The Tacit Appropriation of Hindu Algebra in Renaissance Practical Arithmetic

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Example is better than precept (身教胜于言传 Shēnjiào zhòngyú yānjiào)
Old Chinese proverb

Diophantos would use the rhetorical algebra, the Chinese *Nine Chapters on Arithmetic* would manipulate matrices, and the *Liber abbaci* would find the answer by means of proportions We should hence not ask, as commonly done, whether Diophantos (or the Greek arithmetical environment) was the source of the Chinese or vice versa. There was *no specific source: The ground was everywhere wet*.

Jens Høyrup in *In Measure, Number and Weight*, 1993, p. 98.

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2. Introduction

Comparing a randomly chosen arithmetic or algebra book from sixteenth-century Europe with the *Līlāvatī* by Bhāskara II (written 1150), one cannot escape the feeling that the strong agreement in form and content must stem from a direct influence from Indian sources.[1] However, a direct influence has not been demonstrated and is since more than a century the subject of scientific dispute. Late nineteenth-century histories, in particular Cantor (1880-1908) show an admiration for Hindu algebra but believes it must have originated from Greek sources. [2] Hankel takes a distance from these "humanistic prejudices" and allows for some influence of Hindu algebra on the Greeks.[3] A likely intermediary between Hindu algebra and the abacus tradition is Arab algebra. It is well-established that our positional numbering system with Hindu-Arab ciphers was introduced in Western Europe from India through Arab and Persian translations (*Kitāb al-hisāb al Hind*). Furthermore, there are some rare occasions in which Latin algebraic treatises refer to Indian origins. As a clear influence from Hindu to Islamic mathematics could not be demonstrated, the question has led a continuous debate since the eighteenth century. Cossali concludes his discussion of al-Khwārizmī with "not having taken algebra from the Greeks, he must have

[1] See Table 1. The arithmetic of Jacques Peletier (1552) deals with almost all of the subjects from the *Līlāvatī*. However, the exceptions, such as the rule of inversion and rule of concurrence appear frequently in earlier arithmetic books. Only the Indian treatment of zero, permutations and combinations are absent from sixteenth century arithmetic books.

[2] Cantor (1894, 2nd ed., vol. II) takes every opportunity to point out the Greek influences on Hindu algebra. Some examples are the *Epanthema* (discussed below); "Spuren griechischer Algebra müssen mit griechischer Geometrie nach Indien gedrungen sein und werden sich dort nachweisen lassen" (II, 562); "So glauben wir auch deutlich die griechische Auflösung der quadratischen Gleichung, wie Heron, wie Diophant sie übte, in der mit ihr nicht bloss zufällig übereinstimmenden Regel des Brahmagupta zu erkennen" (II, 584).

[3] Hankel (1874, 212): "Die Vermuthung, dass Diophant unter fremden Einflusse gestanden habe, war kaum mehr abzuweisen" and "Das uns durch die humanistische Erziehung tief eingeprägte Vorurtheil, dass alle höhere geistige Cultur im Orient, insbesondere alle Wissenschaft aus griechischem Boden entsprungen und das einzige geistig wahrhaft productive Volk das griechische gewezen sei, kann uns zwar einen Augenblick geneigt machen, das Vergältniss umzukehren".

either invented it himself, or taken it from the Indians. Of the two, the second appears to me the most probable".[4] This opinion has been echoed by several historians throughout the nineteenth century. Colebrooke, as one of the first Western scholars to have studied Sanskrit mathematical manuscripts discusses at length the influence of Indian astronomy on the Arabs, but falls short in evidence for the Arab use of Indian algebra. He can only assume that al-Khwārizmī "must have learnt from the Hindus the resolution of simple and quadratic equations, or, in short algebra, a branch of their art of computation" (Colebrooke 1817, lxxix). Frederic Rosen, who was the first to translate the *Kitab fī al-jābr wa'l-muqābala* from Arab to English, draws further on Colebrooke and adds the argument that at least the determination of the circumference of the circle is derived from an Indian source (Rosen 1831, viii and 72). Again no substantial evidence is given in relation to algebra. Libri (1838, I, 118) asserts with certainty that "Quant à l'algèbre, tout concourt à prouver que les Arabes l'ont reçue des Indiens" but can only refer to Indian ciphers and Colebrooke on the circumference of the circle in support of his claim. All these in favor for the Indian influence have been criticised by Rodet, Hankel and Cantor, who took more effort in presenting their case. Rodet (1879) gives a detailled comparision of the Arab and Indian algebra with respect to four points: the role of negative terms in polynomials, the way problems are formulated in algebraic terms, the way quadratic problems are solved, and the interpretation of two positive solutions to quadratic problems. For each, he demonstrates that the methods and practices of Brahmagupta and Bhāskara are completely opposite of those of Arab algebra. Hankel and Cantor have always stressed the Greek domination, but Cantor leaves some possibility for Indian influences.[5] Soloman Gandz (1936, 1937) believes al-Khwārizmī is most indebted to the Persian tradition leading back to Egyptian and Babylonian origins.
The current agreement is that "Islamic algebra was in all probability *not* inspired from Indian algebra" (Høyrup 1994, 100). Hindu and Arab algebra are too structurally different to allow for a strong relationship. This brings us back to our original thought.

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[4] Cossali 1779, I, 216-9, cited and translated by Colebrooke 1817, lxxx.

[5] Hankel (1874, 262) points at Diophantus as a likely source for the basic operations of Arab algebra: "So lernte der Araber vielleicht aus griechischen Schriften seine Algebra? Vergleichen wir das Werk des Mohammed mit der einzigen griechischen Quelle, die wir kennen, mit Diophant, so tritt zunächst als eine sehr bemerkenswerthe Analogie hervor, dass hier jene fundamentalen Operationen Mohammed's allerdings an hervorragender Stelle erscheinen". Cantor (1907, I, 679-80): "Was Alchwarizmī gibt kann griechischen, kann indischen Ursprungs sein, kann vielleicht einer aus beiden Quellen gemischten Strömung sein Dasein verdanken, wie wir ja auch in seinem Rechenbuche überwiegend Indisches und daneben einzelne griechische Spuren vorfanden. Wir wollen zu zeigen versuchen, dass, wenn die Algebra überhaupt als eine Mischung zu betrachten ist, jedenfalls griechische Elemente in ihr weitaus vorherrschen".

C	S	Sūtra	**the *Līlāvatī* by Bhāskara**	**Peletier, *l'Arithmetique*, 1552**	Bk; Ch
II	II	10-11	Numeration	Diffinition de Nombre	1; 1
		12-13	Rule of addition and subtraction	De l'Addition / Souztraction des nombres Entiers	1; 3,4
		14-16	Rule of multiplication	De la multiplication des Entiers	1; 5
		17	Rule of division	De la Division des Entiers	
		18-20	Rule for the square of a quantity	De l'extraction de la Racine Quarre	3; 1
		21-22	Rule for the square root		
		23-26	Rule for the cube	De l'extraction des racines cubiques	3; 3
		27-28	Rule for the cube root		
	III	29-30	Simple reduction of fractions	Reduction de diverses Fractions	2; 6
		31-32	Reduction of subdivided fractions	Des Fractions de fractions et de la reduction dicelles	2; 2
		33-35	Rules of quantities increased or decreased by a fraction	La maniere de valuer les Fractions denommees de quelque espece	2; 5
		36-37	Rule for addition and subtraction of fractions	De l'Addition des Fractions De la Souztraction des Fractions	2; 8 2; 9
		38-39	Rule for multiplication of fractions	De La Multiplication des Fractions	2; 10
		40-41	Rule for division of fractions	De la Division des Fractions	2, 11
		42-43	Rule for involution and evolution of fractions		
	IV	44-46	Cipher [rules for zero]		
III	I	47-49	Rule of inversion		
	II	50-54	Rule of supposition [regula falsi]	De la Regle de Faux de deux Positions	4; 6
	III	55-58	Rule of concurrence		
	IV	59-61	Problems concerning squares		
	V	62-69	Rule for assimilation of the root's coefficient		
	VI	70-73	Rule of three terms	De la Regle de Trois	1; 8
		74-78	Rule of three inverse	De la Regle de Trois Everse	1; 9
		79-84	Rule of compound proportion	De la Regle de 6 Quantitez	3; 20
		85-86	Rule of barter		
IV			Investigation of mixture	De la Regle d'Alligation	4; 4
	I	87-93	Interest		
	II	94-95	Fractions [cistern problem]	Aucune questions diversement	4; 10
	III	96-98	Purchase and sale	De la Regle Double	4; 1
	IV	99-100	A present of gems		
	V	101-109	Allegation	De la Regle d'Alligation	4; 4
	VI	110-114	Permutation & combinations		
V			Progressions	De la Progression des Entiers	1; 7
	I	115-126	Arithmetical progressions		
	II	127-132	Geometrical progressions		

Table 1: A comparison of subjects in Indian and European arithmetic books

If there is no direct influence from Hindu algebra on algebraic practice within the abacus tradition, and if Arab algebra did not function as an intermediary, how to explain the strong similarity between Hindu and abacus problems and their solution methods? Høyrup leaves us with an opening for drawing the connection by stating that "below the level of direct scientific import, some influence of Indian Algebra is plausible" (1994, 95). As a scholar working on a wide period of mathematical practice, from Babylonian algebra to the seventeenth century, Høyrup has always given much attention to the more informal transmission of mathematical knowledge which he calls sub-scientific structures. By their nature, sub-scientific structures are not communicated in formal writings and thus their dissemination and influences are very difficult to determine.

This paper is an exploration of the possible paths of transmission from Indian to Western problems and problem solving methods. The study of manuscripts and books passes by on the oral tradition. The narration of stories, riddles and recreational puzzles is the most important factor in the passing of arithmetical problems and their solution methods to other generations and continents. In order to get a grip on the oral tradition we will propose a concrete implementation for sub-scientific knowledge in the form of proto-algebraic rules. A proto-algebraic rule is a procedure or algorithm for solving one specific type of problem. Our main hypothesis is that many recipes or precepts for arithmetical problem solving, in abacus texts and arithmetic books before the second half of the sixteenth century, are based on proto-algebraic rules. We call these rules proto-algebraic because they are or could be based originally on algebraic derivations. Yet their explanation, communication and application does not involve algebra at all. Proto-algebraic rules are disseminated together with the problems to which they can be applied. The problem functions as a vehicle for the transmission of this sub-scientific structure. Little attention has yet been given to sub-scientific mathematics or proto-algebraic rules. However, viewing proto-algebraic rules as solidifications of algebraic problems solving, they function as fossils of algebraic practice in non-algebraic writings. As fossils provide important evolutionary data to paleontologists and archeologists, so do proto-algebraic rules for the historian of mathematics. Analysis and comparison of formulations and variations of these rules allow us to reconstruct possible paths of transmission.

There are only limited sources from which the medieval and Renaissance proto-algebraic rules were derived: Byzantine, Arab, Indian and Chinese. Our hypothesis is that Hindu algebra played an important role in the formulation of proto-algebraic rules for solving linear problems. We will thus limit ourselves to Sanskrit texts as possible sources. Space and time does not allow us to give a complete overview of this subject, but we hope to provide the necessary arguments and a framework for pursuing such study further. We will cover a typical example of linear problems in one, two and more unknowns and point at other types of problems that could be approached from the given framework. We will concentrate on the recipes as they appear in fifteenth and sixteenth-century abacus manuscripts and arithmetic.

3. Terminology in Sanskrit mathematical texts

We face multiple barriers with the interpretation of Hindu algebra Most of our relevant source texts from the Classic period are Sanskrit verses written between 500 and 1200 AD. Some sources, such as the Bakhshālī manuscript, are preserved on beach bark and their transcription and interpretation requires a high level of scholarship in Sanskrit paleography, lexicography and language study. Fortunately, English translations and critical editions have become available for most of the important sources.[6] The faithfulness of the translations and the level of scholarship has increased considerably with recent publications such as Hayashi (1995). The landmark study by Datta and Singh (1935, 1938) provides us with the framework for interpreting the sometimes obscure, formulations of rules. We will mainly follow this reference work as the most reliable outline for the mathematical interpretation. Still, from a conceptual viewpoint, earlier translations show insufficient restraint from modern reading of these ancient texts. The difference in interpretation between multiple translations learns us to be cautious about the use of modern concepts of algebra. For example, in the English rendition of the Sanskrit verses, Colebrooke (1817) refrains from using the terms 'unknown' and 'coefficient' while Dvivedin (1902) does use them. On the other hand, Colebrooke uses 'equation' while this does not appear in Dvivedin's translations. We can therefore cast some doubt about the use of these modern terms in the translations of Sanskrit texts in general. We will give alternative translations when they are available. In the next section the meaning of some of the basic terms will be reviewed. Our approach is from the philosophy of science rather than from a linguistic point of view and we will try to determine the meaning within the context of the conceptual development of algebra.

3.1. Coefficient

In relation to 'coefficient', Datta and Singh (1962, II, 9) claim that "in Hindu algebra there is no systematic use of any special term for the coefficient". This compares with the state of algebra in mid-sixteenth century Europe . The modern interpretation of the term 'coefficient' is inseparable from the study of the structure of equations which emerged with Viète and Girard. Viète (1591) introduced his symbolism of vowels for unknowns and consonants for constants and coefficients in order to be able to represent properties and reveal structures of equations. *A cubus plus B quadratum in A*

[6] For an overview of sources and a list of used abbreviations consult the table in § 10.

aequari B quadratum in Z represents the equation $A^3 + B^2A = B^2Z$ in which the coefficient of *A* should be a perfect square. The relation between the coefficients of the powers of the unknown, their signs and the roots of an equation have become part of algebraic theory developed during the early seventeenth century. As such, it should be used with prudence in contexts before the late sixteenth century. Colebrooke (1817, 344), with some exceptions and Rodet (1879, 44) do not use 'coefficient'. Datta and Singh (1935-38), in spite of their observation, quoted above, frequently use the term in translations.

3.2. Unknowns

The name for the unknown *yā* is derived from the expression *yāvat-tāvat* (as many as) and functions as a symbol for the unknown similar to the cossic symbol which originated in Germany during the fourteenth century.
Datta and Singh (1938, II, 18) cite the Sanskrit lexicographer Amarasiṃha for an interpretation as a measure of quantity. As such, the meaning of the Sanskrit words *yāvat-tāvat* is not very different from the meaning of the Arab name *māl*, for property, possession or money.[7] The term *yāvat-tāvat* was in use in Indian texts before 300 BC. Powers of the unknown are based on combinations of *varga* (square) and *ghana* (cube). Sanskrit texts used the multiplicative system for powers instead of the additive one as used by Diophantus and some Arab sources. The sixth power is thus called *ghana-varga*.
The use of multiple unknowns was already common before Brahmagupta (628). This can be derived from the fact that in the *BSS* Brahmagupta uses colors synonymous with unknowns without a formal explanation (e.g. Colebrooke 1818, 325, 348). The commentator Kṛṣṇa lists 13 colors in his notes to the BG (Colebrooke 1818, 228): *kālaka* (black), *nīlaka* (blue), *pītaka* (yellow), *lohītaka* (red), etc., the latter ones being different shades of black! These colors are used in addition to the *yā* and are abbreviated as *kā*, *nī*, *pī*, etc.[8]
Known quantities are usually represented by *rūpa* as a monetary unit, as we also know from Arab and Byzantine texts.

3.3. Equation

In Indian mathematics, we find expressions such as 'sāmya', 'samatva', and 'samīkaraṇa' meaning 'sameness', 'equality' or 'making equal'. The concept of an equation in the texts of the period from the eleventh century to Nārāyaṇa (1350) is closer to that of the seventeenth century symbolic

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[7] The correspondence of the meanings of the Sanskrit and Arab terms for the unknown was pointed out already by Gandz (1936, 273). For the possible influence of Hindu mathematics on Arab algebra see Heeffer 2006.
[8] A similar scheme of additional unknowns by the letters *A*, *B*, *C* was first introduced in Europe by Stifel (1544).

equation than is European algebra before 1500. However, an equation in these Sanskrit texts is not to be considered identical to our present notion. Especially the lack of a symbolism to express coefficients and relations between coefficients is a handicap. We would compare the equation in Sanskrit texts with the one of mid-sixteenth-century Europe. An advantage of the European symbolism was the use of numbers for powers of the unknown as in Chuquet (1484) where the Hindu texts use names. On the other hand, the Indian notions of an equation were more advanced with regards to negative terms, the equation to zero and the use of zero terms. What was lacking in symbolism in Sanskrit texts was compensated by structure. An organizational plan for writing down the quantities of an equation (*nyāsa*: putting down, to state) was probably in use before equations themselves. The *BM* show some "equations" involving roots without the use of the unknown (Sūtra 50, Hayashi 1995, 333).[9] This corresponds somewhat with the omission of the base of the unknown as done by Chuquet (1484) and later Stevin and Bombelli (1572). The lower right corner of one strip of the *BM* (Figure 1) shows an example of a layout used for the quantities of a problem. This sūtra 52 deals with a problem of consumption of income and stock. A person earns 5 dīnāras every 2 days (*a*) and spends 9 dīnāras every 3 days (*b*). With a stock of 30 dīnāras (*c*), how long will it take to exhaust the stock? Evidently the rule in the transcription of Table 2 corresponds with the equation

$$x = \frac{c}{b-a}$$ the layout used in the example is *a b c*

Using the values of the problem, this results in

$$x = \frac{30}{\left(\frac{5}{2} - \frac{9}{3}\right)} = 60$$

Using abbreviations for the unknown(s), the structure of an equation becomes more distinct. This first appears in the *BSS* by Brahmagupta (628) as in the example of a linear equation which we will discuss below:

$$\begin{array}{ll} ya\ 1 & ru\ 25 \\ ya\ 3 & ru\ 3 \end{array}$$ corresponds with $x + 25 = 3x + 3$

The terms of each side of the equation are ordered by base . Missing terms are denoted by zero, a practice which appears in Europe after Descartes (1637).

