Preferential Semantics using Non-smooth Preference Relations*

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Abstract

This paper studies the properties of eight semantic consequence relations defined from a Tarski-logic **L** and a preference relation $\prec$. They are equivalent to Shoham's so-called preferential entailment for smooth model structures, but avoid certain problems of the latter in non-smooth configurations. Each of the logics can be characterized in terms of what we call *multi-selection semantics*. After discussing this type of semantics, we focus on some concrete proposals from the literature, checking a number of meta-theoretic properties and elaborating on their intuitive motivation. As it turns out, many of their meta-properties only hold in case $\prec$ is transitive. To tackle this problem, we propose slight modifications of each of the systems, showing the resulting logics to behave better at the intuitive level and in metatheoretic terms, for arbitrary $\prec$.

keywords: preferential semantics, smoothness, transitivity, selection function

1 Introduction

In the study of non-monotonic reasoning, preferential semantics have received a lot of attention – we refer to [12] for a general introduction to this field of research. One explanation for the interest in these systems is that the underlying ideas are mathematically straightforward, but nevertheless suffice to obtain strong metatheoretic results.

The general idea behind preferential semantics, as studied in this paper, is to start from a Tarski-logic[1] **L** and a binary relation $\prec$ on its set of models.[2] In this context, $\prec$ is called a *preference relation*: it denotes something like "is more normal than" or "is preferred over". Note though that this is not a technical

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[1]We call a *Tarski-logic* any logic whose consequence relation is reflexive, monotonic and transitive. In many papers on preferential semantics, **L** is also assumed to be compact and supraclassical. We will make it explicit whenever we need such additional properties.

[2]Since the publication of [8], the standard practice in papers on preferential semantics is to consider a set of *states* and a function f that maps this set into the set of models, where f may not be injective. For the current paper, either reading is fine: one may replace the word "model" with "state" throughout the paper and interpret the notation and definitions accordingly. We simply speak of models to avoid additional clutter.

term; we do not assume any properties for $\prec$ other than being a binary relation on models.[3]

On the basis of **L** and $\prec$, we may then define a non-monotonic consequence relation generically. For instance, the relation of *preferential entailment* from Shoham's famous 1987 paper [19] is defined as follows: where Γ is a set of formulas and A a single formula, Γ preferentially entails A iff A is true in all $\prec$-minimal **L**-models of Γ. In the remainder of this paper, we denote this consequence relation by $\Vdash_{\mathbf{S}}$, using **S** to denote the associated logic.

It is shown in [12] that **S** satisfies a number of basic properties such as inclusion, cumulative transitivity, and left and right absorption – these properties are defined in Section 1.2 below. Whenever the set of all models of Γ is $\prec$-smooth[4], some additional properties like cautious monotonicity and right satisfiability also follow (see again Section 1.2 for the exact definitions).

However, without the smoothness property, serious problems arise. Consider first the cases where Γ has no $\prec$-minimal models. Then in view of the preceding, $\Gamma \Vdash_{\mathbf{S}} A$ for every A. So if one wants to model reasoning on the basis of such a premise set, one has to use a different semantic consequence relation.

A more general problem for **S** in non-smooth cases can be explained along the lines of [2]. Suppose that the models of Γ fall apart into two sets: $\mathcal{M}_1$ and $\mathcal{M}_2$, where each model $M \in \mathcal{M}_1$ is incomparable (in view of $\prec$) with respect to any of the models $M' \in \mathcal{M}_2$, and only $\mathcal{M}_1$ is $\prec$-smooth. Suppose moreover that all $\prec$-minimal models in $\mathcal{M}_1$ verify a formula A, whereas none of the models in $\mathcal{M}_2$ verify A. In this case, A will be preferentially entailed by Γ, and hence all models in $\mathcal{M}_2$ are "ignored" by **S**. This outcome seems hardly justified, given that none of these models are comparable to those in $\mathcal{M}_1$.

One example of a construction which does not have the smoothness property is Priest's adaptive paraconsistent logic $\mathbf{LP^m}$ [15], in which $M \prec M'$ holds iff M and M' have the same domain, yet the set of objects that behave inconsistently in M is a proper subset of those that behave inconsistently in M'. The absence of smoothness for this system was first noted in [2]. Similar cases are a specific variant of circumscription logic from [3], and the preferential semantics for open defaults from [1].[5]

In view of such examples, the requirement that the set of models of Γ is $\prec$-smooth seems hardly justifiable, if we take the preference relation to be something more than merely a technical device – i.e. if we presuppose that $\prec$ represents a basic concept such as "is more normal then", or "is more plausible than".[6] It may well be that certain concrete constructions warrant smoothness

[3]In Shoham's approach, $\prec$ is assumed to be a strict partial order. However, as shown in [12], the same basic results (see the next paragraph) can be obtained for arbitrary preference relations $\prec$.

[4]Where X is an arbitrary set, we say that X is $\prec$-*smooth*, or that $\langle X, \prec \rangle$ is *smooth*, iff for each $x \in X$, either x is $\prec$-minimal in X, or there is a $\prec$-minimal $y \in X$ such that $y \prec x$. Note that smoothness is different from well-foundedness, i.e. the property that there are no infinite sequences of ever "better" elements in X. Also, in the absence of transitivity or irreflexivity, even finite sets of models may not be smooth – this will be illustrated by various examples throughout the paper.

[5]In this context, one may also refer to David Lewis' discussion of the so-called Limit Assumption, which is equivalent to the assumption of smoothness in the current context – see [9, pp. 19-21]. In Lewis' system of spheres, $\prec$ expresses similarity of worlds with respect to the actual world. He argues that in many cases, one cannot assume that there is a sphere of non-actual worlds that are "most similar" to the actual world.

[6]See also [7, Section 5.6]: "This [= the smoothness condition] is a difficult condition to

– see e.g. [22] for a rather generic one –, but to assume this for preference relations in general is one bridge too far. Nevertheless, relatively little attention has been paid to preferential semantics that are well-behaved without the presupposition of the smoothness property. One notable exception is Schlechta's Limit Variant [17], which we discuss in Section 4.

The preliminary aim of this paper is to fill this gap. In particular, we will consider three concrete preferential semantics that can deal with smooth and non-smooth cases. To the best of our knowledge, only the second of these, viz. Schlechta's Limit Variant, has been studied in the way we do here (this will be further clarified below). As will be shown, the corresponding three consequence operations are fairly well-behaved and intuitively justified as long as we restrict ourselves to the case where $\prec$ is transitive. However, for non-transitive $\prec$, several problems arise. To solve these, we define variants of the proposed systems which fare better both in terms of intuitions and metatheoretically.

In the remainder of this introduction, it is argued that it is important to have ways to deal with non-transitive preference relations in the first place – a point that seems to have received little attention in the literature on preferential semantics.[7] After that, we spell out a number of metatheoretic properties that will be checked for each of the systems defined in this paper. The section ends with an overview of the rest of the paper.

1.1 Why bother with non-transitive preference relations?

There is a good reason to suspect that preference relations, if taken to represent certain pre-logical notions, are not always transitive. We often assess the normality of different cases in terms of a number of "rules of thumb", intuitive and fairly simple principles that tell us which situations are better than others in a certain respect. In this case, combining these various principles may lead to a preference relation that is well-motivated, yet not transitive.[8]

Let us illustrate this by means of an example from the deontic context.[9] Say we are dealing with two authorities P and Q, which issue the respective commands p_1, p_2, p_3 and q_1, q_2, q_3. The idea is that we want to minimize the number of violations of both groups of commands, and hence select those models that verify as many p_i and q_j as possible.

One way to do so is by minimizing violations of both types as if they are just of one kind. Such a procedure can be characterized as follows. Where M is a model, let $v^P(M) = \{p_i \mid M \not\models p_i\}$ and $v^Q(M) = \{q_i \mid M \not\models q_i\}$; let $v(M) = v^P(M) \cup v^Q(M)$. Then we define $M \prec_{\mathsf{w}} M'$ iff $v(M) \subset v(M')$. This preference relation is a strict partial order, as the reader can easily verify.

However, suppose M is such that it violates less commands than M' from those issued by P, whereas M and M' are incomparable with respect to the commands issued by Q. We may consider this as a sufficient reason to prefer

motivate as natural [...]."

[7] In the literature on preferences in general, this is not so – see e.g. [5, Sections 1.3 and 4.2] for an overview of criticisms on the assumption of transitivity for preference relations.

[8] This argument is similar to one by Voorbraak [23], who considers the combination of various preference relations which are induced by incomplete and possibly conflicting pieces of information. See also Section 4.2 in [5], where it is shown that various seemingly natural ways to combine preference relations do not preserve transitivity.

[9] Our example is structurally similar to one of Schumm [18].

M over M' – after all, if we can please one of the two authorities more, and if our choice makes no difference for the other authority, then why not do this?

To implement this idea, we should define a stronger preference relation $\prec_{\mathsf{s}}$, where $M \prec_{\mathsf{s}} M'$ iff (i) $v^P(M) \subset v^P(M')$ and $v^Q(M') \not\subset v^Q(M)$, or (ii) $v^Q(M) \subset v^Q(M')$ and $v^P(M') \not\subset v^P(M)$. Note that $\prec_{\mathsf{w}} \subset \prec_{\mathsf{s}}$.

Consider now the models in Table 1. Using these models, it easy to verify that $\prec_{\mathsf{s}}$ is both non-transitive and cyclic: we have $M_1 \prec_{\mathsf{s}} M_4 \prec_{\mathsf{s}} M_3 \prec_{\mathsf{s}} M_2 \prec_{\mathsf{s}} M_1$, but e.g. $M_1 \not\prec_{\mathsf{s}} M_3$ and $M_4 \not\prec_{\mathsf{s}} M_2$. Also, note that for no $i, j \in \{1, 2, 3, 4\}$, $M_i \prec_{\mathsf{w}} M_j$. So on the one hand, $\prec_{\mathsf{s}}$ gives us a much stronger criterion to compare models, but on the other hand, we loose certain intuitive properties such as transitivity and a-cyclicity. Obviously, in view of its cyclic behavior, taking the transitive closure[10] of $\prec_{\mathsf{s}}$ would result in a preference relation that is no longer irreflexive, which could in turn be considered as a fairly problematic situation.

	M_1	M_2	M_3	M_4
p_1	-	-	+	+
p_3	-	+	-	+
p_3	+	+	-	-
q_1	-	+	+	-
q_2	+	-	+	-
q_3	+	-	-	+

Table 1: A non-transitive sequence of models. "+" denotes satisfaction of a command, "−" its violation.

Some may argue that even if they sometimes occur in practice, non-transitive clusters are typically a sign of "inconsistency of reasoning", or at least that they point at a problem in the construction of $\prec$. But even if this is so, it does not imply that we can exclude such cases beforehand – compare this to the fact that contradictory convictions are a fact of life, even though some would argue that we should try to avoid them whenever possible. To deny the possibility of reasoning with such ill-behaved preference relations is thus similar to claiming that one cannot make sense of inconsistent theories – a position hardly anyone would still adhere to nowadays.

Technically speaking, it is easy to obtain a transitive $\prec'$ from any $\prec$. One may for instance define $\prec'$ as the transitive closure of $\prec$ – see Section 2 for the formal definition –, or one may define $M \prec' M$ iff $M \prec M$ and M, M' are not members of a non-transitive sequence of models. However, both approaches may sometimes lead to a loss of important information. Consider again the example from Table 1. If we define $\prec'_{\mathsf{s}}$ as the transitive closure of $\prec_{\mathsf{s}}$, then for all $i, j \in \{1, 2, 3, 4\}$, $M_i \prec'_{\mathsf{s}} M_j$. If we obtain $\prec'_{\mathsf{s}}$ by removing non-transitive clusters from $\prec_{\mathsf{s}}$, then for no $i, j \in \{1, 2, 3, 4\}$, $M_i \prec'_{\mathsf{s}} M_j$. Suppose now that our premises are such that we have to choose between M_1 and M_2. Then obviously, both ways to obtain transitivity will result in the fact that we cannot discard M_1, even though there seems to be an intuitive justification for doing so.

Moreover, even if transitivity would be preferable, it is not prima facie clear which way to obtain it is optimal; this may very well depend on the specific

[10]See Section 2.4 for the exact definition of the transitive closure of an arbitrary relation.

type of reasoning we are dealing with. Also, as noted above, just taking the transitive closure of $\prec$ may even result in other unsuitable behavior such as failure of irreflexivity. In any case, as long as we have not settled such issues, we still have to work on the basis of our non-transitive $\prec$ and try to find a suitable way to define our preferential consequence relation from it. Finally, as we will show in this paper, there are several ways to define a well-behaved preferential semantics without assuming transitivity of $\prec$.

1.2 Metatheoretic Properties: A Checklist

Some Preliminaries In the remainder, we use $\mathbf{X}$ as a metavariable for any logic, i.e. $\mathbf{X} : \wp(\mathcal{W}) \to \wp(\mathcal{W})$, where $\mathcal{W}$ is the set of well-formed formulas of a fixed language. $A, B, \ldots$ are metavariables for members of $\mathcal{W}$ and Γ, Δ are metavariables for subsets of $\mathcal{W}$. Where $\Vdash_{\mathbf{X}}$ denotes the semantic consequence relation of $\mathbf{X}$, we let $Cn_{\mathbf{X}}(\Gamma) = \{A \mid \Gamma \Vdash_{\mathbf{X}} A\}$. In the case of our Tarski-logic $\mathbf{L}$, we omit the subscripts, so that $Cn(\Gamma) = \{A \mid \Gamma \Vdash A\}$. Also, we shall simply speak of "models" to refer to $\mathbf{L}$-models. We write $\mathbf{X} \subseteq \mathbf{X}'$ as a shortcut for $\Vdash_{\mathbf{X}} \subseteq \Vdash_{\mathbf{X}'}$. Finally, we use $\mathbf{CL}$ to denote classical propositional logic, where the connectives of $\mathbf{CL}$ are $\neg, \vee, \supset, \wedge$ and its propositional letters are $p, q, \ldots, p_1, \ldots$.

Properties We define the following properties of consequence operations:

Inclusion:	$\Gamma \subseteq Cn_{\mathbf{X}}(\Gamma)$
Right Absorption:	$Cn_{\mathbf{X}}(\Gamma) = Cn(Cn_{\mathbf{X}}(\Gamma))$
Left Absorption:	$Cn_{\mathbf{X}}(Cn(\Gamma)) = Cn_{\mathbf{X}}(\Gamma)$
Cumulative Transitivity:	If $\Gamma' \subseteq Cn_{\mathbf{X}}(\Gamma)$, then $Cn_{\mathbf{X}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{X}}(\Gamma)$
Cautious Monotonicity:	If $\Gamma' \subseteq Cn_{\mathbf{X}}(\Gamma)$, then $Cn_{\mathbf{X}}(\Gamma) \subseteq Cn_{\mathbf{X}}(\Gamma \cup \Gamma')$
Right Satisfiability:	$Cn_{\mathbf{X}}(\Gamma)$ has models if Γ has models
Disjunction Property:	$Cn_{\mathbf{X}}(\Gamma) \cap Cn_{\mathbf{X}}(\Gamma') \subseteq Cn_{\mathbf{X}}(Cn(\Gamma) \cap Cn(\Gamma'))$.

We will now briefly comment on each of these properties – a more elaborate discussion can be found in [12, pp. 42-52].

Note that inclusion of $Cn_{\mathbf{X}}$ corresponds to reflexivity of $\Vdash_{\mathbf{X}}$. It can be easily verified that left absorption and inclusion entail that $Cn(\Gamma) \subseteq Cn_{\mathbf{X}}(\Gamma)$.

