The Reception of Ancient Indian Mathematics by Western Historians

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Summary. While there was an awareness of ancient Indian mathematics in the West since the sixteenth century, historians discuss the Indian mathematical tradition only after the publication of the first translations by Colebrooke in 1817. Its reception cannot be comprehended without accounting for the way new European mathematics was shaped by Renaissance humanist writings. We show by means of a case study on the algebraic solutions to a linear problem how the understanding and appreciation of Indian mathematics was deeply influenced by humanist prejudice that all higher intellectual culture, in particular all science, had risen from Greek soil.

Key words: Algebra, Renaissance, humanism, Greek, Indian, influences [1]

1 The context of Renaissance humanism

Western reception of ancient Indian mathematics during the nineteenth-century is very much biased by the humanist tradition. Reflections and statements of Western historians on Indian mathematics can only be fully understood if this context is known and acknowledged.

During the Middle Ages mathematics was hardly practiced or appreciated by the intellectual elite. The Middle Ages knew two traditions of mathematical practice. On the one hand, there was the scholarly tradition of arithmetic theory, taught at universities as part of the *quadrivium*. The basic text on

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arithmetic, presented as one of the seven liberal arts, was Boethius's *De Institutione Arithmetica* (Friedlein, 1867). The Boethian arithmetic strongly relies on Nichomachus of Gerasa's *Arithmetica* from the 2nd century (Robbins and Karpinski, 1926). This basically qualitative arithmetic deals with properties of numbers and ratios. All ratios have a name and operations or propositions on ratios are expressed in a purely rhetorical form. The qualitative aspect is well illustrated by the following proposition from Jordanus de Nemore's *De elementis arithmetice artis* (c. 1250, Book IX, proposition LXXI; Busard 1991, 199):

> Datis superparticularibus vel multiplicibus superparticularibus multiplices superparticulares et superpartientes et datis superpartientibus aut multiplicibus superpartientibus superpartientes et multiplices superpartientes procreare.

A superparticular has the form $\frac{n+1}{n}$ and thus covers proportions such as the common sesquialter (3/2) and sesquiter (4/3) proportions; a superpartient proportion has the form $\frac{m+n}{n}$ with $m > 1$ and includes proportions such as 8/3. The proposition describes how to create multiple superparticular proportions from a given one. As may be clear from this example treated in the most extensive treatise of the period, arithmetic served little practical purpose and was not applied outside monasteries and universities. It was intended mainly for aesthetic and intellectual pursuit. During the eleventh century a board game named *Rhythmomachia* was designed to meet with these aesthetic aspirations. Originated as the subject of a competition on the knowledge of Boethian arithmetic amongst cathedral schools in Germany (Borst 1987), the game was played until the sixteenth century, when the arithmetic tradition passed into oblivion. Despite its limited applicability, Boethian arithmetic evolved into a specific kind of mathematics, typical for the European Middle Ages, and left its mark on early natural philosophy. Carl Boyer's book on the history of calculus demonstrates how fourteenth-century thinkers such as Bradwardine and Richard Suiseth developed ideas on continuity and acceleration within this framework which influenced the later development of mathematics and natural philosophy (Boyer 1959, ch. 3).

A second tradition concerned arithmetical problem solving, of which Alcuin's *Propositiones ad Acuendos Juvenes* (Propositions for Sharpening Youths) from the ninth century provides us with an extant witness. This collection contains 53 problems of which many are repeated over and over in medieval and Renaissance works. Translations are quite recent. Folkerts (1978) translated Alcuin into German. Hadley provided an English translation, annotated by Singmaster (1992). As the title suggests, the problems were to be used for educational purposes and to be read aloud, copied and solved by students. Arithmetical problem solving became much more advanced with the introduction of Arabic algebra through the Latin translations of al-Khwārizmī's *Algebra* by Robert of Chester (c. 1145), Gerard of Cremona (c. 1150) and Guglielmo de Lunis (c. 1215). With the possible exception of Jean de Murs' *Quadripartitum numerorum* at the Sorbonne (1343, l'Huillier, 1990), algebra was not practiced or even spoken about at universities for the following three centuries. However, algebra flourished and continuously developed within the vernacular tradition of abacus schools in fourteenth- and fifteenth-century Italy. Algebra was not only a foreign invention by its Arabic origin, it was also completely foreign to the scholarly tradition.

