \Diamond^2 r, \neg p \vee \neg q\}$. Note that although we can derive the disjunction of abnormalities $(\Diamond p \wedge \neg p) \check{\vee} (\Diamond^2 q \wedge \neg q)$ in $\mathbf{K}$, there are no Dab-formulas derivable with respect to $\mathbf{K1^r}$. Hence, since $\Diamond p \in \Gamma_{\mathsf{s}}$, $\Gamma_{\mathsf{s}} \vdash_{\mathbf{K1^r}} p$. Hence also $\Gamma_{\mathsf{s}} \vdash_{\mathbf{K1^r}} \neg q$ since we also have $\neg p \vee \neg q \in \Gamma_{\mathsf{s}}$. The only Dab-formula in $Cn_{\mathbf{K1^r}}(\Gamma_{\mathsf{s}})$ with respect to $\mathbf{K2^r}$ is $\Diamond^2 q \wedge \neg q$. Hence, $Cn_{\mathbf{K1^r}}(\Gamma_{\mathsf{s}}) \vdash_{\mathbf{K2^r}} r$ since $\Diamond^2 r \in \Gamma_{\mathsf{s}}$.

Most superpositions of ALs in the literature (see [14, 26, 24, 27, 2, 12, 17]) are defined in such a way that the following holds:

(†) for every $i, j \in I$ such that $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$.

If (†) holds, then we can slightly simplify the two proof theories that are presented below (see Section 6). However, for the sake of generality, we will not assume (†) to hold when defining our generic proof theories for superpositions of ALs.

There are also two more concrete motivations to include other kinds of superpositions. First of all, in some more recent papers [8, 6, 4], logics are developed for which the following holds:

(‡) Where $i, i+1 \in I$: $\Omega_i \subseteq \Omega_{i+1}$.

As shown in [23, 20], such logics have several interesting meta-theoretic properties. Given some additional restrictions, they are cautiously monotonic and cumulatively transitive, just like ALs in standard format. This also means that they are a fixed point, and have the reciprocity property – see again [23, 20] for the details and related results. There it was also shown that these properties fail for **SAL** in the more general case.

Secondly, as shown in [23], logics in the format of *lexicographic adaptive logics* from [21] that have the Minimal Abnormality strategy are often equivalent to a specific class of superpositions for which (‡) holds. Hence, the proof theories for **SAL** presented in the current section can serve as proof theories for those lexicographic ALs as well.

Before we present a proof theory for **SAL**, we list some meta-theoretic properties which are proven in the appendix.

Theorem 2. *For all* $i, i+1 \in I$, $Cn_{\mathbf{SAL_i}}(\Gamma) \subseteq Cn_{\mathbf{SAL_{i+1}}}(\Gamma) \subseteq Cn_{\mathbf{SAL}}(\Gamma)$.

Theorem 3 (**LLL**-closure of **SAL**)**.** $Cn_{\mathbf{SAL}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$.

Corollary 1. *If* $Cn_{\mathbf{LLL}}(\Gamma) = Cn_{\mathbf{LLL}}(\Gamma')$ *then* $Cn_{\mathbf{SAL}}(\Gamma) = Cn_{\mathbf{SAL}}(\Gamma')$.

Theorem 4. *If* $Cn_{\mathbf{LLL}}(\Gamma)$ *not trivial, then* $Cn_{\mathbf{SAL}}(\Gamma)$ *is not trivial.*

3.2 Previous proposals for proof theories for SAL

In this section we briefly take a look at previous proposals for defining adequate proof theories for **SAL**. By "adequate" we mean the following property: $A \in Cn_{\mathbf{SAL}}(\Gamma)$ iff A is (finally) derivable in an **SAL**-proof.

In [1], Diderik Batens proposed an attractive proof theory for a specific class of superpositions of ALs $\langle \mathbf{AL_i^{x_i}} \rangle_{i \in I}$, i.e. superpositions where all $\mathbf{AL_i}$'s have the same adaptive strategy (i.e., either Reliability or Minimal Abnormality) and that satisfy (†). This proof theory is very similar to that of flat ALs: the same generic rules are used, with a conditional rule that allows one to push abnormalities to the condition; a marking definition determines which lines are in and which are out at a given stage s of the proof; the notions of derivability at a stage and final derivability are exactly the same as for flat ALs.

The proof theory from [1] has a certain intuitive appeal. Whether or not a line is marked is defined recursively. If the user of a logic wants to find out whether or not a line is marked or not at stage s, she can follow a sequential marking procedure. Roughly speaking, such a procedure goes as follows: at a stage s, mark lines according to a first marking criterion. This criterion solely depends on the lines that have been derived on the empty condition. In view of the lines that remain unmarked after this first step, we obtain a new marking criterion, which then allows us to determine a third marking criterion, etc. Lines that remain unmarked at the end of the whole procedure are said to be unmarked at stage s.

For superpositions where all logics in the sequence have the Reliability strategy and that obey restriction (†) this proof theory is adequate. However, in the case in which the logics have the Minimal Abnormality strategy adequacy with respect to the consequence relation fails and these proof theories even trivialize some (fairly simple) premise sets– we will return to this point in Section 4.4.

Christian Straßer made a different attempt to characterize some sequential superpositions by a dynamic proof theory in his [25]. On the one hand, Straßer broadens the scope to include superpositions of ALs with mixed strategies. On the other hand, Straßer restricts himself again to logics that obey (†), and only considers the case in which $I = \{1, \ldots, n\}$. Again, for all superpositions in which all logics have the Reliability strategy, this proof theory is adequate, whereas for the Minimal Abnormality-variants and those with mixed strategies, Straßer's proposal faces the same problem as Batens' older proposal.

* * *

In the following sections we will define a proof theory for **SAL** that is characterized in a very generic manner:

1. the index set I is an arbitrary (possibly infinite) set
2. we allow for mixed strategies such that some logics $\mathbf{AL_i^{x_i}}$ may be characterized by the Reliability strategy while other logics $\mathbf{AL_j^{x_j}}$ may be characterized by the Minimal Abnormality strategy
3. there is no restriction on the sequence of sets of abnormalities $\langle \Omega_i \rangle_I$, i.e., some Ω_i and Ω_j may intersect, others may be distinct.

4 A proof theory for superpositions of ALs

4.1 The proof format

The proof format of sequential superpositions which we present here is nearly identical to the one of flat ALs. Again, a line is a quadruple consisting of a line number, a formula, a justification and a condition. The only difference concerns the last element. A condition is not just a set of abnormalities, but instead a *sequence* of sets of abnormalities $\langle \Delta_i \rangle_{i \in I}$ where each Δ_i is a subset of Ω_i. In the following, we write $\mathbf{\Delta}$ for $\langle \Delta_i \rangle_{i \in I}$, $\boldsymbol{\emptyset}$ for the sequence $\langle \emptyset, \emptyset, \ldots \rangle$,[10] $\bigcup \mathbf{\Delta}$ for $\bigcup_{i \in I} \Delta_i$ and $\mathsf{Dab}(\mathbf{\Delta})$ for $\mathsf{Dab}(\bigcup_{i \in I} \Delta_i)$ where $\mathbf{\Delta} = \langle \Delta_1, \ldots, \Delta_m, \emptyset, \ldots \rangle$ in case I is infinite.

Suppose we have the following line in a proof:[11]

$$l \quad A \qquad\qquad k_1, \ldots, k_n; R \quad \langle \Delta_1, \Delta_2, \emptyset, \ldots \rangle$$

where $\Delta_1 \neq \emptyset \neq \Delta_2$. Suppose moreover that line l is unmarked. The idea is that A is derived on the assumption that no abnormality in $\Delta_1 \cup \Delta_2$ is true. Hence, we make use of the defeasible reasoning forms represented by both $\mathbf{AL_1}$ and $\mathbf{AL_2}$. Moreover, in case A is finally derived at line l (see Definition 6 below), then A is a consequence of the superposition of $\mathbf{AL_2}$ on $\mathbf{AL_1}$, since no defeasible assumptions were made that correspond to ALs higher in the sequence of **SAL**.

In order to realize this idea we will again make use of three generic rules and marking definitions.

Similar as in flat adaptive proofs we need to merge the conditions of two or more lines. In the flat case we could just take the union of the respective sets of abnormalities. This idea can easily be generalized to the sequential case in the following way: $\mathbf{\Delta} \Cup \mathbf{\Theta} =_{\mathsf{df}} \langle \Delta_i \cup \Theta_i \rangle_{i \in I}$. For instance,

$$\langle \{A, B\}, \{C\}, \emptyset \rangle \Cup \langle \emptyset, \{D\}, \{E\} \rangle = \langle \{A, B\}, \{C, D\}, \{E\} \rangle$$

[10] The number of members in $\boldsymbol{\emptyset}$ will of course depend on the cardinality of I.
[11] We use R as a metavariable for a generic inference rule.

As in the proof theory of flat ALs, we make use of three generic rules: a premise introduction rule PREM, an unconditional rule RU, and a conditional rule RC. Let us start with the first two:

$$\text{PREM} \quad \text{If } A \in \Gamma: \qquad \frac{\begin{array}{ll} \vdots & \vdots \end{array}}{\begin{array}{ll} A & \boldsymbol{\emptyset} \end{array}}$$

$$\text{RU} \quad \text{If } A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B: \qquad \frac{\begin{array}{ll} A_1 & \boldsymbol{\Delta}_1 \\ \vdots & \vdots \\ A_n & \boldsymbol{\Delta}_n \end{array}}{\begin{array}{ll} B & \boldsymbol{\Delta}_1 \Cup \ldots \Cup \boldsymbol{\Delta}_n \end{array}}$$

As in the flat case, by the rule PREM premises can be introduced on the empty condition (which is now a sequence of empty sets). Also, the unconditional rule RU is analogous to the flat case. In case B is derivable from $A_1, \ldots, A_n$ in the lower limit logic, we may derive B also in an adaptive proof from $A_1, \ldots, A_n$ whereby the conditions $\boldsymbol{\Delta}_i$ on which the A_i's were derived are carried forward and merged to $\boldsymbol{\Delta}_1 \Cup \ldots \Cup \boldsymbol{\Delta}_n$.

The generic conditional rule for our proof theory also closely resembles the conditional rule of Section 2:

$$\text{RC} \quad \text{If } A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \mathbin{\check{\vee}} \mathsf{Dab}(\boldsymbol{\Theta}): \qquad \frac{\begin{array}{ll} A_1 & \boldsymbol{\Delta}_1 \\ \vdots & \vdots \\ A_n & \boldsymbol{\Delta}_n \end{array}}{\begin{array}{ll} B & \boldsymbol{\Delta}_1 \Cup \ldots \Cup \boldsymbol{\Delta}_n \Cup \boldsymbol{\Theta} \end{array}}$$

Suppose we are able to derive $B \mathbin{\check{\vee}} \mathsf{Dab}(\Theta_1 \cup \ldots \cup \Theta_n)$ in **LLL** from $A_1, \ldots, A_n$, where each $\Theta_i \subset \Omega_i$. In that case the proof theory allows us to defeasibly derive B from $A_1, \ldots, A_n$, namely on the assumption that none of the abnormalities in $\Theta_1 \cup \ldots \cup \Theta_n$ is true. This is realized by merging $\boldsymbol{\Theta} = \langle \Theta_1, \ldots, \Theta_n, \emptyset, \ldots \rangle$ with all the conditions on which the A_i's were derived.

We close the discussion on the generic rules with two observations. In case some Ω_i's are intersecting, sometimes B can be derived in various ways on the basis of the same abnormalities that are assumed to be false. Take for instance the case that $C_1 \in \Omega_1 \cap \Omega_2$ and that $C_2 \in \Omega_2 \setminus \Omega_1$. Suppose furthermore that $A_1, A_2 \vdash_{\mathbf{LLL}} B \mathbin{\check{\vee}} (C_1 \mathbin{\check{\vee}} C_2)$. Then the following lines may occur in a proof:

l_1	A_1	$\ldots$	$\boldsymbol{\Delta}_1$
l_2	A_2	$\ldots$	$\boldsymbol{\Delta}_2$
l_3	B	l_1, l_2; RC	$\boldsymbol{\Delta}_1 \Cup \boldsymbol{\Delta}_2 \Cup \langle \{C_1\}, \{C_2\}, \emptyset, \ldots \rangle$
l_4	B	l_1, l_2; RC	$\boldsymbol{\Delta}_1 \Cup \boldsymbol{\Delta}_2 \Cup \langle \emptyset, \{C_1, C_2\}, \emptyset, \ldots \rangle$

Note that RC allows for both inferences, the one at line l_3 and the one at line l_4, and hence leaves room for a choice.[12] We will return to this point at the end of this section, and show that in some cases, it is crucial to warrant

[12] Note that $B \mathbin{\check{\vee}} (C_1 \mathbin{\check{\vee}} C_2)$ corresponds to both $B \mathbin{\check{\vee}} \mathsf{Dab}(\boldsymbol{\Theta})$ and $B \mathbin{\check{\vee}} \mathsf{Dab}(\boldsymbol{\Theta'})$ where $\boldsymbol{\Theta} = \langle \{C_1\}, \{C_2\} \rangle$ and $\boldsymbol{\Theta'} = \langle \emptyset, \{C_1, C_2\} \rangle$.

the completeness of the proof theory with respect to the syntactic consequence relation of **SAL**.