[9] We refer to the Hindu sources by abbreviations as summarized in 10.

Figure 1: One strip of *BM* (f. 60[r] from Kaye 1927, plate XL)

āyavyayaviśeṣaṃ to vibha (ktaṃ) driṣyasaṃguṇaṃ / yal=labdhaṃ sā bhavet=kalaṃ ayaṃ praṣṭe (vya)ye vidhi // uda // dvidine ārjaye paṃca tṛdine nava (bhakṣay)e(t) (bh)āṇḍāgāraṃ tasya tṛnśa kiṃ kālaṃ ārjabhakṣaṇaṃ

di	5	*dinaram*	9	*dr*
di	2	*dina*	3	30

(kara)ṇaṃ / āyavyayaviśeṣan=tu / tatrāyaṃ $\begin{matrix}5\\2\end{matrix}$

(vyayaṃ) $\begin{matrix}\langle 9\rangle\\ \langle 3\rangle\end{matrix}$ (vartyaṃ) $\begin{matrix}\langle 3\rangle\\ \langle 1\rangle\end{matrix}$ anayor=v)i(ś)e(ṣa)m̩

(kṛtvā) $\begin{matrix}\langle 1\rangle\\ \langle 2\rangle\end{matrix}$ (vibhaktaṃ) $\begin{matrix}\langle 2\rangle\\ \langle 1\rangle\end{matrix}$

(dṛśyasaṃguṇaṃ / dṛśyaṃ) <30> (anena guṇitaṃ jātaṃ) <60> (etatkālena ārjanabhakṣaṇaṃ //) (Hayashi 1995, 236)

The difference of the income and the expense is divided (i.e., made into its reciprocal), and multiplied by the visible. What is obtained shall be the time (for consumption). This is a rule for a problem of expense.

A certain man would earn five (dinaras) every other day, and spend nine (dindras) every three days. His stock is thirty (dindras). In what time does the consumption of his earnings (and stock take place)?

5	*dinaras*	9	*dinaras*	*visible*
2	*days*	3	*days*	30

Computation. `The difference of the income and the expense'

(Hayashi 1995, 334)

Table 2: Transcription and English translation by Hayashi (1995)

3.4. Algebra

The first use of algebra in India is found in the Śulba sūtras (800-500 BC). This early algebra was geometrical in nature, similar to Babylonian algebra. The first references to algebra use the word 'bīja-chatuṣtaya' which literally means "the fourfold of seeds", or calculating with one unknown. The title of the treatise of Bhāskara, 'bīja-gaṇita' literally means "calculation with seeds", is the closest term for what we call algebra. The seeds refer to the four types of equations and their representations. Algebra is later also characterized as 'avyakta-gaṇita', the science of calculation with unknowns as opposed to 'vyakta-gaṇita', the science of calculation with known values (Datta and Singh 1962, II, 1). Still later references to algebra use the terms 'ekavarṇa-samīkaraṇa' (equations in one unknown) for linear equations in the *BG* and the *BSS* and 'anekavarṇa-samīkaraṇa' (equations in more than one unknown) in the *BG*. The method for solving quadratic problems is called 'madhyama-āharaṇa', meaning "the elimination of the middle term".

4. Linear problems in one unknown

4.1. Difference of unknowns

We have no general name for this type of problem and therefore use the translation of the corresponding Hindu recipe from Āryabhaṭa, "the difference of beads" or difference of unknown quantities. The general structure of this rule in symbolic form is as follows:

From $ax+b=cx+d$, we may derive $x=\dfrac{d-b}{a-c}$.

We denote the values by (a, b, c, d).

4.1.1. The rule in Hindu algebra (gulikā-antara)

Our first source for a formulation of this rule is from Āryabhaṭa I, 499, (*AB*, ii, 30). The Āryabhaṭīya is one of the basic works of Hindu mathematics and astronomy. The book is written as a collection of 123 verses or stanzas, divided into four sections. The second section, called Gaṇitapāda, covers mathematics in 33 stanzas. The stanzas are rather short and each cover a complete subject such as the rule of three, the sum of an arithmetical progression or interest reckoning. Some of these stanzas are prime representations of what we call proto-algebraic rules. Stanza 30 describes in a very terse form, the equal possessions of two persons each consisting of a known and unknown part:

Clark 1930, 41	Datta & Singh, 1938,II, 40	Rodet 1879, 403
One should divide the difference between the pieces of money possessed by two men by the difference between the objects possessed by them. The quotient will be the value of one of the objects if the wealth of the two men is equal.	The difference of the known "amounts" relating to the two persons should be divided by the difference of the coefficients of the unknown. The quotient will be the value of the unknown, if their possessions be equal.	Par la différence entre des objets divises la différence des roupies que possèdent deux personnes, le quotient est la valeur d'un objet si les fortunes sont égales.

The known part consists of money, or a number of rupies in Rodet's translation. The other part is composed of a number of goods of an unknown value. The commentator Parameśvara, in Kern's edition gives the example of cows (Clark 1930, 41):

> If one man has 100 rupees and 6 cows and the other man has 60 rupees and 8 cows, the value of a cow is 20 rupees provided the wealth of the two men is equal.

Datta and Singh leave out the context and describe the rule in terms of unknowns. In Rodet's translation the rule gives the value of the goods as a general recipe. The three translations and interpretations thus vary widely. Datta and Singh consider this fragment, and more general, the Āryabhaṭīya, as symbolic algebra. They go as far as to state that "the solution of a linear equation in one unknown is found in the *Śulba*" as early as 800 BC and that "classifications of equations" were given in the *Stānānga-sūtra* dated c. 300 BC (Datta and Singh, 1962, 35-6). The translation by Clark is more neutral and, we believe, more convincing. As discussed previously, we consider a solution method algebraic 1) when it uses an unknown as an abstract entity and, 2) when the reasoning proceeds in an analytical way. In a yielding disposition, the we could consider the value of a cow as an unknown, but the text certainly gives no evidence of analytical reasoning. The rule simply prescribes to subtract 60 rupees from 100 and to divide it by 8 cows minus 6 cows to arrive at the price of 20 rupees for a cow.
We find the remains of the same rule in a near pulverized bark leaf of the Bakhshālī manuscript, c700 (Sūtra 51, Hayashi 1995, 333):

> The difference of [the number of] the cows should be made. Make also [the difference of their] properties [in cash].

This formulation compares very well with the *AB*. There is an example added which mentions 10 cows and 8 rupees, but the explanation of the problem is on a leaf which is missing. The *BSS* of Brahmagupta, 628, gives a somewhat different formulation of Āryabhaṭa's rule:

BSS 18.43, Dvivedin, 1902, xviii, 43 (English: Datta & Singh 1938, II, 40)	Colebrooke, 1817, 344
In a [linear] equation in one unknown, the difference of the known terms taken in reverse order, divided by the difference of the coefficients of the unknown [is the value of the unknown]	Rule for a simple equation: The difference of absolute numbers, inverted and divided by the difference of the unknown, is the [value of the] unknown in an equation.

Brahmagupta gives three examples, all related to astronomy. As an illustration of his solution, we take question 13.[10]

[10] Colebrooke (1817, 344) seems to take this as part of the *BSS*. However Hayashi pointed out that the example is from an anonymous commentator (personal communication).

> If four times the twelfth part of one more than the remainder of degrees, being augmented by eight, be equal to the remainder of degrees with one added thereto, tell the elapsed days.
> Here remainder of degrees is put *yāvat-tāvat* viz. *ya* 1. With one added, it is *ya* 1 *ru* 1. Its twelfth part is
>
> $$\frac{ya\ 1\ \ ru\ 1}{12}.$$
>
> This quadrupled is
>
> $$\frac{ya\ 1\ \ ru\ 1}{3}.$$
>
> Augmented by eight absolute, it is
>
> $$\frac{ya\ 1\ \ ru\ 25}{3}.$$
>
> It is equal to remainder of degrees with one added thereto. Statement of both sides tripled,
>
> $$\begin{matrix} ya\ 1 & ru\ 25 \\ ya\ 3 & ru\ 3 \end{matrix}.$$
>
> The difference of [terms of the] unknown is *ya* 2. By this the difference of absolute number, namely 22, being divided, yields the residue of degrees of the sun 11. This residue of degrees must be understood to be in least terms. The elapsed days are to be hence deduced, as before.

Brahmagupta takes the unknown *yā* or *x* for the remainder of degrees. The second part of the equation thus is *x* + 1. The first part is "four times the twelfth part of one more than the remainder of degrees, being augmented by eight", thus

$$4\left(\frac{x+1}{12}\right)+8 \text{ or } \frac{x+25}{3} \text{ resulting in the equation } \frac{x+25}{3}=x+1.$$

Multiplying both sides with three gives the equation for which the rule can be applied $x+25=3x+3$. The difference of the terms of the unknown is 2 *x* and the difference of the known quantities is 22. Applying the rule yields the solution 11.
This fragment provides us with the necessary material to confer on the algebraic character of the solution. The use of *yāvat-tāvat*, with the lexicographical arguments discussed previously, gives a clear determination of an unknown. The solution method is also analytical in the way that the

original problem is reformulated using an unknown to arrive at the equation $x+25=3x+3$. Now, the final part is of interest for our discussion. Datta and Singh (1962, 41) also give the problem and continue with "$2x=22$, therefore $x=11$". But that is not what is written in the original text. In the text, the equation is not resolved by dividing both sides of the equation by two. Instead, quite literally, the final part describes the application of the rule of *gulikāntara* to the values of the known and unknown numbers. Brahmagupta thus employs algebra to reduce a problem in an analytical way to a form in which a general recipe of solution is applicable. He does not use algebra to resolve the value of the unknown. This method of analysis, typical for Hindu algebra is phrased very well by the commentary of Prthūdakasvāmī (*Vāsānābhāshya*, 860), on this specific rule (Colebrooke, 344):

> The value of the unknown quantity, in the example, as proposed by the question, is to be put *yāvat-tāvat;* and, upon that, performing multiplication, division, and other operations as requisite in the instance, two sides are to be carefully made equal. The equation being framed, the rule takes effect. Subtract the [term of the] unknown in the first of those two equal sides from the unknown of the second. The remainder is termed difference of the unknown. The absolute number on the other side is to be subtracted from the absolute number on the first side: and the residue is termed difference of the absolute. The residue of the absolute, divided by the remainder of the unknown, is the value of the unknown

From the eleventh century on, we find descriptions of rules in Indian sources describing operations which are much closer to the manipulation of an equation. Śrīpati formulates the rule as follows (SS, xiv, 14; Sinha 1985, 38):

> The difference of the absolute numbers being taken in the reverse order and divided by the difference [of the coefficients] of the unknown, [the two sides of the equality] should be the measures of the unknown. Alternately, if one wishes [one side] to be added with, decreased by, multiplied, or divided [by an appropriate number], so also is [it] to be performed on the other side.

The rule, as it is here presented, sounds more as the result of a derivation than as a recipe for solution. The additional comment on the possible operations is very much alike the operations on coequal polynomials as we find them in early Arab algebra (between 800 and 1050). After Śrīpati a very similar formulation is given by Bhāskara II, (*AB* 2.30, 1-5):

Dvivedin, 1927, 44 (Datta & Sing 1936, II, 41)	Rodet 1879, 44	Colebrooke, 1817, 185
Subtract the unknown on one side from that on the other and the absolute term on the second from that on the first side. The residual	Les inconnues du premier se retranchent de celles du second, les espèces [sonnantes] du second de celles du premier; par la différence [des	Subtract the unknown quantity of one side from that of the other; and the known number of the one from the other side; then by the remaining

absolute number should be divided by the residual coefficient of the unknown; thus the value of the unknown becomes known	coefficients] des inconnues, on divise la différence des nombre; on connaîtra ainsi la valeur fixe de la quantité inconnue.	unknown divide the remainder of the known quantity: the quotient is the distinct value of the unknown quantity.

Comparing this formulation with the original rule from Āryabhaṭa, we distinguish an important evolution. While both provide a rule for solving the same type of problem, and both rules can be rendered into the same equation, the six centuries that separate the two authors mark a conceptual difference. Āryabhaṭa talks about possessions of two men having a known and an unknown part. Bhāskara's rule is more abstract and deals with unknown quantities or numbers within the structure of an equation. Āryabhaṭa gives a recipe for deriving the value of the objects of the unknown part of the possession. Bhāskara describes a procedure for solving an equation of this particular structure. He does present the solution as a recipe but describes the steps necessary to arrive at a value of the unknown.

4.1.2. The regula augmenti

The rule appears occasionally in Western arithmetic books but less frequently than the negative case discussed below. Johannes Widmann (1489, f. 110^{v}) is the first to coin a name for the rule as the *regula augmenti*.[11] He formulates it as follows:

> Subtrahir die kleyner anzal vonn der grossern und das uberig behalt zu deine*n* teyler. Darnach subtrahir auch die kleyner residuu*m* von dem grossern. Und das uberig geteylt surch deynen vorbehalten teyler bericht die frag des gewichtes ad des gleichen.

Because the rule is poorly represented in other works we suspect that Widmann derived the rule from the negative case. It probably is one of the several variations of mercantile rules to be included in his vast exposition of rules. He adds one example of a man buying cinnamon (*zimmantrinden*). When buying 9 pounds he is left with 13 guilders, when buying 14 pound, one guilder is left. The rule gives an answer for the weight of the goods bought.

4.1.3. The negative case

When we account for the signs of the known values, we get another rule which appears more frequently in Western arithmetic. The general structure of this rule is

[11] This rule should not be confused with the *regula augmentationis* involving two unknowns, discussed below.

$ax + b = cx - d$ with solution $x = \dfrac{b+d}{c-a}$

Rodet (1879, 428) argues that the rule of *gulikāntara* given by Āryabhaṭa is formulated in a general way and that it makes abstraction of the sign. This would mean that the this case is also covered by Āryabhaṭa's rule, discussed above. Without further examples and more explanation from Āryabhaṭa, such assertion is difficult to confirm or disprove. The fact is, that there are indeed no separate rules formulated for this case in Hindu texts. Bhāskara I gives an example of the *gulikāntara* rule with a negative value.[12] The more general formulation by Śrīpati and Bhāskara II indeed allows for the abstraction of the signs. The example is given in the BG of Bhaskara II (c.1150), stanza 103:

> One person has three hundred of known species and six horses. Another has ten horses of like price, but he owes a debt of one hundred of known species. They are both equally rich. What is the price of a horse ?

The solution given in the BG is [13]:

> Example 1st: Here the price of a horse is unknown. Its value is put one so much as (*yāvat-tāvat) ya* 1 and by rule of three, if the price of one horse *yāvat-tāvat,* what is the price of six? Statement: $1\left| ya\ 1 \right| 6 \left|\right.$. The fruit, multiplied by the demand, and divided by the argument, gives the price of six horses, *ya* 6. Three hundred of known species being superadded, the wealth of the first person results; *ya* 6 *ru* 300. In like manner the price of ten horses is *ya* 10.To this being superadded a hundred of known species made negative, the wealth of the second person results; $ya\ 10\ ru\ 1\dot{0}0$. These two persons are equally rich. The two sides, therefore, are of themselves become equal. Statement of them for equal $\begin{matrix} ya\ 6 & ru\ 300 \\ ya\ 10 & ru\ 1\dot{0}0 \end{matrix}$.

"The fruit multiplied by the demand and divided by the argument" is an implicit reference to the *Līlāvatī* where the rule of three is explained in terms of species of fruit, not unlike the expression "comparing apples and oranges" (Colebrooke 1817, 33). Again, as with Bhaskara above, the analytical part of the problems solution reformulates the problem in terms of an

[12] Takao Hayashi pointed my attention to the commentary on the Āryabhaṭīya by Bhāskara in which he cites a Prakrit verse with relation to Āryabhaṭa's example which prescribes the addition and subtraction of negative terms (English translation in Keller 2006, I, 123).
[13] BG stanza 104 (Colebrooke 1817, 188-9). This example is also discussed by Datta and Singh (1962, 42) and by Rodet (1879, 46-). Rodet gives the sanskrit text with a French translation.

unknown, and reduces the expressions to the point at which the general rule can be applied:

> Then, by the rule (§ 101), the unknown of first side being subtracted from the unknown of the other, the residue is *ya* 4. And the known numbers of the second side being subtracted from the known numbers of the first, the remainder is 400. The remainder of known number 400, being divided by the residue of unknown *ya* 4, the quotient is the value in known species, of one so much as (*yavat-tavat*) viz. 100.

Clearly, the final part of the solution is the application of the general rule. Interestingly, the difference between known numbers, meaning 300 and – 100 is 400. So, Hindu authors have no problem adding or subtracting negative values.

Rodet, who was keen in demonstrating that Arab algebra was influenced by India, makes the connection between this problem and the way quadratic problems are treated by Mohammed ibn Mūsa al-Khwārizmī.[14] This rather forced connection can currently not be maintained. However, we share his view that the importance of Hindu algebra for the development in the West has been underestimated. We will make the connection between the *gulikāntara* and proto-algebraic rules which were widely used and circulated in arithmetic books in Western-Europe.

4.1.4. The rule in European arithmetic and algebra treatises

The problem type was named *zuviel – zu wenig* by Tropfke (1980, 601), who cites Chinese, Arab and Hindu sources for its origin. We find three formulations of the problem set in the practical contexts:

1. Distributing figs to children
2. The division of money as payment to workers, soldiers, etc.
3. The payment of bought goods

The distribution of figs to children appears in the *Algorismus Ratisbonensis* (*AR*, c. 1460, Munich Cod. lat. 14908, f 91^v-92^r; Vogel 1954, 75-6) and the *Bamberg Mathematical Manuscript* (*BMM*, c. 1460, Inc. IC I 44, f. 40^v; Schröder 1995, 274-5) with the same values $12x + 37 = 15x - 44$. The solution is a literal application of the Hindu solution recipe. The *BMM* is most specific, referring to the positive and the negative part:

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[14] Rodet 1879, 50, in conclusion of the problem discussed above: "Ces mêmes procédés que nous venons de voir appliquer aux equations du premier degré servent également à transformer les équations du second, pour les amener à la forme toujours trinòme, dans l'Inde, tantôt trinòme, suivant les problèmes, pour les Grecs et les Arabes, ces derniers distinguant, nous l'avons vu, cinq cas (six avec celui du premier degré) de ces équations définitives".