During the fifteenth century Italian humanists eagerly started collecting editions of Greek mathematics. One of the most industrious was Cardinal Bessarion who lived in Venice. By his death in 1472 he had accumulated over five hundred Greek manuscripts (Rose, 1975, 44-46 and 90-109). Regiomontanus, who had befriended Bessarion, began to study these Greek texts around 1463, including Diophantus' *Arithmetica.* He reported his find of the six books of the *Arithmetica* in a letter to Giovanni Bianchini (Curtze, 1902, 256-7). By then he was well-acquainted with the Arabic algebra. He owned a copy of the manuscript on algebra by al-Khwārizmī, possibly from his own pen (MS. Plimpton 188). Highly receptive to influences between traditions, he immediately conjectured a relation. In his *Oratio*, a series of lectures at the University of Padua in 1464, he introduced the idea that Arabic algebra descended from Diophantus' *Arithmetica* (Regiomontanus, 1537). This heralded the initiation of a myth cultivated by humanists for centuries. Diophantus, first considered to be the source of inspiration for Arabic algebra, became the alleged origin of European algebra. Several humanist writers such as Ramus, chose to neglect or reject the Arabic roots of Renaissance algebra altogether (Høyrup, 1998). As a matter of fact, Diophantus had almost no impact on European mathematical practice before the late sixteenth century. Diophantus inspired authors on algebra such as Stevin, Bombelli and Viète because by then symbolic algebra was well established. By overrating the importance of Diophantus and downgrading the achievements of Arabic algebra, humanist writers created a new mythical identity of European mathematics. Suddenly Greek mathematics became European mathematics. However, most Greek sources were unavailable before the sixteenth century. In fact, Greek mathematics was more foreign to the European mathematical practice than Arabic mathematics was; the latter was slowly but surely appropriated with the abacus tradition. Ironically, the medieval qualitative arithmetic, which was a genuine European tradition, became completely forgotten.

Only later, European historians learned about ancient Indian mathematics and what they learned was strongly influenced by the humanist mathematical tradition. We will now give a brief overview of the first assessments of Indian algebra in the West.

2 The first descriptions of Indian algebra

In some sense Wallis's *Treatise on Algebra* (1685) can be considered the first serious historical investigation of the history of algebra. John Wallis was wellinformed about Arabic writings through Vossius and was one of the first to attribute correctly the name algebra to *al-jābr* in *Kitāb fī al-jābr wa'l-muqābala.* He also pointed out the mistaken origin of algebra as Geber's name, which was a common misconception before the seventeenth century (Wallis 1685, 5). Unprecedented, Wallis casts doubts on Diophantus' contribution to modern algebra. He even launched the idea that Arabic algebra may have originated from India (Wallis 1685, 4):

> However, it is not unlikely that the Arabs, who received from the Indians the numeral figures (which the Greeks knew not), did from them also receive the use of them, and many profound speculations concerning them, which neither Latins nor Greeks know, till that now of late we have learned them from thence. From the Indians also they might learn their algebra, rather than from Diophantus.

So, while in the seventeenth century no Sanskrit mathematics had yet been introduced into Europe, scholars by then were aware of the existence of Indian algebra. Wallis's view persisted in eighteenth-century historical studies, which reiterated the influence from Indian mathematics. Pietro Cossali, who wrote an extensive monograph on the history of algebra, concluded his discussion on al-Khwārizmī's *Algebra* with al-Khwārizmī "not having taken algebra from the Greeks,... must have either invented it himself, or taken it from the Indians. Of the two, the second appears to me the most probable"(Cossali 1779, I, 216-9). Hutton, who included a long entry on algebra in his *Mathematical and Philosophical Dictionary*, wrote (Hutton 1795, I, 66):

> But although Diophantus was the first author on algebra that we know of, it was not from him, but from the Moors or Arabians that we received the knowledge of algebra in Europe, as well as that of most other sciences. And it is matter of dispute who were the first inventors of it; some ascribing the invention to the Greeks, while others say that the Arabians had it from the Persians, and these from the Indians.