Finally, note that the generic conditional rule RC only allows for a finite amount of assumptions to be put in the condition column of the proof with each application. As a consequence, for each proof line l with condition $\langle \Delta_i \rangle_{i \in I}$ there is a maximal $k \in I$ such that for all $j > k$, $\Delta_j = \emptyset$. Or, in other words, given that I is infinite we know that for each line of the proof the condition has the format $\langle \Delta_1, \ldots, \Delta_k, \emptyset, \emptyset, \ldots \rangle$ for some $k \in I$.

4.2 Preparing for the marking definitions

Of course, in order to explicate defeasible reasoning it is not enough to be able to apply certain rules conditionally. What is still missing is a mechanism that makes it possible to *retract* defeasible inferences. As in the case of flat ALs, lines in an **SAL**-proof are marked at a certain stage of the proof in order to signify that the corresponding inference is retracted at that stage.

The marking definitions reflect the hierarchical structure of the superposition. For each level $i \in I$ we will state i-marking definitions. That a line is not i-marked for any $i \in I$ indicates that we have no reason to suspect that line. If a line in an **SAL**-proof is i-marked for an $i \in I$, then this means the line is retracted at the given stage of the proof.

Since either $\mathbf{x_i} = \mathbf{r}$ or $\mathbf{x_i} = \mathbf{m}$, and since we also include superpositions of ALs with mixed strategies, we need to state i-marking definitions for both strategies. In order to do so it is useful to define sequential counter-parts to various notions that play a central role for the marking definitions in Section 2.

We first give a sequential account of minimal Dab-formulas resulting in the notion of a minimal Dab_i-formula for each $i \in I$, i.e. a disjunction of members of Ω_i. Similar as the marking at stage s for flat ALs was determined by a set of minimal Dab-formulas relative to s, the i-marking in the sequential case will be determined by a set of Dab_i-formulas relative to s.

Definition 4. Let s be the stage of an **SAL**-proof from Γ and $i \in I$.

- A proof line l with condition $\boldsymbol{\Delta}$ is a $[\leq 0]$-line iff $\boldsymbol{\Delta} = \boldsymbol{\emptyset}$.
- A proof line l with condition $\boldsymbol{\Delta}$ is a $[\leq i]$-line iff $\boldsymbol{\Delta} = \langle \Delta_1, \ldots, \Delta_i, \emptyset, \ldots \rangle$.
- A proof line l is an i-line iff it is a $[\leq i]$-line and not a $[\leq i-1]$-line.
- Where $\Delta \subseteq \Omega_i$, $\mathsf{Dab}(\Delta)$ is a *minimal* Dab_i*-formula at stage* s in case (i) $\mathsf{Dab}(\Delta)$ is derived at some $[\leq i-1]$-line l at stage s, (ii) line l is not marked at stage s (see below for the marking definition), and (iii) for no $\Delta' \subset \Delta$, $\mathsf{Dab}(\Delta')$ is derived at an unmarked $[\leq i-1]$-line at stage s.
- Where $\{\mathsf{Dab}(\Delta_j) \mid j \in J\}$ is the set of the minimal Dab_i-formulas at stage s, let ${}^{\mathbf{C}}\Sigma^i_s(\Gamma) =_{\mathsf{df}} \{\Delta_j \mid j \in J\}$.
- ${}^{\mathbf{C}}U^i_s(\Gamma) =_{\mathsf{df}} \bigcup {}^{\mathbf{C}}\Sigma^i_s(\Gamma)$
- ${}^{\mathbf{C}}\Phi^i_s(\Gamma)$ is the set of minimal choice sets of ${}^{\mathbf{C}}\Sigma^i_s(\Gamma)$

4.3 The i-Marking for Reliability

Now we are able to define the i-marking at a stage s. Let us begin with the marking definition for the Reliability Strategy.

Definition 5 (i-marking for Reliability)**.** A line l with condition $\mathbf{\Delta}$ is i-marked at stage s iff $\Delta_i \cap {}^{\mathbf{C}}U_s^i(\Gamma) \neq \emptyset$.

We say a line is *marked* in case it is i-marked for some $i \in I$ (see also the i-marking Definition 7 for Minimal Abnormality below).

Before we turn to the i-marking definition for Minimal Abnormality, let us illustrate the generic inference rules and the above marking definition by means of a simple example. Recall that the logic $\mathbf{SK2^r}$ from Example 2 is defined as the superposition of the logic $\mathbf{K2^r}$ on the logic $\mathbf{K1^r}$. Now consider the premise set $\Gamma_{\mathsf{p1}} = \{\Diamond p, \Diamond\Diamond q, \Diamond\Diamond r, \neg p \vee \neg r\}$. According to this premise set, p, q and r are all three plausible, but p is more plausible than the other two propositions. However, we also know that either p or r is false. Hence we can expect that the prioritized logic will only allow us to finally derive p, and hence by disjunctive syllogism $\neg r$. Also, since q is not involved in the conflict, we expect it to be finally derivable. This can be done as follows.

We start by introducing the premises on the condition $\langle\emptyset, \emptyset\rangle$:

1	$\Diamond p$	PREM	$\langle\emptyset, \emptyset\rangle$
2	$\Diamond\Diamond q$	PREM	$\langle\emptyset, \emptyset\rangle$
3	$\Diamond\Diamond r$	PREM	$\langle\emptyset, \emptyset\rangle$
4	$\neg p \vee \neg r$	PREM	$\langle\emptyset, \emptyset\rangle$

By the rule RC, we may subsequently derive p, q and r from the first three premises. In order to avoid notational clutter let us from now on abbreviate abnormalities $\Diamond^i A \wedge \neg A$ by $!^i A$. Note that $\Gamma_{\mathsf{p1}} \vdash_{\mathbf{K}} p \check{\vee} !^1 p$, $\Gamma_{\mathsf{p1}} \vdash_{\mathbf{K}} q \check{\vee} !^2 q$ and $\Gamma_{\mathsf{p1}} \vdash_{\mathbf{K}} r \check{\vee} !^2 r$. Hence we can derive e.g. p on the assumption that $!^1 p$ is false. In the adaptive proof this means that we derive p on the condition $\langle !^1 p, \emptyset\rangle$ and similar for q and r:

5	p	1;RC	$\langle\{!^1 p\}, \emptyset\rangle$
6	q	2;RC	$\langle\emptyset, \{!^2 q\}\rangle$
7	r	3;RC	$\langle\emptyset, \{!^2 r\}\rangle$

To understand the rule RU, consider the following continuation of the proof, in which the conditions of lines 5 and 6 are merged:[13]

8	$p \wedge q$	5,6;RU	$\langle\{!^1 p\}, \{!^2 q\}\rangle$

Let us now turn to the marking. We use $\checkmark_i$ to denote that a line is i-marked. To avoid clutter, we only represent the marks at one stage: where k is the last line in the example proof the displayed marks represent marking at stage k.

In order to render line 7 marked, we first have to derive the Dab_2-formula $!^2 r$. This is done as follows:

[13] Note that it is also possible to derive $p \wedge q$ from lines 1 and 2, using the rule RC.

	⋮	⋮	⋮	⋮
	5	p	1;RC	$\langle\{!^1 p\}, \emptyset\rangle$
	6	q	2;RC	$\langle\emptyset, \{!^2 q\}\rangle$
$\checkmark_2$	7	r	3;RC	$\langle\emptyset, \{!^2 r\}\rangle$
	8	$p \wedge q$	5,6;RU	$\langle\{!^1 p\}, \{!^2 q\}\rangle$
	9	$!^1 p \check{\vee} !^2 r$	1,3,4;RU	$\langle\emptyset, \emptyset\rangle$
	10	$!^2 r$	9;RC	$\langle\{!^1 p\}, \emptyset\rangle$

Let us discuss the marking at stage 10 step by step. First of all, note that at stage 10, no Dab_1-formula has been derived on the condition $\langle\emptyset, \emptyset\rangle$.[14] This means that ${}^{\mathbf{C}}\Sigma^1_{10}(\Gamma_{\mathsf{p1}}) = \emptyset$, whence also ${}^{\mathbf{C}}U^1_{10}(\Gamma_{\mathsf{p1}}) = \emptyset$. As a result, no line is 1-marked at stage 10.

Now consider line 10 and its formula $!^2 r$. This is a Dab_2-formula, derived on a condition of the form $\langle\Delta, \emptyset\rangle$. Moreover, line 10 is not 1-marked. As a result, $!^2 r$ is a minimal Dab_2-formula at stage 10. This implies that ${}^{\mathbf{C}}\Sigma^2_{10}(\Gamma_{\mathsf{p1}}) = \{\{!^2 r\}\}$, whence ${}^{\mathbf{C}}U^2_{10}(\Gamma_{\mathsf{p1}}) = \{!^2 r\}$. As a result, line 7 is 2-marked at stage 10, as indicated by the symbol $\checkmark_2$.

We define final derivability for our proof theory exactly in the same way as it was defined for flat adaptive logics in Definition 3.[15]

Definition 6. A is *finally derived* at a line l of a finite stage s in an **SAL**-proof, iff (i) line l is unmarked at stage s, and (ii) every extension of the proof in which l is marked can be further extended in such a way that l is unmarked.

A is *finally derivable* from Γ in **SAL** iff there is a proof from Γ in which A is finally derived. We write $\Gamma \vdash_{\mathbf{SAL}} A$ in case A is finally derivable from Γ.

As a matter of fact, p, q and $p \wedge q$ are finally derived in the proof from Γ_{p1} above. Note that no Dab_1-formula is derivable from this premise set, and the only minimal Dab_2-formula that can be derived from Γ_{p1} is $!^2 r$. This means that in every extension of the proof, the marking of lines 1–10 remains unchanged.

4.4 The i-Marking for Minimal Abnormality

The i-marking for Minimal Abnormality is slightly more complicated:

Definition 7 (i-marking for Minimal Abnormality)**.** A line l with formula A and condition $\mathbf{\Delta}$ is i-marked at stage s iff one of the following conditions hold:

(i) there is no $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ such that $\Delta_i \cap \varphi \neq \emptyset$

(ii) for a $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$: there is no line l' that is not j-marked for some $j < i$ at stage s, with formula A and condition $\langle\Theta_1, \ldots, \Theta_i, \Delta_{i+1}, \Delta_{i+2}, \ldots\rangle$, and $\Theta_i \cap \varphi = \emptyset$.

[14] The formula on line 9 is not a Dab_1-formula, since it contains the abnormality $!^2 r$ which is not a member of $\Omega^{\mathbf{K}}_1$.

[15] In case some $\mathbf{x_i} = \mathbf{m}$ this definition also makes reference to the i-marking for Minimal Abnormality which we define in Section 4.4.

Recall that final derivability as defined in Definition 6 also applies to superpositions that feature ALs with Minimal Abnormality. This completes the technical characterization of our first proof theory for **SAL**. In the appendix we prove its adequacy:

Theorem 5. $\Gamma \vdash_{\mathbf{SAL}} A$ *iff* $A \in Cn_{\mathbf{SAL}}(\Gamma)$

Let us in the remainder of this section illustrate the proof theory and discuss some noteworthy point concerning the marking for Minimal Abnormality.

Requirement (ii) in marking definition for Minimal Abnormality may strike some as surprising. The marking condition has a prospective character since it also takes into account sets of abnormalities in $\mathbf{\Delta}$ that belong to higher levels than i. Naively it may be expected that requirement (ii) reads as follows:

(ii') for a $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$: there is no line l' that is not j-marked for some $j < i$ at stage s, with formula A and condition $\mathbf{\Theta}$ such that $\Theta_i \cap \varphi = \emptyset$.

Let us interpret Definition 7 in terms of an argumentation game. Suppose our proponent derives formula A on the condition $\mathbf{\Delta}$ at stage s. The i-marking concerns the question whether the defeasible assumption that corresponds to level i in the superposition is feasible. The minimal choice sets of ${}^{\mathbf{C}}\Sigma^i_s(\Gamma)$ offer minimally abnormal interpretations (in terms of abnormalities in Ω_i) of the premises with respect to the Dab_i-formulas at the given stage s. That is, they offer possible counter-arguments against the defeasible assumption $\mathbf{\Delta}$ of line l. However, there is a slight complication involved.

The assumptions used in order to derive A may involve abnormalities of lower and higher levels than i. Concerning the lower levels we adopt a bottom-up approach. In case one of the defeasible assumptions at a lower level is not feasible we rely on the marking corresponding to the lower level to retract the line. In this sense the i-marking procedure safely ignores the defeasible assumptions belonging to lower levels. However, the i-marking is sensitive with respect to the defeasible assumptions that belong to higher levels.

The idea is as follows. According to point (i) there should be at least one minimally abnormal interpretation $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ in which the ith defeasible assumption is valid, i.e., $\Delta_i \cap \varphi = \emptyset$. Moreover, for each counter-argument, i.e., for each $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ for which $\Delta_i \cap \varphi \neq \emptyset$, our proponent should be able to defend herself in the following way. She should be able to produce an argument such that the ith defeasible assumption is valid in φ and such that all the higher level defeasible assumptions are the same as in her original argument at line l (see point (ii)).