> Machs also: addiere plus und minus fazit 81, und nimm 12 von 15, bleiben 3. Dividiere 81 per 3, fazit 27 Kinder. Und multipliziere 12 mit 27 und addiere 37, fazit 361 Feigen.

The payment of the rent by students in a *bottega* appears in the *Trattato d'Abacho* of Maestro Benedetto (c. 1470, Florence, Magl. Cl. XI, 76, f. 89^v; Arrighi 1987). It is presented as the first of a series recreational problems ("alchuno chaso di diletto") for which no general rules are formulated. The values for the example are $10x - 30 = 8x + 20$ and the given solution is an application of the general recipe.

The third practical context in which the rule applies, can be found in one of the first printed arithmetic books. Borghi (1484) includes the problem of a master who has to pay wages to his workers with a given sum of money in his purse. When he pays each 12 denari he is left with 50 denari, but paying each 15 denari, he has 70 denari short. The question is how much the master has to pay each.[15]

Borghi writes that the problem can be solved in the following way:

> First add the 50 d. which is left, with the 70 d. which is short. This gives 120 d. Then subtract 12 from 15 which leaves 3. Now divide 120 by 3, which is 40 and that is the number of masters. And to know how much denari he has, because it is told that giving each 12 leaves 50 in the purse, the amount the winegrower has is 480 to which we have to add 50 which gives 530, and this how much he has.

The first part is a direct application of the recipe

$$x = \frac{b+d}{c-a}, \text{ thus } \frac{70+50}{15-12} = 40.$$

The rest is the calculation of the contents of the purse, leading to the value 530. The final part is a test of the validity of the solution.

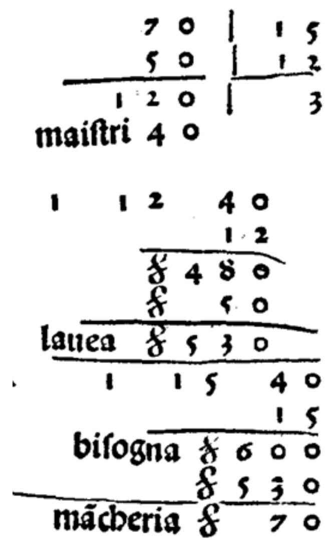

Figure 2 : the *regula augmenti* applied by Borghi (1484)

[15] Borghi, (1484, 112^r): "Esel te fusse dito le uno che ano una quantita de danari e ano apagar maistraze e chostui fa le suo raxon che sel desse azaschun maiestro lire 12, li restera i borsa d. 50 e sel volesse dar azaschun d. 15 li ma*n*cheria d. 70. Adima*n*do qua*n*ti danari laveva e qua*n*ti maistri laveva apagar".

4.1.5. Regula augmenti et decrementi

Again Widmann (1489, f. 112^r) is the first to coin a name for the rule:[16]

> Regula augmenti + decrementi
> An dieser regel soltn dich alszo halten. Subtrahiere die kleyner zal von de*r* grosseren und das ubrige teyl mit der minderung und merung zusam geaddiert un*d* derselben teylung quocient saget dyr zal der person, weliche, szo sy mit gemultiplicirt wirt mit der kleynern anzal un*d* die nymmerung von dem prod*uc*t subtrahirt, wirt ader widerumb das dar noch uberpleybet bericht die ander frag.

Thus for $ax+b=cx-d$, the *merung* and *minderung* refers to *b* and *d* respectively. Evidently, *c* is the larger number and *a* the smaller one. The rule thus prescribes that "the number of persons [receiving a payment]" equals

$$x=\frac{b+d}{c-a}$$

The "other question" about the total payment is $ax+b$.
Widmann adds two examples, one on paying wages to workers with (5, 11, 9, 17) and one about a merchant buying anisette with (12, 37, 15, 44). Some pages further he discusses a rule by the name *Regula Pulchra* III (Widmann 1989, f. 114^v):

> Nu szoltu diesze Regel alszo verfuren. Addir die geminderte zal der d. zur furgelegten zal der d. Und subtrahir die zal des Dinges vo*n* der andern zal yr*s*s gleychen. Unnd dividir die ubrige zal der d. mit der ubrigen zal der gekaufften war. Und der selbigen teylung quocient bericht die frag.

This "pretty rule" is structurally identical with *Regula augmenti et decrementi*, only the context is different. The example provided by Widmann compares the price of 6 eggs minus 5 denari with 4 eggs plus one denari. The price of an egg can be determined with this rule. This compares very well with problem 112 of the *AR* on the price of cheese (f. 83^v-84^r ; Vogel 1954, 63). The *AR* does not mention a name for the rule.

Widmann must have been aware of the redundancy of so many mercantile rules because he applied algebra for this and other types of mercantile problems in the margins of the Dresden C80 manuscript. On folio 355^r he provided an algebraic solution for the problem (Wappler 1899, 544):

> Quidam habuit pueros et denarios et dixit: si cuilibet do 2 d. manent in residuo 5 denarij, si autem cuilibet do 3 d. deficio in 6. Queritur quot sint denarij, et quot sint pueri.

[16] Also discussed by Eneström (1907, 195).

The monk Fridericus Amann included an algebraic solution prior to Widmann in an algebra treatise written in 1461 (Cod. Lat. Monacensis 14908, 153[r]; transcription by Curtze, 1895, 68):

> Quidam habuit laboratores et pecunias. si cuilibet laboratori dedit 5, habundat in 30; si vero daret cuilibet 7, deficiet in 30. Queritur, quot sunt laboratores.
>
> Sit numerus istorum 1 *x*, et fiunt primo 5 *x* et 30, fiet secundo 7 *x* minus 30 equande 5 *x* et 30. 2 *x* equande 60, venit *x* 30

While the rule appears frequently in later arithmetic books, the names coined by Widmann seem to have found little use.

4.1.6. The rule of abstraction

One other alternative name for the *regula augmenti et decrementi* is worth mentioning. A Flemish manuscript by Van Varenbrakens (1532) talks about "Den reghel van abstractie" (the rule of abstraction). He refers to the Latin expression *Regula que dicitur de re*, which we have not found in any extant text. His source must have been an algebraic text as he cites (f. 157[r], Kool 1988):

> Ad sciendum regulam de re scribatur res sub re et numerus sub numero, videlicet in exemplo sequenti. Tunc subtrahur res sub re aut res de re et addatur numerus numero et dividatur numerus perveniens ex illa additione per numerum exemplum.

The terms *res* and *re* apparently refer to the unknowns. In an example comparing the price of four eggs minus 2 denari with one egg plus seven denari, the equation would be

$$4x - 2 = x + 7$$

The rule prescribes the subtraction of the unknown (*re*, *x*) from the unknowns (*res*, *4x*) and the addition of "the numbers" 2 and 7. The solution is the quotient of second by the first. We here witness how a proto-algebraic rule is created from an algebraic solution. Van Varenbrakens (1532) does not mention algebra at all in his text. Most likely he did not master algebra at all. Instead, he recommends his own interpretation of the rule "because not everyone knows Latin, I shall explain this rule in Flemish according to my own understanding" ("Omne dat al de werelt gheen latijn en can, so sal ic u exponeren desen regel in platten vlaemsche na mijnen verstande", f. 157[r]). Van Varenbrakens uses the representation of a cross, typical for many other rules of arithmetic. For the second example, about laborers in a vineyard, he plots the values (12, 37, 15, 44) as follows:

```
    12 | 37                                  12x  + 37
3 --------------- 81    corresponding with
    15 | 44                                  15x – 44
```

Van Varenbrakens takes two pages to explain step by step how to place and manipulate the numbers to arrive at a certain solution. The lack of symbolism is compensated by a specific arrangement of the values in the form of this cross.
Exactly the same values, on distributing figs to children, appears in problem 158 of the *AR* (Vogel 1954, 75) and problem 212 of the Bamberg Inc. Typ. IC I44 (Schröder 1995, 274-5). However, the name for the rule does not appear in these texts. Two centuries later, Isaac Newton includes a problem about the money of beggars in his *Arithmetica Universalis* with $3x - 8 = 2x + 3$ (Newton 1707; 1720, 71).

5. Linear problems in two unknowns

5.1. Regula augmentationis

5.1.1. The rule in Hindu texts

The rule refers to a specific type of problem with two persons asking for a share of the other's money in order to make his sum a given ratio of the other's. In modern symbolism this can be represented as follows:

$$\begin{aligned} x + a &= c(y - a) \\ y + b &= d(x - b) \end{aligned} \qquad (1.1)$$

Other versions involve multiple persons in which the relations are expresses cyclically, which we will not consider here. The earliest source for the two person's version is a Greek epigram attributed to Euclid.[17] A mule and ass are carrying several sacks. The mule tells the ass, "If you gave me one of your sacks, I would have as many as you". The ass replies, "If you gave one of your sacks, I would have twice as many as you". The question is how many sacks they each have. With $x + 1 = y - 1, y + 1 = 2(x - 1)$ the solution is (5, 7).

In Hindu mathematics the first occurrence of this type of problem is in the *BM* (c. 700), Sūtra 10 (Hayashi 1995, 288, 363-4).[18] The rule for solving the problem is formulated as follows:

[17] Heiberg and Menge, 1986, VIII, 286-7. For a history of the problems see Singmaster 1999b, 2000, 2004.
[18] This was previously unnoticed by Kaye (1933, II, 168) writing "I have not yet made out the meaning of this sūtra", and is therefore not covered by Datta and Singh (1938).

> The two multipliers are each increased by a unity, multiplied by the sum of the beggings, divided by the product of the multipliers less unity, and increased by the opposite beggings. This is a rule for [solving] equations involving [two] multipliers.

The example provided for the rule is on a missing leaf, but the context of the problem obviously refers to two beggars asking money from each other. Hayashi give the following rule in modern symbols, which is a literal representation from the text:

$$x = \frac{(c+1)(a+b)}{cd-1} + b$$
$$y = \frac{(d+1)(a+b)}{cd-1} + a \qquad (1.2) \qquad \text{with values } (a, b, c, d)$$

The formulation of the rule is thus very specific to this type of problems. In later arithmetic books we also find this connection between a rule and prototypical problems. A more general approach for solving the problem is taken by Bhāskara II in the *BG* (106, Colebrooke 191):

> Example: One says "give me a hundred, and I shall be twice as rich as you, friend!" The other replies, "if you deliver ten to me, I shall be six times as rich as you." Tell me what was the amount of their respective capitals?
>
> Here, putting the capital of the first $ya\ 2\ \ ru\ 1\dot{0}0$, and that of the second, *ya* 1 *ru* 100; the first of these, taking a hundred from the other, is twice as rich as he is: and thus one of the conditions is fulfilled. But taking ten from the first, the capital of the last with the addition of ten is six times as great as that of the first: therefore multiplying the first by six, the statement is $ya\ 12\ \ ru\ 6\dot{6}0$. Hence by the equation, the value of "so much as" is found, 70. Thence, by "raising" the answer, the original capitals are deduced 40 and 170.

Curiously, the solution is based on the general rule for a linear equation in one unknown (*yavat-tavat*) instead of two. Rather than using the unknown for one of the capitals, Bhāskara II uses 2*x* – 100 for the capital of the first and *x* + 100 for the capital of the second. This agrees with the first condition because if we subtract 100 from the second and add it to the capital of the first he becomes twice as rich as the second. If we use these values for the second condition we arrive at

$$x+100+10 = 6(2x-100-10) \text{ or } x+110 = 12x-660$$

This leads to a value of 70 for the unknown. Using this value in the original expressions of the two capitals we get 40 and 170 as the solution to the problem.
In chapter six of the *BG*, Bhāskara II repeats the problem, but solves it this time by anekavarṇa-samīkaraṇa, or multiple unknowns. These are in addition to *yā*, typically black (*kālaka*), blue (*nīlaka*), yellow (*pītaka*), etc. The second unknown is thus represented by *ka*. The solution goes as follows (§156, Colebrooke 231):

> Example: One says 'give me a hundred, and I shall be twice as rich as you,' &c.
> Let the respective capitals be *ya* 1 *ka* 1. Taking a hundred from the capital of the last, and adding it to that of the first, they become $ya\ 1\ ru\ 100$ and $ka\ 1\ ru\ 1\dot{0}0$. The wealth of the first is double that of the second therefore equating it with twice the second's capital, a value of *yāvat-tāvat* is obtained $\dfrac{ka\ 2\ ru\ 3\dot{0}0}{ya\ 1}$
> Again, ten being taken from the first and added to the capital of the second, there results $ya\ 1\ ru\ 1\dot{0}$ and $ka\ 1\ ru\ 10$. But the second is become six times as rich as the first: wherefore making the second equal to the sextuple of the first, a value of *yāvat-tāvat* is obtained $\begin{matrix} ka\ 1\ ru\ 70 \\ ya\ 6 \end{matrix}$. With these reduced to a common denomination and dropping the denominator, an equation is formed; from which, as being one containing a single color (or character of unknown quantity), the value of *ka* comes out by the foregoing analysis (Ch. 4); viz. 170. With which substituting for *ca*, in the two values of *yāvat-tāvat*, and adding it to the absolute number, and dividing by the appertinent denominator, the value of *yāvat-tāvat* is found, 40.

Bhāskara II constructs the equation $x+100=2(y-100)$ from the problem text and derives that $x=2y-300$. From the second condition he arrives at the identity

$$x=\frac{y+70}{6}.$$

The rest of the solution has unfortunately been omitted, but the standard procedure "reduced to a common denomination and dropping the denominator, an equation is formed" is a recurring expression which refers to the following steps:

- bring the "value" of *x*, $x=2y-300$ to the same denominator as the other "value" of *yā*, thus $x=\dfrac{12y-1800}{6}$

- drop the common denomination and form the equation, leading to $y + 70 = 12y - 1800$
- this is a simple equation in one unknown with the solution $11y = 1870$ or $y = 170$.

Only then Bhāskara uses 'substitution' to replace the value of *ka* in either

$$x = 2y - 300 \text{ or } x = \frac{y+70}{6}$$

which can be solved again by *yāvat-tāvat.* This specific solution reveals two interesting aspects of Hindu algebra. The first unknown has a status which is different from the other unknown(s). Its value is determined by the rule of *yāvat-tāvat* while the other unknown is found through 'analysis of several colors' Secondly, the solution has to follow certain steps in a specific order. The application of *yāvat-tāvat* has to come last. By accident the construction of the first equation gave us a direct expression for the first unknown $x = 2y - 300$. Substituting this in the second equation gives us a value for y which again could be substituted in the first. However, all solutions to linear problems in more than one unknown follow this strict procedure.

Mahāvīra gives a formulation in the *GSS* which refers back to the context of beggars as in the *BM* but generalizes the rule for more than two (stanza 251 ½ , Padmavathamma 2000, 369):

> The sums of the moneys begged are multiplied each by its own corresponding multiple quantity as increased by one. With the aid of these [products] the moneys on hand are arrived at according to the rule given in stanza 241. These quantities [so obtained] are reduced so as to have a common denominator. Then they are [severally] divided by the sum as diminished by the unity of the specified multiple quantities [respectively] divided by [those same] multiple quantities as increased by one. [The resulting quotients] themselves should be understood to be the money on hand [with the various persons].

The rule refers not to the cyclical case but the type of problems in which the ratio is expressed to the sum of all the others. If we use S for the sum of all the shares, the general format of the conditions can be written in modern symbolism as $x_i + a_i = b_i(S - x_i - a_i)$. The rule then gives the following solution

$$x_i = \frac{b_i S}{(b_i + 1)} - a_i$$

One of the two examples is worked out by this rule in the section called 'citrakuṭṭīkāra-miśra', or "various analyses on mixtures".

5.1.2. The rule in European arithmetic

Our problem appears in the early medieval *Problems to Sharpen the Youth* by Alcuin (c. 800, Folkerts 1993) and thus predates Bhāskara II and Mahāvīra. However, the interpretation by Alcuin is somewhat different and comes down to $y-a+c=d(x+a-c)$ providing a simple numerical recipe which shows no resemblance to the Indian rule. Fibonacci (1202) has many versions of the problems and gives no less than five solution methods. Most simple types are solved by the so-called *tree method* which is the rule of single false position dating back to Babylonian sources (Høyrup 2002). The reference to a tree originates from a prototype problem of determining the height of a tree with fractional sums given.[19] The second method he calls the *Regula recta* or "direct method that is used by the Arabs" and this is one of the three occasions in the *Liber Abbaci* where algebra is used outside chapter 15. This solution is all the more unusual because Fibonacci rarely uses algebra for linear problems. For the following problem (Boncompagni 1857, 191; Sigler 2002, 291),

$$\begin{aligned} a+7 &= 5(b-7) \\ b+5 &= 7(a-5) \end{aligned}$$

he poses that the second man has the *thing* plus 7 denari, thus $b = x + 7$. The first man therefore will have $x - 7$. Using the values $x + 7$ and $x - 7$ for the two persons in the second expression yields $x + 12 = 7(5x - 12)$. It is worthwhile to follow Fibonacci's solution to this equation in detail because it is an essential witness account for an early algebraic solution to a linear equation. The text is somewhat confusing because Fibonacci switches from *denari* to *soldi* which we will discount (1 *soldi* = 12 *denari*). He first multiplies the terms so that $x + 12 = 35x - 84$. Then he adds 84 to both parts, and "because if equals are added to equals, then the results will be equal" $x + 96 = 35x$. And "from the above written two parts are subtracted one thing, then those which remain will be equal", thus $34x = 96$. To find the value of the thing you have to divide 96 by 34, resulting in the solution

$$\left(\frac{167}{17},\frac{121}{17}\right).$$

Another instance of the problem is solved in the same way (Sigler 2002, 300). The three other methods apply to a variant of the problem with the following structure (Sigler 2002, 294-9):

[19] For a comprehensive discussion of the method used by Fibonacci and a French translation, see appendix 6 in Spiesser (2003, 641-5).

$$x+a=b(y-a)-c$$
$$y+b=a(x-b)-d$$

These three alternative solutions operate on what Fibonacci calls the least sum ($x + y$), the intermediate sum ($x + y + c$) and the greatest sum ($x + y + d$). Yet another approach for the same problem is taken in chapter 12 using the rule of double false position. However, an application of the Hindu rule for this type of problem does not appear in the *Liber Abbaci*.

