A NOTE ON CONSTRUCTIVE MODALITIES FOR INFORMATION

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1. INTRODUCTION

Kurt Gödel (1933) proved the formal correspondence between the intuitionistic truth of a formula F and its classical provability via modal necessity: if F is derivable in a calculus for intuitionistic logic, a translation of F such that each of its subformulas is prefixed by a necessity operator can be derived in the modal language **S4**. The inverse implication was established few years later by McKinsey and Tarski (1948). The logical equivalence between the notion of necessity from Normal Modal Logics and intuitionistic provability is nonetheless problematic: the appropriate semantics for $\Box F$ in **S4** cannot be in general identified with a provability predicate that ranges existentially over variables *(x,y)* in view of the non-constructive character of the existential quantifier (in an arithmetical model one needs an explicit version of the reflection principle $Proof(n,F) \rightarrow F$, such that it holds explicitly for each natural number *n*, cf. Artemov (2001)). To analyse further this problem, it is appropriate to extend the comparison between necessity and constructive provability in the framework of formal languages for dependent derivations, as in the case of Natural Deduction Calculi (NDC) and Dependent Type Theories (DTT). Where derivability holds under assumptions, Necessitation and Modus Ponens being both allowed rules, the following inference rule becomes unsound (see Hakli, Negri, (ms.)):

$$\frac{\Gamma \vdash A}{\Gamma \vdash \Box A} \qquad (1)$$

The same problem can be formulated with a simple argument in modal logic itself:

> Suppose we want to show X ⊃ Y in some modal axiom system by deriving Y from X. So we add X to our axioms. Say, to make things both concrete and intuitive, that X is "it is raining" and Y is "it is necessarily raining". Since X has been added to the axiom list the necessitation rule applies, and from X we conclude X, that is Y. Then the deduction theorem would allow us to conclude that if it is raining, it is necessarily raining. This does not seem right – nothing would ever be contingent. On the other hand, if we are working in the modal logic K, and we want to see what happens if we strengthen it to T by adding all instances of the scheme X ⊃ X, we certainly want the necessitation rule to apply to these instances. Things are not simple. (Fitting, 2007).

This shows that the notion of dependent derivation requires an appropriate extension on the interpretation of modalities. In forms of contextual reasoning, this is translated in terms of appropriate operations of entering and exiting a context (respectively, making an discharging an assumption). The first approach to this kind of dynamics for dependent derivations has been explored by McCarty (1993) and Buvac et al. (1995) as a *logic of contexts*. Along the very same lines, a dynamics for assumptions is possible by extending modalities for the provability predicate defined in a constructive dependent language. The aim of the research developed in this direction is threefold:

1. to formulate a logical language providing appropriate interpretations of modalities for a constructive notion of provability under assumptions;
2. to extend such a language by describing an appropriate dynamics on contexts;
3. to give intuitive explanations for the involved necessitation and possibility operators for rational agents.

In the present paper, I shall provide the basic steps towards such an analysis in the setting of a modal type-theoretical language. I will start in section 2 by considering an already existing formulation of a natural deduction language with modalities, and use it as a starting point for introducing in section 3 the basics of a type-theoretical interpretation of modalities. In both cases, appropriate rules for the dynamics on modal formulas in contexts shall be provided and an interpretation of modal operators in terms of epistemic states is formulated. In the conclusions, I shall mention the desirable extensions of this language and an appropriate case study.

2. A NATURAL DEDUCTION ANALYSIS

The formulation of dependent languages, since Gentzen's Natural Deduction Calculi, has provided a more realistic formalization of reasoning processes. Recent approaches to epistemic logic have stressed the dynamic nature of reasoning (for an overview, see Gochet, Gribomont (2006)), and the interpretation of a dynamic on contextual reasoning is essential to model the rational behaviour of agents embedded in real environments. Such contextual dynamics requires an appropriate description of epistemic states that distinguishes between the notion of *justified knowledge* and that of *assumed contents*. The corresponding formal explanation can be given in terms of an agent-based, epistemic notion of *information* to describe propositional contents for contextual reasoning (see Primiero (2007)). An extension to modalities of such a framework is a very appropriate formal instrument to make such a language stronger and to derive a number of its properties (see Primiero (forthcoming)). Let us first consider the extension to modalities that can be obtained in Natural Deduction Calculi (see e.g. Simpson (1994)), in order to have an appropriate starting point for our type-theoretical formulation.