In the early nineteenth century, the English orientalist Henry Thomas Colebrooke, who previously published his *Sanskrit Grammar* (1805), undertook the task of translating three classics of Indian mathematics, the *Brāhmasphuṭasiddhānta* of Brahmagupta (628) and the *Līlāvatī* and the *Bījagaṇita* of Bhāskara II (1150) (Colebrooke 1817). At once European historians had something to reflect upon. In a period when mathematics was hardly practiced in Europe and in the Islam regions, there appeared to have existed this Indian tradition in which algebraic problems were solved with multiple unknowns, in which zero and negative quantities were accepted and in which sophisticated methods were used to solve indeterminate methods. In general, nineteenth-century historians showed an admiration for the Hindu tradition. However, whenever explanations were required, scholars became divided into two opposing camps, which we could call the believers and the non-believers. Non-believers did not grant Indian mathematicians the status of original thought. Indian knowledge must have stemmed from the Greeks, the cradle of Western mathematics, or even mathematics as such. The major non-believer was Moritz Cantor who published an influential four-volume work on the history of mathematics (1880-1908). Cantor (1894, II) takes every opportunity to point out the Greek influences on Hindu algebra. Some examples: the Indians learned algebra through traces of algebra within Greek geometry ("Spuren griechischer Algebra müssen mit griechischer Geometrie nach Indien gedrungen sein und werden sich dort nachweisen lassen", Cantor 1894, II, 562); Brahmagupta's solution to quadratic equations has Greek origins ("So glauben wir auch deutlich die griechische Auflösung der quadratischen Gleichung, wie Heron, wie Diophant sie übte, in der mit ihr nicht bloss zufällig übereinstimmenden Regel des Brahmagupta zu erkennen", Cantor 1894, II, 584); or the *Epanthema* as discussed below.

The believers were not convinced by accidental resemblances between Greek and Hindu solution methods and did not see why Indian mathematics could not have been an independent development. In particular, Hankel (1874, 204) touches the sore spot when he writes:

> That by humanist education deeply inculcated prejudice that all higher intellectual culture in the Orient, in particular all science, is risen from Greek soil and that the only mentally truly productive people have been the Greek, makes it difficult for us to turn around the direction of influence for one instant. (Das uns durch die humanistische Erziehung tief eingeprägte Vorurtheil, dass alle höhere geistige Cultur im Orient, insbesondere alle Wissenschaft aus griechischem Boden entsprungen und das einzige geistig wahrhaft productive Volk das griechische gewezen sei, kann uns zwar einen Augenblick geneigt machen, das Verhältniss umzukehren [my translation]).

Soon after Kern (1875) published the Sanskrit edition of the *Āryabhaṭīya* (*AB*), the French orientalist Léon Rodet was the first to provide a translation in a Western language (1877, published in Rodet 1879). Rodet wrote several articles and monographs on Indian mathematics and its relation with earlier and later developments in the Arab and Western world, published in the French *Journal Asiatiques*. He is the scholar who displays the most balanced and subtle views on the relations between traditions. In particular, his appraisal of Hindu and Arabic algebra as two independent traditions is still of value today (see Heeffer 2007b for an assessment). He certainly was a believer. Concerning Āryabhaṭa's inadequate approximation of the volume of a sphere (prop. 7), he writes somewhat cynically that if Āryabhaṭa got his knowledge from the Greeks, then apparently he chose to ignore Archimedes ("Mais elle a, pour l'histoire des mathématiques, d'autant plus de valeur, parce qu'elle nous démonstre que si Āryabhaṭa avait reçu quelque enseignement des Grecs, il ignorait au moins les travaux d'Archimède", Rodet 1879, 409).

George Thibaut who translated several Sanskrit works on astronomy, such as Varāhamihira's *Pañcasiddhāntikā* (1889), also wrote an article on Indian mathematics and astronomy in the *Encylopedia of Indo-Aryan Research*. Concerning influences from Greek mathematics, he takes a middle position. In discussing Hindu algebra he writes that "in all these correspondences does Indian algebra surpass Diophantus" ("In allen diesen Beziehungen erhebt sich die indische algebra erheblich über das von Diophant Geleistete" (Thibaut 1899, 73). As on the origins of Indian mathematics, he points out that Indian algebra, especially indeterminate analysis, is closely intertwined with its astronomy. As he argued on the Greek roots of Indian "scientific" astronomy, his evaluation is that Indian mathematics is influenced by the Greeks through astronomy. However, he adds that several arithmetical and algebraic methods are truly Indian (Thibaut 1899, 76-8).

Despite the existence of several studies and opinions which should provide sufficient counterbalance for Cantor's position as a non-believer, his views remained influential well into the twentieth century. We may say that the "humanist prejudice" is still alive today. The myth that Greek mathematics is our (Western) mathematics has become intertwined with our cultural identity so strongly that it becomes difficult to understand intellectual achievements within mathematics foreign to the Greek tradition.

We will now look in detail at an example that has been one of the main arguments for the advocates of Greek influence. The example clearly shows how historical investigation can be misled through prejudice.