Several abacus manuscripts during the next centuries deal with the problem. The most common solution is by double false position. Some examples are Lucca codex 1754 (c. 1330, Arrighi 1963). Question 5 of the Clm14908 (f. 49^v, Curtze 1895, 43). Pellos (1494, f. 70v) in the *Compenio de l'abaco*. Adam Ries (1572, f. 62^v) formulates the problem with the values

$$a+1=b-1$$
$$b+1=3(a-1)$$

The pseudo Paolo dell'Abbaco of c. 1440 includes two problems. Problem 69, with values (8, 2, 10, 3) and problem 126 with (3, 2, 5, 3) (Arrighi 1964, 63-5, 100-2).

The *Arte Giamata Aresmetica* (c. 1417, Turin, N. III. 53) contains one version of the problem not solved by double false position. The solution method is incomprehensible without the background of the Indian solution recipe (f. 32^v; Rivoli 1982, 41-2). In modern symbolism the problem is:

$$x+9=2(y-9)$$
$$y+11=3(x-11)$$

In her commentary Rivoli (1982, x) classifies the method under false position, possibly because the text gives "per le primo che dice 2 cotanti pone 2/3, per lo segondo che dice 3 cotanti pone ¾". However, we believe this to be erroneous as the author of the manuscript clearly refers to a rule which is different from the rule of double false ("Questa è la sula regula"). Let us attempt to reconstruct the original rule from the numerical application in this example. The text reads as follows:

> ora di': 2/3 e ¾ se trova in 12; li 2/3 de 12 è 8, li ¾ de 12 è 9; ora azonze 8 e 9, fano 17, trane 12, resta 5 e questo è lo partitore. Ora azone le loro domande, zoè 9 e 11, che sono 20, po' di': per lo primo: 8 via 20 fa 160, parte per 5, vene 32, trane 9, resta 23 e tanti d. aveva lo primo; per lo segondo di': 9 per 20 fa 180, parte per 5, vene 36, cava 11, resta 25 e tanti d. avea lo segondo. È fata.

If we use the general formulation of the problem from (1.1) we have for (*a*, *b*, *c*, *d*) the values (9, 11, 2, 3). Using the abstract form, we can reconstruct the rule as follows. The fractional parts 2/3 and ¾ must refer to

$$\frac{c}{c+1} \text{ and } \frac{d}{d+1}.$$

The sum of these fractions is multiplied by 12. This part is most difficult to determine, but from further evidence from the *AR*, discussed below, this stands for the product $(c+1)(d+1)$. Then multiply

$$\left(\frac{c}{c+1}+\frac{d}{d+1}\right) \text{ with } (c+1)(d+1)$$

which gives 8 + 9 or $c(d+1)$ and $d(c+1)$ respectively. Now subtract from this sum equal to 17 the product $(c+1)(d+1)$, or 12, which results in $cd-1$. This is called the *partitore* or divisor. Next, add the parts *a* and *b* together to (*a* + *b*), producing 20. Now for the money of the first, multiply 8 with 20 which results in 160 or $c(d+1)(a+b)$. If we divide this by the divisor we get 32 or

$$\frac{c(a+b)(d+1)}{cd-1}.$$

Finally, 9, or *a*, is subtracted from this and we get the money of the first person. The rule thus amounts to the expression

$$x=\frac{c(a+b)(d+1)}{cd-1}-a.$$

If we bring the this formulation to the same denominator with that of the *BM*,

$$x=\frac{(c+1)(a+b)}{cd-1}+b$$

we see that indeed both are equal to $x=\dfrac{a+ac+bc+bcd}{cd-1}$.

Although formulated slightly different, the two rules thus correspond. The rest of the text formulates a rule for the money of the second person, multiplying 9 with 20, or $d(a+b)(c+1)$. This is divided by the divisor and subtracted with *b* results in the amount of the second. We can therefore conclude that the solution to problem depend on these two rules:

$$\begin{aligned} x &= \frac{c(a+b)(d+1)}{cd-1} - a \\ y &= \frac{d(a+b)(c+1)}{cd-1} - b \end{aligned} \qquad (1.3)$$

The *Trattato dell'Acibra amuchabile* of an anonymous Florence master deals primarily with algebra. The author makes an exception by applying the proto-algebraic rule instead of algebra. This abacus treatise predates the previous one but because of an added complexity, we discuss it here.[20] The text formulates the problem as follows:

> There are two men having a number of denari. The first one tells the second: "if you give me 16 of your denari to me, I have twice as many as you". Tells the second to the first: "if you give me as much denari as the proportion of your part and mine of what you have given to me, I will have three times as much as you. I want to know how much each has.

The added complexity lies in the extra condition that for the amount asked by the second person. This results in the follow formulation using modern symbolism:

$$\begin{aligned} &x + a = 2(y - a) \\ &y + b = 3(x - b) \\ &a = 16 \\ &b = \frac{x}{y} a \end{aligned}$$

The author solves the problem through the application of a 'beautiful rule' which again has to be deciphered from the solution:

> Questa ragione ti voglio io insegniare per una bella reghola per la quale tu potrai fare tutte le somiglianti raxoni. Sapi che tanto è a dire due cotanti, come due terzi di tutta la quantità e tre cotanti è tanto a dire come 3/4 di tutta la quantità. Ora sapi in che numero si truova 2/3 e 3/4, che si truova in 12. Ora piglia i 2/3 di 12, ch'è 8; ora piglia 3/4 di 12 ch'è 9; giugni con 8, fanno 17, chavane 12, in che si truovò il numero, rimane 5 e 5 è nostro partitore. Ora fa il primo al secondo, cioè contra 16, fanno 192, partitolo per 5 che tu serbasti, vienne 38 2/5 e 38 2/5 avia il secondo.

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[20] Florence, Ricc. 2263, problem 35, f. 48v; van Egmond (1980, 151) dates this part of the manuscript c. 1365, based on the watermark. Transcription by Simi (1994, 57-8) : "E sono 2 huomini ch'àno denari. Dicie il primo al secondo: se ttu mi dessi xvj de denari tuoi, io avrei due cotanti di te. Dicie il secondo al primo: se ttu mi dessi tal parte de' tuoi che 'n t'io òe dato a tte de' miei, io avrei tre cotanti di te. Vo' sapere quanto avea chatuno", translation mine.

The first part corresponds fully with the Turin manuscript, to the point of deriving the devisor $cd-1$ equal to 5. Then the 12 is multiplied with 16. This corresponds with *a* times product of the augmented multipliers or $a(c+1)(d+1)$. This divided by the divisor gives immediately the money of the second and the rule amount to:

$$y=\frac{a(c+1)(d+1)}{cd-1} \qquad (1.4)$$

The rest of the text substitutes the numerical value for *y* in the first condition. It remains a mystery by who or how this rule was derived from (1.3) for the relation with the canonical form is not an evident one. If we bring the two equations to the same denominator we get:

$$x=\frac{a(2c+c^2+1)}{cd-1}$$
$$y=\frac{a(c+d+cd+1)}{cd-1}$$

Substituting these in $b=\frac{x}{y}a$ results in

$$b=a\frac{a(2c+c^2+1)}{a(c+d+cd+1)}.$$

Simplifying this to:

$$b=a\frac{(c+1)^2}{(c+1)(d+1)}$$

this results in a value for the proportion and a value for *b*

$$b=a\frac{(c+1)}{(d+1)} \text{ or } b=\frac{3}{4}(16)=12$$

In the *Columbia algorismus* there are four problems of this type (c. 1350, Vogel 1977, 113-7). The most extensive argumentation is offered in relation to problem 104.[21] The problem can be represented as follows:

$$x + 4 = 9(y - 4)$$
$$y + 7 = 5(x - 7)$$

The author starts by assuming the value 7 for *x*. From the first condition, this leads to a value of

y as $5\frac{2}{9}$. Then he adds 7 which gives $y + 7 = 12\frac{2}{9}$.

At this point a tacit rule is applied in which the following happens (Vogel 1977, 11?):

> tanto supra 7, che te nne de' $12\frac{2}{9}$ a parttire per 5 via 9 meno 1, cioiè di 44, e cholui che àne $5\frac{2}{9}$ uno novero quente veni di $12\frac{2}{9}$ a parttire per 44. Dunqua ill'uno aveva 9 e ½ e ll'altro aveva 5 ½. E per questo modo si fanno tucte le similigliante rascioni e di più d' e di più rotto.

Our observation that this solution depends on a general rule stems from the last sentence which recommends this method "for all similar problems". The choice of 7 for *x* has nothing to do with the method of false position but is value *b* in (1.1). Thus the first step is to use *b* as the value of *x* in the first condition to arrive at a value of y_1. This value, added with *b* and divided by $cd - 1$ gives the share of the second. The general rule can therefore be translated in modern symbolism as follows:

$$\begin{aligned} y_1 &= \frac{a+b}{c} + a \\ y &= \frac{y_1 + b}{cd - 1} + y_1 \end{aligned} \qquad (1.5)$$

To compare with the rule of the BM we can substitute y_1 in the second equation:

$$y = \frac{\frac{a+b}{c} + a + b}{cd - 1} + \frac{a+b}{c} + a$$

[21] Columbia Univerisity, X511 A13, f. 51v: "Fammi questa rascione: sonno 2 compagni che ànno d. in borschia. Disse ill'uno all'altro: se mme dai 4, io n'avarò 9 chotanta di tie. Disse ill'altro: se me nne dai 7, io n'avarò 5 chotanta di ti".

Bringing the first two terms to the same denominator gives

$y = \frac{(d+1)(a+b)}{cd-1} + a$ which is the same as the second equation of (1.2).

5.1.2.1. Regula augmentationis

The term *regula augmentationis* originates from the *Algorismus Ratisbonensis* (*AR*) and is mentioned in relation to the problem 138 (c. 1450, Vogel 1954, 70):

$$a+1 = 2(b-1)$$
$$b+1 = 3(a-1)$$

The use of two multipliers allows us to decipher the structure of the rule while the amounts asked for are unfortunately both one. Another example with an application of the rule is problem 220 but also here the amounts are the same.
The application of the rule goes as follows:

> Machss also $\begin{matrix} 1/2 & 1/3 \\ 3 & 4 \end{matrix}$ und mer ydlichen nenner mit einem und multiplicir dy nenner miteinander, facit 12. Nu 2/3 und ¾ von 12 facit 17. Nu zeuch 12 von 17, pleiben 5, [der] divisor. Nu addir 1 und 1 zesam, erit 2. Nu 2/3 von 12 machen 8, multiplicir 8 mit 2 erit 16. Dy dividir mit 5, erit 3 1/5. Nu subtrahir 1, manet 2 1/5 und alz vil hat a.

In his commentary Vogel (1954, 218) explains the name *Regula augmentationis* as derived from the operation of adding one to the coefficient, which also appears in the case with multiple unknowns. However, he provides no further explanation for the rule. Let us therefore reconstruct the general formulation of the solution method. Using the symbolic form from (1.1) we have here for (*a*, *b*, *c*, *d*) the values (1, 1, 2, 3). Using symbols, the rule applied in the *AR* can be reformulated as we did with for the Italian text:

> Add one to each of the multipliers *c* and *d*. Multiply these results and we arrive at $(c+1)(d+1)$. Then multiply $\left(\frac{c}{c+1} + \frac{d}{d+1}\right)$ with $(c+1)(d+1)$ which gives $c(d+1)+d(c+1)$. Now subtract from this result the product $(c+1)(d+1)$ which results in $cd-1$. We will call the divisor. Now add *a* and *b* to (*a* + *b*). Multiply this with $\frac{c}{c+1}\left[(c+1)(d+1)\right]$ which results in

$c(a+b)(d+1)$. If we divide this by the divisor and subtract *a*, we arrive at $x = \frac{c(a+b)(d+1)}{cd-1} - a$.

The *AR* gives also a rule for the other value corresponding with

$y = \frac{d(a+b)(c+1)}{cd-1} - b$ (Vogel 1954,):

> Danarch ¾ von 12, erit 9; multiplicir 9 per 2, erit 18, divide per 5, facit 3 3/5. Subtrahe 1, facit 2 3/5; in tantum habet b und dy zwue zal sein radices dic: da mihi 3, so triplir radices. Spricht du, gib mir 10, so multiplicir radices durch 10, etc.

We can therefore conclude that the solution to problems 138 and 220 depend on the rules formulated in (1.3). From the structure of the solution and the use of the terms *partitore* and *divisor* we believe that the German formulation is derived from this or another fifteenth-century Italian abacus text. The use of *radices* in the German text is intriguing. It seems to be a relic from an earlier algebraic solution.

The rules are rather complex for the fifteenth-century, and they do not appear very frequently in other arithmetic books. The Bamberg Inc. Typ IC 144, (Schröder 1995) which shares 78 problems with the AR, does not include this problem, possibly because it was not easy to understand.

The *Compendy de la Practique des Nombres* of 1476 by Barthelemy of Romans has several versions of the problem (f. 186^{v} – 200^{v}, Spiesser 2003, 295-316). The solution is still different from what we have seen before. Barthelemy gives two rules called *multiplex* and *submultiplex*. The first rule amounts to finding a number x so that the following condition holds:

$$\left(\frac{c}{c+1} + \frac{d}{d+1}\right)x - x = (a+b)$$

The method proposed by Barthelemy is that of single false position. Thus for the problem

$$\begin{aligned} a+7 &= 5(b-7) \\ b+5 &= 7(a-5) \end{aligned} \qquad (1.6)$$

the condition becomes $\left(\frac{5}{6} + \frac{7}{8}\right)x - x = 12$.

He poses $x = 24$ which leads to 17 instead of 12. Applying the rule of three,

$\frac{x}{24} = \frac{12}{17}$ the value for x is $16\frac{16}{17}$.

The *submultiplex* rule asks for a number such that the following condition holds:

$$x - (a + b) = \left(\frac{1}{c+1} + \frac{1}{d+1} \right) x$$

and is solved in the same way.

5.1.2.2. Regula pulchra

This rule is also discussed by Widmann (1489, f. 120^{v}). It is the fourth time in one and the same book in which he uses the name, *Regula pulchra* for a different rule. Instead of using numerical quantities, Widmann formulates the rule in general terms such as ‘parts’, ‘denominator’, ‘sum’ and ‘divisor’, where the Turin text only used ‘divisor’. By doing so Widmann’s text corresponds with the formulation in the Turin manuscript as well as with the *AR*, which is his most likely source.

> An dieser regel soltu also procediren. Secz die teyl in die kleynste zal und multiplicir die nenner zu sammen und addir die teyl des geneynen nenner zusam und von dem aggregat subtrahir den gemein nenner. Pleypt uberig deyn teyler. Darnach addir die czeler der furgab zu sammen. Und das aggregat secz den zeler des ersten gefunden nenner und nym den aber die teyl von den ersten gemeynen nenner und das selbige multiplicir mit dem zeler. Und teyl darnach dieser product mit deynen teyler und von dem das auss solicher teylung kumpt subtrahir den ersten zeler wyder von und pleybt die zal des ersten und also gleycher weys ihr auch den anderen und pleybt zum letzten die zal des anderen und ist gemacht.

Widman presents the example of two men comparing their money. Without such example the rule would be incomprehensible.

> Of the same. One man meets another and tell him “When you give me 1 d. I have twice the money you have”. The other answers the first “if you give me 1 d. from your money I will have three times as much as you have”. The question is how much each of the men had.

The problem is exactly the same as the one in the *AR*, discussed above. With the parts (*teyl*) Widmann refers to

$\frac{c}{c+1}$ and $\frac{d}{d+1}$

in our symbolic formulation. The first line thus describes how to multiply the denominators $(c+1)(d+1)$ and to multiply this with the two parts. From this, subtract the product again, and this is called the divisor. This corresponds with

$$\left(\frac{c}{c+1}+\frac{d}{d+1}\right)(c+1)(d+1)-(c+1)(d+1) \quad \text{which simplifies to } (cd-1).$$

Next, add the multipliers of the problem together as in (*a* + b*).* This is multiplied with the first of the previously found products, results in

$$\left(\frac{c}{c+1}\right)(c+1)(d+1)(a+b) \qquad \text{or} \quad c(d+1)(a+b).$$

Dividing this by the divisor and subtracting the first part *a*, the resulting value is the money of the first person. In conclusion, Widmann's *regula pulchra* IV corresponds with the rules of (1.3).