Let us consider a simple Natural Deduction system expressing the derivability of a formula J depending on the assumptions $A_1,..., A_n$, with the basic intuitionistic restriction on sequents that the consequent is always a single formula. The contextual dynamics is obtained by extending the standard language with modal operators and appropriate operations on the set of assumptions. In the following of this paper capital Greek letters Γ, Δ, ... are used as meta-variables for sets of antecedents in the sequents; meta-variables J_1, ..., J_n stand for the derived judgement in the consequent of the sequents; capital Latin letters A, B, ... are schematic letters for propositions; usual logical connectives are used. Along with the standard axioms of Intuitionistic Logic, modalities for a language **KTB** can be introduced as follows:

M1 $\Box(A \supset B) \supset (\Box A \supset \Box B)$
M2 $(\Box A \,\&\, \Diamond B) \supset \Diamond(A \,\&\, B)$

introducing respectively Distribution for $\Box$ with respect to implication, and for $\Diamond$ with respect to conjunction. To extend these to a stronger axiomatization such as **S4**, one needs the following two further axioms:

M3 $\Box A \supset \Box\,\Box A$
M4 $\Box(A \supset \Diamond B) \supset (\Diamond A \supset \Diamond B)$

Whereas M3 is the standard axiom for iteration of $\Box$, axiom M4 says that the possibility operator is attached to the antecedent of a proved implication whose consequent is a possibility formula. The behaviour of modal operators can be explained in terms of introduction rules (for which here the formulation given in Bellin, de Paiva, Ritter (2001) is presented), along with Modus Ponens and Necessitation Rules. The Box-Rule and the Diamond-Rule are formulated as follows :

$$\frac{\Gamma \vdash J}{\Box\Gamma \vdash \Box J} \text{ (Box-Rule)} \qquad \frac{\Gamma, J_1 \vdash J_2}{\Box\Gamma, \Diamond J_1 \vdash \Diamond J_2} \text{ (Diamond-Rule)}$$

The Box-Rule builds-in the substitution procedure for assumptions, and it works also as a right-introduction rule. Whereas in the rule presented in equation (1) necessitation applies only to the consequent, in the present formulation of the Box-Rule the necessitation of the conclusion is *dependent* on the necessitation (verification) of the antecedents. Correspondingly, the Diamond-Rule preserves the meaning of non-substituted assumptions and it works also as right-introduction rule: non verified antecedents imply *possibility* for the conclusions. Distribution of necessity over conjuncts holds as in any intuitionistic modal logic. Distribution of possibility is debatable (cf. Wijesekera (1990)): the $\Diamond$-operator expresses possibility in a context, for which inconsistency is allowed, which in turn means that $\Diamond\bot \supset \bot$ does not hold; in some versions (see Mendler, de Paiva (2005)) monotonicity on inconsistent states and the rule of Ex Falso Quodlibet are allowed.

In order to formulate the dynamics for contexts of assumptions, the standard language of our Natural Deduction calculus is extended by means of an operation sign $\leftarrow$ to interpret operations on antecedents as actions for "becoming informed" (see Primiero (forthcoming)). This operator is here defined by means of syntactic rules; semantically, it can be seen as a new accessibility relation on possible knowledge states. This allows to have a calculus in which one defines a set of operations on the antecedents. The natural interpretation for deduction becomes now the following:

Definition
A deduction formula of the form $\Gamma \leftarrow \Delta \vdash J$ says that provided the information contained in the antecedents Γ is true under the related extension with Δ, one knows/receives the information that the consequent J is true.

The role of the dynamics affecting the epistemic value of the conclusion J is easily shown by the structural rules determining the behaviour of modal operators $\Box, \Diamond$ in assumptions Γ and Δ (to account respectively for verified and assumed formulas). Intuitionistic definitions of introduction and elimination rules of connectives are defined in a standard way for consequents of the sequents; applications of the Diamond-rule on the antecedents represent a special class of structural rules, intended as operations on information states. We define three different types of such rules:

$$\frac{\Gamma \vdash J}{\Box\Gamma \leftarrow \Diamond\Delta \vdash \Diamond J} \quad \text{(Information Weakening)}$$

It corresponds to extensions of informational states: if J is derivable from the set of assumptions Γ, then by verifying the information in Γ updated with the information in (becoming informed of) Δ, one is informed of the content of J (which means that the alethic value of J remains dependent on the verification of the contents in Δ).

$$\frac{\Gamma, J_1, J_2 \vdash J}{\Box\Gamma \leftarrow (\Diamond J_1 = \Diamond J_2) \vdash \Diamond J} \quad \text{(Information Contraction)}$$

It corresponds to restriction of informational states: if J is derivable from the set of assumptions Γ and from the judgements J_1, J_2, then by verifying the information in Γ updated with the information that (becoming informed) J_1, J_2 are identical, one is informed of the content of J, which means that the alethic value of J remains dependent on the verification of the identity between J_1, J_2.

$$\frac{\Gamma, J_1, \Delta, J_2 \vdash J}{\Box\Gamma \leftarrow \Diamond J_2, \Box\Delta \leftarrow \Diamond J_1 \vdash \Diamond J} \quad \text{(Information Interchange)}$$

It corresponds to a variation on the order of syntactic data in an informational state by substituting items in different repositories Γ and Δ, provided that for any J_i in Γ each J_{i-1} has been formulated (it is therefore essentially meant to restrict the interchange on dependent data).

Provided that all the operations defined for the Natural Deduction Calculi are intuitionistically definable for formulas of a constructive Dependent Type-Theory, we now proceed in showing the definition of equivalent dynamic modal operations in Constructive Type Theory.