3 A case study: the Bloom of Thymaridas

We have demonstrated elsewhere that if there is an influence between Indian algebra and European arithmetic, it should be situated on the level of proto-algebraic solution recipes, orally disseminated through riddles and recreational problems (Heeffer, 2007a). One interesting example in this respect is a class of determinate linear problems in which the partial sums are given and the individual quantities are unknown. We found strong similarities in the rules for solving this type of problem both in Hindu algebra and in Renaissance arithmetic. These rules have a special interest for our discussion as we have both a Greek and a Hindu tradition of their use. There has been a controversy about the possible influence of Greek mathematics on Indian algebra, as defended by Cantor and Kaye and disputed by Rodet. We will here shed more light on the controversy and explain the dispute as a misunderstanding of the rule. We will demonstrate in detail that the Greek and Indian versions are in fact two different rules and that the alleged influence from Greece to India is therefore highly disputable.

3.1 The original formulation in Hindu sources

The first Indian source for a formulation of this rule is from Āryabhaṭa I, 499, (*AB*, ii, 29; Clark 1930, 40) as follows:

> If you know the results obtained by subtracting successively from a sum of quantities each one of these quantities set these results down separately. Add them all together and divide by the number of terms less one. The result will be the sum of all the quantities.

The rule is somewhat obscure and difficult to understand without examples, but some observations can be drawn from the formulation which are central to our further discussion. Firstly, the rule is valid for any number of quantities. It is not limited to two or three quantities. Secondly, the sum of all the quantities is unknown and is provided by the rule. Furthermore, and not evident from the rule, as cited above, is that the partial sums relate to the total of all the quantities, except one. In modern symbolism the general structure of the problem thus is as follows:

Suppose n amounts $(a_1, a_2, ..., a_n)$ with unknown sum S and with the partial sums $(s_1, s_2, ..., s_n)$ given, where $s_i = S - a_i$, then

$$S = \frac{\sum_{i=1}^{n} s_i}{n-1}.$$

The rule and the problems it applies to should not be confused with a similar problem in which the partial sums of two consecutive quantities are given. For three numbers, the problems are evidently the same, but they diverge for more than three quantities. E.g., for five quantities the corresponding equations are:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= s_1 \\ a_1 + a_3 + a_4 + a_5 &= s_2 \\ a_1 + a_2 + a_4 + a_5 &= s_3 \\ a_1 + a_2 + a_3 + a_5 &= s_4 \\ a_2 + a_3 + a_4 + a_5 &= s_5 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} a_1 + a_2 &= s_1 \\ a_2 + a_3 &= s_2 \\ a_3 + a_4 &= s_3 \\ a_4 + a_5 &= s_4 \\ a_5 + a_1 &= s_5 \end{aligned}$$

Let us apply the rule to a simple problem (not discussed by Āryabhaṭa) which can be formulated symbolically as:

$$\begin{aligned} x_1 + x_2 &= 13 \\ x_2 + x_3 &= 14 \\ x_1 + x_3 &= 15 \end{aligned}$$

Applying Āryabhaṭa's rule, the solution would be based on the rule for deriving the sum of all three unknown quantities as follows:

$$x_1 + x_2 + x_3 = \frac{13 + 14 + 15}{3 - 1} = 21.$$

This allows us to determine the value of the quantities by subtracting the partial sums from the total with the solution (7, 6, 8). A commentator of the

Āryabhaṭiya, called Bhāskara I (written 629, not to be confused with Bhāskara II), gives two examples of problems that can be solved with Āryabhaṭa's rule with the partial sums (30, 36, 49, 50) and (28, 27, 26, 25, 24, 23, 21) (Shukla and Sarma 1976, 307-308).

3.2 The derived problem in Hindu sources

From the ninth century we find a derived version of the previous problem in Hindu sources. Mahāvīra gives an elaborate description of the rule in the *Gaṇitasārasaṃgraha* (*GSS*, stanza 233-5, Padmavathamma 2000, 357-9) which we here reproduce:

> The rule for arriving at [the value of the money contents of] a purse which [when added to what is on hand with each of certain persons] becomes a specified multiple [of the sum of what is on hand with the others]:
> The quantities obtained by adding one to [each of the specified] multiple numbers [in the problem and then] multiplying these sums with each other, giving up in each case the sum relating to the particular specified multiple, are to be reduced to their lowest terms by the removal of common factors. [These reduced quantities are then] to be added. [Thereafter] the square root [of this resulting sum] is to be obtained, from which one is [to be subsequently] subtracted. Then the reduced quantities referred to above are to be multiplied by [this] square root as diminished by one. Then these are to be separately subtracted from the sum of those same reduced quantities. Thus the moneys on hand with each [of the several persons] are arrived at. These [quantities measuring the moneys on hand] have to be added to one another, excluding from the addition in each case the value of the money on the hand of one of the persons and the several sums so obtained are to be written down separately. These are [then to be respectively] multiplied by [the specified] multiple quantities [mentioned above]; from the several products so obtained the [already found out] values of the moneys on hand are [to be separately subtracted]. Then the [same] value of the money in the purse is obtained [separately in relation to each of the several moneys on hand].