5.1.3. Algebraic solutions

The first algebraic solution to this type of problem appears in the early fifteenth century in the writing of Giovanni di Bartolo dell'Abacho (1364-1440). There are no extant manuscripts by di Bartolo, but several parts of his treatises have been preserved in other abacus collections. The Siena L. IV. 21 manuscript has problems 13 and 14 as we discussed from the anonymous ms. Florence, Ricc. 2263. Let us look at problem 14:[22]

> Two have a number of denari. The first one tells the second: "if you give me 12 of your denari to me, I have twice as many as you". The second to the first: "if you give me as much denari as the proportion of your part and mine of what you have given to me, I will have three times as much as you. I want to know how much each has.

di Bartolo uses the unknown for the number of denari of the second. In symbolic notation he interprets the problem

[22] Siena L. IV. 21, ff. 435^v-436^r; Pancanti 1982, 25-6: "Due ànno danari; il primo àl sechondo: se ttu mi dessi 12 de' tuoj danari, io arej due chotanti di te. Il sechondo al primo :se ttu mi dessi tal parte de' tuoj danari chente tu chiedesti a mme de' miej, io arej 3 chotanti di te. Adimandasi quanti danari aveva ciaschuno ... da ssé" (the missing part is illegible).

$$\begin{array}{lcl} a+12=2(b-12) & & a+12=2(x-12) \\ b+c=3(a-c) & \text{as} & x+c=3(a-c) \\ c=\dfrac{a}{b}12 & & c=\dfrac{a}{x}12 \end{array}$$

The first, after receiving 12 from the second has twice of what is left for the second, or 2(x – 12). Before receiving the 12, he thus originally had

$$a=2x-36$$

The statement of the problem tells us that the amount given by the first to the second is 12 times the proportion of the two amounts, thus

$$c=12\left(\frac{2x-36}{x}\right) \text{ or } c=\frac{24x-432}{x}.$$

This allows to express the first statement of the problem by the equation

$$x+\frac{24x-432}{x}=3\left(2x-36-\frac{24x-432}{x}\right)$$

Taking several steps to simplify this, di Bartolo arrives at

$$5x=108+\frac{96x-1728}{x}$$

and multiplies both parts to get to the form $5x^2+1728=204x$ which fits his canonical fifth rule (the quadratic equation $ax^2+c=bx$) with solution $28\frac{4}{5}$ for x or the second and $21\frac{3}{5}$ for the first person.

Later during the fifteenth century the problem is frequently discussed but algebraic solutions are an exception. The Florence, Magl. Cl. XI, 76, includes an algebraic solution in one unknown (*quantità*) (f. 100^r; Arrighi 1987). However Rafaello Canacci who wrote an extensive treatise on algebra (c. 1495) finds it necessary to include the proto-algebraic rule after having explained his algebraic solution. His version of the problem is about two men comparing their money. The problem corresponds with the values of Barthelemy (1.6). His alternative solution is an exact reformulation of the proto-algebraic rule.[23]

[23] Florence Pal. 567, f. 69^r-69^v; problem 57; Procissi 1983, 38-9: “sella vuoi per altro modo; si debbi sapere che parte è caschuno di tutta la somma el primo che adimanda 7 d. arà 5

6. Indeterminate linear problems in three or more unknowns

We identified several candidates of proto-algebraic rules for problems involving more than two unknown quantities. Most of these are indeterminate ones. We will discuss one type which has a complex but interesting history. The rule has a special interest for our discussion as we have both a Greek and a Hindu tradition of its use. There has been a controversy on the alleged influence from Greece to India, as defended by Cantor and Kaye and disputed by Rodet. The controversy can be explained as a misunderstanding of the rule. We will demonstrate in detail that the Greek and Indian version are in fact two different rules.

6.1. The original formulation in Hindu sources

The first Indian source for a formulation of this rule is from Āryabhaṭa I, 499, (*AB*, ii, 29) as follows:

Clark 1930, 40	Datta and Singh, 49	Rodet 1879, 402-3
If you know the results obtained by subtracting successively from a sum of quantities each one of these quantities set these results down separately. Add them all together and divide by the number of terms less one. The result will be the sum of all the quantities.	The [given] sums of certain [unknown] numbers, leaving out one number in succession, are added together separately and divided by the number of terms less one; that [quotient] will be the value of the whole.	La somme d'un certain nombre de termes diminuée successivement de chacun de ces termes [forme une série de nombres] qu'on ajoute tous ensemble ; on divise par le nombre de termes moins un, et l'on obtient la valeur de la somme [primitive].

Three observations are central with respect to the type of problems Āryabhaṭa's rule applies to:

1) The rule is valid for any number of quantities
2) The sum of all the quantities is unknown and given by the rule

Not evident from the rule, as cited above, is that

3) The partial sums relate to the total of all the quantities, except one

chotanti di lui si arà 5/6 di tutta la somma meno 7 d. el sechondo che adimanda all ui arà 7 chotanti di lui siara 7/8 di tutta la somma meno 5d. e tramandue arrano 7/8 5/6 di tutta la somma meno 12 d Orsappi in che si truova e 7/8 5/6 chessi truova in 24 e 5/6 di 24 e 20 e 7/8 di 24 e 21 ragiuni insieme 20 e 21 fa 41 trane 24 coe il numero inche si trovò 5/6 e 7/8 resta 17 dove multipricha 12 vie 20 fa 240 e parti p 17 vienne 14 2/7 trane 7 che gle meno resta 7 2/17 e tanti al primo pello sechondo multipricha 12 vie 21 fa 252 e parti p 17 vienne 14 14/17 trane 5 chegli è meno resta 9 14/17 ettanto ae il sechondo e chosì vedi che torna chome dice mmo nella passata ad unche oserva quale modo più ti piace inn elle simile ragione".

In modern symbolism the general structure of the problem thus is:

> Suppose *n* amounts $(a_1, a_2, \ldots, a_n)$ with unknown sum *S*, and with the partial sums $(s_1, s_2, \ldots, s_n)$ given, where $s_i = S - a_i$, then $S = \frac{\sum_{i=1}^{n} s_i}{n-1}$ (1.7)

The rule and the problems it applies to, should not be confused with a similar problem in which the partial sums of two consecutive quantities are given. For three numbers, the problems are evidently the same, but they diverge for more than three quantities. E.g. for five quantities the corresponding equations are:

$$\begin{aligned} a_1 + a_2 + a_3 + a_4 &= s_1 \\ a_1 + a_3 + a_4 + a_5 &= s_2 \\ a_1 + a_2 + a_4 + a_5 &= s_3 \\ a_1 + a_2 + a_3 + a_5 &= s_4 \\ a_2 + a_3 + a_4 + a_5 &= s_5 \end{aligned} \quad \text{and} \quad \begin{aligned} a_1 + a_2 &= s_1 \\ a_2 + a_3 &= s_2 \\ a_3 + a_4 &= s_3 \\ a_4 + a_5 &= s_4 \\ a_5 + a_1 &= s_5 \end{aligned} \quad \text{for consecutive quantities}$$

Kaye (1927, 40, note 2) was the first to point out that “a similar rule” can be found in the *BM*. The rule is formulated in Sūtra N15 as follows (Hayashi 1995, 324):

> Having put down any optional number for the first (unknown), one should subtract successively that (number) and the rest (from the given numbers). One should point out half the sum of the optional number and the (final) remainder as the value of the first (unknown).

The two examples added in the *BM* allow us to understand the meaning of the rule (ibid.):

> Quantities ... There, the mixed quantity of the first and the second is thirteen; the mixed quantity of the second and the third is fourteen; and the mixed quantity of the first and the third is said to be fifteen. What shall be the value of each? Let it be told to me, if you know it.
>
> Computation. “Having put down any optional number for the first” (Sūtra N15). In the present case, the optional is (taken to be) five: 5. This is (supposed to be) the value of the first. Statement: “One should subtract successively that (number) and the rest (from the given numbers)” [Sūtra N15]. Five should be subtracted from the first (given) number; the remainder is 8. This is to be subtracted from fourteen; the remainder is 6. This is to be subtracted from fifteen; the remainder is [9]... .

The example describes the problem, which can be formulated symbolically as:

$$x_1 + x_2 = 13$$
$$x_2 + x_3 = 14$$
$$x_1 + x_3 = 15$$

Applying Āryabhaṭa's rule, the solution would be based on the rule for deriving the sum of the three unknown quantities as follows:

$$x_1 + x_2 + x_3 = \frac{13+14+15}{3-1} = 21$$

This allows to determine the value of the quantities by subtracting the partial sums from the total. Instead, the solution is the *BM* is based on choosing the arbitrary value 5 for x_1 and then calculating the values of the other quantities as $x_2 = 13 - 5 = 8$, $x_3 = 14 - 8 = 6$ and using this value in the third sum to arrive at $15 - 6 = 9$ for the "final remainder". The rule gives the correct value of the first unknown quantity as half the sum of the supposed value and the final remainder, thus $(5 + 9)/2 = 7$.[24] Not only is this rule different from Āryabhaṭa's, its validity also depends on particular number of sums. For an even number of sums the problem becomes indeterminate.

A commentator of the Āryabhaṭīya, called Bhāskara I (written 629, not to be confused with Bhāskara II), gives two examples of problems that can be solved with Āryabhaṭa's rule with the partial sums (30, 36, 49, 50) and (28, 27, 26, 25, 24, 23, 21) (Shukla and Sarma 1976, 307-308).

6.2. The derived problem in Hindu sources

From the ninth century we find a derived version of the previous problem in Hindu sources.
Mahāvīra gives an elaborate description of the rule in the *GSS* (stanza 233-5, pp. 357-9) which we here reproduce:

> The rule for arriving at [the value of the money contents of] a purse which [when added to what is on hand with each of certain persons] becomes a specified multiple [of the sum of what is on hand with the others]:
> The quantities obtained by adding one to [each of the specified] multiple numbers [in the problem and then] multiplying these sums with each other, giving up in each case the sum relating to the particular specified multiple, are to be reduced to their lowest terms by the removal of common factors. [These reduced quantities are then] to be added. [Thereafter] the square root [of this resulting sum] is to be obtained, from which one is [to be

[24] For a proof of the validity, see Hayashi (1995, 405). Several authors misinterpret the rule as the method of false position, e.g. Flegg (1983, 206)

> subsequently] subtracted. Then the reduced quantities referred to above are to be multiplied by [this] square root as diminished by one. Then these are to be separately subtracted from the sum of those same reduced quantities. Thus the moneys on hand with each [of the several persons] are arrived at. These [quantities measuring the moneys on hand] have to be added to one another, excluding from the addition in each case the value of the money on the hand of one of the persons and the several sums so obtained are to be written down separately. These are [then to be respectively] multiplied by [the specified] multiple quantities [mentioned above]; from the several products so obtained the [already found out] values of the moneys on hand are [to be separately subtracted]. Then the [same] value of the money in the purse is obtained [separately in relation to each of the several moneys on hand].

The introductory sentence states that the rule is to be used for determining the value of a purse. The rule is followed by a number of problems that begin as "Four men saw on their way a purse containing money" (ibid. stanza 245 ½, 367). This is the earliest instance, in our investigation of the sources, in which the popular problem of men finding a purse is discussed. While problems with the same structure and numerical values have been formulated before, the context of men finding a purse seems to have originated in India before 850 AD. Formulations with the purse turn up in Arab algebra with al-Karkhī's *Fakhrī* (c. 1050) and in the *Miftāh al-mu'āmalāt* of al-Tabari (c. 1075). Fibonacci has many variations of it in the *Liber Abbaci* (1202) and after that it becomes the most common problem in western arithmetic until the later sixteenth century. For an understanding of the rule, let us look at its application to a given problem (*GSS*, stanza 236-7, pp. 360):

> Three merchants saw [dropped] on the way a purse [containing money]. One [of them] said [to the others], "If I secure this purse, I shall become twice as rich as both of you with your moneys on hand". Then the second [of them] said, "I shall become three times as rich". Then the other, [the third], said, "I shall become five times as rich". What is the value of the money in the purse, as also the money on hand [with each of the three merchants]?

We can represent the problem in symbolic equations as follows:

$$\begin{aligned} x+p&=2(y+z)\\ y+p&=3(x+z)\\ z+p&=5(x+y) \end{aligned} \qquad (1.8)$$

Let us apply the recipe of Mahāvīra's to this problem, step by step. By "adding one to [each of the specified] multiple numbers" we have 3, 4 and 6. "Multiplying these sums with each other" we get 72. This has to be "reduced to their lowest terms by the removal of common factors". This least common multiple is 12. The reduced quantities are then 4, 3 and 2 respectively. Adding these together gives 9. From this the square root is 3. Then the reduced quantities "are to be multiplied by the square root as diminished by one", which is 2. This leads to 8, 6 and 4. The money in hand for each of the persons now is the difference of these values with the sum of the reduced quantities, being 9. The solution thus is 1, 3 and 5. The rest of the rule is an elaborate way to derive the value of the purse. Using the values in any one of the equations of (1.8) immediately leads to 15 for the value of the purse. Mahāvīra provides no explanation or derivation of the rule. However, a mathematical argumentation for the validity of the rules goes as follows. Given that

$$
\begin{aligned}
x + p &= a(y+z) \\
y + p &= b(x+z) \\
z + p &= c(x+y)
\end{aligned}
$$

the sum of the three shares with the purse is

$$x + y + z + p = (a+1)(y+z) = (b+1)(x+z) = (c+1)(x+y)$$ which we call *S*.

Thus

$$
\begin{aligned}
(b+1)(c+1) &= \frac{(a+1)(b+1)(c+1)}{S}(y+z) \\
(a+1)(c+1) &= \frac{(a+1)(b+1)(c+1)}{S}(x+z) \\
(a+1)(b+1) &= \frac{(a+1)(b+1)(c+1)}{S}(x+y)
\end{aligned}
$$

Adding these three equations together leads to

$$(a+1)(b+1) + (a+1)(c+1) + (b+1)(c+1) = 2(x+y+s)\frac{(a+1)(b+1)(c+1)}{S}$$

which we denote as *T*.

Multiplying each of the three previous equations by two and subtracting the result from *T* this leads us to three new equations:

$$2x\frac{(a+1)(b+1)(c+1)}{S} = T - (b+1)(c+1)$$
$$2y\frac{(a+1)(b+1)(c+1)}{S} = T - (a+1)(c+1)$$
$$2z\frac{(a+1)(b+1)(c+1)}{S} = T - (a+1)(b+1)$$

Thus *x*, *y*, and *z* are in proportion to each other as

$$T - 2(b+1)(c+1) : T - 2(a+1)(c+1) : T - 2(a+1)(b+1)$$

By removing the common factors in these proportions, the procedure arrives at the smallest integral solution, as in the example 1 : 3 : 5.

6.3. The problem in Greek sources

6.3.1. The Bloom of Thymaridas

We know almost nothing about Thymaridas of Paros, but he is supposed to have lived between 400 and 350 BC (Tannery 1887, 385-6). The only extant witness is Iamblichus, in his comments on the *Introduction to Arithmetic* by Nicomachus of Gerasa. The best known source for *The Bloom of Thymaridas* is Heath's classic on Greek mathematics. Heath (1921, 94) does not formulate the rule, he only observes that "the rule is very obscurely worded" and writes out the equations. The text from Iamblichus was first published in Holland with a Latin translation by Samuel Tennulius (1668) from the Paris manuscript BNF Gr. 2093. A critical edition, based on multiple manuscripts was published by Pistelli (1884). Nesselmann (1842, 233) quotes the Greek text and the Latin translation from Tennulius, who translated the method as *florida sententia*. We give here the Pistelli version and our own literal translation from the Latin and German.

eντευθεν καί ή εφοδος του Θυμαριδείου επανθήματος ελήφθη. ώρισμένων γαρ η αορίστων μερισαμένων ώρισμένον τι καί ένος ούτινοσουν τοίς λοιποίς καθ' έκαστον συντεθέντος, τό εκ πάντων αθροισθέν πλήθος επί μεν τριών μετά τήν εξ αρχής όρισθείσαν ποσότητα όλον τω συγκριθέντι προσνέμει τ' αφ' ού τό λεί.πον καθ' έκαστον τών λοιπών άφαιρεθήσεται, επί δέ τεσσάρων τό ήμίσυ και επί πέντε τό τρίτον καί επί έξ τό τέταρτον και αεί άκολουθως.

From this we are also acquainted with the method of the *Epanthema*, passed down to us by Thymaridas. Indeed, when a given quantity divides into determined and unknown parts, and the unknown quantity is paired with each of the others, so will the sum of these pairs, diminished by the sum [of all the quantities] be equal to the unknown quantity in case of three quantities. With four quantities it will be half of it, with five it will be the third, with six, the fourth and do on.

Thymaridas' rule (from Pistelli 1884, 62)

The rule is not as obscure as considered by Heath. Let us extract the basic elements of the rule, and compare these with the version of Āryabhaṭa:

- The rule applies, to any number of quantities as does Āryabhaṭa's.
- The sum is given in the problem. The rule is described as the division of a known quantity in determined and undetermined parts. In Āryabhaṭa's rule the sum is what is looked for.
- The partial sums are the sums of the pairs of the unknown part with each of the known quantities. In Āryabhaṭa's rule the partial sums include all the numbers except one.

In short, this rule is different from Āryabhaṭa's in two important aspects. Its intention is to find one unknown part of a determined quantity. Āryabhaṭa's rule is meant for finding the sum of numbers of which the partial sums of all minus one is given. Even in the case of three numbers, when the partial sums are the same, the rules have different applications. To make it clear to the modern eye, here is a symbolic version in the general case:

$$\left.\begin{cases} x + a_1 + a_2 + \ldots + a_{n-1} = s \\ x + a_1 = s_1 \\ x + a_2 = s_2 \\ \vdots \\ x + a_{n-1} = s_{n-1} \end{cases}\right\} x = \frac{\sum_{i=1}^{n-1} s_i - s}{n-2}$$

6.3.2. Diophantus

In the first book of the *Arithmetica* of Diophantus we find four instances of the problem type. Problems 16 and 17 are of the original type as covered by Āryabhaṭa's rule. Let us first look at problem 17 with four unknown quantities. In modern symbolism, the problem reads as follows:[25]

$$\begin{aligned} a + b + c &= 20 \\ b + c + d &= 22 \\ a + c + d &= 24 \\ a + b + d &= 27 \end{aligned}$$

[25] We use Ver Eecke (1926, 22) as the best translation of the *Arithmetica*: "Trouver quatre nombres qui, additionnés trois à trois, forment des nombres proposés. Il faut toutefois que le tiers de la somme des quatre nombres soit plus grand que chacun d'eux. Proposons donc que les trois nombres, additionnés à la suite à partir du premier, forment 20 unités; que les trois à partir du second forment 22 unités, que les trois à partir du troisième forment 24 unités, et que les trois à partir du quatrième forment 27 unités".