It is crucial that in her defense, the proponent uses the same higher level defeasible assumptions as in her original argument. Let us demonstrate this by a simple example. As before, we use a **K**-based prioritized logic with only two levels of abnormalities. This time however, we consider the Minimal Abnormality-variant, i.e. $\mathbf{SK2^m}$ characterized by the sequence $\langle \mathbf{K1^m}, \mathbf{K2^m} \rangle$.

Let $\Gamma_{\mathsf{p2}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \Diamond\Diamond s, \neg p \vee \neg q, \neg p \vee \neg r, \neg q \vee \neg s\}$. Note that the following disjunctions of abnormalities are **K**-derivable from Γ_{p2}:

(i) $!^1p \check{\vee} !^1q$
(ii) $!^1p \check{\vee} !^2r$
(iii) $!^1q \check{\vee} !^2s$

However, (ii) and (iii) are neither Dab_1-formulas nor Dab_2-formulas. The following $\mathbf{SK2^m}$-proof shows how we can derive Dab_2-formulas from Γ_{p2}:

	1	$\Diamond p$	PREM	$\langle\emptyset,\emptyset\rangle$
	2	$\Diamond q$	PREM	$\langle\emptyset,\emptyset\rangle$
	3	$\Diamond\Diamond r$	PREM	$\langle\emptyset,\emptyset\rangle$
	4	$\Diamond\Diamond s$	PREM	$\langle\emptyset,\emptyset\rangle$
	5	$\neg p \vee \neg q$	PREM	$\langle\emptyset,\emptyset\rangle$
	6	$\neg p \vee \neg r$	PREM	$\langle\emptyset,\emptyset\rangle$
	7	$\neg q \vee \neg s$	PREM	$\langle\emptyset,\emptyset\rangle$
	8	$!^1p \check{\vee} !^1q$	1,2,5;RU	$\langle\emptyset,\emptyset\rangle$
	9	$!^1p \check{\vee} !^2r$	1,3,6;RU	$\langle\emptyset,\emptyset\rangle$
	10	$!^1q \check{\vee} !^2s$	2,4,7;RU	$\langle\emptyset,\emptyset\rangle$
	11	$!^2r \check{\vee} !^2s$	9;RC	$\langle\{!^1p\},\emptyset\rangle$
	12	$!^2r \check{\vee} !^2s$	10;RC	$\langle\{!^1q\},\emptyset\rangle$
$\checkmark_1$	13	$!^2r$	9;RC	$\langle\{!^1p\},\emptyset\rangle$
$\checkmark_1$	14	$!^2s$	10;RC	$\langle\{!^1q\},\emptyset\rangle$

Note that ${}^{\mathbf{C}}\Sigma^1_{14}(\Gamma_{\mathsf{p2}}) = \{\{!^1p, !^1q\}\}$, whence ${}^{\mathbf{C}}\Phi^1_{14}(\Gamma_{\mathsf{p2}}) = \{\{!^1p\},\{!^1q\}\}$. Hence, at the current stage of our proof there are two minimally abnormal interpretations with respect to the abnormalities in Ω_1: one according to which $!^1p$ is the only true abnormality, and another one according to which $!^1q$ is the only true abnormality. This means that we cannot finally derive $!^2r$ on the condition $\langle\{!^1p\},\emptyset\rangle$, since we cannot derive $!^2r$ on an assumption that is valid in the minimally abnormal interpretation offered by means of the minimal choice set $\{!^1q\}$ (see condition (ii) in Definition 7).

For the same reason, we cannot finally derive $!^2s$. Both lines 13 and 14 are 1-marked. However, the disjunction of both level 2-abnormalities is finally derived at stage 12. This follows immediately from the fact that $\mathsf{Dab}(\Delta_1)$ where $\Delta_1 = \{!^1p, !^1q\}$ is the only minimal Dab_1-consequence of Γ_{p2}. Also, it can easily be verified that $\mathsf{Dab}(\Delta_2)$ where $\Delta_2 = \{!^2r, !^2s\}$ is the *only* minimal Dab_2-consequence of $Cn_{\mathbf{K1^m}}(\Gamma_{\mathsf{p2}})$.

In view of the preceding, it is easy to see that the sets ${}^{\mathbf{C}}\Sigma^1_s(\Gamma_{\mathsf{p2}})$ and ${}^{\mathbf{C}}\Sigma^2_s(\Gamma_{\mathsf{p2}})$ remain stable from stage 12 on. Put differently, in every further stage s of the proof,

($\dagger_1$) ${}^{\mathbf{C}}\Phi^1_s(\Gamma_{\mathsf{p2}}) = {}^{\mathbf{C}}\Phi^1_{12}(\Gamma_{\mathsf{p2}}) = \{\{!^1p\},\{!^1q\}\}$
($\dagger_2$) ${}^{\mathbf{C}}\Phi^2_s(\Gamma_{\mathsf{p2}}) = {}^{\mathbf{C}}\Phi^2_{12}(\Gamma_{\mathsf{p2}}) = \{\{!^2r\},\{!^2s\}\}$

Let us now return to the prospective character of clause (ii) in Definition 7. Consider the following extension, in which the (arbitrarily chosen) formula t is derived:

9	$!^1p \check{\vee} !^2r$	1,3,6;RU	$\langle\emptyset,\emptyset\rangle$
10	$!^1q \check{\vee} !^2s$	2,4,7;RU	$\langle\emptyset,\emptyset\rangle$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

	15	$t \check{\vee} !^1p \check{\vee} !^2r$	9;RU	$\langle\emptyset,\emptyset\rangle$
$\checkmark_1$	16	t	15;RC	$\langle\{!^1p\},\{!^2r\}\rangle$
	17	$t \check{\vee} !^1q \check{\vee} !^2s$	10;RU	$\langle\emptyset,\emptyset\rangle$
$\checkmark_1$	18	t	17;RC	$\langle\{!^1q\},\{!^2s\}\rangle$

Since we obtained lines 15 and 17 by the rule of addition, we can make a similar move with any formula A instead of t. Let $\mathbf{\Theta}$ be the condition of line 16. The following facts hold:

(i.t.1) there is a $\varphi \in {}^{\mathbf{S}}\Phi^1_{18}(\Gamma_{\mathsf{p2}})$ such that $\Theta_1 \cap \varphi = \emptyset$ (viz. $\psi_1 = \{!^1q\}$)

(ii.t.1)' for every $\varphi \in {}^{\mathbf{S}}\Phi^1_{18}(\Gamma_{\mathsf{p2}})$, t is derived on a condition $\mathbf{\Theta'}$ such that $\Theta'_1 \cap \varphi = \emptyset$ at stage 18

(i.t.2) there is a $\varphi \in {}^{\mathbf{S}}\Phi^2_{18}(\Gamma_{\mathsf{p2}})$ such that $\Theta_2 \cap \varphi = \emptyset$ (viz. $\psi_2 = \{!^2s\}$)

(ii.t.2)' for every $\varphi \in {}^{\mathbf{S}}\Phi^2_{18}(\Gamma_{\mathsf{p2}})$, t is derived on a condition $\mathbf{\Theta'}$ such that $\Theta'_2 \cap \varphi = \emptyset$ at stage 18

In other words, replacing clause (ii) with (ii)' in the definition of i-marking for Minimal Abnormality, would imply that line 16, and by an analogous argument line 18, are not marked at stage 18 of the proof. Moreover, in view of ($\dagger_1$) and ($\dagger_2$), these lines would not be marked in any further extension of the proof.

This is where the prospective character of item (ii) in Definition 7 comes into play. Take for instance line 16. It is not the case that for every $\varphi \in {}^{\mathbf{S}}\Phi^1_{18}(\Gamma_{\mathsf{p2}})$, t is derived on a condition $\langle\Delta,\{!^2r\}\rangle$ such that $\Delta \cap \varphi = \emptyset$ – this requirement fails for $\{!^1p\}$, which is a minimal choice set of level 1. According to item (ii) t would also have to be derived on the condition $\langle\{!^1q\},\{!^2r\}\rangle$ in order for line 16 not to be 1-marked. An analogous argument applies to line 18. As a result, lines 16 and 18 are 1-marked at stage 18. Moreover, there is no way to extend this proof such that these lines are not 1-marked.

Recall the remark in Section 3.2 that the proof theories proposed in [1] and [25] are not adequate with respect to the consequence relation of $\mathbf{SAL^m}$. This negative result holds even in very simple (finite) cases and under the assumption that for every $i, j \in I$ such that $i \neq j$, $\Omega_i \cap \Omega_j = \emptyset$. The above example is one of those cases. What was lacking in those earlier proposals, is precisely the prospective character of (ii) in the marking for Minimal Abnormality.

The following continuation of the proof shows how the formula $(p \wedge s) \vee (q \wedge r)$ can be finally $\mathbf{SK2^m}$-derived from Γ_{p2}. In this case, requirement (ii) of Definition 7 is fulfilled for both $i = 1$ and $i = 2$, whence lines 21–24 are neither 1-marked nor 2-marked.

	⋮	⋮	⋮	⋮
$\checkmark_1$	19	$p \wedge s$	1,4;RC	$\langle\{!^1p\},\{!^2s\}\rangle$
$\checkmark_1$	20	$q \wedge r$	2,3;RC	$\langle\{!^1q\},\{!^2r\}\rangle$
	21	$(p \wedge s) \vee (q \wedge r)$	19;RU	$\langle\{!^1p\},\{!^2s\}\rangle$
	22	$(p \wedge s) \vee (q \wedge r)$	20;RU	$\langle\{!^1q\},\{!^2r\}\rangle$
	23	$(p \wedge s) \vee (q \wedge r)$	9;RU	$\langle\{!^1p\},\{!^2r\}\rangle$
	24	$(p \wedge s) \vee (q \wedge r)$	10;RU	$\langle\{!^1q\},\{!^2s\}\rangle$

4.5 The need for sequences of abnormalities as conditions

In Section 6, we will show how the **SAL**-proof theory can be simplified whenever restriction (†) from Section 3.1 holds. More specifically, given this restriction, we can just use sets of abnormalities for the conditions, instead of sequences of such sets. However, as we will now show, we need sequences in the more general case.

Consider the superposition-logic **SKP**, whose consequence relation defined as follows: $Cn_{\mathbf{SKP}}(\Gamma) =_{\mathrm{df}} Cn_{\mathbf{K1^r}}(Cn_{\mathbf{K2^r}}(Cn_{\mathbf{K1^r}}(\Gamma)))$.

Note that in this specific superposition, $\Omega_1 = \Omega_3 = \Omega_1^{\mathbf{K}}$. Let $\Gamma_{\mathsf{p3}} = \{\Diamond p, \Diamond q, \Diamond\Diamond r, \neg p \vee \neg q, \neg p \vee \neg r\}$. The following are minimal Dab-consequences of Γ_{p3}:

$$!^1 p \,\check{\vee}\, !^1 q \tag{1}$$

$$!^1 p \,\check{\vee}\, !^2 r \tag{2}$$

In view of (1), both $!^1 p$ and $!^1 q$ are unreliable for the first logic in the superposition. This means that we cannot finally derive $!^2 r$ on the condition $\{!^1 p\}$ in a $\mathbf{K1^r}$-proof from Γ_{p3}. More generally, $!^2 r \notin Cn_{\mathbf{K1^r}}(\Gamma_{\mathsf{p3}})$. Hence $!^2 r$ is a reliable abnormality in view of the second logic in the superposition. Since also $!^1 p \,\check{\vee}\, !^2 r \in Cn_{\mathbf{K1^r}}(\Gamma)$, it follows that we can derive $!^1 p$ on the condition $!^2 r$ in a $\mathbf{K2^r}$-proof from $Cn_{\mathbf{K1^r}}(\Gamma_{\mathsf{p3}})$. But then $!^1 p \,\check{\vee}\, !^1 q$ is no longer a minimal Dab-formula for the *third* logic in the superposition, whence q is finally $\mathbf{K1^r}$-derivable from $\Diamond q$ on the condition $\{!^1 q\}$, and hence $q \in Cn_{\mathbf{SKP}}(\Gamma_{\mathsf{p3}})$.