The introductory sentence states that the rule is to be used for determining the value of a purse. The rule is followed by a number of problems that begin as "Four men saw on their way a purse containing money" (ibid. stanza $245\frac{1}{2}$, 367). This is the earliest instance, in our investigation of the sources, in which the popular problem of men finding a purse is discussed. While problems with the same structure and numerical values have been formulated before, the context of men finding a purse seems to have originated in India before 850 AD. Formulations with the purse turn up in Arabic algebra with al-Karkhī's *Fakhrī* (c. 1050) and in the *Miftāh al-mu`āmalāt* of al-Tabari (c. 1075). Fibonacci has many variations of it in the *Liber Abbaci* (1202) and after that it becomes the most common problem in western arithmetic until the later sixteenth century. For an understanding of the rule, let us look at its application to a given problem (*GSS*, stanza 236-7, pp. 360):

> Three merchants saw [dropped] on the way a purse [containing money]. One [of them] said [to the others], “If I secure this purse, I shall become twice as rich as both of you with your moneys on hand”. Then the second [of them] said, “I shall become three times as rich”. Then the other, [the third], said, “I shall become five times as rich”. What is the value of the money in the purse, as also the money on hand [with each of the three merchants]?

We can represent the problem in symbolic equations as follows:

$$x + p = 2(y + z)$$
$$y + p = 3(x + z)$$
$$z + p = 5(x + y)$$

Let us apply the recipe of Mahāvira to this problem, step by step. By “adding one to [each of the specified] multiple numbers” we have 3, 4 and 6. “Multiplying these sums with each other” we get 72. This has to be “reduced to their lowest terms by the removal of common factors”. This least common multiple is 12. The reduced quantities are then 4, 3 and 2 respectively. Adding these together gives 9. From this the square root is 3. Then the reduced quantities “are to be multiplied by the square root as diminished by one”, which is 2. This leads to 8, 6 and 4. The money in hand for each of the merchants now is the difference of these values with the sum of the reduced quantities, being 9. The solution thus is 1, 3 and 5. The rest of the rule is an elaborate way to derive the value of the purse. Using the values in any one of the equations immediately leads to 15 for the value of the purse. Mahāvīra provides no explanation or derivation of the rule. For a mathematical argument for the validity of the rules see Heeffer (2007a).

3.3 The problem in Greek sources

3.3.1 The Bloom of Thymaridas

We know almost nothing about Thymaridas of Paros, but he is supposed to have lived between 400 and 350 BC (Tannery 1887, 385-6). The only extant witness is Iamblichus, in his comments on the *Introduction to Arithmetic* by Nichomachus of Gerasa. The best known source for *The Bloom of Thymaridas* is Heath’s classic on Greek mathematics. Heath (1921, 94) does not formulate the rule, he only observes that “the rule is very obscurely worded” and writes out the equations. The text from Iamblichus was first published in Holland with a Latin translation by Samuel Tennulius (1668) from the Paris manuscript BNF Gr. 2093. A critical edition, based on multiple manuscripts was published by Pistelli (1884). Nesselmann (1842, 233) quotes the Greek text and the Latin translation from Tennulius, who translated the method as *florida sententia*. We give here the our own literal translation from Pistelli (1884, 62):

> From this we are also acquainted with the method of the Epanthema, passed down to us by Thymaridas. Indeed, when a given quantity divides into determined and unknown parts, and the unknown quantity is paired with each of the others, so will the sum of these pairs, diminished by the sum [of all the quantities] be equal to the unknown quantity in case of three quantities. With four quantities it will be half of it, with five it will be a third, with six, a fourth and so on.

The rule is not as obscure as considered by Heath. Let us extract the basic elements of the rule, and compare these with the version of Āryabhaṭa:

- The rule applies to any number of quantities, as does Āryabhaṭa's.
- The sum is given in the problem. The rule is described as the division of a known quantity in determined and undetermined parts. In Aryabhaṭa's rule the sum is what is looked for.
- The partial sums are the sums of the pairs of the unknown part with each of the known quantities. In Āryabhaṭa's rule the partial sums include all the numbers except one.