If $Cn_{\mathbf{X}}$ is both cautiously monotonic and cumulatively transitive, we say that it is *cumulative*. This property can be motivated as follows: if we show that certain formulas $A_1, A_2, \ldots$ follow from our theory Γ, then we should be able to rely on these as though they are premises, when we try to derive additional information from Γ. However, to do so, we should have a warrant that such additions to Γ will not have an impact on whatever else is derivable – they should not allow us to derive more than what we could derive from Γ alone, but they should also not block the derivation of other formulas that follow from Γ.

In the presence of inclusion, cumulative transitivity is sufficient to obtain the *fixed point* property, i.e. $Cn_{\mathbf{X}}(Cn_{\mathbf{X}}(\Gamma)) = Cn_{\mathbf{X}}(\Gamma)$.[11] Note also that cumulative transitivity, inclusion and left absorption together entail right absorption.[12]

[11] See e.g. [12, p. 43].

[12] To see why, suppose $Cn_{\mathbf{X}}$ does not satisfy right absorption. Let $A \in Cn(Cn_{\mathbf{X}}(\Gamma)) - Cn_{\mathbf{X}}(\Gamma)$. Then by left absorption and inclusion, $A \in Cn_{\mathbf{X}}(Cn_{\mathbf{X}}(\Gamma))$. Hence $Cn_{\mathbf{X}}(Cn_{\mathbf{X}}(\Gamma)) \not\subseteq Cn_{\mathbf{X}}(\Gamma)$.

If $\mathbf{CL} \subseteq \mathbf{L}$, then right satisfiability entails that $Cn_{\mathbf{X}}(\Gamma)$ is consistent (relative to $\mathbf{CL}$) whenever Γ is so. More generally, if $\mathbf{L}$ has no trivial models[13] and in the presence of right satisfiability, $Cn_{\mathbf{X}}(\Gamma)$ is non-trivial whenever Γ is non-trivial.[14]

The importance of the disjunction property can be illustrated by a lemma from [12]:

Lemma 1 ([12], Observation 2.3.3, p. 47) *If* $\mathbf{CL} \subseteq \mathbf{L}$ *and* $\mathbf{X}$ *satisfies left and right absorption and the disjunction property, then each of the following holds:*

1. $Cn_{\mathbf{X}}(\Gamma \cup \{A\}) \cap Cn_{\mathbf{X}}(\Gamma \cup \{B\}) \subseteq Cn_{\mathbf{X}}(\Gamma \cup \{A \vee B\})$. *(Disjunction in the Antecedent)*
2. $Cn_{\mathbf{X}}(\Gamma \cup \{A\}) \cap Cn_{\mathbf{X}}(\Gamma \cup \{\neg A\}) \subseteq Cn_{\mathbf{X}}(\Gamma)$. *(Proof By Cases)*
3. *If* $B \in Cn_{\mathbf{X}}(\Gamma \cup \{A\})$, *then* $A \supset B \in Cn_{\mathbf{X}}(\Gamma)$. *(Deduction Theorem)*

Before we proceed, let us insert a short disclaimer: we do not consider any of the above properties as a prerequisite to speak of a "sensible" logic, or as ways to prove the superiority of certain consequence relations over others. In fact, various scholars have put forward interesting types of reasoning in which the failure of some of these properties is a desideratum, rather than a problem – see e.g. [13] for the case of right satisfiability and right absorption and [23] for cumulativity. Our aim is more modest: we just want to check for each of the above properties whether the logics we present below have them or not, to get a better understanding of those logics.

Properties that hold for S The following two corollaries summarize the results from the literature concerning $\mathbf{S}$ and the properties from our above checklist:

Corollary 1 *$Cn_{\mathbf{S}}$ satisfies inclusion, left and right absorption, and cumulative transitivity. If* $\mathbf{CL} \subseteq \mathbf{L}$, *then* $Cn_{\mathbf{S}}$ *also has the disjunction property.*

Corollary 2 *If for all* Γ, $\langle \mathcal{M}(\Gamma), \prec \rangle$ *is smooth, then* $Cn_{\mathbf{S}}$ *satisfies cautious monotonicity and right satisfiability.*

As mentioned in the introduction, cautious monotonicity and right satisfiability fail for $Cn_{\mathbf{S}}$ in the general, non-smooth case – see [12] for a counterexample.

1.3 Overview of this Paper

In Section 2, we define so-called multi-selection semantics. This offers a generic characterization of all semantics studied in this paper. We provide a representation theorem for the class of all multi-selection semantics and establish sufficient conditions for certain metatheoretic properties. In addition, we introduce the notion of local transitive closure, which is put to work in several subsequent sections.

[13] We call a model trivial if it verifies every formula of the language. Obviously, most logics do not have trivial models; Priest's logic **LP** is a notable exception.

[14] We call Δ $\mathbf{L}$-trivial iff $Cn_{\mathbf{L}}(\Delta)$ is the set of all formulas of the language.

In the three subsequent sections, we discuss three families of concrete consequence relations based on **L** and $\prec$. Each time, we start with a logic that has already been discussed or mentioned in the literature: an idea from [2] (Section 3), the Limit Variant from [17] (Section 4), and finally an approach based on ideas from modal logic [4, 6] (Section 5). As we shall argue, each of these consequence relations face certain problems, both at the intuitive and the metatheoretic level, in cases where $\prec$ is not transitive. We will show how one can slightly modify the definitions of each of the consequence relations, and thereby obtain variants that score better in several respects. The variants are equivalent to their original counterparts whenever $\prec$ is transitive.

In Section 6, we give an overview of our metatheoretic results, and compare the various logics in terms of their logical strength. We end the paper with some general comments and prospects for future research.

2 Setting the Stage

In this section we introduce some concepts that are central to the paper. We start with some notational conventions, after which we introduce the notion of multi-selection semantics. After that, a number of metatheoretic properties are established for consequence relations based on this type of semantics. Finally, we discuss the idea of local transitive closure which is put to work in each of the following three sections.

2.1 Notational Conventions

We use $\mathcal{M}(\Gamma)$ as shorthand for the set of all (**L**-)models of Γ, i.e. all models in which every member of Γ is valid. $\models$ refers to validity in a model; $\Vdash$ to the semantic consequence relation (of **L**), which is defined as usual: $\Gamma \Vdash A$ iff for all $M \in \mathcal{M}(\Gamma)$, $M \models A$. $\Gamma \Vdash_{\mathbf{X}} \Gamma'$ is a shortcut for $\Gamma \Vdash_{\mathbf{X}} A$ for all $A \in \Gamma'$.

L is *compact* iff each of the following holds: (i) if $\Gamma \Vdash_{\mathbf{L}} A$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \Vdash_{\mathbf{L}} A$ and (ii) if every finite $\Gamma' \subseteq \Gamma$ has **L**-models, then Γ has **L**-models.[15]

In Sections 4 and 5, we will sometimes refer to slightly weaker variants of cautious monotonicity, resp. cumulative transitivity. Let Γ' be a finite set for which $\Gamma' \subseteq Cn_{\mathbf{X}}(\Gamma)$. Then these properties read as follows:[16]

Finitary Cautious Monotonicity: $Cn_{\mathbf{X}}(\Gamma) \subseteq Cn_{\mathbf{X}}(\Gamma \cup \Gamma')$
Finitary Cumulative Transitivity: $Cn_{\mathbf{X}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{X}}(\Gamma)$

Many results from this paper will be spelled out in purely set-theoretic terms. We use $X, Y, \ldots$ for arbitrary sets and $x, y, \ldots$ for their elements. We will use $\prec$ as a metavariable for binary relations. $\preceq$ is defined from $\prec$ in the usual way: $x \preceq y$ iff $x \prec y$ or $x = y$. Let in the remainder

$$\mathsf{min}_{\prec}(X) =_{\mathsf{df}} \{x \in X \mid \text{ for no } y \in X, y \prec x\}$$

[15] If one can define a logical falsum $\perp$ in **L**, then (ii) follows from (i). Also, if the language contains a classical negation, then (ii) implies (i). However, in some logics, like e.g. Priest's **LP**, this is not the case.

[16] We do not presuppose that Γ is finite. Also, recall that the underlying logic **L** need not contain a classical conjunction; hence Γ' cannot always be reduced to a single formula.

The proof of the following is safely left to the reader:

Fact 1 *If $X \subseteq Y$ and $x \in X \cap \mathsf{min}_{\prec}(Y)$, then $x \in \mathsf{min}_{\prec}(X)$.*

Finally, we will often use the concepts of $\prec$-lowerness and $\prec$-density when talking about sets. These are defined as follows:[17]

(i) Y is $\prec$-*dense* in X iff $Y \subseteq X$ and for every $x \in X - Y$, there is a $y \in Y$ such that $y \prec x$.
(ii) Y is $\prec$-*lower* in X iff $Y \subseteq X$ and for every $x \in X, y \in Y$: if $x \prec y$, then $x \in Y$.

Note that $\mathsf{min}_{\prec}$ is always $\prec$-lower in X, and if $\langle X, \prec\rangle$ is smooth, then $\mathsf{min}_{\prec}(X)$ is also $\prec$-dense in X. Also, if Y is $\prec$-dense in X and $X \neq \emptyset$, then obviously $Y \neq \emptyset$.

2.2 Multi-selection Semantics

In this section, we introduce the notion of multi-selection semantics as a generalization of selection semantics. The latter have been studied in their most general form in [10]. The idea of selection semantics is that one selects a set $\mathcal{M} \subseteq \mathcal{M}(\Gamma)$, where $\mathcal{M}$ is thought of as the models that are "best", "minimal", "safe", or something alike. Semantic consequence is then defined as validity in all the selected models, hence in all models $M \in \mathcal{M}$.

More formally, let a *selection function* be any function $\psi : \wp(X) \to \wp(X)$, where $\psi(Y) \subseteq Y$ for all $Y \in \wp(X)$. We have:

Definition 1 *Where $\psi : \wp(\mathcal{M}(\emptyset)) \to \wp(\mathcal{M}(\emptyset))$ is a selection function, $\Gamma \Vdash_{\psi} A$ ($A \in Cn_{\psi}(\Gamma)$) iff A is true in every $M \in \psi(\mathcal{M}(\Gamma))$.*[18]

In this paper, we will restrict the focus to selection functions ψ that are defined on the basis of $\prec$. The most familiar example of such a function is $\mathsf{min}_{\prec}$, which gives us the system **S** as discussed in the introduction. Various other examples are discussed in Section 3.

From the preceding definitions, one can easily verify that any operation Cn_{ψ} satisfies left and right absorption and inclusion. Moreover, simple criteria can be used to establish additional properties. For instance,[19]

Lemma 2 ([17], p. 32) *Each of the following holds:*

1. *Cn_{ψ} is cautiously monotonic whenever ψ satisfies the condition:*
   if $\psi(X) \subseteq Y \subseteq X$, then $\psi(Y) \subseteq \psi(X)$

[17] In some of Schlechta's papers, Y is called *downward closed* in X whenever (according to our terminology) Y is $\prec$-lower in X. In [1], $\prec$-lowerness is dubbed $\succ$-closedness and $\prec$-density is dubbed $\prec$-completeness. In the same paper, Y is called $\succ$-*dense* in X whenever (in our terms) Y is both $\prec$-lower and $\prec$-dense in X.

[18] In fact, to obtain a well-defined semantic consequence relation, it suffices to have a function $\psi' : \Upsilon \to \wp(\mathcal{M}(\emptyset))$, where $\Upsilon =_{\mathsf{df}} \{\mathcal{M}(\Gamma) \mid \Gamma \subseteq \mathcal{W}\}$. However, all selection functions from the current paper are defined generically for arbitrary sets, and hence we avoid the additional clutter that such a restriction would bring along.

[19] In fact, it suffices that ψ obeys these conditions for all sets $X, Y \in \Upsilon$ – see also footnote 18. Again, we avoid such restrictions in this paper as it turns out that we can always guarantee the conditions wherever we need them, without restrictions on the domain.

2. *Cn_ψ is cumulatively transitive whenever ψ satisfies the condition:*

   if $\psi(X) \subseteq Y \subseteq X$, then $\psi(X) \subseteq \psi(Y)$

3. *Cn_ψ satisfies right satisfiability whenever ψ satisfies the condition:*

   if $X \neq \emptyset$, then $\psi(X) \neq \emptyset$

As announced, multi-selection semantics generalize the idea behind selection semantics: instead of selecting one set $\mathcal{M} \subseteq \mathcal{M}(\Gamma)$, one picks various such sets $\mathcal{M}_i \subseteq \mathcal{M}(\Gamma)$ ($i \in I$), where the semantic consequence relation is defined as validity in all $M \in \mathcal{M}_i$ for an $i \in I$. As we shall see at the end of section 3, this has several advantages in case $\langle \mathcal{M}(\Gamma), \prec \rangle$ is not smooth.

Let us make the idea of multi-selection formally precise.

Definition 2 *A* multi-selection function *is any function $\pi : \wp(X) \rightarrow \wp(\wp(X))$, where for every $Y \subseteq X$:*

(i) $\pi(Y) \neq \emptyset$ *and*
(ii) $Z \subseteq Y$ *for all* $Z \in \pi(Y)$

Note that (i) does not imply that every $Z \in \pi(Y)$ is non-empty. So there should be at least one selection, but just as in a selection semantics, it may well be that no model at all is selected.[20] Obviously, every selection function can be rephrased as a multi-selection function with $\pi(Y) = \{Z\}$ for all $Y \subseteq X$.

A simple, and arguably not very interesting example of a multi-selection function is the following: let $Y \in \Lambda(X)$ iff Y is $\prec$-dense in X.[21] Given this multi-selection function Λ, we can define $\Gamma \Vdash_\Lambda A$ iff A is true in all $M \in \mathcal{M}$, for an $\mathcal{M} \in \Lambda(\mathcal{M}(\Gamma))$. With this definition, A follows from Γ iff for every model M of Γ, there is an $M' \in \mathcal{M}(\Gamma)$ such that $M' \preceq M$ and $M' \models A$. It can easily be verified that the resulting consequence operation has inclusion and left absorption, but that right satisfiability, right absorption, and both directions of cumulativity fail for it.

In general, we define the semantic consequence relation based on a multi-selection function as follows:

Definition 3 *Where $\pi : \wp(\mathcal{M}(\emptyset)) \rightarrow \wp(\wp(\mathcal{M}(\emptyset)))$ is a multi-selection function, $\Gamma \Vdash_\pi A$ ($A \in Cn_\pi(\Gamma)$) iff there is an $\mathcal{M} \in \pi(\mathcal{M}(\Gamma))$ such that $M \models A$ for all $M \in \mathcal{M}$.*

As we will see below, multi-selection semantics may sometimes behave in a rather non-standard way, compared to selection semantics – for instance, right absorption does not always hold, and proving right satisfiability can be considerably more tough for certain multi-selection functions. On the other hand, by means of multi-selection semantics, we can get a very strong consequence relation, since it suffices to have a selected set $\mathcal{M}_A$ for every consequence A, whereas in a regular selection semantics every consequence is based on exactly one selected set, viz. $\psi(\mathcal{M}(\Gamma))$.

[20] Again, one may restrict π to the domain Υ – see footnote 18 – and still obtain a well-defined consequence relation from it. For the same reasons as before, we will not do so in this paper.

[21] Since $X \in \Lambda(X)$, requirement (i) is trivially fulfilled. Also, in view of the definition of $\prec$-density in X, every $Y \in \Lambda(X)$ is a subset of X, which guarantees that (ii) holds.