Diophantus' solution is not based on a proto-algebraic rule but has all the characteristics of algebra. He uses the *arithmos* as an abstract quantity for the unknown, to represent the sum of the four quantities (Ver Eecke 1926, 22):

> Posons que la somme des quatre nombres est 1 arithme. Dès lors, si nous retranchons les trois premiers nombres, c'est-à-dire 20 unités, de 1 arithme, il nous restera, comme quatrième nombre, 1 arithme moins 20 unités. Pour les mêmes raisons, le premier nombre sera 1 arithme moins 22 unités; lé second sera 1 arithme moins 24 unités, et le troisième 1 arithme moins 27 unités. Il faut enfin que les quatre nombres additionnés deviennent égaux à 1 arithme. Mais, les quatre nombres additionnés forment 4 arithmes moins 93 unités; ce que nous égalons à 1 arithme, et l'arithme devient 31 unités.

If $a + b + c + d = x$, then the four numbers not included in the partial sums are $x - 20$, $x - 22$, $x - 24$, and $x - 27$ respectively. Adding these four together is equal to their sum or x, thus $4x - 93 = x$ and $x = 31$.
This problem in the *Arithmetica* is followed by problems 18 and 19, of a related type, but not the one covered by Mahāvīra's formulation. We show here only the symbolic translation of problem 19:

$$\begin{aligned} a+b+c &= d+20 \\ b+c+d &= a+30 \\ a+c+d &= b+40 \\ a+b+d &= c+50 \end{aligned}$$

The solution is similar as the previous problem but depends on the choice of $2x$ for the sum of the four numbers.

6.3.3. The extended rule from Iamblichus

Iamblichus extends the rule of Thymaridas to another problem type which will become very popular during the next centuries. In modern symbolism this amounts to the set of equations:

$$\begin{aligned} x+p &= a(y+z) \\ y+p &= b(x+z) \qquad (1.9) \\ z+p &= c(x+y) \end{aligned}$$

Iamblichus gives two examples of the problem. The first example can be formulated symbolically as follows:[26]

[26] Nesselmann (1842, 234-5) gives the literal German translation from the Greek. We will follow Nesselmann rather than Heath's reconstruction.

$$a+b=2(c+d)$$
$$a+c=3(b+d)$$
$$a+d=4(b+c)$$
$$a+b+c+d=5(b+c)$$

The problem is formulated in a way that reminds us of Diophantus: "Find four numbers such that …". Although Diophantus's *Arithmetica* has no problems like this, problems 18 to 20 of the first book are variations on the original *epanthema* problem. Iamblichus's own variation is in some way analogous to the versions of the *Arithmetica* and might be influenced by it. However, while Diophantus's solution is algebraic, this one depends on a proto-algebraic rule. The fourth expression in the problem formulation is superfluous and also recognized as such by Iamblichus, as he adds "this follows directly from the previous statements". It is added to facilitate the application of the rule. The procedure is explained by Iamblichus in three steps:

1) Set the sum of the four numbers equal to the number found by multiplying the four factors together. Thus 2 . 3 . 4 . 5 = 120.

   Iamblichus does not explain why this is necessary, but it can be demonstrated in the following way: Completing the left side of the equations in (1.9) to the sum of the four numbers we arrive at:

   $$x+y+z+p=(a+1)(y+z)$$
   $$x+y+z+p=(b+1)(x+z)$$
   $$x+y+z+p=(c+1)(x+y)$$

   Therefore, the sum of the four integers must be divisible by (a + 1), (b + 1) and (c + 1). This can be represented by means of the least common multiple s. Now Iamblichus does not use s but 2s for a reason that will become apparent later. In the example the least common multiple is 60, therefore 2s is 120. So, Let us suppose that $x+y+z+p=2s$.

2) The sum of each pair can be found by taking $\frac{a}{a+1}$, $\frac{b}{b+1}$ and $\frac{c}{c+1}$ from the sum 2s respectively.

   This becomes apparent from

   $$x+p=a(y+z)$$
   $$(a+1)(x+p)=a(x+y+z+p)$$

The three sums ($x + p$), ($y + p$) and ($z + p$) in the example become 80, 90 and 96.

3) Only now, Iamblichus refers to the use of the *Epanthema* rule. Indeed, we have the partial sums ($x + p$), ($y + p$), ($z + p$) and we have the total sum $2s$. The *Epanthema* therefore determines the common part p as follows:

$$p = \frac{(x+p)+(y+p)+(z+p)-2s}{2} \quad \text{or} \quad p = \frac{80+90+96-120}{2} = 73$$

which leads to the other values as 7, 17 and 23.

The reason why Iamblichus used $2s$ instead of the least common multiple s, is that s would lead to the non-integral solution:

$$p = \frac{40+45+48-60}{2} = 36\frac{1}{2}$$

In summary, we discern two important factors which are relevant for the understanding of the controversy that follows.
1) Our only source for the *Epanthema* is Iamblichus. There are at least six centuries between Thymaridas and the extant witness. In the absence of any written source we should consider Iamblichus's discussion of the method as a late interpretation of Pythagorean number theory. The formulation of the rule with determined and unknown quantities fits better the context of third century Greek analysis than it can be brought into agreement with the world of Pythagorean number mysticism.
2) The extended problem, which has become know as the problem of men finding a purse, is in itself quite different from the original problem to which the *Epanthema* rule applies. The problem, devised by Iamblichus, could be considered a variation such as several others in the *Arithmetica* of Diophantus. Iamblichus gives the rules to reduce the problem to a form in which the *Epanthema* can be used. This distinction is important because many have identified the men-find-a-purse problem wrongfully with the Bloom of Thymaridas.

6.3.4. The controversy

We now come to the discussion of the relevance of the *Epanthema* method and the controversy on the influences to and from Indian mathematics. As there are two aspects of the discussion we will deal with the issues separately. Firstly, the historical question of the main source of the men-find-a-purse problem. Secondly, the more philosophical question of the relevance of the Bloom on the conceptual development of algebra.

6.3.4.1. The origin of linear problems of men finding a purse

Nesselmann (1842) restrains from much comments or interpretations on the Bloom of Thymaridas in his *Algebra of the Greeks*. He treats the method with full respect for the extant Greek text by Iamblichus. After Nesselmann, the problem was discussed by several scholars in relation to Hindu algebra. Rodet (1879), in his French translation of the Āryabhaṭa's treatise, does not mention the *Epanthema*. Rodet was no believer of the influence of Greek mathematics in Asia. We can assume that he did not discuss the *Epanthema* because, in from his point of view, there simply is no relation with Āryabhaṭa's rule.
On the other hand, Cantor (II, 584) after discussing Āryabhaṭa's stanza 29, remarks "Wir fürchten keinen Widerspruch, wenn wir in dieser Aufgabe und in dem Epantheme des Thymaridas so nahe Verwandte erkenne, dass an einen Zufall nicht zu denken ist". Kaye (1927, 40, note 2) writes "The examples in the text are undoubtedly akin to the 'Epanthema'" and cites Cantor and Heath. Tropfke (1980, 399) words it more sharply and considers the formulation of Āryabhaṭa's stanza 29 "equivalent with the Epanthema of Thymaridas" and *BM* "contains problems of the same sort".[27]
All the suppositions of the Greek influence are solely based on the alleged resemblance of the problems. As we have shown, Āryabhaṭa's rule is very different from the *Epanthema*. The argument that both are equivalent is plainly wrong. The suggestion that the *Epanthema* provides evidence for an influence of Greek mathematics on Hindu algebra has very little to base on. The motivation for the influence seems to stem from normative grounds on the superiority of Greek culture. Let us now proceed to the second question on conceptual influences.

6.3.4.2. A case of Pythagorean algebra?

This single problem, which became known to us through Iamblichus, six centuries after Thymaridas, has convinced many that Greek algebra originated with the Pythagoreans. After writing out the equations, Cantor (1894, I, 148) concludes:

> Das ist, wie man sieht, volständig gesprochene Algebra, welcher nur Symbole fehlen, um mit einer modernen Gleichungsauflösung durchaus übereinzustimmen, und insbesondere ist mit Recht auf die beiden Kunstausdrücke der *gegebene* und *unbekannten* Grösse aufmerksam gemacht worden.

Heath's interpretation is copied in many other works including Smith (1925, 91), Cajori (59), van der Waerden (1988, 116), Flegg (1983, 205) and Kaplan

[27] In the original edition, Tropfke (1937, III, 42) is more prudent: "Āryabhaṭa bietet einige solcher Wortgleichungen, unter denen uns eine wegen ihren Änlichkeit mit dem Epanthem des Thymaridas ausffält". Apparently it is Kurt Vogel, who edited the 1980 edition, who believes in a strong connection.

(2001, 62). Cajori finds in the Thymaridas "investigations of subjects which are really algebraic in their nature". Van der Waerden goes as far as to claim that "we see from this that the Pythagoreans, like the Babylonians, occupied themselves with the solution of systems of equations with more than one unknown". Instead, Klein (1968, 36) sees in the problem an intent to "determine special relations between numbers" and places it as "the counterparts in the realm of 'pure' units of the computational problems proper to practical logistic". We agree with Klein's interpretation. Even if Iamblichus's depiction of the problem from Thymaridas is faithful, the six centuries separating the two require an interpretation accounting for two different contexts. Pythagoreans were concerned with the properties of numbers and relations between numbers. Lacking any further evidence we cannot attribute an algebraic interpretation to Pythagorean number theory. On the other hand, the context of the late Greek period of Diophantus and Iamblichus does allow for an algebraic reading. The Bloom is thus an old number problem revived and extended in an algebraic context.

6.3.5. The rules in European arithmetic books

Because of the vast number of manuscripts and books in which these problems are discussed, we will not attempt to give an overview. We refer to Singmaster (2004, 7.R.1) for a complete history. Also Kurt Vogel (1940, 1954) provided a systematic treatment of some of these problem types. We will limit our further discussion to the demonstration that both types of Hindu recipes are represented in European arithmetic books.

First the rule of Āryabhaṭa as described by (1.7). This rule is applied as a recipe at many occasions. We present one example not included in Singmaster's overview. Francés Pellos applies the rule to solve problem 20 of the *Compendion de l'abaco*, written in the occitane dialect (1492, f. 50v; Lafont 1967, 140):

> Item sont quatre merchans che han d'argent ieu non sabi cant, mays ieu dieuc que ay audit dire que tres d'ellos sensa lo prumier merchant han 30 ducats, et tres sensa lo segont merchant han 36 ducats, et tres sensa lo ters merchant han 40 ducats, he tres sensa lo quart han 41 ducats. Ademandi quantos ducats ha cascun d'aquellos merchans per solet, en maniera che aquesta question sy trobe vera. Fay ensins: ajustas aquestas quatre summas ensemble, coma 30, 36, 40, 41 et fan ducats 147, et aquesta regula vol che tu deves la principala summa tostemps partir per un ponch mens che non son los companhons. Empero partas 147 per 3, et troberas che ven 49, de laqual summa leva 30 et restan 19, et tantos ducats ha lo prumier. Apres leva 36 de 49, et restan 13, et tantos ducats ha lo segont. Apres leva 40 de 49, et restan 9, et tantos ducats ha lo ters. Apres leva 41 de 49, et restan 8, et tantos ducats ha lo quart merchant, et ensins aves fach.

The problem in symbolic notation amounts to

$$
\begin{aligned}
x_1 + x_2 + x_3 &= 41 \\
x_2 + x_3 + x_4 &= 30 \\
x_1 + x_3 + x_4 &= 36 \\
x_1 + x_2 + x_4 &= 40
\end{aligned}
$$

Pellos adds the four partial sums together and divides it by "one less then the number of companions" which leads to the sum of all four as prescribed by Āryabhaṭa. Subtracting the partial sums leads to the shares of each.

The derived problem from Hindu algebra with men finding a purse is even better represented in European arithmetic. Recipes very close in formulation to the one of Mahāvīra appear frequently in the fifteenth century such as the *Memoriale* of Bartoli (Sesiano 1984a, 136-7). We take an example from the *AR* (f. 84^r; Vogel 1954, 63-4):

> Item: Es sein 3 gesellen, dy haben 1 peutel gefunden mit gelt. Nu spricht der erst zw den anderen zwaijen: het ich daz gelt, daz in dem peutel ist, so het ich alz vil als ir paid. Spricht der ander zw den zwayen: het ich daz gelt, daz in dem peutel ist, so het ich zwir als vil alz ir peud. Spricht der drit zw den andern 2: het ich daz gelt, daz in dem peutel ist, het ich 3 mol als vil alz ir paid. Queritur, wye vil ydlicher peij im hat gehapt und wie vil in dem peutel ist.
>
> Daz setz also augmentaliter: ½, 2/3, ¾. Nu vind 1 zal, in der du hast 1/2 1/3 1/4, daz ist 24. Nu 1/2 von 24 ist 12 und 2/3 von 24 ist 16 und 3/4 ist 18. Addirß zesamm, facit 46. Nu zeuch dy du gefunden hast, daz ist 24, da von, pleibt 22. So vil ist gewesen in dem peutel. Nu wist tu wissen, wievil ydlicher hat gehabt, daz mach also: duplir 12, ist 24. Da von zeuch 22, da pleibt [2]. So vil hat der erst gehabt. Darnach duplir 16, wirt 32, da von zeuch 22, pleibt 10. Daz hat der ander gehabt. Darnach duplir 18, wirt 36, dovon zeuch 22, pleibt 14. So hat der drit gehabt.

Except for the reference to the square root, which is an oddity in Mahāvīra's rule, all other steps of the Hindu procedure are repeated in this formulation of the recipe. Vogel (1954, 217) wrongly refers to Iamblichos as he claims that "Diese unbestimmte Problem hat seinen Ursprung in ältester griechischer Algebra". Comparing the text of the *AR* with our analysis of Mahāvīra and Iamblichos above, the connection points in the direction of the Hindu sources.

7. Other problems

Not all recipes are from Hindu origin. Some common types of problems that appear throughout the Middle-Ages are solved with recipes for which we have no corresponding rules in Hindu sources. A typical example is about two men buying some commodity which none of them can pay. They need a fractional part of the other's money to buy the commodity. This problem corresponds with the equations

$$\begin{aligned} x+\frac{1}{b}y &= p \\ y+\frac{1}{a}x &= p \end{aligned} \quad \text{with values } (a, b, p)$$

The recipe provides a procedure to determine the value for the first or second unknown quantity, corresponding with the equations:

$$x=\frac{a(b-1)}{ab-1}p \qquad \text{and} \qquad y=\frac{b(a-1)}{ab-1}p$$

The most common values are (2, 3, 20) and found in al-Karkhī's *Fakhrī* (Woepcke 80),
Our first source for the recipe is a Byzantine text (c. 1305, Paris, Suppl. Gr. 387, f. 120^r; Vogel 1968, 27-9) for the problem with values (3, 4, 100). It appears with the same values and the same recipe as problem 173 in the *Algorismus Ratisbonensis* (Munchen, f. 98; Vogel 1954, 82). We list here the text explaining the recipe for the two unknown values:

> Fac sic: multiplica denominatores, facit 12. Post hoc multiplica numeratores, facit 1, illud subtrahe ab 12, manet 11, divisor. Darnach subtrahir ab communi denominatore, hoc est 12, daz der erst begert, hoc est 1/3. Nu 1/3, ab 12 est 4, dy nym von 12, manet 8, dein merer. Den mer oder multiplicir in 100 vnd waz da kumpt, daz dividir per 11, facit 72 8/11 [sic]. Alz vil hat der erst. Post hoc subtrahir ab 12 id est communi denominatore 1/4. Nu 1/4, ab 12 est 3, subtrahir ab 12, manet 9. Illa multipicir per 100 et quod manet, divide per 11, facit 81 9/11 habet secundus.

Although the recipe is found in arithmetic books of the next two centuries, the problem is later solved by the rule of double false or by algebra.

7.1. Rules for other linear problems

An overview of determinate linear problems in one, two and three unknowns is shown in Table 3. We have not undertaken a study of these rules in relation to the Hindu sources. We shall briefly describe rules known from Indian algebra and provide some clues for corresponding proto-algebraic rules in European arithmetic.

7.1.1. Rule of inversion (*viparītakarma*)

This rule applies to linear problems in one unknown of the type

$$(((a_1x-b_1)a_2-b_2)\ldots)a_n-b_n=c$$

This type of a problem was popular in Europe before the end of the sixteenth century. The problem class is called *Schachtelaufgaben* by Tropfke (1980, 592) but better known under its twentieth-century name "monkey and coconut problem". The name is derived from a short story by Ben Ames Williams, published in a newspaper in 1926.[28] It is about five sailors that are shipwrecked on an island. They gather coconuts all day. During the night each of the sailors takes one fifth of the remaining coconuts and gives one to the monkey. In the morning they each get one fifth, again leaving one coconut for the monkey. The question is how many coconuts they had gathered. The problem can easily be solved by reversing the order and calculating backwards. In Indian mathematics the method was called *viparītakarma* or *Rule of inversion*. Āryabhaṭa prescribes the *viparītakarma* for the four arithmetical operations, but this was extended to include squaring and roots by Brahmagupta and Bhāskara. The *BM* has examples formulated as business trips (Hayashi 1995, Sūtra C1).
Many European arithmetic books describe the rule in the same way as it was first formulated by Āryabhaṭa. Folkerts (1978) found 17 instances of this type of problems in fourteenth and fifteenth-century manuscripts. The *AR* has 2 examples of the problem (c. 1450, Vogel 1964, prob. 185 and 187). All important Renaissance treatises deal with it: Bartoli's *Memoriale* (Sesiano 1984, 138, 148), Cod. lat. Mon. 14684 (Curtze 1895a, prob. 5), Chuquet (1484, prob. 30-33), Calandri (1491, F. 66v, 74r), Pacioli (1494, prob. 22), Köbel (1514), Ghaligai (1521), Tunstall (1522, question 43 and 44), Riese (1524, *aufgabe* 53), Tartaglia (1556, Book 17, art 9 and 20), Trenchant (1558, prob. 6) and Buteo (1559, prob. 21). In the *Initius algebras* the rule is called *regula Salomonis* and attributed to king Salomon (Curtze 1902, 461). The name could be a corruption of *regula sermonis* (Cantor 1892, II, 247). Widmann (1489) called the method *Regula pulchra* I or *Regula transversa*.