3. AN INTERPRETATION OF MODALITIES FOR CONSTRUCTIVE TYPE THEORY

Fitting (2007) analyses the problem of dependent derivability by giving a new interpretation of the notion of assumption based on the following distinction:
1. *Global Assumptions*: formulas that the agent assumes in every model or state of his knowledge base (premises and axioms);
2. *Local Assumptions*: formulas that the agent assumes at a given state of his knowledge process to update her global assumptions.
Given such a distinction, the previous dynamic explanation of derivability reduces to an account of derivations from a set of global assumptions *updated* with a set of local ones. An agent assuming globally the contents contained in Γ and updating with local assumptions contained in Δ performs the epistemic operations of becoming informed at some world/time of the contents

contained in the new set of local assumptions. The modal operators are now needed in order to express the appropriate epistemic nature of assumptions, in terms of the global/local distinction. The use of the necessitation operator corresponds to global assumptions, or *proved contents used in contexts*; the possibility operator explains the use of local assumptions, i.e. *propositional contents assumed without being proved* (see also Primiero (forthcoming2) for the notion of epistemic possibility in a constructive philosophy of logic).

In this new setting, the justification in terms of the deduction theorem holds only for the boxed formulas (global assumptions and verified local ones). The obtained derivability relation cannot be a truth-relation, because it is not preserved under the implication $A \vdash \Box A$ (whenever A is true there may be an accessible possible world in which $\Box A$ is false). On the other hand, the inference relation holds between judgements preserving knowledge of propositional contents: indeed, when A is known to be true, then A is necessarily true. This holds in the constructivist tradition: a judgement "A is true" is evident if proposition A has a proof/is known (Martin-Löf, (1996)). By extension, reasoning under assumptions consists in saying that the conclusion is known whenever the assumptions are known, which in turn means to prove the assumed contents and turn them into proper premises. If one assumes to know that A is true and to have a proof of it, it would seem that one is entitled to derive the conclusion A: such knowability relation is closer to a form of reasoning in which the underlying concept is validity rather than truth, because the assumption concerns the provability of A, not its truth.

The translation to a type-theoretical language is given by assumptions formulated as expressions of the *context* type, each assigning a type to distinct variables. Judgements are formulas of the form $\Gamma \vdash a{:}A$, meaning that *a* is a proof A assuming formulas in Γ to be verified. For such an expression the corresponding type-theoretical formulation is $(\Gamma)a{:}A$. The standard language of Constructive Type Theory (see Primiero (2008) for an introduction) can be extended with the use of modalities to express the epistemic value on the messages contained in the contexts. Their intuitive meaning can be given as follows:

$(\Box x_i{:}A_i)a{:}A$ = being proved content A_i, proposition A is known to be true;
$(\Diamond x_i{:}A_i)a{:}A$ = receiving information A_i, a construction 'a' holds which makes proposition A true.

The dynamic rules that determine the behaviour of modal formulas within contexts shall be the exact counterparts of the ones previously introduced in natural deduction style:

Information Weakening: $(\Box\Gamma \leftarrow \Diamond\Delta)\ \Diamond J$
Information Contraction: $(\Box\Gamma \leftarrow \Diamond x_1 = \Diamond x_2{:}A_i)\ \Diamond J$
Information Interchange: $((\Box\Gamma \leftarrow \Diamond x_i{:}A_i), (\Box\Delta \leftarrow \Diamond x_j{:}A_j)\ \Diamond J$

The typing rules for modalities are the obvious ones:

$$\frac{(\Box\Gamma)a{:}A \quad (\Gamma)b{:}B}{(\Box\Gamma)b{:}B} \quad \text{(Box-Rule)}$$

$$\frac{(\Box\Gamma)a{:}A \quad (\Diamond\Delta)b{:}B \quad (\Box\Gamma \leftarrow \Diamond B)c{:}C}{(\Diamond\Gamma)c{:}C} \quad \text{(Diamond Rule)}$$

The formulation of a complete calculus shall spell out the properties of these rules and the result in terms of a comparison with Normal Modal Logics.

4. FURTHER EXTENSIONS

The first basic extension of this language is in terms of multi-modalities for expressing multi-agent settings. This will allow to express appropriate formulas for *distributed knowledge,* including the epistemic weakening offered by the possibility operator. This analysis can be compared with the known results for *common knowledge* presented for example in Artemov (2006) and Capretta (2007). Another possible extension is represented by the use of formulas corresponding to multi-conclusion modal intuitionistic sequents as introduced in de Paiva, Pereira (2005). By such extension one would be able to describe multi-agent knowledge processes that have multiple resulting knowledge states. The main application for our type-theoretical language shall be devoted to the phenomenon of *information cascades*, situations in which each agent performs epistemic actions based on other agent's knowledge, ignoring private messages. To this aim one further requires the interpretation of various meta-theoretical notions such as the one of *trust* and *trusted nets*.

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