Nevertheless, some properties like e.g. inclusion can be easily shown to hold for all multi-selection semantics in general. This is done in Section 2.3. Moreover, we can formulate simple sufficient conditions for other properties like right absorption and cumulativity, which greatly simplify proofs about concrete systems of this type.

The idea of a multi-selection semantics was obtained by a generalization of the Limit Variant from [17] – see Section 4.[22] The stronger Boutilier-variant which we present in Section 5 can also be characterized in terms of a multi-selection semantics.

An altogether different example of a multi-selection semantics are the so-called normal selections in the adaptive logic framework.[23] One may describe these roughly as follows. First, a set of "abnormalities" $\Omega \subseteq \mathcal{W}$ is defined, which allows us to compare models in a structural way. For every model M, we define its "abnormal part" $Ab(M) = \{A \in \Omega \mid M \models A\}$. Next, we let $\tau(\Gamma) = \mathsf{min}_{\subset}\{Ab(M) \mid M \in \mathcal{M}(\Gamma)\}$. So $\tau(\Gamma)$ consists of the $\subset$-minimal abnormal parts within the set of all abnormal parts of the models of Γ. Finally, we define π^{ns} as follows:[24]

$$\pi^{ns}(\mathcal{M}(\Gamma)) = \{\{M \in \mathcal{M}(\Gamma) \mid Ab(M) = \Delta\} \mid \Delta \in \tau(\Gamma)\}$$

Using this function, we have $\Gamma \Vdash_{\pi^{ns}} A$ iff there is at least one set of models $\mathcal{M} \subseteq \mathcal{M}(\Gamma)$ such that (i) all models $M \in \mathcal{M}$ have the same abnormal part; (ii) this abnormal part is moreover $\subset$-minimal within the set of abnormal parts of models of Γ, and (iii) A is true in every $M \in \mathcal{M}$. As shown e.g. in [20], the resulting consequence set may not be satisfiable, but is nevertheless non-trivial whenever Γ is non-trivial.

2.3 Some Metatheory of Multi-selection Semantics

This section is of a rather technical nature. We will first provide a representation theorem for the class of all consequence operations that can be defined in terms of a multi-selection semantics. After that, we establish a number of sufficient conditions for certain meta-properties of consequence operations based on multi-selection functions.

Representation Theorem for Multi-selection Semantics In [10], Lindström shows that all operations $Cn_{\mathbf{X}}$ characterized by a selection semantics are inclusive and satisfy left and right absorption. Moreover, whenever an operation $Cn_{\mathbf{X}}$ has these three properties, it can be characterized by a selection semantics. In this paragraph, we will establish a similar representation theorem

[22] In the introduction of [16], Schlechta also considers the generalization of semantics in terms of a single selection function f to semantics in terms of a set F of such functions, where semantic consequence is then defined as validity in at least one set $f(\mathcal{M}(\Gamma))$ for an $f \in F$. It can be easily verified that our approach covers Schlechta's idea as a special case. Apart from this, Schlechta's results concern only one specific type of multi-selection semantics based on a preference relation, viz. his Limit Variant – see Section 4 below.

[23] Adaptive logics using this type of semantics have been developed to characterize various sorts of defeasible reasoning, including abduction [11], credulous acceptance in abstract argumentation [21], and the universal Rescher-Manor consequence [14]. The general metatheory of these systems is studied in [20].

[24] Where $\mathcal{M} \neq \mathcal{M}(\Gamma)$ for any $\Gamma \subseteq \mathcal{W}$, we can simply let $\pi^{ns}(\mathcal{M}) = \{\mathcal{M}\}$. See also the previous footnotes concerning restrictions on the domain of π.

for multi-selection semantics. As this class is more general, the properties that characterize it will obviously be weaker.

First, it can be easily checked that every operation Cn_π satisfies inclusion and left absorption. Inclusion is immediate in view of the reflexivity of **L**, and conditions (i) and (ii) on multi-selection functions. Left absorption follows from the fact that $\mathcal{M}(\Gamma) = \mathcal{M}(Cn(\Gamma))$.

Moreover, the following property also holds for $Cn_{\mathbf{X}} = Cn_\pi$:[25]

Singular Right Absorption: $Cn(\Gamma \cup \{A\}) \subseteq Cn_{\mathbf{X}}(\Gamma)$ for all $A \in Cn_{\mathbf{X}}(\Gamma)$

That is, suppose $A \in Cn_\pi(\Gamma)$. Hence there is an $\mathcal{M} \in \pi(\mathcal{M}(\Gamma))$ such that for all $M \in \mathcal{M}$, $M \models A$. So $\mathcal{M} \subseteq \mathcal{M}(\Gamma \cup \{A\})$. Suppose now that $B \in Cn(\Gamma \cup \{A\})$. It follows that every $M \in \mathcal{M}$ verifies B. As a result, for every such B, $B \in Cn_\pi(\Gamma)$.

We now show that these three basic properties also fully characterize the set of all operations Cn_π:

Theorem 1 *$Cn_{\mathbf{X}}$ satisfies inclusion, left absorption and singular right absorption iff it is equivalent to an operation Cn_π, where π is a multi-selection function.*

Proof. In view of the preceding, it suffices to prove the left-right direction. So suppose the antecedent holds. Define $\pi^{\mathbf{x}} : \wp(\mathcal{M}(\emptyset)) \to \wp(\wp(\mathcal{M}(\emptyset)))$ as follows:

(0) $\pi^{\mathbf{x}}(\mathcal{M}) = \{\mathcal{M}\}$ iff there is no $\Gamma \subseteq \mathcal{W}$ such that $\mathcal{M} = \mathcal{M}(\Gamma)$;
(1) $\pi^{\mathbf{x}}(\mathcal{M}(\Gamma)) =_{\mathsf{df}} \{\{M \in \mathcal{M}(\Gamma) \mid M \models A\} \mid A \in Cn_{\mathbf{X}}(\Gamma)\}$ if $Cn_{\mathbf{X}}(\Gamma) \neq \emptyset$;
(2) $\pi^{\mathbf{x}}(\mathcal{M}(\Gamma)) =_{\mathsf{df}} \{\mathcal{M}(\emptyset)\}$ otherwise.

We first check that $\pi^{\mathbf{x}}$ is function. Suppose $\mathcal{M} = \mathcal{M}'$. In the case of (0), we trivially have that $\pi^{\mathbf{x}}(\mathcal{M}) = \pi^{\mathbf{x}}(\mathcal{M}')$. Suppose $\mathcal{M} = \mathcal{M}(\Gamma) = \mathcal{M}' = \mathcal{M}(\Gamma')$. Then $Cn(\Gamma) = Cn(\Gamma')$, and hence by left absorption, $Cn_{\mathbf{X}}(\Gamma) = Cn_{\mathbf{X}}(\Gamma')$. In view of the construction, it follows that $\pi^{\mathbf{x}}(\mathcal{M}(\Gamma)) = \pi^{\mathbf{x}}(\mathcal{M}(\Gamma'))$.

Next, we show that $\pi^{\mathbf{x}}$ is a multi-selection function. Requirement (i) is trivial for cases (0) and (2). For case (1), let $A \in Cn_{\mathbf{X}}(\Gamma)$. It follows that $\{M \in \mathcal{M}(\Gamma) \mid M \models A\} \in \pi^{\mathbf{x}}(\mathcal{M}(\Gamma))$. Requirement (ii) holds trivially in all three cases.

It remains to prove that $Cn_{\pi^{\mathbf{x}}}(\Gamma) = Cn_{\mathbf{X}}(\Gamma)$ for all Γ. First, suppose $Cn_{\mathbf{X}}(\Gamma) \neq \emptyset$. "$\subseteq$" Suppose $A \in Cn_{\pi^{\mathbf{x}}}(\Gamma)$. Hence there is a $\mathcal{M} \in \pi^{\mathbf{x}}(\mathcal{M}(\Gamma))$ such that all $M \in \mathcal{M}$ verify A. By the construction, there is a $B \in Cn_{\mathbf{X}}(\Gamma)$, such that $\mathcal{M} = \mathcal{M}(\Gamma \cup \{B\})$. It follows that $\Gamma \cup \{B\} \Vdash A$. Hence by singular right absorption, $A \in Cn_{\mathbf{X}}(\Gamma)$.

"$\supseteq$" Suppose $A \in Cn_{\mathbf{X}}(\Gamma)$. Let $\mathcal{M}_A = \{M \in \mathcal{M}(\Gamma) \mid M \models A\}$. Note that $\mathcal{M}_A \in \pi^{\mathbf{x}}(\mathcal{M}(\Gamma))$. Hence $A \in Cn_{\pi^{\mathbf{x}}}(\Gamma)$.

If $Cn_{\mathbf{X}}(\Gamma) = \emptyset$, then by inclusion and left absorption, $Cn(\Gamma) = \emptyset$. By the construction, also $Cn_{\pi^{\mathbf{x}}}(\Gamma) = Cn(\emptyset)$ and we are done. ■

[25] It was XXX who proposed the name singular right absorption (in personal correspondence). It is easy to check that this property does not follow from inclusion and left absorption: for a fixed letter p, let $Cn_{\mathbf{X}}(\Gamma) = Cn(\Gamma) \cup \{p\}$ for all Γ. To see why singular right absorption and left absorption do not imply inclusion, let $Cn_{\mathbf{X}}(\Gamma) = \emptyset$ for all Γ. To see why singular right absorption and inclusion do not imply left absorption, let $Cn_{\mathbf{Y}}(\Gamma) = Cn(\Gamma \cup \{q\})$ whenever $p \in \Gamma$, and $Cn_{\mathbf{Y}}(\Gamma) = Cn(\Gamma)$ otherwise (for fixed p, q).

Conditions for Other Properties All systems studied in this paper are either based on a multi-selection semantics, or can be equivalently reformulated using one. Hence, the above insights allow us to skip inclusion, left absorption and singular right absorption from our checklist of metatheoretic properties. In the remainder of this paper, we will thus focus on the following properties: right satisfiability, right absorption, cautious monotonicity, cumulative transitivity, and the disjunction property.

For the first four of these, we shall be using the following conditions on $\pi : \wp(\mathcal{M}(\emptyset)) \to \wp(\wp(\mathcal{M}(\emptyset)))$:[26]

NT Where $X \neq \emptyset$: if $Y \in \pi(X)$, then $Y \neq \emptyset$.

RS Where $X \neq \emptyset$: if $Y_1, \ldots, Y_n \in \pi(X)$, then $Y_1 \cap \ldots \cap Y_n \neq \emptyset$

RA Where $X, Y \in \pi(Z)$: $X \cap Y \in \pi(Z)$

CM If $Y \in \pi(X)$ and $Y \subseteq Z \subseteq X$, then $Y \in \pi(Z)$.

CT If $X \in \pi(Z_1)$, $Y \in \pi(Z_2)$, and $X \subseteq Z_2 \subseteq Z_1$, then $X \cap Y \in \pi(Z_1)$.

Fact 2 **CT** *implies* **RA**.[27]

Fact 3 *(***NT** *and* **RA***) implies* **RS***.*

Lemma 3 *(Compactness of* **L** *and* **RS***) imply right satisfiability of* Cn_π*.*

Proof. Suppose **L** is compact and **RS** holds. Suppose moreover that $\mathcal{M}(\Gamma) \neq \emptyset$. It suffices to show, for all $A_1, \ldots, A_n \in Cn_\pi(\Gamma)$, that $\{A_1, \ldots, A_n\}$ has models. For every A_i ($1 \leq i \leq n$), let $\mathcal{M}_i \in \pi(\mathcal{M}(\Gamma))$ be such that A_i is true in every $M \in \mathcal{M}_i$. By **RS**, $\bigcap_{1\leq i\leq n} \mathcal{M}_i \neq \emptyset$. So let $M \in \bigcap_{1\leq i\leq n} \mathcal{M}_i$. It follows that $M \models A_i$ for all $i \in \{1, \ldots, n\}$. ■

Lemma 4 *(Compactness of* **L** *and* **RA***) imply right absorption of* Cn_π*.*

Proof. Suppose **L** is compact and **RA** holds. "$Cn(Cn_\pi(\Gamma)) \subseteq Cn_\pi(\Gamma)$" Let $B \in Cn(Cn_\pi(\Gamma))$. By compactness, there are $A_1, \ldots, A_n \in Cn_\pi(\Gamma)$, such that (†) $\{A_1, \ldots, A_n\} \Vdash B$. For every A_i ($1 \leq i \leq n$), let $\mathcal{M}_i \in \pi(\mathcal{M}(\Gamma))$ be such that A_i is true in every $M \in \mathcal{M}_i$. By **RA**, $\bigcap_{1\leq i\leq n} \mathcal{M}_i \in \pi(\mathcal{M}(\Gamma))$. By (†), $M \models B$ for all $M \in \bigcap_{1\leq i\leq n} \mathcal{M}_i$. Hence $B \in Cn_\pi(\Gamma)$.

"$Cn_\pi(\Gamma) \subseteq Cn(Cn_\pi(\Gamma))$" Immediate in view of the reflexivity of **L**. ■

In view of Lemma 4, an obvious way to obtain an operation $Cn_{\pi'}$ that satisfies right absorption, from any given operation Cn_π, is by closing π under finite intersections. That is, for all X, let $\pi'(X) \supseteq \pi(X)$ be minimal such that for all $Y_1, \ldots, Y_n \in \pi'(X)$, also $Y_1 \cap \ldots \cap Y_n \in \pi'(X)$. It can easily be verified that (i) π' is a multi-selection function whenever π is, and that (ii) $\Vdash_{\pi'}$ satisfies right satisfiability whenever $\Vdash_\pi$ does. Moreover, in view of its construction, $Cn_{\pi'}$ satisfies right absorption.

Lemma 5 *(***RA** *and* **CM***) imply finitary cautious monotonicity of* Cn_π*.*

[26] As with selection semantics, we may restrict these conditions to all $X, Y, Z, Z_1, Z_2 \in \Upsilon$ – see also footnote 18 – and obtain the same metatheoretic properties of the corresponding consequence relations.

[27] To see why this holds, suppose **CT** holds, and let $Z_1 = Z_2 = Z$.

Proof.[28] Suppose that **RA** and **CM** hold, $\Gamma \Vdash_\pi \Gamma'$ for a finite Γ', and $\Gamma \Vdash_\pi B$. We have to prove that $\Gamma \cup \Gamma' \Vdash_\pi B$.

For all $A \in \Gamma'$, let $\mathcal{M}_A \in \pi(\mathcal{M}(\Gamma))$ be such that all $M \in \mathcal{M}_A$ verify A. Let $\mathcal{M}_{\Gamma'} = \bigcap_{A \in \Gamma'} \mathcal{M}_A$. By **RA**, $\mathcal{M}_{\Gamma'} \in \pi(\mathcal{M}(\Gamma))$ (†).

Let $\mathcal{M}_B \in \pi(\mathcal{M}(\Gamma))$ be such that all $M \in \mathcal{M}_B$ verify B. By **RA** and (†), $\mathcal{M}_{\Gamma'} \cap \mathcal{M}_B \in \pi(\mathcal{M}(\Gamma))$. Note that $\mathcal{M}_{\Gamma'} \cap \mathcal{M}_B \subseteq \mathcal{M}(\Gamma \cup \Gamma') \subseteq \mathcal{M}(\Gamma)$. Hence by **CM**, $\mathcal{M}_{\Gamma'} \cap \mathcal{M}_B \in \pi(\mathcal{M}(\Gamma \cup \Gamma'))$. Since all $M \in \mathcal{M}_{\Gamma'} \cap \mathcal{M}_B$ verify B, we can infer that $\Gamma \cup \Gamma' \Vdash_\pi B$. ■

Lemma 6 **CT** *implies finitary cumulative transitivity of Cn_π.*

Proof. Suppose **CT** holds, $\Gamma \Vdash_\pi \Gamma'$ for a finite Γ', and $\Gamma \cup \Gamma' \Vdash_\pi B$. We have to prove that $\Gamma \Vdash_\pi B$. Note that by (i) and Fact 2, also **RA** holds.