7.1.2. Rule of concurrence (*sankramana*)

This simple rule for finding two quantities given their sum and their difference appears first in the *BSS* (Colebrooke 1818, 340). Brahmagupta formulates the rules as "the sum set down twice and having the difference added and subtracted and being in both instances halved, the moieties are the residues" (ibid. 376), and applies it to a problem from astronomy. The rule corresponds with the symbolic representation:

$$\begin{aligned} x+y&=a \\ x-y&=b \end{aligned} \quad \text{with} \quad x=\frac{a+b}{2} \text{ and } y=\frac{a-b}{2}$$

[28] The story about the publication and the many reactions it provoked is described by Martin Gardner (1958). See also Singmaster, 1997, and 2004.

We have not looked for remnants of this rule in European arithmetic books.

	Meta-description	Name of the rule	Problem type	Hindu sources
1	$ax+c=bx+d$	Difference of unknowns (*gulikāntara*)	If I had x from you..	Āryabhaṭa I, 499 Bakhshālī, c. 700 Bhāskara II, 1150
	$ax+c=bx-d$	Regula augmenti + decrementi	Figs for children Paying wages	Bhāskara II, 1150
	$(((a_1x-b_1)a_2-b_2)\ldots)a_n$ $-b_n=c$	Rule of inversion (*viparītakarma*)	Monkey and coconut	Āryabhaṭa I, 499 Brāhmagupta, 628 Bakhshālī, c. 700 Śrīdhara, c. 900
2	$x+y=c$ $ax-by=d$		Lazy worker	
	$x+y=a$ $x-y=b$	Rule of concurrence (*sankramana*)		Brāhmagupta, 628 Bhāskara, I, 18
	$x+ap=cy$ $y+bp=dx$		Two men find a purse	
	$x-ay=y-bx=c$			Mahāvīra, 850
	$x+a=c(y-a)$ $y+b=d(c-b)$	Regula augmentationis	Mule and ass Geben und Nehmen	Bakhshālī, c700 Mahāvīra, 850 Bhāskara II, 1150
	$ax+by=c$ $bx+ay=d$ $a+b=e$		Buying cloth Price of goods	Mahāvīra, c 850 Bhāskara II, 1150 Euler, 1770, Ch. 4
	$ax=by+c$ $dx=ey-f$	Rekeninghe van Plus ende Min		
	$x+y=a$ $\frac{x}{y}=b$		Split a number in two parts	Diophantus Al-Kwārizmi
	$x+p=ay$ $y+p=bx$		two cups and one cover	
3	$x+p=a(y+z)$ $y+p=b(x+z)$ $z+p=c(x+y)$	Epanthema	Men find a purse	Iamblichos, c 300 Aryabhaṭa I, 499 Mahāvīra, c 850
	$x+y=a$ $x+z=b$ $y+z=c$			Diophantus Bakhshālī, c 700
	$ax=by=cz$ $x+y+z=d$			Bhāskara II, 1150

Table 3: An overview of recipes possibly derived from rules of Hindu algebra

7.1.3. Price of goods

By lack of a better name, we call this the rule for determining the price of goods. Mahāvīra calls it vyastārghapaṇyapramāṇānayanasūtram in the *GSS*, or a "rule for arriving at the measure of two given commodities whose prices are interchanged" (Padmavathamma 2000, 314).

The rule has the following form:

$$\begin{aligned} ax+by&=c\\ bx+ay&=d\\ a+b&=e \end{aligned}$$

The Hindu rule consists of reducing the problem to a form in which the rule of concurrence can be applied. Indeed, adding the first two equations together gives us

$$(a+b)(x+y)=c+d$$

Dividing this by the third leads to

$$x+y=\frac{c+d}{e}$$

Subtracting the first two equations gives

$$(a-b)(x-y)=c-d \quad \text{or} \quad x-y=\frac{c-d}{a-b}$$

With the sum and the difference given, this allows to apply the *sankramana*.

7.2. Rules for linear indeterminate problems

7.2.1. Regula coecis or regula virginum

The problem known as "the hundred fowls problem" is definitely of Chinese origin. The problem gives the total number of birds bought, the price of each bird and the total sum paid. The question is how many of each bird are bought. With more than two types of birds, the problem is indeterminate. A typical problem involves three kinds and amounts to the equations:

$$\begin{aligned} x+y+z&=d\\ ax+by+cz&=d \end{aligned} \quad \text{or} \quad \begin{aligned} x+y+z&=d\\ ax+by+cz&=e \end{aligned}$$

We have several Chinese sources for the problem, starting with *Nine chapters of the mathematical art* (*Jiŭ zhāng suàn shù* 九章算術) of the first century (Vogel, 1968). Its first appearance in Hindu sources is through the *BM*. The first occurrence in Europe is through Alcuin's *Propositiones ad Acuendos Juvenes* (Propositions for Sharpening Youths) of the ninth century, with no less than eight problems. The most likely route of transmission from China is through India.

Meta-description	Name of the rule	Problem type	Hindu sources
$ax - by = c$	Pulversizer (kuṭṭaka)	Problem of remainders	Āryabhaṭa I, 499 Bhāskara I, 522
$ax = by \pm 1$	Constant pulverizer (sthira- kuṭṭaka)		Bhāskara I, 522 Brahmagupta, 628 Bhāskara II, 1150
$x + y + z = d$ $ax + by + cz = d$	Regula Coecis, Type I		Bakhshālī, c 700 Śrīdhara, c 750 Mahāvira, 850 Bhāskara II, 1150
$x + y + z = d$ $ax + by + cz = e$	Regula Coecis, Type II	100 fowls problem	Bakhshālī, c 700 Śrīdhara, c 750 Mahāvîra, c 850 Śrīpati, 1039 Bhāskara II, 1150
		Selling different amounts	Bakhshālī, c700 Śrīdhara, c 750 Mahāvira, 850 Fibonacci, 1202
$a_1x_1 + r_1 = a_2x_2 + r_2 = \ldots$	Regula Ta-yen	Chinese Remainder problem Basket of eggs	Āryabhaṭa I, 499 Bhāskara I, 522 Mahāvīra, 850 Śrīpati, 1039 Bhāskara II, 1150
$xy = ax + by$	bhāvita		Bakhshālī, c 700 Brahmagupta, 628 Śrīpati, 1039 Bhāskara II, 1150

Table 4: Similar recipes for indeterminate problems in multiple unknowns

Interestingly, a sixteenth-century German and Latin text, which includes the problem, refers to Indian sources (Gottingen Codex Philos. 30). The *Prologus in algebram* contains a short history in which the author claims that Arab algebra was translated and brought to India during Alexandrian times. This is one of the few references to Indian algebra we can find in extant texts. The hundred fowls problem is introduced with special reference to Indian algebra (Curtze 1902, 449):

> Von disetn Text schrifftlich ein vorstentnus zu geben, so ist zu wissen das ALIABRAS, der Indus, gepraucht an vielen orten die so gemelte Regeln, und khumbt ursprunglich aus den communicirenden zaln, als wir figuriren wollen exemplariter, damit wir auf den vorstandt des texts khomen.
> Es hat einer willen zu kaufen 100 stuckh viehes umb 100 fl., nemlich schaf, Esel und Ochsen, das dann die Kinder ratende vorgeschlagen, aber uns hieher zu geprauchen. Erofnen wir, souil vns not ist der rationalischen zaln halber, welcher kauft 20 schaf vor 1 fl., 1 Esel pro 1 fl. und 1 Ochs pro 3 fl., und solch summe der fl. ist 100 und ' des viches ist auch 100 stukh.

The solution text, cited in chapter 2, corresponds with the Hindu recipe.

7.2.2. Tangible arithmetic

There is one particular formulation of the *regula coecis* which deserves mentioning. It demonstrates how proto-algebraic rules can become tangible operations and actions.
The most common version of the hundred fowls problem in European arithmetic is about a number of men, woman and children who have to pay the bill at a tavern. This formulation, which explains the name *Regula virginum*, first appears in the Parmiers manuscript written around 1430 (Sesiano 1984b). The first arithmetic book written in Swedish (Aurelius, 1614) describes a rule for solving such problems by specific actions with the left and right hand: [29]

> Thus is the way to use this rule. Put the number of men at the left hand and what is eaten at the right hand. In between you must place every person and the money they spent. Then you must "resolve" all the spent money you have; then you must multiply the smallest coin with the number of all the persons; the product you must subtract from all the spent money, the rest will be put aside.. At last you always must subtract the lesser number from the greatest; and the rest will be the divisor. When this is done, take the number which was put aside and divide it into certain parts.

While the procedure described by the rule can be matched with the one from the *GSS* by Mahāvīra (Sūtra 143 ½; Padmavathamma 2002, 303), a depiction by manual actions is apparently a European invention. Through several intermediaries we could trace the original formulation back to the second arithmetic by Adam Ries (1522, 135) under the name *Regula cecis oder*

[29] Aurelius, Aegidius Matthiae (1622) *Arithmetica eller Een Kort och Eenfaldigh Räknebook uthi heele och brutne Taal medh lustige och sköne Exempel med Eenfaldigom som til thenne Konst lust och behagh hafwe kortelighen och eenfaldelighen til Nytto och Gagn författat och tilsammandraghen aff Aegideo Aurelio*, Eshillo Matthiae, Upsala. The book was reprinted in 1622, 1628, 1633, 1636, 1642, 1665. Staffan Rodhe pointed me to the reference of Aurelius on occasion of the conference *The Origins of Algebra: From al-Khwārizmī to Descartes* at the University of Pompeu Fabra, Barcelona, March 27-29, 2003. He provided the translation from Swedish.

virginum. In a later edition of 1574, the operation is described as *Zech rechnen* (Ries 1574, f. 69^v):

> Schreib vor dich gegen der lincken handt die anzahl de Personen. Gegen der rechten handt / wie viel sie vertruncken / und in die mitte / wie vil einjegliche Person / jeglichs geschlechts in sonderheit gibt. Darnach mach das gelt dem menigsten uberall gleich / als dann multiplicir das kleinest an der bezahlung mit den Personen / und nimb von dem das sie vertruncten haben/ Das da bleibt ist die zahl / welche getheilt sol werden.

The problems added as illustration correspond with the equations :

$$\begin{aligned} x+y&=21 \\ 3x+5y&=81 \end{aligned} \quad \text{and} \quad \begin{aligned} x+y+z&=20 \\ 3x+2y+z&=40 \end{aligned}$$

The same tanglible procedure is discussed in the Flemish arithmetic of Vander Gucht (1569, ff. 80^v-81^v, 1594, ff. 70^r-71^r, small differences in formulation):

> Om desen reghel te maken soo schrijft tghetal der persoonen altijt teghen der luchter ofte lincker handt en tghelt dat zij verdroncken hebben tegen de rechter hand en int middel settet jeghelicks ghelt dat zy verdroncken hebben. Daer naer neemt t'minste ghetal vande meesten. De reste settet bezijden. Twelcke deelders werden. Alsdan multipliceert den ondersten ghetale (welck die cleynste is inder betalinghe) met die persoonen. Dat product substraheert vanden ghelde dat zij betalen souden. En de reste deelt gelijck in soo veeldeelen als deelders voorhouden in sulcher voughen dat een jeghelick ghetal met zijnen behoorlicken deelder ghedeelt ghelijck op sta. En what uit zuclcken deelinghe comt beteekent den ghetal der persoonen eens jeghelicks gheslachts besonder uitghenomen de persoonen die dat minste gheven, welken ghi dan hehtelicken bevinden cont wanneer ghij die ghetalen der persoonen die uit der deelinghe commen tezamen addeert. En dat zelve van tghetal der persoonen teghen de lincker handt substraheert, en de reste beteeckent de persoonen de welcke het minste gheven.

Vander Gucht adds four examples:

$$\begin{aligned} x+y&=16 \\ 30x+18y&=348 \end{aligned} \qquad \begin{aligned} x+y+z&=34 \\ 48x+24y+12z&=816 \end{aligned} \qquad \begin{aligned} x+y+z&=27 \\ 18x+16y+8z&=348 \end{aligned}$$

$$\begin{aligned} x+y+z+u&=80 \\ 12x+10y+8z+6u&=35\frac{1}{2} \end{aligned}$$

8. Conclusion

We have selected three types of recipes from European arithmetic and algebra treatises and demonstrated their structural correspondence with rules for solving algebraic problems in the Hindu tradition. Some proto-algebraic rules from Renaissance texts are strikingly similar to rules from Hindu algebra. Others vary slightly in formulation and structure. Still other recipes, in addition to three examples worked out, suggest a high correlation with rules from Hindu algebra. What can we infer from such striking correspondence?

No textual evidence substantiates an influence from Hindu algebra on fourteenth to sixteenth-century European arithmetic. For example, the Indian rules relating to linear problems with multiple unknowns are formulated with reference to colors. No such reference to colors could be found in Rennaisance texts.

Jens Høyrup (2002, 96) used the term 'sub-scientific source traditions' as opposed to the scientific source traditions. Only the latter are studied by historians, by means of extant texts. He discusses commercial calculation and practical geometry as two disciplines which thrived in sub-scientific grounds. Such knowledge is disseminated orally through stories, riddles, tangible and mnemonic aids. Because of the lack of written evidence, science historians have neglected these traditions. Høyrup has always recognized the importance of these traditions and his assessment of Islamic science and mathematics is partially based on such influences. The wertern appropriation of rules from Hindu algebra may be situated within the sub-scientific tradition of solving recreational problems. As we have shown for the 'men find a purse"-problem, the practical context in which some of these recreational problems are formulated, is Indian. If, in addition, recipes for solving the riddles correspond with the rules from the scientific tradition of Hindu algebra, it becomes warranted to think in lines of an effective influence. Algebraic procedures from Hindu algebra have migrated to the West in the form of proto-algebraic rules connected with and imbedded in recreational and practical problems.

What is the relevance of these proto-algebraic rules is for the development of Renaissance algebra? Probably there is no direct influence from Hindu algebra on the abacus and cossist traditions. However, the procedures for problem solving, transmitted through proto-algebraic rules, hold important clues for an algebraic approach to these problems. A recipe for solving the problem of men finding a purse, depending on the sum of the shares of the persons and the value of the purse, leads to the idea of using the unknown for that sum in abacus algebra.[30] Proto-algebraic rules have therefore a heuristic value for algebraic problem solving.

9. Acknowledgments

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10. List of abbreviations

	Title	Author	Date	Edition	Translation
AB	Āryabhaṭīya	Āryabhaṭa I	499	Kern 1875 Shukla 1976	Rodet 1879 Clark 1930 Shukla 1976
BG	Bījagaṇita	Bhāskara II	1150	Dvivedin 1927	Colebrooke 1817
BM	Bakhshālī Manuscript	Unknown	c 700	Kaye 1927, 1933	Hayashi 1995
BP	Bījapallavam (commentory on BG)	Kṛṣṇadaivajña		Śastrī 1958	
BSS	Brāhmasphuṭasiddhānta	Brāhmagupta	628	Dvivedin 1902	Colebrooke 1817
GS	Gaṇitasāra	Thakkura pherū	c 1300		
GSS	Gaṇitasārasaṃgraha	Mahāvīra	850	Rangācārya 1912	Padmavathamma 2000
L	Līlāvatī	Bhāskara II	1150	Sharma 1975	Colebrooke 1817 Pandit 1992
PG	Pāṭīgaṇita	Śrīdhara	c 750	Shukla 1959	Shukla 1959
PS	Pañcasiddhāntikā	Varāhamihira	505	Dvivedin 1889	Thibaut 1889
SP	Sūryaprakāśa (commentory on BG)	Sūryadāsa	1538	Jain 2001	Jain 2001
SS	Siddhāntaśekhara	Śrīpati	1039	Misra 1932	Sinha 1985, 1986
SSM	Siddhāntaśiromaṇi	Bhāskara		Śastrī 1999	Wilkinson 1861 Patte 2004
Tr	Triśatikā	Śrīdhara	c 750	Dvivedin 1899	Kaye 1912

11. References

11.1. Cited Manuscripts

Avignon, Archives départementales du Vaucluse, cote 1F 54 (transcription by Sesiano, 1984)

Bamberg, Staatsbibliothek, Inc Typ. IC144 (transcription and facsimile by Schröder, 1995)

[30] This recipe is not discussed here. Such algebraic solution is found in the *chasi dilettevoli* by Maestro Benedetto, Cod. Magl. XI 76; Arrighi 1987, 357.