The following proof illustrates the fact that q is not $\mathbf{K1^r}$-derivable from Γ_{p3}, but only from $Cn_{\mathbf{K2^r}}(Cn_{\mathbf{K1^r}}(\Gamma_{\mathsf{p3}}))$, whence it is **SKP**-derivable from Γ_{p3}:

	1	$\Diamond p$	PREM	$\langle\emptyset,\emptyset,\emptyset\rangle$
	2	$\Diamond q$	PREM	$\langle\emptyset,\emptyset,\emptyset\rangle$
	3	$\Diamond\Diamond r$	PREM	$\langle\emptyset,\emptyset,\emptyset\rangle$
	4	$\neg p \vee \neg q$	PREM	$\langle\emptyset,\emptyset,\emptyset\rangle$
	5	$\neg p \vee \neg r$	PREM	$\langle\emptyset,\emptyset,\emptyset\rangle$
	6	$!^1 p \,\check{\vee}\, !^1 q$	1,2,4;RU	$\langle\emptyset,\emptyset,\emptyset\rangle$
	7	$!^1 p \,\check{\vee}\, !^2 r$	1,3,5;RU	$\langle\emptyset,\emptyset,\emptyset\rangle$
	8	$!^1 p$	7;RC	$\langle\emptyset,\{!^2 r\},\emptyset\rangle$
$\checkmark_1$	9	q	2;RC	$\langle\{!^1 q\},\emptyset,\emptyset\rangle$
	10	r	3;RC	$\langle\emptyset,\{!^2 r\},\emptyset\rangle$
	11	q	2;RC	$\langle\emptyset,\emptyset,\{!^1 q\}\rangle$

Note that ${}^{\mathbf{C}}U^1_{11}(\Gamma_{\mathsf{p3}}) = \{!^1 p, !^1 q\}$. This explains why line 9 is 1-marked: the first member of its condition contains the abnormality $\{!^1 q\}$, which is unreliable at level 1. Since ${}^{\mathbf{C}}\Sigma^2_{11}(\Gamma_{\mathsf{p3}}) = \emptyset$, lines 8 and 10 are not 1- or 2-marked. But this means that $!^1 p$, the formula derived on line 8, is a Dab_3-formula at stage 11 of the proof. Hence, $!^1 p \,\check{\vee}\, !^1 q$ is no longer a minimal Dab_3-formula at stage 11, whence ${}^{\mathbf{C}}U^3_{11}(\Gamma_{\mathsf{p3}}) = \{!^1 p\}$. The last crucial move takes place at line 11. Here, q is derived, but this time by pushing $!^1 q$ to the *third* set in the condition – note that this is perfectly in line with the generic rule RC, which leaves room for choice in this case. Since $\{!^1 q\} \cap {}^{\mathbf{C}}U^3_{11}(\Gamma_{\mathsf{p3}}) = \emptyset$, line 11 is unmarked and will remain so in every further extension of the proof.

5 A Bottom-Up Proof Theory

In this section we define a proof theory that is more in the "bottom-up" sequential spirit of **SAL**. Taking over the proof format of Section 4.1, the variant is realized in three respects:

1. The generic rule RC of the proof theory in Section 4 has a holistic character in the sense that defeasible assumptions corresponding to various ALs in the sequence $\langle \mathbf{AL_i} \rangle_{i \in I}$ can be applied in one and the same inference step. In what follows we replace RC by generic conditional rules RC_i for each $\mathbf{AL_i}$ in the sequence. Where $l_1, \ldots, l_n$ are $[\leq i]$-lines, $\Theta \subseteq \Omega_i$, and $\boldsymbol{\Theta}$ is such that $\Theta_i = \Theta$ and $\Theta_j = \emptyset$ for all $j \neq i$:

$$\text{RC}_i \quad \text{If } A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} \mathsf{Dab}(\Theta): \quad \begin{array}{ll} A_1 & \boldsymbol{\Delta}_1 \\ \vdots & \vdots \\ A_n & \boldsymbol{\Delta}_n \\ \hline B & \boldsymbol{\Delta}_1 \Cup \ldots \Cup \boldsymbol{\Delta}_n \Cup \boldsymbol{\Theta} \end{array}$$

   Each conditional rule RC_i forces the user to make inferences that make use of one defeasible reasoning form at a time, i.e., those modeled by $\mathbf{AL_i}$. This way the proofs are more analytic and transparent in explicating the reasoning processes leading to the **SAL**-consequences.
   We employ the rules PREM and RU just as before.
2. The i-marking definitions for Reliability and Minimal Abnormality only apply to i-lines. This is realized by making efficient use of the markings gained in the previous steps in the sense that these markings get inherited to higher levels.
3. This allows us furthermore to replace the prospective character of requirement (ii) in the definition for the marking with Minimal Abnormality by its simplified version (ii') (see Section 4.4).

The latter two points can be summarized as follows: the i-marking of a line l with condition $\boldsymbol{\Delta}$ concerns for both adaptive strategies only i-lines and depends only on Δ_i. In contrast, before we also had to check whether some i-line is j-marked for any $j < i$. This may introduce some complexity overhead especially if the strategy is Minimal Abnormality. In this variant we use a less expensive forward-chaining of the markings from previous inference steps that belong to lower levels: If the justification of an i-line l calls upon a marked j-line l' (where $j < i$), the marking of l' is inherited to l. Similarly, if the justification of l calls upon a line l' and l' already inherits some marking from a lower level, this marking is inherited to l (see Definition 10 below).

We say that a line is *marked* iff it is i-marked for some $i \in I$ according to Definitions 8 or 9 (depending on the strategy), or it is inh-marked according to Definition 10, where:

Definition 8 (i-marking for Reliability)**.** An i-line l with condition $\boldsymbol{\Delta}$ is i-marked at stage s iff $\Delta_i \cap {}^{\mathbf{C}}U^i_s(\Gamma) \neq \emptyset$.

Definition 9 (i-marking for Minimal Abnormality)**.** An i-line l with formula A and condition $\boldsymbol{\Delta}$ is i-marked at stage s iff one of the following conditions hold:

(i) there is no $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ such that $\Delta_i \cap \varphi \neq \emptyset$

(ii) for a $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$: there is no unmarked $[\leq i]$-line l' at stage s with formula A and condition $\boldsymbol{\Theta}$ such that $\Theta_i \cap \varphi = \emptyset$.

Definition 10 (inh-marking of lines)**.** An i-line l with condition $\boldsymbol{\Delta}$ and justification $l_1, \ldots, l_n; R$ is inh-marked in case some l_j (where $1 \leq j \leq n$) is (i) k-marked for some $k < i$, or (ii) inh-marked.

This completes the characterization of our variant. Final derivability and the derivability relation $\vdash_{\mathbf{SAL}}$ are defined as in Definition 6. In the appendix we prove the following adequacy result:

Theorem 6. *$\Gamma \vdash_{\mathbf{SAL}} A$ iff $A \in Cn_{\mathbf{SAL}}(\Gamma)$*

In order to illustrate our new variant we take again a look at our premise set Γ_{p1} (we skip the introduction of premises in lines 1–7, see Section 4.4):

	⋮	⋮	PREM	$\langle\emptyset,\emptyset\rangle$
	8	$!^1p \check{\vee} !^1q$	1,2,5;RU	$\langle\emptyset,\emptyset\rangle$
	9	$!^1p \check{\vee} !^2r$	1,3,6;RU	$\langle\emptyset,\emptyset\rangle$
	10	$!^1q \check{\vee} !^2s$	2,4,7;RU	$\langle\emptyset,\emptyset\rangle$
	11	$!^2r \check{\vee} !^2s$	9;RC$_1$	$\langle\{!^1p\},\emptyset\rangle$
	12	$!^2r \check{\vee} !^2s$	10;RC$_1$	$\langle\{!^1q\},\emptyset\rangle$
$\checkmark_1$	13	$!^2r$	9;RC$_1$	$\langle\{!^1p\},\emptyset\rangle$
$\checkmark_1$	14	$!^2s$	10;RC$_1$	$\langle\{!^1q\},\emptyset\rangle$
	15	$t \check{\vee} !^1p \check{\vee} !^2r$	9;RU	$\langle\emptyset,\emptyset\rangle$
$\checkmark_1$	16	$t \check{\vee} !^2r$	15;RC$_1$	$\langle\{!^1p\},\emptyset\rangle$
inh	17	t	16;RC$_2$	$\langle\{!^1p\},\{!^2r\}\rangle$
	18	$t \check{\vee} !^1q \check{\vee} !^2s$	10;RU	$\langle\emptyset,\emptyset\rangle$
$\checkmark_1$	19	$t \check{\vee} !^2s$	18;RC$_1$	$\langle\{!^1q\},\emptyset\rangle$
inh	20	t	19;RC$_2$	$\langle\{!^1q\},\{!^2s\}\rangle$

Note that t cannot be inferred in one step on the condition $\langle\{!^1p\},\{!^2r\}\rangle$, but we need first to derive $t \check{\vee} !^2r$ on the condition $\langle\{!^1p\},\emptyset\rangle$ by means of RC$_1$ at line 16. Then we can derive t by means of RC$_2$ at line 17. An analogous argument applies to the derivation of t at line 20. What ensures the marking of lines 17 and 20 is the inh-marking. Note that e.g. line 16 gets 1-marked due to the fact that $t \check{\vee} !^2r$ is not derived on the condition $\langle\{!^1q\},\emptyset\rangle$ (recall that $\Phi^1_{20}(\Gamma) = \{\{!^1p\},\{!^1q\}\}$). This marking carries forward to line 17 since it calls upon the 1-marked line 16. This ensures that the arbitrary formula t is not derivable.

Were we only to proceed along the lines of points 2 and 3 and hence use a generic conditional rule RC as in the proof theory of Section 4, then we would immediately be confronted with problems. In that case we would be able to produce the following proof fragment:

15	$t \check{\vee} !^1 p \check{\vee} !^2 r$	9;RU	$\langle \emptyset, \emptyset \rangle$
16	t	15;RC	$\langle \{!^1 p\}, \{!^2 r\} \rangle$
17	$t \check{\vee} !^1 q \check{\vee} !^2 s$	10;RU	$\langle \emptyset, \emptyset \rangle$
18	t	17;RC	$\langle \{!^1 q\}, \{!^2 s\} \rangle$

Recall that the arbitrarily introduced formula t is not $\mathbf{SK2^m}$-derivable. Hence, lines 16 and 18 should get marked. According to the marking definition 9, we only have to check whether lines 16 and 18 are 2-marked, since both lines 16 and 18 are 2-lines. However, neither line is 2-marked and hence t would be finally derivable.

6 A Simplification for Non-Intersecting Ω_i's

We will now consider the special case in which (†) for all $i, j \in I$ for which $i \neq j$ we have $\Omega_i \neq \Omega_j$. In this case the logical form of an abnormality A unambiguously determines an $i \in I$ such that $A \in \Omega_i$. This means in turn that we do not need to represent the condition of lines in the proof in terms of sequences of sets of abnormalities but can instead just represent them by means of sets of abnormalities in $\bigcup_{i \in I} \Omega_i$.

To implement this simplification, we need to slightly adjust our terminology. Let $i \in I$ and l be a proof line with condition Δ. First, we say line l is a $[\leq 0]$-*line* iff $\Delta = \emptyset$, it is a $[\leq i]$-*line* iff $\Delta \subseteq \Omega_1 \cup \ldots \cup \Omega_i$. Given these adjustments, we can use the same definitions of ${}^{\mathbf{C}}U^i_s(\Gamma)$ and ${}^{\mathbf{C}}\Phi^i_s(\Gamma)$ as before – see Definition 4.

Let us now turn to the proof theory from Section 4. The generic rules are very straightforward: just take the generic rules of the standard format, but treat the sets Δ_i and Θ as metavariables for subsets of $\bigcup_{i \in I} \Omega_i$. The marking definitions are adjusted as follows:

Definition 11 (i-marking for Reliability, special case)**.** A line l with condition Δ is i-marked at stage s iff $\Delta \cap {}^{\mathbf{C}}U^i_s(\Gamma) \neq \emptyset$.

Definition 12 (i-marking for Minimal Abnormality, special case)**.** A line l with formula A and condition Δ is i-marked at stage s iff one of the following conditions hold:

(i) there is no $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ such that $\Delta \cap \varphi \neq \emptyset$, or

(ii) there is a $\varphi \in {}^{\mathbf{C}}\Phi^i_s(\Gamma)$ such that there is no line l' that is not j-marked for some $j < i$ at stage s, with formula A and condition Θ such that $\Theta \cap \varphi = \emptyset$, and $\Theta \cap (\Omega_{i+1} \cup \Omega_{i+2} \cup \ldots) = \Delta \cap (\Omega_{i+1} \cup \Omega_{i+2} \cup \ldots)$.

Note that even in this special case, we cannot do without the prospective character of the marking definition for Minimal Abnormality – this follows immediately from the example Γ_{p2} which we discussed in Section 4.4.

To spell out the simplification for the proof theory of Section 5, we first redefine the generic rule RC_i. Where for each Δ_j $(1 \leq j \leq n)$, $\Delta_j \subseteq \Omega_1 \cup \ldots \cup \Omega_i$, and where $\Theta \subseteq \Omega_i$, we have:

$$\text{RC}_i \quad \text{If } A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \check{\vee} \mathsf{Dab}(\Theta): \quad \begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

Next, we adjust the marking definitions, in the same way as this was done with the first proof theory. In this case, it suffices to just replace $\boldsymbol{\Delta}$ and Δ_i with Δ, and $\boldsymbol{\Theta}$ and Θ_i with Θ in Definitions 8–10. We leave it to the reader to check that this proof theory is equivalent to the original version from Section 5. Also, by the same example as the one spelled out in Section 4.5, it follows that in some cases where the restriction (†) fails, one still needs sequential conditions in order to obtain an adequate proof theory along the lines of that defined in Section 5.

7 Outlook and Conclusion

In this paper we have presented two proof theories for superpositions of ALs in standard format. We bring it to a closing by indicating some interesting further developments to which the research presented here provides a fruitful basis.

First of all, as pointed out in Section 3.1, there is a specific class of superpositions that are very interesting from a metatheoretic perspective, i.e. those for which (‡) $\Omega_1 \subseteq \Omega_2 \subseteq \ldots$. It remains to be seen whether we can also simplify the **SAL**-proof theory for these superpositions in a way similar to the simplification we provided in Section 6.