For all $A \in \Gamma'$, let $\mathcal{M}_A \in \pi(\mathcal{M}(\Gamma))$ be such that all $M \in \mathcal{M}_A$ verify A. Let $\mathcal{M}_{\Gamma'} = \bigcap_{A \in \Gamma'} \mathcal{M}_A$. By **RA**, $\mathcal{M}_{\Gamma'} \in \pi(\mathcal{M}(\Gamma))$ (†).

Let $\mathcal{M}'_B \in \pi(\mathcal{M}(\Gamma \cup \Gamma'))$ be such that all $M \in \mathcal{M}'_B$ verify B. Note that $\mathcal{M}_{\Gamma'} \subseteq \mathcal{M}(\Gamma \cup \Gamma') \subseteq \mathcal{M}(\Gamma)$. Hence by **CT** and (†), $\mathcal{M}_{\Gamma'} \cap \mathcal{M}'_B \in \pi(\mathcal{M}(\Gamma))$. Since all $M \in \mathcal{M}_{\Gamma'} \cap \mathcal{M}'_B$ verify B, we have shown that $\Gamma \Vdash_\pi B$. ■

2.4 Local Transitive Closure

In this short section, we introduce the notion of local transitive closure of $\prec$, relative to a base set X. This will turn out to be useful in the remainder, when defining certain variants of (multi-)selection functions. Let us start with some definitions. First, we define the (global) transitive closure of $\prec$ in the standard way, as follows: $x \prec^{\mathsf{tr}} y$ iff there are $z_1, \ldots, z_n$ such that $x \prec z_1 \preceq \ldots \preceq z_n \preceq y$.[29] The local transitive closure, relative to a set Z, is obtained in a similar way, but restricting the scope to those $z_1, \ldots, z_n$ that are in Z. Formally:

Definition 4 *$x \prec^{\mathsf{tr}}_Z y$ iff $x, y \in Z$ and $x \prec y$ or there are $z_1, \ldots, z_n \in Z$ such that $x \prec z_1 \ldots z_n \prec y$.*

We sometimes write that x is *below* y in Z to indicate that $x \prec^{\mathsf{tr}}_Z y$. In line with the preceding, we let $x \preceq^{\mathsf{tr}}_Z y$ iff $x \prec^{\mathsf{tr}}_Z y$ or $(x, y \in Z$ and $x = y)$. The following facts highlight some properties of $\prec^{\mathsf{tr}}_Z$:

Fact 4 *Each of the following hold:*

1. *If ($x \prec y$ and $x, y \in Z$), then $x \prec^{\mathsf{tr}}_Z y$.*
2. *If $x \prec^{\mathsf{tr}}_Z x'$ and $x' \prec^{\mathsf{tr}}_Z x''$, then $x \prec^{\mathsf{tr}}_Z x''$.*
3. *Let $\prec$ be transitive. Then $x \preceq^{\mathsf{tr}}_Z y$ iff ($x, y \in Z$ and $x \preceq y$).*
4. *Where $Z \subseteq Z'$: if $x \prec^{\mathsf{tr}}_Z y$, then $x \prec^{\mathsf{tr}}_{Z'} y$.*

Fact 5 $\mathsf{min}_{\prec^{\mathsf{tr}}_X}(X) = \mathsf{min}_\prec(X)$.

It is fairly obvious that $\prec^{\mathsf{tr}}_X$ is not always identical to $\prec^{\mathsf{tr}}$. More surprisingly, the difference between both also has an impact on the various multi-selection semantics defined from them. That is, replacing $\prec$ with $\prec^{\mathsf{tr}}_X$ in the definition of a given function π, does not give us the same result as replacing $\prec$ with $\prec^{\mathsf{tr}}$. This will be illustrated throughout the paper.

We end this section with a property that is perhaps less obvious, although its proof turns out to be fairly straightforward:

[28] This proof and the next one are based on those for the two directions of [17, Fact 3.4.5].

[29] As before, we let $x \preceq^{\mathsf{tr}} y$ iff $x \prec^{\mathsf{tr}} y$ or $x = y$.

Lemma 7 *Y is $\prec$-lower in X iff Y is $\prec_X^{\mathsf{tr}}$-lower in X iff Y is $\prec^{\mathsf{tr}}$-lower in X.*

Proof. We only prove the first equivalence. To show that Y is $\prec$-lower in X iff Y is $\prec^{\mathsf{tr}}$ X-lower in X, it suffices to everywhere replace $\prec_X^{\mathsf{tr}}$ with $\prec^{\mathsf{tr}}$ in our proof.

($\Leftarrow$) Suppose Y is not $\prec$-lower in X. So there is a $y \in Y$, $x \in X - Y$: $x \prec y$. Hence also $x \prec_X^{\mathsf{tr}} y$. It follows that Y is not $\prec_X^{\mathsf{tr}}$-lower in X.

($\Rightarrow$) Suppose Y is $\prec$-lower in X; let $x \in X$ and $y \in Y$ be such that $x \prec_X^{\mathsf{tr}} y$. Hence there are $z_1, \ldots, z_n \in X$ ($n \geq 0$) such that $x \prec z_1 \prec \ldots \prec z_n \prec y$. If $n > 0$, then by the supposition, (1) $z_n \in Y$. If $n > 1$, then by (1) and the supposition, also $z_{n-1} \in Y$. Applying this reasoning n times, we can derive that $z_1 \in Y$, and hence by the supposition, also $x \in Y$. ■

3 Selection Semantics: Safe Selection

In this section, we take a closer look at three specific consequence relations based on a selection semantics. We start with the definition of what we call safe selections, after which we consider two variants that behave differently in cases where $\prec$ is non-transitive. We end with a general problem for the approach in terms of selection functions in the absense of smoothness, which motivates the more general framework of multi-selection semantics from the next section.

Recall that, as mentioned in the previous section, all consequence relations obtained from a selection semantics satisfy left and right absorption and inclusion. Hence in the current section, we will not discuss those properties any further.

3.1 Safe Selections

The Definition In the conclusion of his [2], Batens mentions a straightforward solution for preferential entailment whenever $\langle \mathcal{M}(\Gamma), \prec \rangle$ is not smooth. However, he does not discuss the metatheoretic properties of this variant of **S**, as his main concern is with proving that his own systems warrant smoothness anyway. We will use the term *safe selection* for the selection function used in Batens' proposal.

The idea behind safe selections can be explained as follows. When selecting a set $\mathcal{M} \subseteq \mathcal{M}(\Gamma)$, we should be able to justify, for each $M \in \mathcal{M}(\Gamma) - \mathcal{M}$, why this model is "discarded". That there is an $M' \in \mathcal{M}(\Gamma)$ for which $M' \prec M$ is in itself not a sufficient reason for ignoring M, since it may well be that M' is itself "beaten" by yet another model $M'' \in \mathcal{M}(\Gamma)$. According to the safe selection, M' has to be $\prec$-*minimal* in $\mathcal{M}(\Gamma)$, in order to justify the fact that M is not selected.

Formally, we have:

Definition 5 $\Psi^0(X) = \{x \in X \mid$ *for no* $y \in \mathsf{min}_\prec(X), y \prec x\}$.

Following Definition 1, we define $\Gamma \Vdash_{\Psi^0} A$ ($A \in Cn_{\Psi^0}(\Gamma)$) iff A is true in every $M \in \Psi^0(\mathcal{M}(\Gamma))$.

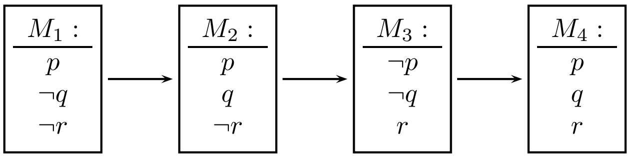

Figure 1: Models of Γ_{a}.

Metatheoretic properties of $\Vdash_{\Psi^0}$ By item 3 of Lemma 2, the following lemma suffices to obtain right satisfiability for this variant:

Lemma 8 *If $X \neq \emptyset$, then $\Psi^0(X) \neq \emptyset$.*

Proof. Suppose $X \neq \emptyset$. Case 1: $\mathsf{min}_\prec(X) = \emptyset$. Then $\Psi^0(X) = X$. Case 2: $\mathsf{min}_\prec(X) \neq \emptyset$. Let $x \in \mathsf{min}_\prec(X)$. Obviously, there is no $y \in \mathsf{min}_\prec(X)$ such that $y \prec x$. Hence $x \in \Psi^0(X)$. ■

More generally, $\Psi^0(X)$ is $\prec$-dense in X, as only those $x \in X$ are discarded for which there is a $y \in \mathsf{min}_\prec(X)$ such that $y \prec x$. However, $\Psi^0(X)$ is not in general $\prec$-lower in X – we return to this point at the end of this section.

Let us now consider cautious monotonicity. In view of Lemma 2.1, it suffices to prove the following:

Lemma 9 *If $\Psi^0(X) \subseteq Y \subseteq X$, then $\Psi^0(Y) \subseteq \Psi^0(X)$.*

Proof. Suppose the antecedent holds and $x \notin \Psi^0(X)$. Hence, there is a $y \in \mathsf{min}_\prec(X)$ with $y \prec x$. Note that $y \in \Psi^0(X)$ and hence by the supposition, $y \in Y$. So $y \in Y \cap \mathsf{min}_\prec(X)$. By Fact 1, $y \in \mathsf{min}_\prec(Y)$. Hence, also $x \notin \Psi^0(Y)$. ■

Corollary 3 *Cn_{Ψ^0} is cautiously monotonic.*

It can be shown that whenever $\prec$ is transitive, then Cn_{Ψ^0} is also cumulatively transitive. This is an immediate corollary of Fact 6.4 and Theorem 2 in Section 3.2.

The failure of Cumulative Transitivity in the more general case can be shown by means of a four-model example. We only use three propositional variables in the language of **CL**. Let $\Gamma_{\mathsf{a}} = \{p \vee \neg q, p \vee r, \neg p \vee \neg r \vee q\}$. Figure 1 represents the four models of this premise set, and the preference order on them.

Note that $\Psi^0(\mathcal{M}(\Gamma_{\mathsf{a}})) = \{M_1, M_2, M_4\}$. Hence $p \in Cn_{\Psi^0}(\Gamma_{\mathsf{a}})$.

Consider now $\mathcal{M}(\Gamma_{\mathsf{a}} \cup \{p\}) = \{M_1, M_2, M_4\}$. Note that $\Psi^0(\mathcal{M}(\Gamma_{\mathsf{a}} \cup \{p\})) = \{M_2, M_4\}$ – this time M_1 is removed, since M_2 is $\prec$-minimal in the smaller set of models. As a result, $q \in Cn_{\Psi^0}(\Gamma_{\mathsf{a}} \cup \{p\})$.[30]

The disjunction property fails for Cn_{Ψ^0} even in cases where $\prec$ is transitive. We will show by means of a concrete example that the deduction theorem does not hold for systems Ψ^0 that are based on **CL**; in view of Lemma 1.3 and the fact that Cn_{Ψ^0} satisfies left and right absorption, it follows immediately that also the disjunction property fails.

As before, we let $\mathbf{L} = \mathbf{CL}$, but this time working on the basis of the infinite letter set $\{p_i \mid i \in \mathbb{N}\} \cup \{q, r\}$. Let $\Gamma_{\mathsf{b}} = \{p_i \vee p_j \vee q \mid i, j \in \mathbb{N}, i \neq j\}$. We can distinguish between various subsets of $\mathcal{M}(\Gamma_{\mathsf{b}})$:

[30]In fact, this example shows that not only cumulative transitivity fails, but also the weaker fixed point property. We leave the verification of this to the reader.

$$\boxed{\mathcal{M}_{q,\neg r}} \Longrightarrow \boxed{\mathcal{M}_{q,r}} \Longrightarrow \boxed{\mathcal{M}_0} \Longrightarrow \boxed{\mathcal{M}_1} \Longrightarrow \boxed{\mathcal{M}_2} \Longrightarrow \ldots$$

Figure 2: Models of Γ_{b}. $\mathcal{M} \Rightarrow \mathcal{M}'$ should be read as: for every $M \in \mathcal{M}$ and every $M' \in \mathcal{M}'$, $M' \prec M$.

$$\begin{array}{l}
\mathcal{M}_{q,\neg r} = \{M \in \mathcal{M}(\Gamma_{\mathsf{b}}) \mid M \models q, \neg r\} \\
\mathcal{M}_{q,r} = \{M \in \mathcal{M}(\Gamma_{\mathsf{b}}) \mid M \models q, r\} \\
\mathcal{M}_0 = \{M \in \mathcal{M}(\Gamma_{\mathsf{b}}) \mid M \models \neg q, p_i \text{ for all } i \in \mathbb{N}\} \\
\mathcal{M}_1 = \{M \in \mathcal{M}(\Gamma_{\mathsf{b}}) \mid M \models \neg q, \neg p_1, p_i \text{ for all } i \in \mathbb{N} - \{1\}\} \\
\mathcal{M}_2 = \{M \in \mathcal{M}(\Gamma_{\mathsf{b}}) \mid M \models \neg q, \neg p_2, p_i \text{ for all } i \in \mathbb{N} - \{2\}\} \\
\vdots
\end{array}$$

Figure 2 represents a (transitive and modular) partial order $\prec$ on $\mathcal{M}(\Gamma_{\mathsf{b}})$.

Note that $\mathcal{M}(\Gamma_{\mathsf{b}} \cup \{q\}) = \mathcal{M}_{q,\neg r} \cup \mathcal{M}_{qr}$. As a result, $r \in Cn_{\Pi^0}(\Gamma_{\mathsf{b}} \cup \{q\})$. However, there are models $M \in \Psi^{\prec}(\mathcal{M}(\Gamma_{\mathsf{b}}))$ with $M \models q, \neg r$, viz. all models $M \in \mathcal{M}_{q,\neg r}$. These models cannot be discarded, since there are no $\prec$-minimal models below them. As a result, $\Gamma \not\Vdash_{\Psi^0} q \supset r$.

Towards the Variants We have seen in the preceding that in non-transitive cases, Cn_{Ψ^0} is not cumulatively transitive. This in itself motivates the search for variants that allow us to preserve cumulative transitivity yet lead to non-trivial consequence sets in the absence of smoothness. However, there is also a more fundamental problem with Ψ^0, viz. that it does not warrant $\prec$-lowerness unless $\prec$ is transitive.[31]

Consider again the example depicted in Figure 1. Recall that $\Psi^0(\mathcal{M}(\Gamma_{\mathsf{a}})) = \{M_1, M_2, M_4\}$. Let us now focus on M_2 and M_3. On the one hand, M_2 is selected, since it is not beaten by any $\prec$-minimal model (i.e. by M_1). On the other hand, M_3 is *not* selected, notwithstanding the fact that $M_3 \prec M_2$. This may strike some as counterintuitive: if M_2 cannot be ignored, then why is it "safe" to ignore M_3? Or alternatively, if we can ignore M_3, then why not do the same with the even less preferred model M_2?