Florence, BNCF, Palatino 567 (transcription by Procissi, 1983)
Florence, BNCF, Magliabechiano Cl. XI, 76, (partial transcription by Arrighi, 1987)
Florence, BNCF, Magliabechiano Cl. XI, 86 (transcription by Arrighi, 1964)
Florence, Biblioteca Riccardiana, 2263 (transcription by Simi, 1994)
Ghent, Universiteitsbibliotheek, Hs. 2141 (transcription by Kool, 1988)
Lucca, Biblioteca Statale, Ms. 1754 (partial transcription by Arrighi, 1973)
Munich, Bayerischen Staatsbibliothek, Cod. Lat. Monacensis 14684 (transcription by Curtze, 1895a)
Munich, Bayerischen Staatsbibliothek, Cod. Lat. Monacensis 14908, *Deutsche Algebra* (ff. 133^{v}-158^{r}, transcription by Curtze, 1895b)
Munich, Bayerischen Staatsbibliothek, Cod. Lat. Monacensis 14908, *Algorismus Ratisbonensis* (transcription by Vogel, 1954)
New York, Columbia University, X511 A13 (transcription by Vogel, 1977)
Paris, BNF, Français 1346 (transcription by Marre, 1880, partial translation in Flegg e.a., 1985)
Parma, Biblioteca Palatina, 78 (transcription by Gregori and Grugnetti, 2001)
Siena, Biblioteca Communale, L.IV.21, *Certi Chasi* (ff. 431^{v}-451^{r}, transcription by Pancanti, 1982)
Turin, Biblioteca Nazionale, Codice N.III.53 (transcription by Rivoli, 1983)

11.2. Primary Sources

Böschenstain, Johannes (1518) *Ain neugeordnet Rechenbüchlein mitt den zyffern*, Augsburg.
Buteo, Johannes (1559) *Logistica, quae et arithmetica vulgo dicitur, in libros quinque digesta*, Lyon.
Calandri, Philppo (1491) Arimethrica, Lorenzo de Morgiani & Giovanni Thedesco da Maganza, Florence.
Ghaligai, Francisco di Lionardo (1521) *Summa de arithmetica*, Bernardo Zucchetta, Florence.
Jamblichus (1668) *Jamblichus chalcidensis ex Cœle-Syria In Nicomachi geraseni Arithmeticam introductionem, et De fato. Nunc primum editus, in latinum sermonem conversus, notis perpetuis illustratus à Samuele Tennulio, ; accedit Joachimi Camerarii explicatio in duos libros Nicomachi, cum indice rerum & verborum locupletissimo*, prostant apud Joh. Fredericum Hagium, Arnhem.
Köbel, Jacob (1514) *Ain Neu Geordnet Rechen Biechlein Auf Linien Mit Rechenpfeningen* Augsburg.
Newton, Isaac (1707) *Arithmetica universalis; sive De compositione et resolutione arithmetica liber. Cui accessit Halleiana æquationum radices arithmetice inveniendi methodus. In usum juventutis Academicæ*, Typis Academicis, Cantabrigiæ; Impensis B. Tooke, Bibliopolae, London.
Newton, Isaac (1720) *Universal arithmetick; or, A treatise of arithmetical composition and resolution. To which is added, Dr. Halley's Method of finding the roots of equations arithmetically. Translated from the Latin by the late Mr. Raphson, and revised and corrected by Mr. Cunn*,Printed for J. Senex, W. Taylor, T. Warner, and J. Osborn, London.
Pacioli, Luca (1494) *Summa de arithmetica geometria proportioni: et proportionalita. Continetia de tutta lopera*, Paganino de Paganini, Venice.
Pellos, Francesco (1492) *Sen fegue de la art de arithmeticha et semblātment de ieumetria dich ho nominatz cōpendiō de lo abaco*, Benedeti and Jacobino, Turin (see also Lafont, 1967).

Ries, Adam (1522) *Rechenung auff der linihen und fdern in zal, mass und gewicht auff alleley handierung*, Mathes Maler, Erfurt.
Silíceo, Juan Martínez (1519) *Arithmetica Ioannis Martini, Scilicei : in theoricen, et praxim scissa / nuper ab Orontio Fine, Delphinate, summa diligentia castigata, longeq[ue] castigatius q[uam] prius, ispo cura[n]te impressa*, Henrici Stephani, Paris.
Tartaglia, Nicolo (1551-60) *La prima* [sesta] *parte del general trattato di nvmeri et misvre* (6 vols.), per Curtio Troiano de i Navò, Venice.
Trenchant, Jean (1561) *L'Arithmétique, départie en trois livres. Ensemble un petit discours des changes, avec l'art de calculer aux getons*, M. Jove, Lyon.
Tunstall, Cuthbert (1522) *De Arte Supputandi Libri Quattuor*, Impress. in aedibus R. Pynsoni, London.
Widmann, Johannes (1489) *Behende vnd hubsche Rechenung auff allen Kaufmanschafft*, Leipzig.

11.3. Secondary Sources

Arrighi, Gino (1963) "Metodi di calcolo in un codice Lucchese del trecento. Regola del 'Chatuino' e problemo di secondo grado", *Bolletino della Unione Matematica Italiano*, S. III, 18, pp. 433-45.
Arrighi, Gino (ed.) (1964) *Paolo dell'Abaco, Trattato d'Aritmetica. Secondo la lezione del Codice Magliabechiano XI,86 della Biblioteca Nazionale di Firenze*, Domus Galileana, Pisa.
Arrighi, Gino (ed.) (1973) *Scuola lucchese, Libro d'abaco, Dal Codice 1754 (sec. XIV) della Biblioteca Statale di Lucca*, Cassa di Risparmio di Lucca, Lucca.
Arrighi, Gino (1987) "'Chasi dilettevoli' di M° Benedetto da Firenze, dal Cod. Magl. XI 76 della Biblioteca Nazionale di Firenze", *Atti dell'Academia lucchese di scienze, lettere ed arti*, II, t. 19-20, pp. 97-120.
Clark, Walter Eugene (ed.) (1930) *The* Āryabhaṭīya *of* Āryabhaṭa*. An Ancient Indian Work on Mathematics and Astronomy*, University of Chicago Press, Chicago.
Cantor, Moritz (1880-1908) *Vorlesungen über Geschichte der Mathematik* (4 vols.), vol I (1880): "Von den ältesten Zeiten bis zum Jahre 1200 n. Chr."; vol. II (1892): "Von 1200-1668" (2nd ed. 1894, 1900), Teubner, Leipzig.
Colebrooke, Henry Thomas (1817) *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bhāscara*, John Murray, London (Vaduz, Lichtenstein: Sandig Reprint Verlag, 2001).
Cossali, Pietro (1797-9) *Origine, trasporto in Italia, primi progressi in essa dell'algebra. Storia critica di nuove disquisizioni analitiche e metafisiche arricchita* (2 vols.), Parma.
Curtze, Maximilian (1895a) "Aritmetische Scherzaufgaben aus dem 14. Jahrhundert", *Bibliotheca Mathematica*, Folge 2, 9, pp. 77-88.
Curtze, Maximilian (1895b) "Ein Beitrag zur Geschichte der Algebra in Deutschland im fünfzehnten Jahrhundert", *Abhandlungen zur Geschichte der Mathematik*, (supplement of *Zeitschrift für Mathematik und Physik*) 5, pp. 31-74.
Datta, Bibhutibhusan and Avadhesh Narayan Singh (1935, 1938) *History of Hindu Mathematics, a Source Book*, Asia Publishing House, Bombay (reprint in 2 vols., Bharatiya Kala Prakashan, Delhi, 2001).
Dvivedin, Mahamahopadhyaya Sudhākara (1889), *Pañcasiddhāntikā*, Benares (2nd ed. Chowkhamba Sanskrit Series 68, Vārānasī, 1968).
Dvivedin, Mahamahopadhyaya Sudhākara (1889) *Triśatikā*, Benares.
Dvivedin, Mahamahopadhyaya Sudhākara (1902) *The* Brāhmasphuṭasiddhānta *and* Dhyānagrahapadeśādhyāya, Medial Hall Press, Benares.

Dvivedin, Mahamahopadhyaya Sudhākara (1927) *The* Bījagaṇita *of Bhāskara II*, Benares (revised by Muralidhara Jha with the commentary, entitled Navānkura of Kṛṣṇa by D. Apte, Anandasrama Sanskrit Series, Poona, 1930).

Flegg, Graham (1983) *Numbers. Their History and their Meaning*, Andre Deutsch, London.

Flegg, Graham, Cynthia Hay, and Barbara Moss (eds.) (1985) *Nicolas Chuquet, Renaissance Mathematician*, Reidel, Dordrecht.

Folkerts, Menso (1978) "Die älteste mathematische Aufgabensammlung in lateinischer Spräche: Die Alkuin zugeschriebenen Propositiones ad Acuendos Iuvenes", *Denkschriften der Österreichischen Akademie der Wissenschaften Mathematische-naturwissenschaftliche Klasse*, 116 (6), pp. 13-80.

Gandz, Solomon (1936) "The Sources of Al-Kwowārizmī's Algebra" *Osiris*, 1, pp. 265-77.

Gandz, Solomon (1937) "The Origin and Development of the Quadratic Equations in Babylonian, Greek, and early Arabic algebra", *Osisris*, 3, pp. 405-556.

Gregori, Silvano and Lucia Grugnetti (2001) *Anonimo (sec. XV) Trattato d'abaco, dal Manucritto Parmense 78 della Biblioteca Palatina di Parma*, Università degli Studi di Parma, Facoltà di Scienze Mathematiche, Fisische e Naturali, Dipartimento di Matematica, Parma.

Grover, Sudarshan (1993) *History and Development of Mathematics in India*, Atma Ram & Sons, Delhi.

Hankel, Herman (1874) *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, Teubner, Leipzig.

Hayashi, Takao (1995) *The Bakhshālī Manuscript. An Ancient Indian Mathematical Treatise*, Egbert Forsten, Groningen.

Heiberg, I.L. and H. Menge (eds.) (1896) *Euclidis Opera omnia*, Vol. 8: *Euclidis Phaenomina et scripta musica*, Teubner, Leipzig.

Høyrup, Jens (2002) *Lengths, Widths, Surfaces: a Portrait of Old Babylonian Algebra and its Kin*, Springer, Heidelberg.

Jain, Pushpa Kumari (2001) *The Sūryaprakāśa of Sūryadāsa (A commentory on Bhāskarācārya's* Bījagaṇita*)*, Oriental Institute, Vadodara.

Kaplan, Robert (2001) *The Nothing That Is: A Natural History of Zero*, Oxford University Press, Oxford.

Kaye, G.R. (1912) "The *Triśatikā* of Śrīdharācārya", *Bibliotheca Mathematica*, 13 (3), pp. 203-17.

Kaye, G.R. (1927, 1933) *The Bakhshāli Manuscript. A Study in Mediaeval Mathematics*, (3 parts in 2 vols.), Vol 1: *Archaeological Survey of India, Calcutta*; Vol 2: *Delhi*, 1933, (reprint Cosmo Publications, New Delhi, 1981).

Keller, Agathe (2006) *Expounding The Mathematical Seed*, (2 vols.), Vol. 1: *The Translation: A Translation Of Bhaskara I On The Mathematical Chapter Of The* Āryabhaṭīya, Vol. 2: *The Supplements: A Translation of Bhaskara I on the Mathematical Chapter of the* Āryabhaṭīya, Science Networks. Historical Studies, 31, Birkhäuser, Boston, Ma.

Kern, H (1875) *The* Āryabhaṭīya*, with the commentary* Bhaṭadīpikā *of Paramādīçvara*, Brill, Leiden.

Kool, Marjolein (ed.) (1988) *Christianus van Varenbrakens "Die Edel Conste Arithmetic",* Omiral, UFSAL, Brussel.

Lafont, Robert (ed.) (1967) *Francés Pellos. Compendion de l'Abaco*, Université de Montpellier, Montpellier.

Marre, Aristide (1880) "Le Triparty en la Science des Nombres par Maistre Nicolas Chuquet, Parisien d'apres le manuscrit Fond francaise n° 1346", *Bulletino di*

Bibliografia e di Storia della Scienze Matematiche e Fisiche, 13, pp. 593-658, pp. 693-814.

Marre, Aristide (1881) "Appendice au Triparty en la science des nombres de Nicolas Chuquet Parisien", *Bulletino di Bibliografia e di Soria della Scienze Matematiche e Fisiche*, 14, pp. 413-59.

Nesselmann, Georg Heinrich Ferdinand (1842) *Versuch einer kritischen Geschichte der Algebra.* Vol 1. *Die Algebra der Griechen*, Berlin, (Reprint by Minerva, Frankfurt, 1969).

Padmavathamma and Rao Bahadur M. Rangācārya (eds.) (2000) *The* Gaṇitasārasangraha *of Sri Mahāvīrācārya with English tranliteration, Kannada translation and notes*, Sri Siddhāntakīrthi Granthamāla, Hombuja.

Pancanti, Marisa (ed.) (1982) *Giovanni di Bartolo. Certi chasi nella trascelta a cura di Maestro Benedetto secondo la lezione del Codice L.IV.2l (sec. XV) della Biblioteca degli Intronati di Siena*, Quaderni del Centro Studi della Matematica Medioevale, 2, Servizio editoriale dell'università di Siena, Siena.

Patte, François (2004) *Le siddhāntaśiromani: l'oeuvre mathématique et astronomique de Bhāskarācārya* (2 vols.) Droz, Genève.

Pistelli, H. (ed.) (1984) *Iamblichus Nichomachi arithmeticam introductionem liber*, Teubner, Leipzig (reprint Teubner, Leipzig, 1975).

Procissi, Angiolo (ed.) (1983) *Rafaello Canacci. Ragionamenti d'Algebra I Problemi dal Codice Pal. 567 della Biblioteca Nazionale di Firenze*, Quaderni del Centro Studi della Matematica Medioevale, 7, Servizio editoriale dell'Università di Siena, Siena

Rangācārya, Rao Bhadur M. (1912) Gaṇitasārasaṃgraha, Madras.

Rivoli, Maria Teresa (ed.) (1983) *Anonimo Maestro Lombardo. Arte Giamata Aresmetica. Un'antologia dal codice N.III.53 della Biblioteca Nazionale di Torino*, Quaderni del Centro Studi della Matematica Medioevale, 8, Servico Editoriale dell'Università di Siena, Siena.

Rodet, Léon (1879) "Leçons de calcul d'Aryabhaṭa", *Journal Asiatique*, 13 (7), pp. 393-434.

Rosen, Frederic (1831) *The Algebra of Mohammed Ben Musa*, The Oriental Translation Fund, London (reprint Olms Verlag, 1986).

Śastrī, Pandit Bapu Deva (ed.) (1861) *Siddhāntaśiromani de Bhāskara, translated by the late Lancelot Wilkinson*, The Baptist Mission Press, Calcutta.

Schröder, Eberhard (1995) *Ein mathematisches Manuscript aus dem 15. Jahrhundert*, Algorismus, 16, Studien zur Geschichte der Mathematik und der Naturwissenschaften, Insititut für Geschichte der Naturwissenschaften, Munich.

Sesiano, Jacques (1984a) "Les problèmes mathématiques du 'Memoriale' de F. Bartoli", *Physis*, 26 (1), pp. 129-50.

Sesiano, Jacques (1984b) "Une Arithmétique médiévale en langue provençale", *Centaurus*, 27, pp. 26-75

Shukla, Kripa Shankar and K. V. Sarma (transl. and eds.) (1976) Āryabhaṭīya *of Āryabha□a*, Indian National Science Academy, New Delhi.

Sigler, Laurence (2002) *Fibonacci's Liber Abaci. A Translation into Modern English of Leonardo Pisano's Book of Calculation*, Springer, Heidelberg.

Singmaster, David (1999a) "Some Diophantine Recreations", in E. Berlekamp and T. Rodgers (eds.) *The Mathemagician and the Pied Puzzler: A Collection in Tribute to Martin Gardner*, A. K. Peters, MA, Boston, pp. 219-35.

Singmaster, David (1999b) "A Variation of the Ass and Mule Problem", *Crux Mathematicorum*, 28, pp. 236-8.

Singmaster, David (2004) *Sources in Recreational Mathematics, An Annotated Bibliography*, Eighth Preliminary Edition (unpublished, electronic copy from the author).
Sinha, Kripa Nath (1985) "Śripati: An Eleventh-Century Indian Mathematician", *Historia Mathematica*, 12, pp. 25-44.
Sinha, Kripa Nath (1986) "Algebra of Sripati : an eleventh century Indian mathematician", *Ganita Bharātī* , 8, 1-4, pp. 27-34.
Simi, Annalisa (ed.) (1994) *Annonimo (sec.XIV): Trattato dell'Alcibra Amuchabile (dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze)*, Quaderni del Centro Studi della Matematica Medioevale, 22, Università degli Studi di Siena, Siena.
Smith, David Eugene (1923, 1925) *History of Mathematics* (2 vols.), Ginn and company, Boston (Dover edition 1958, 2 vols.).
Tannery, Paul (1920a) *Pour l'histoire de la science Hellène*, Gauthier-Villars, Paris, (reprint of the 2nd ed. by Jacques Gabay, Sceaux, 1990).
Tannery, Paul (1920b) "Sciences exactes chez les Byzantins", *Mémoires scientifiques*, 4, Gauthier-Villars, Paris, pp. 118-86.
Tropfke, Johannes (1930-40) *Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter*, Vol. I (1930): *Rechnen*, Vol. II (1933): *Allgemeine Arithmetik*, Vol. III (1937): *Proportionen, Gleichungen*, Vol. IV (1940): *Ebene Geometrie*, (3rd ed.), Walter de Gruyter, Leipzig.
Tropfke, Johannes (1980) *Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter*, Vol. I: *Arithmetik und Algebra*, revised by Kurt Vogel, Karin Reich and Helmuth Gericke, Walter de Gruyter, Berlin.
Thibaut G. & Mahamahopadhyaya Sudhākara Dvivedin (1889) *The Panchasiddhantika. The Astronomical Work of Varaha Mihira*, Benares.
Vogel, Kurt (1940) "Zur Geschichte der linearen Gleichungen mit mehreren Unbekannten", *Deutsche Mathematik*, 5 (3), pp. 217-40.
Vogel, Kurt (1954) "Griechische Algebra in Rechenbüchern des Mittelalters", *Mathematisch-physikalische Semesterberichte*, 2, pp. 122-30.
Vogel, Kurt (1977) *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A 13)*, Veröffentlichungen des Forschungsinstituts des Deutschen Museums für die Geschichte der Naturwissenschaften und der Technik. Reihe C, 33, Munich.
Waerden, Bartel Leendert van der (1954) *Science Awakening*, (5th ed. 1988, Kluwer, Dortrecht).
Velukutty, K.K. (1997) *Heritages to and from Aryabhata*, Sahithi, Palakkad.