In [21] the authors presented a generalization of the standard format to reason with prioritized defeasible assumptions, resulting in so-called lexicographic ALs. There they also presented proof theories for Reliability and Minimal Abnormality, which have the same generic rules as those of the standard format. The marking definitions are analogous to the marking definitions of the standard format, with the difference that they also take into account priorities among the abnormalities. Altogether, the structural similarity between ALs in standard format and lexicographic ALs makes it plausible that we may use the same techniques as presented in this paper in order to define proof theories for *superpositions* of lexicographic ALs.

We also intend to adjust the presented proof theories to other (non-standard) strategies such as the normal selections strategy [10, 25] or the counting strategy [18, 11].

APPENDIX

Preliminaries Due to the more technical nature of the appendix we drop the supposition $\Gamma \subseteq \mathcal{W}$ that was used throughout the paper. In the remainder $\mathcal{W}^+$ denotes all well-formed formulas in the extension of the language $\mathcal{L}$ by the checked classical connectives (see Section 2). From now on, if not stated differently, $\Gamma \subseteq \mathcal{W}^+$.

Let **SAL** be a superposition of the ALs $\mathbf{AL_i}$ in the sequence $\langle \mathbf{AL_i} \rangle_{i \in I}$ with lower limit logic **LLL** and set of abnormalities Ω_i. All $\mathbf{AL_i}$'s are in standard format.

Let $\Omega = \bigcup_{i \in I} \Omega_i$. Where $\Delta \subseteq \Omega$ is finite, we say that a sequence $\mathbf{\Delta} = \langle \Delta_i \rangle_{i \in I}$ *corresponds* to Δ iff (i) $\bigcup \mathbf{\Delta} = \Delta$ and (ii) there is an $i \in I$ such that for all $j \in I$ for which $j > i$, $\Delta_j = \emptyset$.

Lemma 1. *The set of* $\mathbf{\Delta}$ *that correspond to* Δ *is countable.*

Proof. We prove the Lemma for the case $I = \mathbb{N}$. The other cases are trivial. Let $i \in I$ be minimal such that $\Delta \subseteq \Omega_1 \cup \ldots \cup \Omega_i$. Where $n \geq i$, since Δ is finite also the set of all $\mathbf{\Delta} = \langle \Delta_1, \ldots, \Delta_n, \emptyset \rangle$ where $\Delta_n \neq \emptyset$ that correspond to Δ is finite (possibly empty). Let $\mathbf{\Delta}_1^n, \ldots, \mathbf{\Delta}_{n_m}^n$ be a list of all these $\mathbf{\Delta}$'s. Altogether, $\mathbf{\Delta}_1^i, \ldots, \mathbf{\Delta}_{i_m}^i, \mathbf{\Delta}_1^{i+1}, \ldots, \mathbf{\Delta}_{(i+1)_m}^{i+1}, \mathbf{\Delta}_1^{i+2}, \ldots, \ldots$ is a list of all $\mathbf{\Delta}$'s that correspond to Δ. □

We say that $\mathsf{Dab}(\Delta)$ is a *minimal* Dab_i*-consequence* of Γ iff $\Delta \subseteq \Omega_i$, $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$, and for all $\Delta' \subseteq \Delta$: if $\mathsf{Dab}(\Delta') \in Cn_{\mathbf{LLL}}(\Gamma)$ then $\Delta' = \Delta$. Where $\mathsf{Dab}(\Delta_1), \mathsf{Dab}(\Delta_2), \ldots$ are the minimal Dab_i-consequences from Γ, let $\Sigma^i(\Gamma) =_{\mathrm{df}} \{\Delta_1, \Delta_2, \ldots\}$. Let $\Phi^i(\Gamma)$ be the set of all minimal choice sets of $\Sigma^i(\Gamma)$ and $U^i(\Gamma) =_{\mathrm{df}} \bigcup \Sigma^i(\Gamma)$.

In the remainder, **AL** is a flat AL in standard format with lower limit logic **LLL** and the set of abnormalities Ω. $\mathsf{Dab}(\Delta)$ is a *minimal* Dab*-consequence* of Γ iff $\Delta \subseteq \Omega$, $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$ and for all $\Delta' \subseteq \Delta$: if $\mathsf{Dab}(\Delta') \in Cn_{\mathbf{LLL}}(\Gamma)$ then $\Delta' = \Delta$. Where $\mathsf{Dab}(\Delta_1), \mathsf{Dab}(\Delta_2), \ldots$ are all the minimal Dab-consequences of Γ, $\Sigma(\Gamma) =_{\mathrm{df}} \{\Delta_1, \Delta_2, \ldots\}$. $\Phi(\Gamma)$ is the set of minimal choice sets of $\Sigma(\Gamma)$ and $U(\Gamma) =_{\mathrm{df}} \bigcup \Sigma(\Gamma)$.

In the following, $\check{\vee}\, \mathsf{Dab}(\Delta)$ denotes the empty string in case $\Delta = \emptyset$. For the sake of convenience we will sometimes speak about the empty proof, meaning the "proof" which consists of 0 lines. We denote this proof by $\mathcal{P}_\varepsilon$.

In what follows we will first show the adequacy for the proof theory of Section 4 and in the last subsection we will show the adequacy of the proof theory presented in Section 5. Hence, before this last subsection, whenever we refer to a **SAL**-proof we mean a proof in the system presented in Section 4.

A complete proof stage g In the following it will be very useful to speak about the extension of a given (possibly empty) **AL**-, resp. **SAL**-proof $\mathcal{P}$ in which (a) A is derived on the condition Δ whenever $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta)$ resp. A is derived on the condition $\mathbf{\Delta}$ whenever $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta)$ where $\mathbf{\Delta}$ corresponds to Δ, and (b) A is derived on the condition $\emptyset$ resp. $\boldsymbol{\emptyset}$ whenever $\Gamma \vdash_{\mathbf{LLL}} A$. We dub a corresponding stage $\mathsf{g}(\mathcal{P})$ a *complete stage*.

This stage exists and can be constructed along the following lines (we show the variant for **SAL**, the one for **AL** is analogous). Note that each well-formed formula has a Gödel-number. From this it follows immediately that $\Theta = \{A \mid \Gamma \vdash_{\mathbf{LLL}} A\}$ is enumerable, e.g. $\Theta = \{B_1, B_2, \ldots\}$. Moreover, due to the compactness of **LLL**, for each $B_i \in \Theta$ there are some $\{A_1, \ldots, A_n\}$ such that $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B_i$. Hence, for each $B_i \in \Theta$ we have the following proof $\mathcal{P}_i$:

l_1^i	A_1	PREM	$\boldsymbol{\emptyset}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
l_n^i	A_n	PREM	$\boldsymbol{\emptyset}$
l_{n+1}^i	B_i	$l_1^i, \ldots, l_n^i$;RU	$\boldsymbol{\emptyset}$

In case B_i is of the form $A \check{\vee} \mathsf{Dab}(\Delta)$ we add some further lines. By Lemma 1 there is a list $\mathbf{\Delta}_1^i, \mathbf{\Delta}_2^i, \ldots$ of all $\mathbf{\Delta}$'s that correspond to Δ. We add a line l_{n+1+j}^i for each $\mathbf{\Delta}_j^i$ that corresponds to Δ:

l^i_{n+j+1} A l^i_{n+1};RC $\mathbf{\Delta}$

Where $\mathcal{P}$ consists of lines $l^0_1, l^0_2, \ldots$, we now combine the proofs $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \ldots$ to a proof $\mathcal{P}'$ that extends $\mathcal{P}$ to the stage $\mathsf{g}(\mathcal{P})$ by means of listing the respective lines as follows (and by renumbering the lines accordingly):

$$l^0_1, l^0_2, l^1_1, l^1_2, l^0_3, l^1_3, l^2_1, l^2_2, l^2_3, l^0_4, \ldots, l^2_4, l^3_1, \ldots, l^3_4, l^0_5, \ldots, l^3_5, l^4_1, \ldots, l^4_5, \ldots \quad (3)$$

Fact 1. *If a line l is marked at stage $\mathsf{g}(\mathcal{P})$, then it is marked in every further extension. Hence, the markings remain stable from stage $\mathsf{g}(\mathcal{P})$ on.*

Note that the marking at a stage is determined by the minimal Dab_i-formulas derived at this stage (where $i \in I$). Since in $\mathsf{g}(\mathcal{P})$ every possible Dab_i-formula is derived on every possible condition, the marking remains stable from $\mathsf{g}(\mathcal{P})$ on.

Some results for flat ALs In order to prove Lemma 8 and Corollary 2 it is useful to first prove some lemmas about flat ALs.

The following fact holds for the extension of an **AL** proof $\mathcal{P}$ to the stage $\mathsf{g}(\mathcal{P})$:

Fact 2. $\Sigma_{\mathsf{g}(\mathcal{P})}(\Gamma) = \Sigma(\Gamma)$ *and hence* $U_{\mathsf{g}(\mathcal{P})}(\Gamma) = U(\Gamma)$ *and* $\Phi_{\mathsf{g}(\mathcal{P})}(\Gamma) = \Phi(\Gamma)$.

The following fact follows immediately by the reflexivity, the monotonicity, and the transitivity of **LLL**.

Fact 3 (Fixed point property for **LLL**). $Cn_{\mathbf{LLL}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{LLL}}(\Gamma)$

The following two lemmas are known to hold where $\Gamma \subseteq \mathcal{W}$ (see [5]). In what follows it is useful to show that they also hold where $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$.[16]

Lemma 2. *Where* $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$ *or* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta)$ *and* $\Delta \cap U(\Gamma) = \emptyset$ *then* $A \in Cn_{\mathbf{AL^r}}(\Gamma)$.

Proof. The case $\Gamma \subseteq \mathcal{W}$ has been proven in [5]. Suppose $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$ and that the antecedent is true. Suppose first $A \in \Omega$. Since $\Delta \cap U(\Gamma) = \emptyset$ and $A \check{\vee} \mathsf{Dab}(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$, also $A \in Cn_{\mathbf{LLL}}(\Gamma) = \Gamma$. Hence we can finally derive A in one step by means of PREM. Suppose now that $A \notin \Omega$. Note that $A \check{\vee} \mathsf{Dab}(\Delta) \in \Gamma$. Hence we can prove A on the condition Δ in two steps: in line 1 we introduce the premise $A \check{\vee} \mathsf{Dab}(\Delta)$ by PREM, in line 2 we derive A on the condition Δ by RC. Since $A \notin \Omega$, line 2 is not marked. Suppose it is marked in an extension $\mathcal{P}$ of the proof, then we can further extend the proof to stage $\mathsf{g}(\mathcal{P})$. In this stage line 2 is not marked due to the supposition, Fact 2 and the marking Definition 1. □

Lemma 3. *Where* $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$ *or* $\Gamma \subseteq \mathcal{W}$*: if for every* $\varphi \in \Phi(\Gamma)$ *there is a* Δ_φ *such that* $\varphi \cap \Delta_\varphi = \emptyset$ *and* $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta_\varphi)$*, then* $A \in Cn_{\mathbf{AL^m}}(\Gamma)$.

Proof. The case $\Gamma \subseteq \mathcal{W}$ is proven in [5]. The other case is similar to the previous proof and left to the reader. □

The following two Lemmas have been proven in [5]:

[16] Note that they do not hold for just any premise set that also contains formulas with "checked" symbols as the example in [25] (for Minimal Abnormality), [9] (for Minimal Abnormality), and [20] (for both strategies) show.

Lemma 4. *If $A \in Cn_{\mathbf{AL^r}}(\Gamma)$ then there is a $\Delta \subseteq \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

Lemma 5. *If $A \in Cn_{\mathbf{AL^m}}(\Gamma)$ then for every $\varphi \in \Phi(\Gamma)$ there is a $\Delta_\varphi \subseteq \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta_\varphi)$ and $\Delta_\varphi \cap \varphi = \emptyset$.*

Lemma 6. *Where $\Gamma = Cn_{\mathbf{LLL}}(\Gamma)$ or $\Gamma \subseteq \mathcal{W}$: $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$.*

Proof. The left-right direction is trivial due to the reflexivity of **LLL**. Suppose $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$. By the compactness of **LLL** there are $B_1, \ldots, B_n \in Cn_{\mathbf{AL}}(\Gamma)$ such that $B_1, \ldots, B_n \vdash_{\mathbf{LLL}} A$. Suppose the strategy of **AL** is Reliability. By Lemma 4, for each $i \leq n$ there is a $\Delta_i \subseteq \Omega$ such that $\Delta_i \cap U(\Gamma) = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} B_i \check{\vee} \mathsf{Dab}(\Delta_i)$. Hence $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta_1 \cup \ldots \cup \Delta_n)$. Since $(\Delta_1 \cup \ldots \cup \Delta_n) \cap U(\Gamma) = \emptyset$, by Lemma 2 also $A \in Cn_{\mathbf{AL}}(\Gamma)$. The case for Minimal Abnormality is similar and left to the reader (we use Lemmas 3 and 5 instead of Lemmas 2 and 4). □

Lemma 7 (Dab-conservatism of **AL**)**.** *If $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$ then $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$.*

Proof. Suppose $\Gamma \vdash_{\mathbf{AL^x}} \mathsf{Dab}(\Delta)$. *Case 1:* $\mathbf{x} = \mathbf{r}$. By Lemma 4, $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta) \check{\vee} \mathsf{Dab}(\Theta)$, for $\Theta \subseteq \Omega \backslash U(\Gamma)$. Let $\Delta' \subseteq \Delta$ and $\Theta' \subseteq \Theta$ be minimal such that $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta') \check{\vee} \mathsf{Dab}(\Theta')$. It follows that $\Delta' \cup \Theta' \subseteq U(\Gamma)$. If $\Theta' \neq \emptyset$, then $\Theta \cap U(\Gamma) \neq \emptyset$ — a contradiction. Hence $\Theta' = \emptyset$, which means that $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta')$, and by **CL**-properties, $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta)$.