At least intuitively, it seems that the requirement that for any selection function ψ that is based on $\prec$, $\psi(X)$ should be $\prec$-lower in X, makes perfect sense.

In Sections 3.2 and 3.3, we will consider two different ways to warrant $\prec$-lowerness in non-transitive cases, resulting in two alternative selection functions Ψ^1 and Ψ^2. The first of the two yields a *smaller* selection, i.e. it only picks the last element from every smooth sequence in $\langle X, \prec\rangle$. So e.g. we will have $\Psi^1(\mathcal{M}(\Gamma_{\mathsf{a}})) = \{M_4\}$. This is done by defining the selection in terms of the local transitive closure of $\prec$ in X, rather than in terms of $\prec$ itself. Hence, the resulting consequence operation is stronger. As will be shown, this variant is cumulatively transitive, but this time we loose cautious monotonicity.

The second variant yields a *larger* selection than Ψ^0, viz. by extending $\Psi^0(X)$ in such a way that we obtain a $\prec$-lower superset of it. So e.g. we have

[31] If $\prec$ is transitive, we can easily show that $\Psi^0(X)$ is always $\prec$-lower in X. That is, assume that (i) $x \in \Psi^0(X)$, (ii) $y \in X - \Psi^0(X)$ and (iii) $y \prec x$. Then by (ii), there is a $z \in \mathsf{min}_{\prec}(X)$ with $z \prec y$. However, by (iii) and the transitivity of $\prec$, also $z \prec x$, and hence $x \notin \Psi^0(X)$, contradicting (i).

$\Psi^2(\mathcal{M}(\Gamma_{\mathsf{a}})) = \{M_1, M_2, M_3, M_4\}$. As a result, $\Vdash_{\Psi^2}$ is weaker than $\Vdash_{\Psi^0}$ and $\Vdash_{\Psi^1}$, yet it preserves both cautious monotonicity and cumulative transitivity.

3.2 Variant 1: Local Transitive Closure

The idea behind the first variant of Ψ^0 is to select a subset of $\Psi^0(X)$ which is $\prec$-lower. When doing so, we try to keep as many elements from $\Psi^0(X)$ as possible. To see how this can be done, consider first an arbitrary selection function $\psi : \wp(X) \to \wp(X)$. Then we may define:

Definition 6 $\hat{\psi}(X) = \{x \in \psi(X) \mid$ *for no* $y \in X - \psi(X), y \prec^{\mathsf{tr}}_X x\}$.

So whenever ψ does not select a certain $x \in X$, then all y that are above x are also ignored by $\hat{\psi}$. Put differently, whenever $\psi(X)$ contains a "gap" in view of $\langle X, \prec\rangle$, then $\hat{\psi}$ removes all elements that are above this gap. The following lemma shows that this is the most conservative way to obtain a $\prec$-lower subset of $\psi(X)$:

Lemma 10 *$\hat{\psi}(X)$ is the unique biggest set $Y \subseteq \psi(X)$ such that Y is $\prec$-lower in X.*

Proof. We first prove that $\hat{\psi}(X)$ is $\prec$-lower in X. So suppose $x \in \hat{\psi}(X)$, $y \in X$ and $y \prec x$. Since $\hat{\psi}(X) \subseteq X$ it follows that $y \prec^{\mathsf{tr}}_X x$. Assume that $y \notin \hat{\psi}(X)$. Hence there is a $z \in X - \psi(X)$ such that $z \prec^{\mathsf{tr}}_X y$. So also $z \prec^{\mathsf{tr}}_X x$. But then $x \notin \hat{\psi}(X)$ — a contradiction.

Assume now that (i) $Z \subseteq \psi(X)$, (ii) Z is $\prec$-lower in X, and (iii) $Z \not\subseteq \hat{\psi}(X)$. Let $x \in Z - \hat{\psi}(X)$. By (ii), $x \in \psi(X)$. Since $x \notin \hat{\psi}(X)$, there is a $y \in X - \psi(X)$ such that $y \prec^{\mathsf{tr}}_X x$. Hence there are $z_1, \ldots, z_n \in X$ ($n \geq 1$) such that $y \preceq z_1 \prec \ldots \prec z_n \prec x$. But then, by an obvious induction and (ii), each of $z_n, \ldots, z_1, y$ are also in Z. However, that $y \in Z$ contradicts the fact that $y \in X - \psi(X)$ and $Z \subseteq \psi(X)$. ■

So the idea behind the first variant of Cn_{Ψ^0} is to use the selection function $\hat{\Psi}_0$ instead of Ψ^0. However, for pragmatic reasons, we shall use the following, slightly more reader-friendly definition:

Definition 7 $\Psi^1(X) = \{x \in X \mid$ *for no* $y \in \mathsf{min}_\prec(X), y \prec^{\mathsf{tr}}_X x\}$.

Lemma 11 *$\Psi^1(X) = \hat{\Psi}_0(X)$.*

Proof. "$\subseteq$" Suppose that $x \in \Psi^1(X) - \hat{\Psi}_0(X)$. So there is a $y \in X - \Psi^0(X)$ such that $y \prec^{\mathsf{tr}}_X x$. It follows that, for a $z \in \mathsf{min}_\prec(X)$, $z \prec y$. Hence also $z \prec^{\mathsf{tr}}_X x$. But then $x \notin \Psi^1(X)$ — a contradiction.

"$\supseteq$" Suppose that $x \in \hat{\Psi}_0(X) - \Psi^1(X)$. So there is a $y \in \mathsf{min}_\prec(X)$ such that $y \prec^{\mathsf{tr}}_X x$. Note that by the supposition, $x \in \Psi^0(X)$. So $y \not\prec x$. Hence there are $z_1, \ldots, z_n$ ($n \geq 1$) such that $y \prec z_1 \prec \ldots \prec z_n \prec x$. Note that $z_1 \notin \Psi^0(X)$ and $z_1 \prec^{\mathsf{tr}}_X x$. But then $x \notin \hat{\Psi}_0(X)$ — a contradiction. ■

Lemma 12 *Each of the following holds:*

1. *$\Psi^1(X)$ is $\prec$-lower in X.*

2. $\Psi^1(X)$ *is* $\prec^{\mathsf{tr}}_X$*-lower in* X*.*

Proof. Ad 1. Immediate in view of Lemmas 10 and 11. *Ad 2.* Immediate in view of item 1 and Lemma 7. ■

The consequence relation $\Vdash_{\Psi^1}$ is obtained from Ψ^1 in the usual way – see Definition 1. The following fact summarizes the relation between $\Vdash_{\Psi^1}$ and $\Vdash_{\Psi^0}$:

Fact 6 *Each of the following holds:*

1. $\Psi^1(X) \subseteq \Psi^0(X)$.
2. $\Vdash_{\Psi^0} \ \subseteq \ \Vdash_{\Psi^1}$.
3. *If* $\prec$ *is transitive, then* $\Psi^1(X) = \Psi^0(X)$.
4. *If* $\prec$ *is transitive, then* $\Vdash_{\Psi^0} \ = \ \Vdash_{\Psi^1}$.

It can easily be checked that Cn_{Ψ^1} is often stronger than Cn_{Ψ^0}. For instance, consider again the example Γ_{a} from page 15 (see also Figure 1). Here $\Psi^0(\mathcal{M}(\Gamma_{\mathsf{a}})) = \{M_1, M_2, M_4\}$ whereas $\Psi^1(\mathcal{M}(\Gamma_{\mathsf{a}})) = \{M_4\}$. As a result, $\Gamma_{\mathsf{a}} \not\Vdash_{\Psi^0} q$ whereas $\Gamma_{\mathsf{a}} \Vdash_{\Psi^1} q$.

Metatheoretic Properties Right satisfiability is immediate, as $\Psi^1(X)$ is $\prec^{\mathsf{tr}}_X$-dense in X and by item 3 of Lemma 2.

In view of item 4 of Fact 6, the disjunction property fails for Cn_{Ψ^1}, since it fails for Cn_{Ψ^0} even in the transitive case.

On the positive side, Cn_{Ψ^1} is in general cumulatively transitive. By Lemma 2.2, it suffices to prove the following:

Lemma 13 *Where* $\Psi^1(X) \subseteq Y \subseteq X$*:* $\Psi^1(X) \subseteq \Psi^1(Y)$*.*

Proof. Suppose the antecedent holds, and let $x \in \Psi^1(X)$. Assume that $x \notin \Psi^1(Y)$. Hence, there is a $y \in \min_\prec(Y)$ such that $y \prec^{\mathsf{tr}}_Y x$, and hence also $y \prec^{\mathsf{tr}}_X x$. Since $x \in \Psi^1(X)$, we can derive that $y \notin \min_\prec(X)$, and hence there is a $z \in X$ with $z \prec y$. It follows that $z \prec^{\mathsf{tr}}_X x$, and hence by Lemma 12.2, $z \in \Psi^1(X)$. But then by the supposition, $z \in Y$, and hence $y \notin \min_\prec(Y)$ — a contradiction. ■

By Lemma 2.2, we immediately have:

Theorem 2 $\Vdash_{\Psi^1}$ *is cumulatively transitive.*

So we have obtained a variant of $\Vdash_{\Psi^0}$ which warrants cumulative transitivity. However, this time we loose cautious monotonicity. This follows from the same example as the one we used to disprove the cumulative transitivity of Cn_{Ψ^0}, viz. the premise set Γ_{a} on page 15, in combination with the preference relation depicted in Figure 1.

Note first that $\Psi^1(\mathcal{M}(\Gamma_{\mathsf{a}}) = \{M_4\}$. Hence $p, q, r \in Cn_{\Psi^1}(\Gamma_{\mathsf{a}})$. However, consider now $\Gamma_{\mathsf{a}} \cup \{p\}$. Note that M_3 is not a model of this premise set. Moreover, we have $M_4 \not\prec^{\mathsf{tr}}_{\{M_1,M_2,M_4\}} M_2$ – we really need the model M_3 to have a link between M_4 and M_2. It follows that $M_2 \in \Psi^1(\mathcal{M}(\Gamma_{\mathsf{a}} \cup \{p\}))$. Since $M_2 \not\models r$, also $r \notin Cn_{\Pi^1}(\Gamma_{\mathsf{a}} \cup \{p\})$.

This seems to point at a general feature of preferential consequence relations that use the transitive closure of $\prec$ relative to $\mathcal{M}(\Gamma)$. That is, if we add certain consequences to our premise set, then we restrict the range of connections that can be drawn between different models, and hence we might no longer be able to argue for other consequences of the initial premise set. In our case, Ψ^1 will remove all $x \in X$ such that, for a $y \in \mathsf{min}_\prec(X)$, $y \prec^{\mathsf{tr}}_X x$. If we now remove some such x, then we cut through the connections between other x' for which $y \prec^{\mathsf{tr}}_X x'$. As a result, we may have $\Psi^1(X) \subseteq Y \subset X$, yet $\Psi^1(Y) \not\subseteq \Psi^1(X)$.

3.3 Variant 2: $\prec$-Lower Extension of Ψ^0

We now turn to the second strategy to ensure that $\psi(X)$ is $\prec$-lower. This strategy boils down to taking all $x \in \Psi^0(X)$, but enforcing that whenever $y \in X$ is below a selected $z \in X$, then y is itself selected as well. Before we turn to the selection function Ψ^2, we again show how such a variant can be obtained from any arbitrary selection function $\psi : \wp(X) \to \wp(X)$.

Definition 8 $\breve{\psi}(X) =_{\mathsf{df}} \{x \in X \mid$ *there is a* $y \in \psi(X) : x \preceq^{\mathsf{tr}}_X y\}$.

Lemma 14 *$\breve{\psi}(X)$ is the unique smallest set $Y \subseteq X$ such that $\psi(X) \subseteq Y$ and Y is $\prec$-lower in X.*

Proof. We first prove that $\breve{\psi}$ is $\prec$-lower in X. So suppose that $x \in \breve{\psi}(X)$ and $y \in X$ with $y \prec x$. It follows that there is a $z \in \psi(X)$ such that $x \preceq^{\mathsf{tr}}_X z$. Hence also $y \prec^{\mathsf{tr}}_X z$ and hence $y \in \breve{\psi}(X)$.

Assume now that Z is such that (i) $\breve{\psi}(X) \not\subseteq Z$, (ii) $\psi(X) \subseteq Z$ and (iii) Z is $\prec$-lower in X. Let $x \in \breve{\psi}(X) - Z$. Let $y \in \psi(X)$ be such that $x \preceq^{\mathsf{tr}}_X y$. So there are $z_1, \ldots, z_n \in X$ such that $x \preceq z_1 \prec \ldots \prec z_n \prec y$. By (ii), $y \in Z$. Hence by (iii) and a straightforward mathematical induction, each of $z_n, \ldots, z_1 \in Z$ and also $x \in Z$ — a contradiction. ■

So $\breve{\psi}$ turns out to be something like the dual of $\hat{\psi}$, for any given selection function ψ. Accordingly, we can obtain a "dual" of Ψ^1:

Definition 9 $\Psi^2(X) = \{x \in X \mid$ *there is a* $y \in \Psi^0(X) : x \preceq^{\mathsf{tr}}_X y\}$.

Corollary 4 *$\Psi^2(X)$ is $\prec^{\mathsf{tr}}_X$-lower in X.*

In view of Lemma 14, Ψ^2 is in fact the *strongest* selection function ψ such that for all X, $\Psi^0(X) \subseteq \psi(X)$ and $\psi(X)$ is $\prec$-lower in X. Also, since $\Psi^0(X) \subseteq \Psi^2(X) \subseteq X$, and since $\Psi^0(X)$ is $\prec$-dense in X, we have:

Fact 7 *Ψ^2 is $\prec$-dense in X.*

Fact 8 summarizes the relation between $\Vdash_{\Psi^0}$ and $\Vdash_{\Psi^2}$:[32]

Fact 8 *Each of the following holds:*

1. $\Psi^0(X) \subseteq \Psi^2(X)$.
2. $\Vdash_{\Psi^2}\ \subseteq\ \Vdash_{\Psi^0}$.
3. *If $\prec$ is transitive, then* $\Psi^0(X) = \Psi^2(X)$.
4. *If $\prec$ is transitive, then* $\Vdash_{\Psi^2}\ =\ \Vdash_{\Psi^0}$.

[32] Perhaps item 3 of this fact is not that immediate. To see why it holds, suppose $\prec$ is transitive, and $x \in \Psi^2(X) - \Psi^0(X)$. Hence (i) x is below a $y \in \Psi^0(X)$, but (ii) there is a $z \in \mathsf{min}_\prec(X)$ such that $z \prec x$. By transitivity, $z \prec y$ as well, and hence $y \notin \Psi^0(X)$ — a contradiction.

Metatheoretic Properties Right satisfiability follows by Fact 7 and item 3 of Lemma 2. Also, we can infer from Fact 8.4 that $\Vdash_{\Psi^2}$ does not have the disjunction property, since this property fails for $\Vdash_{\Psi^0}$ in the transitive case. For the proof of cumulativity, we rely on items 1 and 2 of Lemma 2:

Lemma 15 *If $\Psi^2(X) \subseteq Y \subseteq X$, then $\Psi^2(X) = \Psi^2(Y)$.*

Proof. Suppose the antecedent holds.