In short, this rule is different from Āryabhaṭa's in two important aspects. Its intention is to find one unknown part of a determined quantity. Āryabhaṭa's rule is meant for finding the sum of numbers of which the partial sums of all minus one is given. Even in the case of three numbers, when the partial sums are the same, the rules have different applications. To make it clear to the modern eye, here is a symbolic version in the general case:

$$\left\{\begin{array}{l} x + a_1 + a_2 + ... + a_{n-1} = s \\ x + a_1 = s_1 \\ x + a_2 = s_2 \\ \vdots \\ x + a_{n-1} = s_{n-1} \end{array}\right\} \quad x = \frac{\sum_{i=1}^{n-1} s_i - s}{n-2}$$

3.3.2 Diophantus

In the first book of the *Arithmetica* of Diophantus we find four instances of the problem type. Problems 16 and 17 are of the original type as covered by Āryabhaṭa's rule. Let us first look at problem 17 with four unknown quantities. We use Ver Eecke (1926, 22) as the best translation of the *Arithmetica*:

> Trouver quatre nombres qui, additionnés trois à trois, forment des nombres proposés. Il faut toutefois que le tiers de la somme des quatre nombres soit plus grand que chacun d'eux. Proposons donc que les trois nombres, additionnés à la suite à partir du premier, forment 20 unités; que les trois à partir du second forment 22 unités, que les trois à partir du troisième forment 24 unités, et que les trois à partir du quatrième forment 27 unités".

In modern symbolism, the problem reads as follows:

$$a + b + c = 20$$
$$b + c + d = 22$$
$$a + c + d = 24$$
$$a + b + d = 27$$

Diophantus's solution is not based on a proto-algebraic rule but has all the characteristics of algebra. He uses the *arithmos* as an abstract quantity for the unknown, to represent the sum of the four quantities (Ver Eecke 1926, 22):

> Posons que la somme des quatre nombres est 1 arithme. Dès lors, si nous retranchons les trois premiers nombres, c'est-à-dire 20 unités, de 1 arithme, il nous restera, comme quatrième nombre, 1 arithme moins 20 unités. Pour les mêmes raisons, le premier nombre sera 1 arithme moins 22 unités; lé second sera 1 arithme moins 24 unités, et le troisième 1 arithme moins 27 unités. Il faut enfin que les quatre nombres additionnés deviennent égaux à 1 arithme. Mais, les quatre nombres additionnés forment 4 arithmes moins 93 unités; ce que nous égalons à 1 arithme, et l'arithme devient 31 unités.

If $a+b+c+d = x$, then the four numbers not included in the partial sums are $x - 20$, $x - 22$, $x - 24$, and $x - 27$ respectively. Adding these four together is equal to their sum or x, thus $4x - 93 = x$ and $x = 31$. This problem in the *Arithmetica* is followed by problems 18 and 19, of a related type, but not the one covered by Mahāvīra's formulation. We show here only the symbolic translation of problem 19:

$$a + b + c = d + 20$$
$$b + c + d = a + 30$$
$$a + c + d = b + 40$$
$$a + b + d = c + 50$$

The solution is similar to the previous problem but depends on the choice of $2x$ for the sum of the four numbers.

3.3.3 The extended rule from Iamblichus

Iamblichus extends the rule of Thymaridas to another problem type which will become very popular during the next centuries. In modern symbolism this amounts to the set of equations:

$$x + p = a(y + z) \quad (1)$$
$$y + p = b(x + z) \quad (2)$$
$$z + p = c(x + y) \quad (3)$$

Iamblichus gives two examples of the problem. The first example can be formulated symbolically as follows. Nesselmann (1842, 234-5) gives the literal

German translation from the Greek. We will follow Nesselmann's rather than Heath's reconstruction:

$$\begin{aligned} a+b &= 2(c+d) \\ a+c &= 3(b+d) \\ a+d &= 4(b+c) \\ a+b+c+d &= 5(b+c) \end{aligned}$$

The problem is formulated in a way that reminds us of Diophantus: "Find four numbers such that ...". Although Diophantus's *Arithmetica* has no problems like this, problems 18 to 20 of the first book are variations on the original *epanthema* problem. Iamblichus's own variation is in some way analogous to the versions of the *Arithmetica* and might be influenced by it. However, while Diophantus's solution is algebraic, this one depends on a proto-algebraic rule. The fourth expression in the problem formulation is superfluous and is recognized as such by Iamblichus, where he adds: "this follows directly from the previous statements". It is added to facilitate the application of the rule. The procedure is explained by Iamblichus in three steps:

1) Set the sum of the four numbers equal to the number found by multiplying the four factors together. Thus $2 \,.\, 3 \,.\, 4 \,.\, 5 = 120$.
   Iamblichus does not explain why this is necessary, but it can be demonstrated in the following way: Completing the left side of the equations (1, 2, 3) to the sum of the four numbers we arrive at:

   $$\begin{aligned} x+y+z+p &= (a+1)(y+z) \\ x+y+z+p &= (b+1)(x+z) \\ x+y+z+p &= (c+1)(x+y) \end{aligned}$$

   Therefore, the sum of the four integers must be divisible by $(a+1), (b+1)$ and $(c+1)$. This can be represented by means of the least common multiple s. Now Iamblichus does not use s but $2s$ for a reason that will become apparent later. In the example the least common multiple is 60, therefore $2s$ is 120. So, let us suppose that $x+y+z+p \;=\; 2s$.

2) The sum of each pair can be found by taking $\frac{a}{a+1}$, $\frac{b}{b+1}$ and $\frac{c}{c+1}$ from the sum $2s$ respectively. This becomes apparent from

   $$\begin{aligned} x+p &= a(y+z) \\ (a+1)(x+p) &= a(x+y+z+p). \end{aligned}$$

   The three sums $(x+p), (y+p)$ and $(z+p)$ in the example become $80, 90$ and 96.

3) Only now, Iamblichus refers to the use of the *Epanthema* rule. Indeed, we have the partial sums $(x+p), (y+p), (z+p)$ and we have the total sum $2s$. The *Epanthema* therefore determines the common part p as follows:

$$p = \frac{(x+p)+(y+p)+(z+p)-2s}{2}$$

or

$$p = \frac{80+90+96-120}{2} = 73,$$

which leads to the other values as 7, 17 and 23. The reason why Iamblichus used $2s$ instead of the least common multiple s, is that s would lead to the non-integral solution:

$$p = \frac{40+45+48-60}{2} = 36\frac{1}{2}.$$

In summary, we discern two important factors which are relevant for the understanding of the controversy that follows.

1) Our only source for the *Epanthema* is Iamblichus. There are at least six centuries between Thymaridas and the extant witness. In the absence of any written source we should consider Iamblichus's discussion of the method as a late interpretation of Pythagorean number theory. The formulation of the rule with determined and unknown quantities suits the context of third century Greek analysis better than it would fit in Pythagorean number mysticism.
2) The extended problem, which has become known as the problem of men finding a purse, is in itself quite different from the original problem to which the *Epanthema* rule applies. The problem, devised by Iamblichus, could be considered a variation such as several others in the *Arithmetica* of Diophantus. Iamblichus gives the rules to reduce the problem to a form in which the *Epanthema* can be used. This distinction is important because many have wrongly identified the men-find-a-purse problem with the *Bloom of Thymaridas*.

3.3.4 The controversy

We now come to the discussion on the relevance of the *Epanthema* method and the controversy about the influences on and from Indian mathematics. As there are two aspects of the discussion, we will deal with the issues separately. Firstly, we address the historical question of the main source of the men-find-a-purse problem. Secondly, we discuss the more philosophical question of the relevance of the *Bloom* on the conceptual development of algebra.

3.3.4.1 The origin of linear problems of men finding a purse

Nesselmann (1842) restrains from comments on the *Bloom of Thymaridas* in his *Algebra of the Greeks*. He treats the method with full respect for the extant Greek text by Iamblichus. After Nesselmann, the problem was discussed, by several scholars, in relation to Hindu algebra. Rodet (1879), in his French translation of the Āryabhaṭa's treatise, does not mention the *Epanthema*. Rodet was no believer in the influence of Greek mathematics in Asia. We can assume that he did not discuss the *Epanthema* because, in his point of view, there simply is no relation to Āryabhaṭa's rule.

On the other hand, Cantor (II, 584), after discussing Āryabhaṭa's stanza 29, remarks, "We do not fear any disagreement, if in this problem and in the *Epanthema* of the Thymaridas, we recognize a relation which is so close that a coincidence is not imaginable" ("Wir fürchten keinen Widerspruch, wenn wir in dieser Aufgabe und in dem Epantheme des Thymaridas so nahe Verwandte erkenne, dass an einen Zufall nicht zu denken ist"). Citing Cantor and Heath, Kaye (1927, 40, note 2) writes "The examples in the text are undoubtedly akin to the '*Epanthema*' ". Tropfke (1980, 399) words it more sharply and considers the formulation of Āryabhata's stanza 29 "equivalent with the *Epanthema* of Thymaridas"and states that the *BM* "contains problems of the same sort". However, in the original edition, Tropfke (1937, III, 42) is more prudent: "Āryabhaṭa bietet einige solcher Wortgleichungen, unter denen uns eine wegen ihren Änlichkeit mit dem Epanthem des Thymaridas ausffält". Apparently it is Kurt Vogel, who edited the 1980 edition, who believes in a strong connection.