Case 2: $\mathbf{x} = \mathbf{m}$. By Lemma 5, for every $\varphi \in \Phi(\Gamma)$, there is a $\Theta_\varphi \subseteq \Omega \setminus \varphi$ such that $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta) \check{\vee} \mathsf{Dab}(\Theta_\varphi)$. Let each $\Delta_\varphi \subseteq \Delta$ and $\Theta'_\varphi \subseteq \Theta_\varphi$ be minimal such that $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta_\varphi) \check{\vee} \mathsf{Dab}(\Theta'_\varphi)$. It follows that each $\mathsf{Dab}(\Delta_\varphi \cup \Theta'_\varphi)$ is a minimal Dab-consequence of Γ.

Assume now that each $\Theta'_\varphi \neq \emptyset$. Let ψ be a minimal choice set of $\{\Theta'_\varphi \mid \varphi \in \Phi(\Gamma)\}$ and let ψ' be a minimal choice set of $\{\Lambda \in \Sigma(\Gamma) \mid \Lambda \cap \psi = \emptyset\}$. It can be easily verified that $\psi \cup \psi'$ is a minimal choice set of $\Sigma(\Gamma)$.[17] It follows that there is a $\Theta_{\psi \cup \psi'} \subseteq \Omega \setminus (\psi \cup \psi')$ such that $\Gamma \vdash_{\mathbf{LLL}} \mathsf{Dab}(\Delta) \check{\vee} \mathsf{Dab}(\Theta_{\psi \cup \psi'})$. But in view of the construction, there is a $B \in \Theta_{\psi \cup \psi'}$ such that $B \in \psi$,— a contradiction. Hence, some $\Theta'_\varphi = \emptyset$. Thus, $\mathsf{Dab}(\Delta_\varphi)$ is a minimal Dab-consequence and hence $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$. □

Some results for SAL and $\mathbf{SAL_i}$

Lemma 8 (**LLL**-closure of $\mathbf{SAL_i}$)**.** *Where $\Gamma \subseteq \mathcal{W}$: $Cn_{\mathbf{SAL_i}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$.*

Proof. "$i = 1$": This follows by Lemma 6. "$i \Rightarrow i{+}1$": Note that $Cn_{\mathbf{SAL_{i+1}}}(\Gamma) = Cn_{\mathbf{AL_{i+1}}}(Cn_{\mathbf{SAL_i}}(\Gamma))$. By the induction hypothesis and Lemma 6, $Cn_{\mathbf{SAL_{i+1}}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL_{i+1}}}(Cn_{\mathbf{SAL_i}}(\Gamma))) = Cn_{\mathbf{SAL_{i+1}}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_{i+1}}}(\Gamma))$. □

Corollary 2. *Where $\Gamma \subseteq \mathcal{W}$: $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$ iff there is a $\Delta \subseteq \Omega_i$ such that $A \check{\vee} \mathsf{Dab}(\Delta) \in Cn_{\mathbf{SAL_{i-1}}}(\Gamma)$ and*

1. *where $\mathbf{x_i} = \mathbf{r}$, $\Delta \cap U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma)) = \emptyset$, or*
2. *where $\mathbf{x_i} = \mathbf{m}$, there is a $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ such that $\varphi \cap \Delta = \emptyset$ and for each $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ there is a $\Theta \subseteq \Omega_i$ such that $A \check{\vee} \mathsf{Dab}(\Theta) \in Cn_{\mathbf{SAL_{i-1}}}(\Gamma)$ and $\Theta \cap \varphi = \emptyset$.*

Proof. "$i = 1$" follows directly Lemmas 2, 3, 4, 5 and the fact that $Cn_{\mathbf{SAL_1}}(\Gamma) = Cn_{\mathbf{AL_1}}(\Gamma)$. "$i \Rightarrow i + 1$": Note that $Cn_{\mathbf{SAL_{i+1}}}(\Gamma) = Cn_{\mathbf{AL_{i+1}}}(Cn_{\mathbf{SAL_i}}(\Gamma))$ and by Lemma 8 $Cn_{\mathbf{SAL_i}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$. Thus, the corollary follows by Lemmas 2, 3, 4, and 5. □

[17] The reasoning proceeds wholly analogous to the proof of Lemma 6 in [21].

Now we are also able to prove Theorem 3 which we restate here.[18]

Theorem 3 (**LLL**-closure of **SAL**). *Where* $\Gamma \subseteq \mathcal{W}$: $Cn_{\mathbf{SAL}}(\Gamma) = Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$.

Proof. The left-right direction follows immediately due to the reflexivity of **LLL**. Suppose now $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL}}(\Gamma))$. By the compactness of **LLL** there is a finite $\Gamma' \subseteq Cn_{\mathbf{SAL}}(\Gamma)$ such that $A \in Cn_{\mathbf{LLL}}(\Gamma')$. For each $B \in \Gamma'$ there is a $i \in I$ such that $B \in Cn_{\mathbf{SAL_i}}(\Gamma)$. Let k be the maximal such i. By Theorem 2, $\Gamma' \subseteq Cn_{\mathbf{SAL_k}}(\Gamma)$. By Lemma 8, $A \in Cn_{\mathbf{SAL_k}}(\Gamma)$ and hence $A \in Cn_{\mathbf{SAL}}(\Gamma)$. □

The following corollary follows immediately by Lemma 7 and Lemma 8.

Corollary 3 (Dab-conservatism of $\mathbf{SAL_i}$). *Where* $\Delta \subset \Omega_i$: *if* $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$ *then* $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{SAL_{i-1}}}(\Gamma)$.

The adequacy of the proof theory from Section 4 As we will see below, given some proof $\mathcal{P}$, a formula A is derived on an unmarked line at stage $\mathsf{g}(\mathcal{P})$ iff $A \in Cn_{\mathbf{SAL}}(\Gamma)$.

Lemma 9. *Where* $\Gamma \subseteq \mathcal{W}$ *and* $\mathcal{P}$ *is a* **SAL**-*proof from* Γ, *each of the following holds for every* $i \in I$:

1a. ${}^{\mathbf{C}}\Sigma^i_{\mathsf{g}(\mathcal{P})}(\Gamma) = \Sigma^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$, *whence also*
1b. ${}^{\mathbf{C}}U^i_{\mathsf{g}(\mathcal{P})}(\Gamma) = U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ *and*
1c. ${}^{\mathbf{C}}\Phi^i_{\mathsf{g}(\mathcal{P})}(\Gamma) = \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$
2a. there is a line l with formula A and condition $\langle\Delta_j\rangle_{j\in I}$ *that is not j-marked for all* $j \leq i$ *at stage* $\mathsf{g}(\mathcal{P})$ *iff* $A \check{\vee} \mathsf{Dab}(\Delta_{i+1} \cup \Delta_{i+2} \cup \ldots) \in Cn_{\mathbf{SAL_i}}(\Gamma)$, *and hence*
2b. there is a line l with formula A and a condition $\langle\Delta_1, \ldots, \Delta_i, \emptyset, \ldots\rangle$ *that is not marked at stage* $\mathsf{g}(\mathcal{P})$ *iff* $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$.

Proof. "$i = 1$": *Ad 1.* Immediate in view of Fact 3.

Ad 2. Case $\mathbf{x_1} = \mathbf{r}$. There is a line with formula A and with a condition $\mathbf{\Delta}$ that is 1-unmarked iff [by the construction of stage $\mathsf{g}(\mathcal{P})$ and Definition 5] $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\mathbf{\Delta})$ and $\Delta_1 \cap {}^{\mathbf{C}}U^1_{\mathsf{g}(\mathcal{P})}(\Gamma) = \emptyset$ iff [by 1.] $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\mathbf{\Delta})$ and $\Delta_1 \cap U^1(Cn_{\mathbf{LLL}}(\Gamma)) = \emptyset$ iff [by Fact 3] $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\mathbf{\Delta})$ and $\Delta_1 \cap U^1(\Gamma) = \emptyset$ iff [by Corollary 2.1] $A \check{\vee} \mathsf{Dab}(\Delta_2 \cup \Delta_3 \cup \ldots) \in Cn_{\mathbf{SAL_1}}(\Gamma)$.

Case $\mathbf{x_1} = \mathbf{m}$. There is a line with formula A and with a condition $\mathbf{\Delta}$ that is 1-unmarked iff [by Definition 7] for each $\varphi \in {}^{\mathbf{C}}\Phi^i_{\mathsf{g}(\mathcal{P})}(\Gamma)$ there is a line with formula A and a condition $\langle\Theta_\varphi, \Delta_2, \Delta_3, \ldots\rangle$ such that $\varphi \cap \Theta_\varphi = \emptyset$ and $\Delta_1 = \Theta_\varphi$ for some $\varphi \in {}^{\mathbf{C}}\Phi^i_{\mathsf{g}(\mathcal{P})}(\Gamma)$ iff [by the construction of stage $\mathsf{g}(\mathcal{P})$ and 1.] for each $\varphi \in \Phi^1(Cn_{\mathbf{LLL}}(\Gamma))$ there is a Θ_φ such that $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Theta_\varphi \cup \Delta_2 \cup \ldots)$ iff [by Fact 3 and Corollary 2.2] $A \check{\vee} \mathsf{Dab}(\Delta_2 \cup \Delta_3 \cup \ldots) \in Cn_{\mathbf{SAL_1}}(\Gamma)$.

"$i \Rightarrow i{+}1$": *Ad 1.* Where $\Delta \subset \Omega_{i+1}$, the following are equivalent in view of (1) the definition of ${}^{\mathbf{C}}\Sigma^i_{\mathsf{g}(\mathcal{P})}(\Gamma)$, (2) item 2b of the induction hypothesis, (3) Lemma 8 and (4) the definition of $\Sigma^i(\Gamma)$:

- $\Delta \in {}^{\mathbf{C}}\Sigma^{i+1}_{\mathsf{g}(\mathcal{P})}(\Gamma)$
- $Dab(\Delta)$ is derived at an unmarked $[\leq i]$-line at stage $\mathsf{g}(\mathcal{P})$ and for no $\Delta' \subset \Delta$: $Dab(\Delta')$ is derived at an unmarked $[\leq i]$-line at stage $\mathsf{g}(\mathcal{P})$

[18] Note that throughout the main paper we presupposed that $\Gamma \subseteq \mathcal{W}$. This supposition was dropped in the Appendix, whence the slightly different formulation of the Theorem here.

- $Dab(\Delta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$, and for no $\Delta' \subset \Delta$: $Dab(\Delta') \in Cn_{\mathbf{SAL_i}}(\Gamma)$
- $Dab(\Delta) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$, and for no $\Delta' \subset \Delta$: $Dab(\Delta') \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$
- $\Delta \in \Sigma^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$.