"$\subseteq$". Assume $x \in \Psi^2(X) - \Psi^2(Y)$. Let $y \in \Psi^0(X)$ be such that $x \preceq_X^{\mathsf{tr}} y$. Let $z_1, \ldots, z_n \in X$ be such that $x \prec z_1 \preceq \ldots \preceq z_n \preceq y$. Note that $y \in \Psi^2(X)$ by Fact 8.1. Hence, by Corollary 4, we obtain that each of $z_n, \ldots, z_1, x$ are in $\Psi^2(X)$. So by the supposition, $z_n, \ldots, z_1 \in Y$, and hence also $x \preceq_Y^{\mathsf{tr}} y$. Since $x \notin \Psi^2(Y)$, it follows that $y \notin \Psi^0(Y)$. Hence there is a $z \in \mathsf{min}_\prec(Y)$ such that $z \prec y$.

Case 1: $z \in \mathsf{min}_\prec(X)$. Then, since $z \prec y$, $y \notin \Psi^0(X)$ – a contradiction.

Case 2: $z \notin \mathsf{min}_\prec(X)$. Let $v \in X$ be such that $v \prec z$. Note that, since $z \prec y$, $v \prec_X^{\mathsf{tr}} y$. Hence, since $y \in \Psi^0(X)$, it follows that $v \in \Psi^2(X)$. But then also $v \in Y$ by the supposition, and hence $z \notin \mathsf{min}_\prec(Y)$ — a contradiction.

"'$\supseteq$". Note first that, by the supposition, Fact 8.1, and Lemma 9, (†) $\Psi^0(Y) \subseteq \Psi^0(X)$. Suppose $x \in \Psi^2(Y)$. So there is a $y \in \Psi^0(Y)$ such that $x \preceq_Y^{\mathsf{tr}} y$, hence also $x \preceq_X^{\mathsf{tr}} y$. By (†), also $y \in \Psi^0(X)$. It follows immediately that $x \in \Psi^2(X)$. ∎

Corollary 5 *Cn_{Ψ^2} is cumulatively transitive and cautiously monotonic.*

3.4 Formulas that hold in the limit

We have seen in the preceding that $\Vdash_{\Psi^2}$ has a rather strong metatheory, and hence so has $\Vdash_{\Psi^0}$ in the transitive case. $\Vdash_{\Psi^1}$ is not cautiously monotonic, but it is significantly stronger than $\Vdash_{\Psi^0}$ in the non-transitive case where some sequences of models in $\mathcal{M}(\Gamma)$ have a least element.

Nevertheless, at the intuitive level, these three consequence relations sometimes seem rather weak in the absence of smoothness. Let us briefly explain why this is so.

For the present discussion, we can let $\prec$ be transitive, so that the three safe selection-variants coincide. Let $\mathcal{M}(\Gamma) = \{M_i \mid i \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}\}$ and let $M_j \prec M_i$ whenever $j > i$. Moreover, let A be such that for all $j \geq 0$, $M_i \models p$. Since $\mathsf{min}_\prec(\mathcal{M}(\Gamma)) = \emptyset$, we can derive that $\Psi^0(\mathcal{M}(\Gamma)) = \mathcal{M}(\Gamma)$. As a result, $\Gamma \not\Vdash_{\Psi^0} p$.

In view of this example, one might claim that $\Vdash_{\Psi^0}$ and its variants are too weak to deal with non-smooth preferential systems. That is, from a certain point in the sequence $\langle \ldots, M_{-1}, M_0, M_1, \ldots \rangle$ on, all models verify p. So even if the sequence has no last element, does this not indicate that p is "normally" true whenever Γ is? And hence, should we not define the preferential consequence relation in such a way that Γ preferentially entails p?

However, to take this into account, yet still obtain a non-trivial consequence set, seems hard to do by means of a selection semantics. That is, the structure $\langle \mathcal{M}(\Gamma), \prec \rangle$ is symmetric in all the elements M_i. So it seems that if $\psi(\mathcal{M}(\Gamma))$ is a function of $\prec$ and $\mathcal{M}(\Gamma)$, then either all these models should be selected, or none. To the best of our knowledge, no selection semantics has been proposed that allows us to obtain p in this and similar cases.[33] As will become clear in the next section, this problem is solved by moving to the more general framework of multi-selection semantics.

4 Multiselection Semantics: The Limit Variant

In this section, we introduce the Limit Variant from [17], showing it to have a rather weak metatheory in cases where $\prec$ is not transitive (Section 4.1). This motivates our discussion of two variants: one obtained by letting $\prec^{\mathrm{tr}}_{\mathcal{M}(\Gamma)}$ play the role of $\prec$ in the original Limit Variant (Section 4.2) and another obtained by imposing an additional criterion on the sets that are selected (Section 4.3).

4.1 The Limit Variant

The Definition We will now focus on (variations of) one particular multi-selection function Π^0, which was originally defined by Bossu & Siegel [3] in the context of circumscription logic. More recently, their idea has been generalized by Schlechta [17], giving rise to his so-called *Limit Variant*.

In Schlechta's work, the sets $\mathcal{M} \in \Pi^0(\mathcal{M}(\Gamma))$ are called *minimizing initial segments* of $\mathcal{M}(\Gamma)$. In our terminology they are defined as follows:

Definition 10 $\Pi^0(X) = \{Y \subseteq X \mid Y$ *is* $\prec$ *-lower and* $\prec$ *-dense in* $X\}$.

Note that Π^0 is a multi-selection function. Hence we can obtain a preferential consequence relation $\Vdash_{\Pi^0}$ from it, by letting $\Gamma \Vdash_{\Pi^0} A$ iff there is a $\mathcal{M} \in \Pi^0(\mathcal{M}(\Gamma))$, such that A is true in every $M \in \mathcal{M}$.

Let us briefly return to the example from the preceding section that motivated our shift towards multi-selection semantics. Recall that we had $\mathcal{M}(\Gamma) = \{\ldots, M_{-1}, M_0, M_1, \ldots\}$, and $M_j \prec M_i$ whenever $j > i$. Also, we assumed that for all M_i with $i \geq 0$, $M_i \models p$. It can be easily verified that

$$\Pi^0(\mathcal{M}(\Gamma)) = \{\mathcal{M}_i \mid i \in \{\ldots, -1, 0, 1, \ldots\}\}$$

Where each $\mathcal{M}_i = \{M_j \mid j \geq i\}$. Since, for instance, every $M \in \mathcal{M}_0$ verifies p, it follows that $\Gamma \Vdash_{\Pi^0} p$.

Note that in this example, $\bigcap \Pi^0(\mathcal{M}(\Gamma)) = \emptyset$. Nevertheless, each $\mathcal{M}_i \in \Pi^0(\mathcal{M}(\Gamma))$ is necessarily non-empty, in view of the $\prec$-density of each selected set of models in $\mathcal{M}(\Gamma)$. This holds for all the multi-selection functions which we shall consider in this paper, even if we restrict ourselves to the transitive case. As a result, the proofs of right satisfiability and right absorption will always rely on the assumption of compactness – see also Lemmas 3 and 4 from Section 2.

[33] Of course, one could define ψ in an ad hoc manner, e.g. by letting $M \in \psi(\mathcal{M}(\Gamma))$ iff $M \in \mathcal{M}(\Gamma)$ and $M \models A$ for all $A \in Cn_{\mathbf{X}}(\Gamma)$, where $\mathbf{X}$ is one of the systems introduced in the next section. However, even if it would turn out to be well-behaved or even equivalent to $\mathbf{X}$, the resulting system can hardly be called insightful, and we would need a multi-selection semantics to provide more insight into it. so our point here does not concern mathematical possibility, but rather mathematical elegance.

$$\begin{tikzcd} \boxed{M_1 : p, \neg q, r} \arrow[r] \arrow[d] & \boxed{M_3 : \neg p, q, r} \arrow[d] \\ \boxed{M_2 : p, \neg q, \neg r} \arrow[r] & \boxed{M_4 : p, q, r} \end{tikzcd}$$

Figure 3: Models of Γ_{c}

Metatheory of $\Vdash_{\Pi^0}$ We start with right satisfiability.

Lemma 16 Π^0 *satisfies* **RS**.[34]

Proof. Let $X \neq \emptyset$ and $Y_1, \ldots, Y_n \in \Pi^0(X)$. We prove by an induction that $\bigcap_{1 \leq i \leq j} Y_i \neq \emptyset$ for all $j \leq n$.

"$j = 1$" Let $x \in X$. Since Y_1 is $\prec$-dense in X, there is a $y \in Y_1$ such that $y \preceq x$.

"$j \Rightarrow j + 1$" Let $x \in Y_1 \cap \ldots \cap Y_j$ (this x exists in view of the induction hypothesis). Since $x \in X$ and since Y_{j+1} is $\prec$-dense in X, there is a $y \in Y_{j+1}$ such that $y \preceq x$. Since each Y_k ($k \leq n$) is $\prec$-lower in X and since $x \in Y_l$ for all $l \leq j$, also $y \in Y_l$ for all $l \leq j$. Hence $y \in Y_1 \cap \ldots \cap Y_{j+1}$. ■

By Lemma 3, we have:

Corollary 6 *If* **L** *is compact, then Cn_{Π^0} satisfies right satisfiability.*

When $\prec$ is not transitive, right *absorption*, cumulative transitivity and cautious monotonicity may sometimes fail for the Limit Variant, even in finite, simple cases. This was already noted before – see e.g. [16, Section 3.2]. We will illustrate each of these facts by means of one simple example that is similar to those presented by Schlechta.

Suppose our language only contains the letters p, q, r. Let $\mathbf{L} = \mathbf{CL}$ and $\Gamma_{\mathsf{c}} = \{p \vee q, \neg q \vee r\}$. This premise set has only four models, all of which are represented in Figure 3. As the arrows indicate, $M_4 \prec M_3, M_2$ and $M_2, M_3 \prec M_1$, but $M_4 \not\prec M_1$ (so $\prec$ is not transitive).

It can be easily verified that $\{M_2, M_4\}$ and $\{M_3, M_4\}$ are both $\prec$-dense and $\prec$-lower subsets of $\mathcal{M}(\Gamma_{\mathsf{c}})$. Hence both p and q are in $Cn_{\Pi^0}(\Gamma_{\mathsf{c}})$. However, $p \wedge q \notin Cn_{\Pi^0}(\Gamma_{\mathsf{c}})$, since M_4 is not a $\prec$-dense subset of $\{M_1, \ldots, M_4\}$ (as $M_4 \not\prec M_1$). So $Cn_{\Pi^0}(\Gamma_{\mathsf{c}})$ is not closed under **CL**. From this it follows immediately that $\Vdash_{\Pi^0}$ is also not cumulatively transitive in this case. That is, by left absorption, $p \wedge q \in Cn_{\Pi^0}(\Gamma_{\mathsf{c}} \cup \{p, q\})$, and hence adding p and q to Γ_{c} results in *more* consequences.

To see why cautious monotonicity fails for Cn_{Π^0}, consider $\Gamma_{\mathsf{c}} \cup \{p\}$. This set has only three models, viz. M_1, M_2 and M_4. There are two $\prec$-dense and $\prec$-lower subsets of $\mathcal{M}(\Gamma_{\mathsf{c}} \cup \{p\})$, viz. $\{M_2, M_4\}$ and $\{M_1, M_2, M_4\}$. Note that M_2 does not verify q, and hence $q \notin Cn_{\Pi^0}(\Gamma_{\mathsf{c}} \cup \{p\})$.

One can also easily construct an example for which the deduction theorem fails. Let $\Gamma_{\mathsf{d}} = \{p \vee q\}$, where p and q are the only variables of our language. Let

[34] We found no proof of this lemma in the literature. In view of our proof, this can only be explained by the fact that usually, only the transitive case is considered for Π^0. For that case, Lemma 16 follows immediately from [17, Lemma XX, item (3)].

M_1, M_2, M_3 be the models of Γ_{d}, such that $M_1 \models \neg p, q$, $M_2 \models p, \neg q$, $M_3 \models p, q$. Finally, let $M_3 \prec M_2 \prec M_1$. We safely leave it to the reader to verify that $\Gamma_{\mathsf{d}} \not\Vdash_{\Pi^0} p \supset q$, whereas $\Gamma_{\mathsf{d}} \cup \{p\} \Vdash_{\Pi^0} q$.

As we have seen in Section 2.3, one can obtain a variant of $\Vdash_{\Pi^0}$ that satisfies right absorption, simply by closing Π^0 under finite intersections. However, the resulting system seems to have little additional motivation apart from the fact that it gives us exactly this property. Also, it does not have cumulative transitivity or cautious monotonicity. Hence we shall not include this variant in our discussion.

If $\prec$ is transitive, we can preserve a number of properties such as right absorption and finitary cumulativity for $\Vdash_{\Pi^0}$. These properties follow immediately from Fact 10.5 and Corollary 12 in Section 4.3 below.[35] If **L** contains a classical disjunction, one can also prove the disjunction property for $\Vdash_{\Pi^0}$, along the lines of [17]. Nevertheless, the infinitary versions of cumulative transitivity and cautious monotonicity do not hold for $\Vdash_{\Pi^0}$, even in the case where $\prec$ is transitive – we refer to [17] for a simple counterexample.

4.2 Variant 1: Local Transitive Closure

In Section 3, we saw that by means of the idea of local transitive closure, we obtained a variant of safe selections that warrants cumulative transitivity and is significantly stronger than the original variant. So one may ask whether a similar technique can be applied to obtain a variant of Π^0, and how the resulting consequence operation behaves. This gives us the following multi-selection function:

Definition 11 $\Pi^1(X) = \{Y \subseteq X \mid Y$ *is* $\prec^{\mathsf{tr}}_X$ *-lower and* $\prec^{\mathsf{tr}}_X$ *-dense in* $X\}$.

As we will show below, the operation Cn_{Π^1} satisfies both right absorption and finitary cumulative transitivity for arbitrary $\prec$, in contrast to Cn_{Π^0}. Also, by the following lemma, the local transitive closure-variant is at least as strong as its original counterpart:

Lemma 17 *Each of the following holds:*

1. $\Pi^0(X) \subseteq \Pi^1(X)$.
2. *If* $\prec$ *is transitive, then* $\Pi^0(X) = \Pi^1(X)$.

Proof. Ad 1. Suppose that (i) $Y \subseteq X$, (ii) for all $x \in X$, there is a $y \in Y$ with $y \preceq x$, and (iii) Y is $\prec$-lower in X. By (i) and (ii), we have that for all $x \in X$, there is a $y \in Y$ with $y \preceq^{\mathsf{tr}}_X x$. So Y is $\prec^{\mathsf{tr}}_X$-dense in X. Finally, by (iii) and Lemma 7, Y is also $\prec^{\mathsf{tr}}_X$-lower in X.

Ad 2. Immediate in view of Fact 4.3. ■

Corollary 7 $\Vdash_{\Pi^0} \subseteq \Vdash_{\Pi^1}$. *If* $\prec$ *is transitive, then* $\Vdash_{\Pi^0} = \Vdash_{\Pi^1}$.

For an illustration of the difference between $\Vdash_{\Pi^0}$ and $\Vdash_{\Pi^1}$, we can refer to the example Γ_{c} on page 22. Note that $M_4 \prec^{\mathsf{tr}}_{\mathcal{M}(\Gamma_{\mathsf{c}})} M_1$, and hence $\{M_4\}$

[35]This result generalizes Fact 3.4.3 from [17], as Schlechta only proves cumulativity for single formulas at the right and left hand side of the turnstile and restricts the scope to the case where $\mathbf{L} = \mathbf{CL}$.

is $\prec^{\mathsf{tr}}_{\mathcal{M}(\Gamma_{\mathsf{c}})}$-dense in $\mathcal{M}(\Gamma_{\mathsf{c}})$. Since there is no $M \in \mathcal{M}(\Gamma)$ below M_4, $\{M_4\}$ is also $\prec^{\mathsf{tr}}_{\mathcal{M}(\Gamma_{\mathsf{c}})}$-lower in $\mathcal{M}(\Gamma_{\mathsf{c}})$. As a result, $\{M_4\} \in \Pi^1(\mathcal{M}(\Gamma_{\mathsf{c}}))$ and hence $p \wedge q \in Cn_{\Pi^1}(\Gamma_{\mathsf{c}}) - Cn_{\Pi^0}(\Gamma_{\mathsf{c}})$.