All the suppositions of the Greek influence are based solely on the alleged resemblance of the problems. As shown above, Āryabhaṭa's rule is very different from the *Epanthema*. The argument that both are equivalent is plainly false. The suggestion that the *Epanthema* provides evidence of an influence of Greek mathematics on Hindu algebra has very little substance. Instead, it seems that the argument is biased by normative beliefs about the superiority of Greek culture. Let us now proceed to the second question on conceptual influences.

3.3.4.2 A case of Pythagorean algebra?

This single problem, which became known to us through Iamblichus, six centuries after Thymaridas, has convinced many that Greek algebra originated with the Pythagoreans. After writing out the equations, Cantor (1894, I, 148) concludes:

> This is, as one can see, all rhetorical algebra, in which only the symbols are missing in order to agree completely with the modern way of solving equations, and specifically the expressions of the given and unknown quantities was rightly emphasized. (Das ist, wie man sieht, volständig gesprochene Algebra, welcher nur Symbole fehlen, um mit einer modernen Gleichungsauflösung durchaus übereinzu-stimmen, und insbesondere ist mit Recht auf die beiden Kunstausdrücke der *gegebene* und *unbekannten* Grösse aufmerksam gemacht worden)

Heath's interpretation is copied in many other works including Smith (1925, 91), Cajori (59), van der Waerden (1988, 116), Flegg (1983, 205) and

Kaplan (2001, 62). Cajori finds in the Thymaridas "investigations of subjects which are really algebraic in their nature". Van der Waerden goes as far as to claim that "we see from this that the Pythagoreans, like the Babylonians, occupied themselves with the solution of systems of equations with more than one unknown". Instead, Klein (1968, 36) sees in the problem an intent to "determine special relations between numbers" and places it as "the counterparts in the realm of 'pure' units of the computational problems proper to practical logistic". We agree with Klein's interpretation. Even if Iamblichus's depiction of the problem from Thymaridas is faithful, the six centuries separating these two mathematicians require an interpretation that accounts for two different contexts. Pythagoreans were concerned with the properties of numbers and with the relations between numbers. Lacking any further evidence, we cannot attribute an algebraic interpretation to Pythagorean number theory. On the other hand, in the context of the late Greek period of Diophantus and Iamblichus, an algebraic reading is warranted. Thus, the *Bloom* is an old number problem, revived and extended in an algebraic context.

4 Conclusion: the ground was everywhere wet

The humanist project of reviving ancient Greek science and mathematics played a crucial role in the creation of an identity for the European intellectual tradition. While Greek mathematics was hardly known or practiced before the fifteenth century, humanist mathematicians identified themselves with this tradition. When Regiomontanus declared that algebra was invented by Diophantus, humanist writers rejected the Arabic roots of algebra, though it was practiced and turned into an independent tradition for two centuries in Italian cities such as Florence and Sienna. The newly created identity of mathematics descending from ancient Greek thinkers blurred historical perception. When Indian algebra and arithmetic was introduced into Europe, the leading historians of the nineteenth-century could only see its alleged relation with Greek mathematics. The *Bloom of Thymaridas* is an excellent illustration of distorted historical investigation. Not only was it wrongly inferred that the Indian method for solving determined linear problems depended on Iamblichus, historians forced a connection between third-century Greek analysis and Pythagorean number theory. The origin of the algebra of Diophantus still needs an explanation, but it is very doubtful that it is to be found in Pythagoras.

Apparently nineteenth-century historians found it difficult to accept that mathematics is a human intellectual activity endeavored across cultures within societies that needed and supported the achievements of mathematical practice. A true history of mathematics should take into account contributions of all origins. Jens Høyrup, who studied the evolution and transmission of mathematics between cultures, formulates it as follows (Høyrup 1993, 98):

Diophantos would use the rhetorical algebra, the Chinese *Nine Chapters on Arithmetic* would manipulate matrices, and the *Liber abbaci* would find the answer by means of proportions We should hence not ask, as commonly done, whether Diophantos (or the Greek arithmetical environment) was the source of the Chinese or vice versa. There was *no specific source: The ground was everywhere wet*.

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