Ad 2. Case $\mathbf{x_{i+1}} = \mathbf{r}$. At stage $\mathsf{g}(\mathcal{P})$, each of the following are equivalent in view of (1) Definition 5, (2) item 1b and (3) item 2a of the induction hypothesis, (4) Corollary 2.1 and Lemma 8:

- there is a line l with formula A and condition $\langle \Delta_j \rangle_{j \in I}$ that is not j-marked for any $j \leq i{+}1$
- there is a line l with formula A and condition $\langle \Delta_j \rangle_{j \in I}$ that is not j-marked for any $j \leq i$, and $\Delta_{i+1} \cap {}^{\mathbf{C}}U^{i+1}_{\mathsf{g}(\mathcal{P})}(\Gamma) = \emptyset$
- there is a line l with formula A and condition $\langle \Delta_j \rangle_{j \in I}$ that is not j-marked for any $j \leq i$, and $\Delta_{i+1} \cap U^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \emptyset$
- there are $\Delta_{i+1} \subset \Omega_{i+1}, \Delta_{i+2} \subset \Omega_{i+2}, \ldots$, such that $A \check{\vee} Dab(\Delta_{i+1} \cup \Delta_{i+2} \cup \ldots) \in Cn_{\mathbf{SAL_i}}(\Gamma)$ and $\Delta_{i+1} \cap U^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \emptyset$
- there are $\Delta_{i+2} \subset \Omega_{i+2}, \Delta_{i+3} \subset \Omega_{i+3}, \ldots$, such that $A \check{\vee} Dab(\Delta_{i+2} \cup \Delta_{i+3} \cup \ldots) \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$

Case $\mathbf{x_{i+1}} = \mathbf{m}$. At stage $\mathsf{g}(\mathcal{P})$, each of the following are equivalent in view of (1) Definition 7, (2) item 1c, (3) item 2a of the induction hypothesis and (4) Corollary 2.2 and Lemma 8:

- there is a line l with formula A and condition $\langle \Delta_j \rangle_{j \in I}$ that is not j-marked for any $j \leq i{+}1$
- there is a line l with formula A and condition $\langle \Delta_j \rangle_{j \in I}$ such that
  (a) l is not j-marked for any $j \leq i$,
  (b) $\Delta_{i+1} \cap \varphi = \emptyset$ for a $\varphi \in {}^{\mathbf{C}}\Phi^{i+1}_{\mathsf{g}(\mathcal{P})}(\Gamma)$, and
  (c) for every $\varphi \in {}^{\mathbf{C}}\Phi^{i+1}_{\mathsf{g}(\mathcal{P})}(\Gamma)$: A is derived on a line l_φ with condition $\langle \Theta_1, \ldots, \Theta_{i+1}, \Delta_{i+2}, \Delta_{i+3}, \ldots \rangle$ such that $\Theta_{i+1} \cap \varphi = \emptyset$, and each line l_φ is not j-marked for any $j \leq i$
- there is a line l with formula A and condition $\langle \Delta_j \rangle_{j \in I}$ such that
  (a) l is not j-marked for any $j \leq i$,
  (b) $\Delta_{i+1} \cap \varphi = \emptyset$ for a $\varphi \in \Phi^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$, and
  (c) for every $\varphi \in \Phi^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$: A is derived on a line l_φ with condition $\langle \Theta_1, \ldots, \Theta_{i+1}, \Delta_{i+2}, \Delta_{i+3}, \ldots \rangle$ such that $\Theta_{i+1} \cap \varphi = \emptyset$, and each line l_φ is not j-marked for any $j \leq i$
- There are $\Delta_{i+2} \subset \Omega_{i+2}, \Delta_{i+3} \subset \Omega_{i+3}, \ldots$, such that for every $\varphi \in \Phi^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$, $A \check{\vee} Dab(\Theta_{i+1} \cup \Delta_{i+2} \cup \Delta_{i+3} \cup \ldots) \in Cn_{\mathbf{SAL_i}}(\Gamma)$ for a $\Theta_{i+1} \subseteq \Omega_{i+1} \setminus \varphi$
- $A \check{\vee} Dab(\Delta_{i+2} \cup \Delta_{i+3} \cup \ldots) \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$ □

Lemma 10. *Where* $\Gamma \subseteq \mathcal{W}$ *and* $\mathcal{P}$ *is a* **SAL***-proof from* Γ*:* $A \in Cn_{\mathbf{SAL}}(\Gamma)$ *iff* A *is derived at an unmarked line at stage* $\mathsf{g}(\mathcal{P})$*.*

Proof. $A \in Cn_{\mathbf{SAL}}(\Gamma)$ iff there is an $i \in I$ for which $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$ iff [by Lemma 9.2b] A is derived on an unmarked line l at stage $\mathsf{g}(\mathcal{P})$ with some condition $\boldsymbol{\Delta} = \langle \Delta_1, \ldots, \Delta_i, \emptyset, \emptyset, \ldots \rangle$ iff A is derived at an unmarked line at stage $\mathsf{g}(\mathcal{P})$. □

Theorem 7. *Where* $\Gamma \subseteq \mathcal{W}$*: if* $\Gamma \vdash_{\mathbf{SAL}} A$*, then* $A \in Cn_{\mathbf{SAL}}(\Gamma)$*.*

Proof. Suppose $\Gamma \vdash_{\mathbf{SAL}} A$. By Definition 6, A is derived at an unmarked line l of a finite **SAL**-proof $\mathcal{P}$ from Γ. Suppose we extend $\mathcal{P}$ to stage $\mathsf{g}(\mathcal{P})$. By Fact 1, if l

is marked in this extension, then l is marked in every further extension of the proof, which contradicts the fact that A is finally derived at line l. Hence line l is unmarked at stage $\mathsf{g}(\mathcal{P})$. By Lemma 10, $A \in Cn_{\mathbf{SAL}}(\Gamma)$. □

Theorem 8. *Where $\Gamma \subseteq \mathcal{W}$: if $A \in Cn_{\mathbf{SAL}}(\Gamma)$, then $\Gamma \vdash_{\mathbf{SAL}} A$.*

Proof. Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$. By Lemma 10, (†) A is derived at an unmarked line l with condition $\mathbf{\Delta}$ at stage $\mathsf{g}(\mathcal{P}_\varepsilon)$. In view of Lemma 9.1 and the marking definitions 5 and 7, we can infer that for all $i \in I$ and any **SAL**-proof $\mathcal{P}'$ from Γ:

($\dagger^{\mathbf{r}}$) where $\mathbf{x_i} = \mathbf{r}$: $\Delta_i \cap U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma)) = \Delta_i \cap {}^{\mathbf{C}}U^i_{\mathsf{g}(\mathcal{P}')}(\Gamma) = \Delta_i \cap {}^{\mathbf{C}}U^i_{\mathsf{g}(\mathcal{P}_\varepsilon)}(\Gamma) = \emptyset$

($\dagger^{\mathbf{m}}$) where $\mathbf{x_i} = \mathbf{m}$: $\Delta_i \cap \varphi = \emptyset$, for a $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma)) = {}^{\mathbf{C}}\Phi^i_{\mathsf{g}(\mathcal{P}')}(\Gamma) = {}^{\mathbf{C}}\Phi^i_{\mathsf{g}(\mathcal{P}_\varepsilon)}(\Gamma)$

By the construction of stage $\mathsf{g}(\mathcal{P}_\varepsilon)$, $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\mathbf{\Delta})$, whence by the compactness of **LLL**, there is a $\Gamma' = \{B_1, \ldots, B_m\} \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\mathbf{\Delta})$.

Let the **SAL**-proof $\mathcal{P}$ be constructed as follows. At line 1 we introduce the premise B_1 by PREM, …, and at line m we introduce the premise B_m by PREM. At line $m{+}1$ we derive A by RC on the condition $\mathbf{\Delta}$. Let s be the stage consisting of lines 1 up to $m{+}1$.

Since $\Gamma' \subseteq \Gamma \subseteq \mathcal{W}$, for every $i \in I$, all Dab_i-formulas that are derived at stage s (if any) are singletons $C \in \Omega_i$. Moreover, by the reflexivity of each logic $\mathbf{SAL_i}$, for every such C, $C \in Cn_{\mathbf{SAL_{i-1}}}(\Gamma)$, whence also $C \in U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ and $C \in \varphi$ for every $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$. Hence for every $i \in I$:

($\ddagger^{\mathbf{r}}$) ${}^{\mathbf{C}}U^i_s(\Gamma) \subseteq U^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$

($\ddagger^{\mathbf{m}}$) $\bigcup {}^{\mathbf{C}}\Phi^i_s(\Gamma) \subseteq \varphi$ for every $\varphi \in \Phi^i(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$

By ($\dagger^{\mathbf{r}}$), ($\dagger^{\mathbf{m}}$), ($\ddagger^{\mathbf{r}}$) and ($\ddagger^{\mathbf{m}}$), we can infer that there is no $i \in I$ such that line $m{+}1$ is i-marked at stage s. Suppose that line $m{+}1$ is marked in an extension $\mathcal{P}'$ of the proof. In that case, we may further extend the proof to stage $\mathsf{g}(\mathcal{P}')$. Hence in view of ($\dagger^{\mathbf{r}}$) and ($\dagger^{\mathbf{m}}$), line $m{+}1$ is unmarked in the second extension. By Definition 6, A is finally derived at stage s. □

The adequacy of the proof theory from Section 5 First we have to show how we construct a complete stage $\mathsf{g}(\mathcal{P})$ for a given **SAL**-proof $\mathcal{P}$ from Γ. Let again $\Theta = \{B_1, B_2, \ldots\} = \{A \mid \Gamma \vdash_{\mathbf{LLL}} A\}$. For each $B_i \in \Theta$ we have the following proof $\mathcal{P}_i$:

l^i_1	A_1	PREM	$\emptyset$
⋮	⋮	⋮	⋮
l^i_n	A_n	PREM	$\emptyset$
l^i_{n+1}	B_i	$l^i_1, \ldots, l^i_n$;RU	$\emptyset$

In case B_i is of the form $A \check{\vee} \mathsf{Dab}(\Delta)$ we extend the proof further. By Lemma 1 there is a list $\{\mathbf{\Delta}^1, \mathbf{\Delta}^2, \ldots\}$ of all $\mathbf{\Delta}$'s that correspond to Δ. For each $\mathbf{\Delta}^j = \langle \Delta^j_1, \ldots, \Delta^j_m, \emptyset, \ldots \rangle$ we append a sub-proof $\mathcal{P}^j_i$ to $\mathcal{P}_i$ as follows:

$l^{i,j}_1$	$A \check{\vee} \mathsf{Dab}(\Delta^j_2 \cup \ldots \cup \Delta^j_m)$	l^i_{n+1}; RC$_1$	$\langle \Delta^j_1, \emptyset, \ldots \rangle$
⋮	⋮	⋮	⋮
$l^{i,j}_{m-1}$	$A \check{\vee} \mathsf{Dab}(\Delta^j_m)$	$l^{i,j}_{m-1}$; RC$_{m-1}$	$\langle \Delta^j_1, \ldots, \Delta^j_{m-1}, \emptyset, \ldots \rangle$
$l^{i,j}_m$	A	$l^{i,j}_m$; RC$_m$	$\langle \Delta^j_1, \ldots, \Delta^j_m, \emptyset, \ldots \rangle$

We combine the proofs $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \ldots$ by means of the construction in (3) to an extension of $\mathcal{P}$ at stage $\mathsf{g}(\mathcal{P})$.

Analogous to Fact 1 we get:

Fact 4. *The markings of the lines at stage $\mathsf{g}(\mathcal{P})$ remain stable in every further extension of the proof at stage $\mathsf{g}(\mathcal{P})$.*

Lemma 11. *Where $\Gamma \subseteq \mathcal{W}$ and $\mathcal{P}$ is a* **SAL***-proof from Γ, we have for each $i \in I$:*

1. ${}^{\mathbf{C}}\Sigma^{i}_{\mathsf{g}(\mathcal{P})}(\Gamma) = \Sigma^{i}(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ *and hence* ${}^{\mathbf{C}}U^{i}_{\mathsf{g}(\mathcal{P})}(\Gamma) = U^{i}(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$ *and* ${}^{\mathbf{C}}\Phi^{i}_{\mathsf{g}(\mathcal{P})}(\Gamma) = \Phi^{i}(Cn_{\mathbf{SAL_{i-1}}}(\Gamma))$*;*
2. *there is a $[\leq i]$-line l with formula A and that is unmarked at stage $\mathsf{g}(\mathcal{P})$ iff $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$.*

Proof. "i=1": *Ad 1.* Immediate in view of Fact 3.

Ad 2. Case $\mathbf{x_1} = \mathbf{r}$. There is a $[\leq 1]$-line l with formula A and condition $\boldsymbol{\Delta}$ that is unmarked iff [by the construction of stage $\mathsf{g}(\mathcal{P})$ and Definition 8] $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\boldsymbol{\Delta})$ and $\bigcup \boldsymbol{\Delta} \cap {}^{\mathbf{C}}U^{1}_{\mathsf{g}(\mathcal{P})}(\Gamma) = \emptyset$, iff [by 1.] $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\boldsymbol{\Delta})$ and $\bigcup \boldsymbol{\Delta} \cap U^{1}(Cn_{\mathbf{LLL}}(\Gamma)) = \emptyset$, iff [by Fact 3] $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\boldsymbol{\Delta})$ and $\bigcup \boldsymbol{\Delta} \cap U^{1}(\Gamma) = \emptyset$, iff [by Lemma 2 and 4] $A \in Cn_{\mathbf{AL_1}}(\Gamma)$, iff $A \in Cn_{\mathbf{SAL_1}}(\Gamma)$.

Case $\mathbf{x_1} = \mathbf{m}$. The proof is similar and left to the reader.

"$i \Rightarrow i{+}1$": *Ad 1.* Where $\Delta \subseteq \Omega_{i+1}$, we have: $\Delta \in {}^{\mathbf{C}}\Sigma^{i+1}_{\mathsf{g}(\mathcal{P})}(\Gamma)$ iff $\mathsf{Dab}(\Delta)$ is derived at an unmarked $[\leq i]$-line and there is no $\Delta' \subset \Delta$ such that $\mathsf{Dab}(\Delta')$ is derived at an unmarked $[\leq i]$-line, iff [by 2. and the induction hypothesis] $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$ and for no $\Delta' \subset \Delta$, $\mathsf{Dab}(\Delta') \in Cn_{\mathbf{SAL_i}}(\Gamma)$, iff [by Lemma 8] $\mathsf{Dab}(\Delta) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$ and for no $\Delta' \subset \Delta$, $\mathsf{Dab}(\Delta') \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{SAL_i}}(\Gamma))$, iff $\Delta \in \Sigma^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma))$.