Metatheory of $\Vdash_{\Pi^1}$ Since each $Y \in \Pi^1(X)$ is $\prec^{\mathsf{tr}}_X$-dense in X, we immediately have:

Fact 9 Π^1 *satisfies* **NT**.

This fact and the following lemma suffice to obtain all our positive results about $\Vdash_{\Pi^1}$:

Lemma 18 Π^1 *satisfies* **CT**.

Proof. Suppose that $X \in \Pi^1(Z_1)$, $Y \in \Pi^1(Z_2)$, and $X \subseteq Z_2 \subseteq Z_1$. We have to prove that $X \cap Y \in \Pi^1(Z_1)$, hence that (i) $X \cap Y$ is $\prec^{\mathsf{tr}}_{Z_1}$-lower in Z_1, and (ii) $X \cap Y$ is $\prec^{\mathsf{tr}}_{Z_1}$-dense in Z_1.

Ad (i). Such that $x \in X \cap Y$, and that $y \in Z_1$ is such that $y \prec^{\mathsf{tr}}_{Z_1} x$. So there are $z_1, \ldots, z_n \in Z_1$ such that $y \prec z_1 \preceq \ldots \preceq z_n \preceq x$. Since X is $\prec^{\mathsf{tr}}_{Z_1}$-lower in Z_1 and $x \in X$, each of $z_n, \ldots, z_1, y$ are in X. Hence since $X \subseteq Z_2$, also $y \prec^{\mathsf{tr}}_{Z_2} x$. Since Y is $\prec^{\mathsf{tr}}_{Z_2}$-lower in Z_2, we have that $y \in Y$ and hence $y \in X \cap Y$.

Ad (ii). Suppose $x \in Z_1$. Since X is $\prec^{\mathsf{tr}}_{Z_1}$-dense in Z_1, there is a $y \in X$: (1) $y \preceq^{\mathsf{tr}}_{Z_1} x$. Since $X \subseteq Z_2$ and Y is $\prec^{\mathsf{tr}}_{Z_2}$-dense in Z_2, there is a $z \in Y : z \preceq^{\mathsf{tr}}_{Z_2} y$, and hence (2) $z \preceq^{\mathsf{tr}}_{Z_1} y$. By (1) and (2), $z \preceq^{\mathsf{tr}}_{Z_1} x$. Hence, since x is $\prec^{\mathsf{tr}}_{Z_1}$-lower in Z_1, $z \in X$. It follows that $z \in X \cap Y$. ■

In view of Facts 2, 3 and 9, we have:

Corollary 8 Π^1 *satisfies* **RS** *and* **RA**.

Corollary 9 *If* **L** *is compact, then* Cn_{Π^1} *satisfies right satisfiability and right absorption.*

Corollary 10 Cn_{Π^1} *satisfies finitary cumulative transitivity.*

Finitary cautious monotonicity and the disjunction property both fail for Cn_{Π^1}. For cautious monotonicity, this can be shown by the same example as the one we used to show the failure of this property for Cn_{Ψ^1} – see page 18. The problem is essentially the same: by adding certain consequences of Γ to this premise set, some models (i.c. the model M_3) are removed, and as a result certain connections (viz. the one between M_2 and M_4) are lost. So we have $\Gamma_{\mathsf{a}} \Vdash_{\Pi^1} p, r$ but $\Gamma_{\mathsf{a}} \cup \{p\} \nVdash_{\Pi^1} r$.

Our counterexample for the disjunction property relies on the same principle, though it looks slightly different. We only need two variables p and q; let $\Gamma = \{p \vee q\}$. This set has three models, viz. M_1, M_2, M_3, where $M_1 \models p, \neg q$, $M_2 \models \neg p, q$ and $M_3 \models p, q$. Let $M_1 \prec M_2 \prec M_3 \prec M_1$ (hence $\prec$ is cyclic over $\mathcal{M}(\Gamma)$). In view of this, $\Pi^1(\mathcal{M}(\Gamma)) = \{\mathcal{M}(\Gamma)\}$ (we cannot ignore any model of Γ), and hence $\Gamma \nVdash_{\Pi^1} p \supset q$. However, $\prec$ is not cyclic over $\mathcal{M}(\Gamma \cup \{p\})$, and $\{M_3\} \in \Pi^1(\mathcal{M}(\Gamma \cup \{p\}))$. As a result, $\Gamma \cup \{p\} \Vdash_{\Pi^1} q$.

4.3 Variant 2: $\prec$-Clearness

To introduce the second variant of Π^0, it is helpful to take a closer look at those cases where either cautious monotonicity, or cumulative transitivity fail for Cn_{Π^0}. These are typically cases where the selection of models cuts right through non-transitive chains. E.g., where we have $\{M_1, M_2, M_3\}$ with $M_1 \prec M_2 \prec M_3$ but $M_1 \not\prec M_3$, Π^0 allows us to select only M_1 and M_2 but not M_3.

As mentioned in the introduction, one may consider intransitive clusters to be a sign of inconsistency of reasoning. Now if you select M_1, M_2 but not M_3, this structural property - the inconsistency - gets lost in your selection: it in a sense "irons out" the inconsistency.

The idea behind the second variant is that the selection $Y \subseteq X$ is not allowed to cut through non-transitive sequences; non-transitive clusters have to be either in, or out of the selection (as a whole). This is done by imposing one further restriction, apart from the fact that Y is $\prec$-lower and $\prec$-dense in X.

Definition 12 *Y is* $\prec$-clear *in X iff $Y \subseteq X$ and for all $x, y \in Y$ and all $z \in X - Y$: if $x \prec y \prec z$, then $x \prec z$.*

Definition 13 $\Pi^2(X) = \{Y \subseteq X \mid Y$ *is* $\prec$ *-lower,* $\prec$ *-dense and* $\prec$ *-clear in* $X\}$.

Fact 10 *Each of the following holds:*

1. $\Pi^2(X) \subseteq \Pi^0(X)$.
2. $\Vdash_{\Pi^2} \subseteq \Vdash_{\Pi^0}$.
3. *If $\prec$ is transitive, then every $Y \subseteq X$ is $\prec$-clear in X.*
4. *If $\prec$ is transitive, then $\Pi^2(X) = \Pi^0(X)$.*
5. *If $\prec$ is transitive, then $\Vdash_{\Pi^2} = \Vdash_{\Pi^0}$.*

We can again refer to the premise set Γ_{c} introduced on page 22, to highlight the difference between this second variant and $\Vdash_{\Pi^0}$. In the case of Π^2, we cannot ignore the model M_1. The reason is obvious: we have $M_4 \prec M_2 \prec M_1$, but $M_4 \not\prec M_1$; likewise, $M_4 \prec M_3 \prec M_1$ but $M_4 \not\prec M_1$. Hence either each of M_1, M_2 and M_3 have to be selected, or all three of them have to be disselected. However, since we need an $M \prec M_1$ to obey $\prec$-density, we cannot disselect both M_2 and M_3. As a result, $\Pi^2(\mathcal{M}(\Gamma_{\mathsf{c}})) = \{\mathcal{M}(\Gamma_{\mathsf{c}})\}$ and hence $\Gamma_{\mathsf{c}} \nVdash_{\Pi^2} p, q$.

Metatheory of $\Vdash_{\Pi^2}$ By Definition 13, $\Pi^2(\mathcal{M}(\Gamma))$ is $\prec$-dense in $\mathcal{M}(\Gamma)$. So we immediately have:

Fact 11 Π^2 *satisfies* **NT**.

By Fact 10.5, and the fact that Cn_{Π^0} is not cumulatively transitive or cautiously monotonic even when $\prec$ is transitive, it follows that the same applies to Cn_{Π^2}. However, we will now proceed by showing that Cn_{Π^2} does satisfy the *finitary* versions of both properties. The proof relies essentially on the following two lemmata.[36]

Lemma 19 *Each of the following holds.*

[36] In view of Fact 10, items 5 and 6 are equivalent to items (1) and (2) from Fact 3.4.3 in [17, p. 141] in the transitive case; however, for the general case where $\prec$ is arbitrary, they are stronger.

1. *If* Y *is* $\prec$*-lower in* X *and* $Y \subseteq Z \subseteq X$*, then* Y *is* $\prec$*-lower in* Z*.*
2. *If* Y *is* $\prec$*-dense in* X *and* $Y \subseteq Z \subseteq X$*, then* Y *is* $\prec$*-dense in* Z*.*
3. *If* Y *is* $\prec$*-clear in* X *and* $Y \subseteq Z \subseteq X$*, then* Y *is* $\prec$*-clear in* Z*.*

Proof. Ad 1. Suppose the antecedent holds. Let $z \in Z$ and $y \in Y$, with $z \prec y$. Note that $z \in X$. Hence, since Y is $\prec$-lower in X, also $z \in Y$ and we are done.

Ad 2. Suppose the antecedent holds. Let $z \in Z$. Hence also $z \in X$. Since Y is $\prec$-dense in X, it follows at once that there is a $y \in Y$ such that $y \preceq z$.

Ad 3. Suppose the antecedent holds. Let $z \in Z - Y$, and $x, y \in Y$ with $x \prec y \prec z$. It follows that also $z \in X - Y$. Hence, since Y is $\prec$-clear in X, $x \prec z$ as well. ■

Lemma 20 Π^2 *satisfies* **CM***,* **CT** *and* **RA***.*

Proof. "**CM**" Immediate in view of Lemma 19.

"**CT**" Let $X \in \Pi^2(Z_1)$, $Y \in \Pi^2(Z_2)$, and $X \subseteq Z_2 \subseteq Z_1$. We have to check three things:

"$X \cap Y$ is $\prec$-dense in Z_1": Suppose $x \in Z_1 - (X \cap Y)$. Case 1: $x \in X$. Hence $x \notin Y$. Note that since $X \subseteq Z_2$, and since Y is $\prec$-dense in Z_2, there is a $y \in Y$ with $y \prec x$. Since $Y \subseteq Z_1$, and since X is $\prec$-lower in Z_1, also $y \in X$. So $y \in X \cap Y$ and we are done.

Case 2: $x \notin X$. Since X is $\prec$-dense in Z_1, there is a $y \in X$: $y \prec x$. If $y \in Y$, we are done. So suppose $y \notin Y$. Note that $y \in Z_2$, and hence since Y is $\prec$-dense in Z_2, there is a $z \in Y$ such that $z \prec y$. Note that also $z \in Z_1$, since $Y \subseteq Z_2 \subseteq Z_1$. Since X is $\prec$-lower in Z_1, $z \in X \cap Y$. So we have: $z \prec y \prec x$, and $z, y \in X$, and $x \in Z_1 - X$. Hence, since X is $\prec$-clear in Z_1, $z \prec x$ and we are done.

"$X \cap Y$ is $\prec$-lower in Z_1": suppose $x \in X \cap Y$ and $y \in Z_1$, $y \prec x$. Since $x \in X$ and since X is $\prec$-lower in Z_1, also $y \in X$. It follows that also $y \in Z_2$. Since $x \in Y$ and Y is $\prec$-dense in Z_2, also $y \in Y$. Hence $y \in X \cap Y$ and we are done.

"$X \cap Y$ is $\prec$-clear in Z_1": Suppose $x, y \in X \cap Y$, $z \in Z_1 - (X \cap Y)$, and $x \prec y \prec z$. Case 1: $z \notin X$. Then, since x is $\prec$-clear in Z_1, and since $x, y \in X$: $x \prec z$.

Case 2: $z \in X$. Hence $z \notin Y$, but also $z \in Z_2$. Since Y is $\prec$-clear in Z_2, and since $x, y \in Y$, $x \prec z$.

"**RA**" Immediate in view of the preceding item and Fact 2. ■

By Lemma 20, Fact 11 and Lemmas 3-6, we have:

Corollary 11 *If* **L** *is compact, then* Cn_{Π^2} *satisfies right absorption and right satisfiability.*

Corollary 12 Cn_{Π^2} *finitary cautious monotonicity and finitary cumulative transitivity.*

5 The Boutilier-variant

We now turn to the third family of preferential semantic consequence relations. We start with a proposal that was briefly mentioned in [12], showing it to yield a stronger consequence relation $\Vdash_{\Phi^0}$ than the Limit Variant. After that, we briefly discuss a simple variant of $\Vdash_{\Phi^0}$, which is obtained by taking the transitive closure of $\prec$.

5.1 The Boutilier-variant

The Definition In [4], Boutilier uses a specific semantic clause for object-level expressions of the type "A preferentially entails B", within a possible world semantics in terms of a transitive and reflexive accessibility relation. In his overview article, Makinson [12] mentions this paper, noting that one may apply Boutilier's idea also at the meta-level. This gives us the following definition [12, p. 74]:[37]

Definition 14 *$\Gamma \Vdash_{\Phi^0} A$ ($A \in Cn_{\Phi^0}(\Gamma)$) iff for every $M \in \mathcal{M}(\Gamma)$, there is an $M' \in \mathcal{M}(\Gamma)$ such that each of the following holds:*

(1) $M' \preceq M$
(2) for all $M'' \in \mathcal{M}(\Gamma)$ such that $M'' \preceq M'$: $M'' \models A$

Makinson himself does not discuss this consequence relation in detail, but restricts his overview to the properties of **S**. More recently, Horty applied essentially the same idea as Boutilier's in the context of deontic stit logic, where he uses it to define the concept of obligatory actions, in view of a transitive preference relation on actions – see [6]. To the best of our knowledge, this is the first paper where the application of this mechanism at the meta-level is studied. We shall henceforth call this consequence relation the *Boutilier-variant*.

Below, we will see how one can characterize $\Vdash_{\Phi^0}$ in terms of a multi-selection semantics. However, we will first focus on the original definition of $\Vdash_{\Phi^0}$ and illustrate its behavior in simple examples.

At first sight, it may seem that the Boutilier-variant and the Limit Variant are equivalent. For instance, in the critical example Γ_{c} from page 22, we can easily verify that both $\Gamma_{\mathsf{c}} \Vdash_{\Phi^0} p$ and $\Gamma_{\mathsf{c}} \Vdash_{\Phi^0} q$, but nevertheless $\Gamma_{\mathsf{c}} \not\Vdash_{\Phi^0} p \wedge q$. That is, there is no model $M \in \{M_1, M_2, M_3, M_4\}$ that is as good as or better than M_1, such that $M \models p \wedge q$. This also shows that $\Vdash_{\Phi^0}$ does not guarantee right absorption, as was the case with $\Vdash_{\Pi^0}$.

In general, we have:

Theorem 3 $\Vdash_{\Pi^0} \subseteq \Vdash_{\Phi^0}$.