Ad 2. Case $\mathbf{x_{i+1}} = \mathbf{r}$. Let l be some $[\leq i{+}1]$-line with formula A and condition $\boldsymbol{\Delta}$. Suppose line l is unmarked. If l is a j-line with $j \leq i$ we get $A \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$ due to the induction hypothesis and Theorem 2. Thus, suppose l is a $i{+}1$-line. We prove the statement by another induction on the number of steps j needed to derive A. "$j = 1$": Only premises can be introduced in one inference step, but this does not lead to a $i{+}1$-line. "$j = 2$": The proof looks as follows: A is derived by RC_{i+1} from some line l' at which some B is introduced as a premise and $B \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta_{i+1})$. Since l is unmarked at stage $\mathsf{g}(\mathcal{P})$, $\Delta_{i+1} \cap {}^{\mathbf{C}}U^{i+1}_{\mathsf{g}(\mathcal{P})}(\Gamma) = \emptyset$ and hence by 1., $\Delta_{i+1} \cap U^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \emptyset$. By Corollary 2.1, $A \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$. "$j \Rightarrow j+1$": Suppose A is derived with the justification $l_1, \ldots, l_n; R$ where $R \in \{\mathrm{RU}, \mathrm{RC}_{i+1}\}$ and each line l_k (where $1 \leq k \leq n$) features a formula A_k and a condition $\boldsymbol{\Delta}^k$. By the definition of RU and RC_{i+1}, (†) $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Delta'_{i+1})$ for some (possibly empty) $\Delta'_{i+1} \subseteq \Delta_{i+1} \subset \Omega_{i+1}$, and $\Delta'_{i+1} \cup \Delta^{1}_{i+1} \cup \ldots \cup \Delta^{n}_{i+1} = \Delta_{i+1}$. Since l is unmarked, (a) by Definition 8 and 1., $\Delta'_{i+1} \cap U^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \Delta'_{i+1} \cap {}^{\mathbf{C}}U^{i+1}_{\mathsf{g}(\mathcal{P})}(\Gamma) = \emptyset$, (b) by the definition of inh-marking each of the lines l_k is neither o-marked for any $o \leq i$ nor inh-marked, (c) neither line l_k is $i{+}1$-marked since $\Delta^{k}_{i+1} \subseteq \Delta_{i+1}$ and by (a). By our induction hypothesis, (b) and (c), $A_k \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$ and by (†) and Lemma 8 also (‡) $A \check{\vee} \mathsf{Dab}(\Delta'_{i+1}) \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$. By (a), $\check{\neg} \mathsf{Dab}(\Delta'_{i+1}) \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$ and hence by Lemma 8 and (‡), $A \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$.

For the other direction suppose $A \in Cn_{\mathbf{SAL_{i+1}}}(\Gamma)$. By Corollary 2.1 there is a $\Delta \subseteq \Omega_{i+1}$ for which $A \check{\vee} \mathsf{Dab}(\Delta) \in Cn_{\mathbf{SAL_i}}(\Gamma)$ and $\Delta \cap U^{i+1}(Cn_{\mathbf{SAL_i}}(\Gamma)) = \emptyset$. By the induction hypothesis there is an unmarked $[\leq i]$-line l at which $A \check{\vee} \mathsf{Dab}(\Delta)$ is derived on some condition $\boldsymbol{\Delta}$. By the construction of stage $\mathsf{g}(\mathcal{P})$ there is a line l' with formula A, justification $l; \mathrm{RC}_{i+1}$ and condition $\langle \Delta_1, \ldots, \Delta_i, \Delta, \emptyset, \ldots \rangle$. Since by 1., $\Delta \cap {}^{\mathbf{C}}U^{i+1}_{\mathsf{g}(\mathcal{P})}(\Gamma) = \emptyset$, line l' is not marked according to the $i{+}1$-marking with Reliability. Moreover, since l is unmarked, l' is also not inh-marked.

Let $\mathbf{x_{i+1}} = \mathbf{m}$. The proof is similar and left to the reader. □

Since $A \in Cn_{\mathbf{SAL}}(\Gamma)$ iff there is an $i \in I$ such that $A \in Cn_{\mathbf{SAL_i}}(\Gamma)$, we get by item 2 of the previous lemma:

Corollary 4. *Where $\Gamma \subseteq \mathcal{W}$ and $\mathcal{P}$ is a* **SAL***-proof from Γ: $A \in Cn_{\mathbf{SAL}}(\Gamma)$ iff A is derived at an unmarked line at stage $\mathsf{g}(\mathcal{P})$.*

Theorem 9. *Where $\Gamma \subseteq \mathcal{W}$: if $\Gamma \vdash_{\mathbf{SAL}} A$ then $A \in Cn_{\mathbf{SAL}}(\Gamma)$.*

Proof. Suppose $\Gamma \vdash_{\mathbf{SAL}} A$. Hence, there is a finite **SAL**-proof $\mathcal{P}$ in which A is finally derived at some line l. We extend $\mathcal{P}$ to stage $\mathsf{g}(\mathcal{P})$. By Definition 6 and Fact 4, line l is unmarked and hence $A \in Cn_{\mathbf{SAL}}(\Gamma)$ by Corollary 4. □

Suppose A is derived at an unmarked line at stage $\mathsf{g}(\mathcal{P}_\varepsilon)$. Let $\underline{\Delta} = \{\mathbf{\Delta}_1, \mathbf{\Delta}_2, \ldots\}$ be the set of conditions on which A is derived at stage $\mathsf{g}(\mathcal{P}_\varepsilon)$ at an unmarked line. We say that $\mathbf{\Delta}$ is *a minimal sequence for A* iff $\mathbf{\Delta} \in \min_{\prec}(\underline{\Delta})$ where $\prec$ is the partial order defined as follows: $\mathbf{\Delta} \prec \mathbf{\Theta}$ iff there is a $k \in I$ such that (i) for all $j \in I$ for which $j > k$, $\Delta_j = \Theta_j$, and (ii) $\emptyset = \Delta_k \subset \Theta_k$.[19] Obviously,

Fact 5. *If A is derived at an unmarked line at stage $\mathsf{g}(\mathcal{P}_\varepsilon)$, then there is a minimal such sequence for A (there may be many).*

Lemma 12. *Where $\mathbf{\Theta} = \langle \Theta_1, \ldots, \Theta_m, \emptyset, \ldots \rangle$ is a minimal sequence for A: there are no $l, k \leq m$ such that $l \leq k$, $\{A\} \cup \Theta_l \cup \ldots \cup \Theta_m \subseteq \Omega_k$, and $\Theta_k \neq \emptyset$.*

Proof. Assume there are l and k such that $\{A\} \cup \Theta_l \cup \ldots \cup \Theta_m \subseteq \Omega_k$, $l \leq k$ and $\Theta_k \neq \emptyset$. *Case $k < m$:* Evidently also (†) $\{A\} \cup \Theta_{k+1} \cup \ldots \cup \Theta_m \subseteq \Omega_k$. Since, by the construction of stage $\mathsf{g}(\mathcal{P}_\varepsilon)$ and Lemma 11.2, $A \check{\vee} \mathsf{Dab}(\Theta_{k+1} \cup \ldots \cup \Theta_m) \in Cn_{\mathbf{SAL_k}}(\Gamma)$, by Corollary 3 and (†) also $A \check{\vee} \mathsf{Dab}(\Theta_{k+1} \cup \ldots \cup \Theta_m) \in Cn_{\mathbf{SAL_{k-1}}}(\Gamma)$. Hence, by Lemma 11.2, there is an unmarked $[\leq k{-}1]$-line at stage $\mathsf{g}(\mathcal{P}_\varepsilon)$ at which $A \check{\vee} \mathsf{Dab}(\Theta_{k+1} \cup \ldots \cup \Theta_m)$ is derived on some condition $\langle \Delta_1, \ldots, \Delta_{k-1}, \emptyset, \ldots \rangle$. Note that also $\mathbf{\Delta} = \langle \Delta_1, \ldots, \Delta_{k-1}, \emptyset, \Theta_{k+1}, \ldots, \Theta_m, \emptyset, \ldots \rangle$ is a sequence for A. But then, $\mathbf{\Delta} \prec \mathbf{\Theta}$ since (i) for all $i > k$, $\Delta_i = \Theta_i$ and (ii) $\emptyset = \Delta_k \subset \Theta_k$. Thus, $\mathbf{\Theta}$ is not a minimal sequence for A,— a contradiction.

Case $k = m$: Evidently also $A \in \Omega_m$. Since by Lemma 11.2, $A \in Cn_{\mathbf{SAL_m}}(\Gamma)$, by Corollary 3, also $A \in Cn_{\mathbf{SAL_{m-1}}}(\Gamma)$. Hence, by Lemma 11.2, there is an unmarked $[\leq m{-}1]$-line at stage $\mathsf{g}(\mathcal{P}_\varepsilon)$ at which A is derived on a condition $\mathbf{\Delta} = \langle \Delta_1, \ldots, \Delta_{m-1}, \emptyset, \ldots \rangle$. Note that also $\mathbf{\Delta}$ is a sequence for A. But then $\mathbf{\Delta} \prec \mathbf{\Theta}$ since (i) for all $i > m$, $\Delta_i = \Theta_i = \emptyset$, and (ii) $\emptyset = \Delta_m \subset \Theta_m$. Thus, $\mathbf{\Theta}$ is not a minimal sequence for A,— a contradiction. □

Theorem 10. *Where $\Gamma \subseteq \mathcal{W}$: if $A \in Cn_{\mathbf{SAL}}(\Gamma)$ then $\Gamma \vdash_{\mathbf{SAL}} A$.*

Proof. Suppose $A \in Cn_{\mathbf{SAL}}(\Gamma)$. Hence by Corollary 4, A is derived at an unmarked line at stage $\mathsf{g}(\mathcal{P}_\varepsilon)$. Let (†) $\mathbf{\Theta} = \langle \Theta_1, \ldots, \Theta_n, \emptyset, \ldots \rangle$ be a minimal sequence for A. Since $\Gamma \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Theta_1 \cup \ldots \cup \Theta_n)$ and by the compactness of **LLL**, there are $B_1, \ldots, B_o \in \Gamma$ such that $\{B_1, \ldots, B_0\} \vdash_{\mathbf{LLL}} A \check{\vee} \mathsf{Dab}(\Theta_1 \cup \ldots \cup \Theta_n)$. We now construct a **SAL**-proof $\mathcal{P}$ for A as follows:

[19] E.g., where $\Delta_1, \Delta_2, \Delta_3, \Delta_4 \neq \emptyset$, we have $\langle \Delta_1, \Delta_2, \Delta_3, \emptyset, \emptyset, \ldots \rangle \prec \langle \Delta_1, \emptyset, \Delta_3, \Delta_4, \emptyset, \emptyset, \ldots \rangle$ and $\langle \Delta_1, \emptyset, \emptyset, \Delta_4, \emptyset, \emptyset, \ldots \rangle \prec \langle \Delta_2, \Delta_3, \emptyset, \Delta_4, \emptyset, \emptyset, \ldots \rangle$.

1	B_1	PREM	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
o	B_0	PREM	$\emptyset$
$o{+}1$	$A \check{\vee} \mathsf{Dab}(\Theta_1 \cup \ldots \cup \Theta_m)$	$1, \ldots, o$; RU	$\emptyset$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$o{+}n$	$A \check{\vee} \mathsf{Dab}(\Theta_n)$	$o{+}n{-}1$; RC_{n-1}	$\langle \Theta_1, \ldots, \Theta_{n-1} \rangle$
$o{+}n{+}1$	A	$o{+}n$; RC_n	$\langle \Theta_1, \ldots, \Theta_n \rangle$

Let s be the stage of our proof. Since $\Gamma \subseteq \mathcal{W}$, the only Dab-formulas in $\{B_1, \ldots, B_o\}$ are abnormalities and hence for every $j \in I$, $\{B_1, \ldots, B_o\} \cap \Omega_j \subseteq {}^{\mathbf{C}}U^j_{\mathsf{g}(\mathcal{P})}(\Gamma) = {}^{\mathbf{C}}U^j_{\mathsf{g}(\mathcal{P}_\varepsilon)}(\Gamma)$; and for every $\varphi \in {}^{\mathbf{C}}\Phi^j_{\mathsf{g}(\mathcal{P})}(\Gamma) = {}^{\mathbf{C}}\Phi^j_{\mathsf{g}(\mathcal{P}_\varepsilon)}(\Gamma)$, $\{B_1, \ldots, B_o\} \cap \Omega_j \subseteq \varphi$. By Lemma 12 and (†), for all $j \leq n$ there is no j-Dab-formula at any line $o + j'$ where $j' \leq j$ and $\Theta_j \neq \emptyset$. From these facts, one can easily infer that line $o{+}n{+}1$ is unmarked.

Suppose line $o{+}n{+}1$ is marked in an extension of the proof resulting in the proof $\mathcal{P}'$. We can extend the proof further to stage $\mathsf{g}(\mathcal{P}')$. That line $o{+}n{+}1$ is unmarked is an immediate consequence of the construction of line $o{+}n{+}1$, (†), and the fact that by Lemma 11 ${}^{\mathbf{C}}\Phi^j_{\mathsf{g}(\mathcal{P}')}(\Gamma) = {}^{\mathbf{C}}\Phi^j_{\mathsf{g}(\mathcal{P}_\varepsilon)}(\Gamma)$ and ${}^{\mathbf{C}}U^j_{\mathsf{g}(\mathcal{P}')}(\Gamma) = {}^{\mathbf{C}}U^j_{\mathsf{g}(\mathcal{P}_\varepsilon)}(\Gamma)$. □

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