Proof.[38] Suppose $\Gamma \Vdash_{\Pi^0} A$. Let $\mathcal{M} \subseteq \mathcal{M}(\Gamma)$ be a $\prec$-dense and $\prec$-lower subset of $\mathcal{M}(\Gamma)$, such that for all $M \in \mathcal{M}$, $M \models A$. Now consider an arbitrary $M \in \mathcal{M}(\Gamma)$. Since $\mathcal{M}$ is a $\prec$-dense subset of $\mathcal{M}(\Gamma)$, there is an $M' \in \mathcal{M}$ such that $M' \preceq M$. Note that $M' \models A$ in view of the supposition. Now consider an arbitrary $M'' \in \mathcal{M}(\Gamma)$ such that $M'' \preceq M'$. Since $\mathcal{M}$ is a $\prec$-lower set of $\mathcal{M}(\Gamma)$, also $M'' \in \mathcal{M}$. As a result, $M'' \models A$ and we are done. ■

Nevertheless, the converse of Theorem 3 does not hold. This is an immediate consequence of the fact that $\Vdash_{\Phi^0}$ does not obey right satisfiability – this will

[37] For the time being one may read $\Vdash_{\Phi^0}$ as primitive. However, further on in this section, it is shown that $\Vdash_{\Phi^0}$ can be characterized in terms of a multi-selection semantics, using a function Φ^0.

[38] This theorem could also be proved in purely set-theoretic terms, relying on Theorem 4 below. In that case, one would first prove that $\Pi^0(X) \subseteq \Phi^0(X)$ – see page 28 where the function Φ^0 is defined.

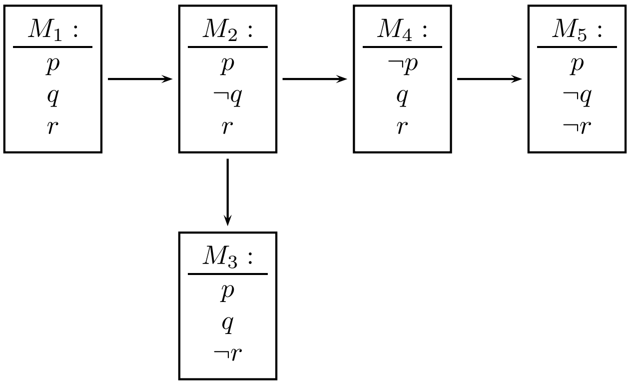

Figure 4: Models of Γ_{e}.

be shown in Section below –, whereas Π^0 does. However, there are also cases in which $\Vdash_{\Phi^0}$ does lead to a satisfiable consequence set, which is nevertheless stonger than the Π^0-consequence set. We will now take a look at one such case to illustrate what is going on.

Let the language schema only contain three propositional letters, i.e. p, q, r, and let $\Gamma_{\mathsf{e}} = \{p \vee (q \wedge r)\}$. It can be easily verified that $\mathcal{M}(\Gamma_{\mathsf{e}})$ contains exactly five models, which are depicted in Figure 4. Note that M_1, M_2, M_3 and M_5 verify p, whereas $M_4 \not\models p$.

It can be easily verified that all $\prec$-dense and $\prec$-lower subsets of $\mathcal{M}(\Gamma_{\mathsf{e}})$ are supersets of $\{M_2, \ldots, M_5\}$. We cannot do without M_2, since this is the only model below M_1. Since all of M_3, M_4, M_5 are below, these are also included in each $\mathcal{M} \in \Pi_{\prec}(\mathcal{M}(\Gamma_{\mathsf{e}}))$. Finally, as $M_4 \not\models p$, we have that $p \notin Cn_{\Pi^0}(\Gamma_{\mathsf{e}})$.

It remains to check that $p \in Cn_{\Phi^0}(\Gamma_{\mathsf{e}})$. To do so, we look at each model $M \in \mathcal{M}(\Gamma_{\mathsf{e}})$ separately and show that there is an M' such that $M' \preceq M$, and for all M'' with $M'' \preceq M$, $M'' \models p$. Where $M = M_3$ and $M = M_5$, this holds trivially so, since both models are $\prec$-minimal. For $M = M_1$, we can let $M' = M = M_1$; since $M_2 \models p$, there is no problem. For $M = M_2$, we can take $M' = M_3$, since the latter is $\prec$-minimal and verifies p. Finally, for $M = M_4$ we can take $M' = M_5$.

For the example to work, it is essential that $\prec$ is not transitive. Otherwise, $\{M_3, M_5\}$ would be a $\prec$-dense and $\prec$-lower subset of $\mathcal{M}(\Gamma_{\mathsf{e}})$, and hence p would be a Π^0-consequence as well. More generally, if $\prec$ is transitive, then $\Vdash_{\Phi^0}$ is equivalent to $\Vdash_{\Pi^0}$ – see Corollary 13 on page 31. For that reason, we only consider the more general case in this section.

Characterizing $\Vdash_{\Phi^0}$ by a multi-selection semantics In view of Definition 14, we can characterize $\Vdash_{\Phi^0}$ by means of a multi-selection semantics. This requires that we define a notion of strong $\prec$-density:

Definition 15 *Y is* strongly $\prec$-dense *in X iff $Y \subseteq X$ and for all $x \in X$ there is a $y \in Y$ such that (i) $y \preceq x$ and (ii) for all $z \in X$ with $z \prec y$, $z \in Y$.*

Let $\Phi^0(X) =_{\mathsf{df}} \{Y \subseteq X \mid Y \text{ is strongly } \prec\text{-dense in } X\}$. Then we have:

Theorem 4 *$\Gamma \Vdash_{\Phi^0} A$ iff there is a $\mathcal{M} \in \Phi^0(\mathcal{M}(\Gamma))$ such that, for all $M \in \mathcal{M}$: $M \models A$.*

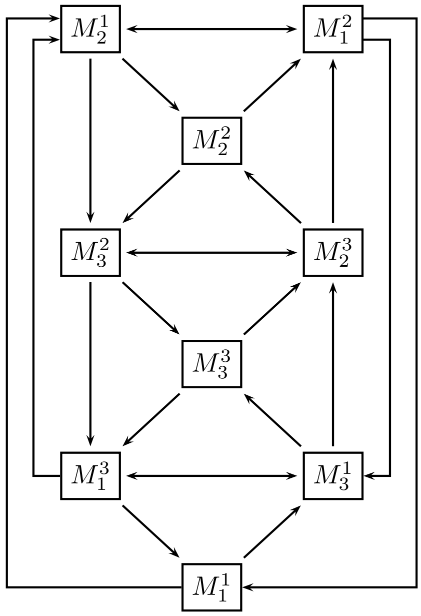

Figure 5: Models of Γ_{f}

Proof. ($\Rightarrow$) Suppose the antecedent holds. For every $M \in \mathcal{M}(\Gamma)$, let $M' \in \mathcal{M}(\Gamma)$ be such that (i) $M' \preceq M$, and for all $M'' \in \mathcal{M}(\Gamma)$ with $M'' \preceq M'$, $M'' \models A$. Let each $\mathcal{M}_M = \{M'' \in \mathcal{M}(\Gamma) \mid M'' \preceq M'\}$. In view of the construction, for all $M \in \mathcal{M}(\Gamma)$ and all $M''' \in \mathcal{M}_M$, $M''' \models A$. We safely leave it to the reader to check that $\bigcup_{M \in \mathcal{M}(\Gamma)} \mathcal{M}_M$ is strongly $\prec$-dense in $\mathcal{M}(\Gamma)$, whence $\mathcal{M} \in \Phi^0(\mathcal{M}(\Gamma))$.

($\Leftarrow$) Suppose the consequent holds. Let $\mathcal{M}_A \in \Phi^0(\mathcal{M}(\Gamma))$ be arbitrary such that all $M \in \mathcal{M}_A$ verify A. Let M_1 be arbitrary in $\mathcal{M}(\Gamma)$. Since $\mathcal{M}_A$ is strongly $\prec$-dense, there is an $M_2 \in \mathcal{M}_A$ such that (i) $M_2 \preceq M_1$ and (ii) for all $M_3 \in \mathcal{M}_A$ with $M_3 \prec M_2$, $M_3 \in \mathcal{M}_A$. The rest is immediate in view of Definition 14. ■

Metatheory of $\Vdash_{\Phi^0}$ Without transitivity of $\prec$, $\Vdash_{\Phi^0}$ has none of the metatheoretic properties from our (reduced) checklist. That is, in such cases, one cannot warrant even satisfiability of $Cn_{\Phi^0}(\Gamma)$ whenever Γ is satisfiable. In the next few paragraphs, we will show why this is so.

The language schema of our example is restricted to only four letters: p_1, p_2, p_3, q. Let $\Gamma_{\mathsf{f}} = \{p_1 \vee p_2 \vee p_3, \neg p_1 \vee \neg p_2 \vee \neg p_3, p_1 \vee p_2 \vee q, p_2 \vee p_3 \vee q, p_3 \vee p_1 \vee q\}$. This set has nine models: $\mathcal{M}(\Gamma_{\mathsf{f}}) = \{M^i_j \mid i, j \in \{1, 2, 3\}\}$, where for all M^i_j, each of the following holds:

(i) $M^i_j \models p_i, p_j$ but $M^i_j \not\models p_k$ for all $k \in \{1, 2, 3\} - \{i, j\}$;
(ii) If $i < j$, then $M^i_j \models q$;
(iii) If $j < i$, then $M^i_j \models \neg q$;
(iv) If $i = j$, then $M^i_j \models q$.

We define $\prec$ as follows: $M^j_k \prec M^i_j$ iff ($i \neq j$ or $j \neq k$). So for instance $M^1_3 \prec M^2_1$, $M^1_3 \prec M^3_1$, and $M^3_1 \prec M^1_3$, but $M^2_2 \not\prec M^2_2$.

Now let $M = M^i_j$ be an arbitrary model in $\mathcal{M}(\Gamma_{\mathsf{f}})$, and let k be arbitrary in $\{1,2,3\}$. In view of the construction, $M^j_k \preceq M^i_j$. So there is a model $M' \in \mathcal{M}(\Gamma_{\mathsf{f}})$, with $M' \preceq M$ and $M' \models p_k$. Suppose now that, for an $M'' \in \mathcal{M}(\Gamma_{\mathsf{f}})$, $M'' \prec M'$. In view of the construction, $M'' = M^k_l$ for an $l \in \{1,2,3\}$. Hence also $M'' \models p_k$ and we are done.

So $\Gamma_{\mathsf{f}} \Vdash_{\Phi^0} p_1, p_2, p_3$. However, since Cn_{Φ^0} is inclusive, also $\Gamma_{\mathsf{f}} \Vdash_{\Phi^0} \neg p_1 \vee \neg p_2 \vee \neg p_3$. It follows immediately that $Cn_{\Phi^0}(\Gamma_{\mathsf{f}})$ is not satisfiable.

As figure 5 shows, the $\prec$ displays cycles in $\mathcal{M}(\Gamma_{\mathsf{f}})$. For instance, $M^1_3 \prec M^3_1 \prec M^1_3$, and also $M^1_3 \prec M^3_2 \prec M^2_1 \prec M^1_3$. However, one can easily construct an infinite model-structure where $\prec$ is both irreflexive and acyclic, which leads to the same semantic consequences as $\mathcal{M}(\Gamma_{\mathsf{f}})$; it suffices to add just a bit more labels to each model.

A natural question one may then ask is, what conditions on $\prec$ and Γ guarantee that $Cn_{\Phi^0}(\Gamma)$ is satisfiable? It can be easily verified that the following holds:

Fact 12 *If Γ has $\prec$-minimal models, then $Cn_{\Phi^0}(\Gamma)$ is satisfiable.*

To see why Fact 12 holds, let $M \in \mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$. Let A be arbitrary in $Cn_{\Phi^0}(\Gamma)$. Note that the only $M' \in \mathcal{M}(\Gamma)$ with $M' \preceq M$ *is* M, and hence $M \models A$. So M verifies all the members of $Cn_{\Phi^0}(\Gamma)$.

We now turn to the other properties: right absorption, cumulative transitivity, cautious monotonicity and the disjunction property. As a matter of fact, even for premise sets that have $\prec$-minimal models, each of these properties fail. For the first three properties, we can use exactly the same example Γ_{c} and the same reasoning that showed their failure for $\Vdash_{\Pi^0}$ – see page 22 and Figure 3. Also for the disjunction property we can re-use an old example, viz. the set Γ_{d} from page 4.1. Here again the reasoning is the same as before, so we get $\Gamma_{\mathsf{d}} \nVdash_{\Phi^0} p \supset q$, whereas $\Gamma \cup \{p\} \Vdash_{\Phi^0} q$.

5.2 Variant: Local Transitive Closure

As in the previous sections, we will now check what happens to Φ^0 if we replace $\prec$ with its local transitive closure. This gives us the following definition:

Definition 16 $\Gamma \Vdash_{\Phi^1} A$ *($A \in Cn_{\Phi^0}(\Gamma)$) iff for every $M \in \mathcal{M}(\Gamma)$, there is an $M' \in \mathcal{M}(\Gamma)$ such that each of the following holds:*

(1) $M' \preceq^{\mathsf{tr}}_{\mathcal{M}(\Gamma)} M$
(2) for all $M'' \in \mathcal{M}(\Gamma)$ such that $M'' \preceq^{\mathsf{tr}}_{\mathcal{M}(\Gamma)} M'$: $M'' \models A$

By Fact 4.3, we have:

Fact 13 *If $\prec$ is transitive, then $\Vdash_{\Phi^1} \; = \; \Vdash_{\Phi^0}$.*

In line with the preceding, we may also characterize $\Vdash_{\Phi^1}$ in terms of a multi-selection semantics. That is, let $\Phi^1(X) =_{\mathsf{df}} \{Y \subseteq X \mid Y$ is strongly $\prec^{\mathsf{tr}}_X$-dense in $X\}$. Then $\Gamma \Vdash_{\Phi^1} A$ iff there is a $\mathcal{M} \in \Phi^1(\mathcal{M}(\Gamma))$ such that all $M \in \mathcal{M}$ verify A. The proof is essentially the same as the one for Theorem 4 – it suffices to replace $\prec$ by $\prec^{\mathsf{tr}}_{\mathcal{M}(\Gamma)}$, resp. $\preceq$ by $\preceq^{\mathsf{tr}}_{\mathcal{M}(\Gamma)}$ everywhere in the proof. So we get:

Theorem 5 $\Gamma \Vdash_{\Phi^1} A$ *iff there is a $\mathcal{M} \in \Phi^1(\mathcal{M}(\Gamma))$ such that, for all $M \in \mathcal{M}$: $M \models A$.*

Equivalence with $\Vdash_{\Pi^1}$ Before considering the metatheory of Φ^1 and its relation to Φ^0, it is convenient to show the following:

Lemma 21 *Each of the following holds:*[39]

1. $\Pi^1(X) \subseteq \Phi^1(X)$.
2. *If* $Y \in \Phi^1(X)$*, then there is a* $Z \subseteq Y$ *such that* $Z \in \Pi^1(X)$.

Proof. Ad 1. Suppose (i) Y is $\prec^{\mathsf{tr}}_X$-dense and (ii) Y is $\prec^{\mathsf{tr}}_X$-lower in X. Let $x \in X$ be arbitrary. By (i), there is a $y \in Y$ such that $y \preceq^{\mathsf{tr}}_X x$. Let $z \in X$ be such that $z \prec^{\mathsf{tr}}_X y$. Then by (ii), also $z \in Y$. It follows that Y is strongly $\prec^{\mathsf{tr}}_X$-dense in X.

Ad 2. Suppose the antecedent holds. For all $x \in X$, let $x' \in Y$ be such that (i) $x' \preceq^{\mathsf{tr}}_X x$ and (ii) for all $y \in X$ with $y \prec^{\mathsf{tr}}_X x'$, $y \in Y$. Furthermore, let each $Z_x = \{x'\} \cup \{y \in X \mid y \prec^{\mathsf{tr}}_X x'\}$. It follows that each $Z_x \subseteq Y$. Let $Z = \bigcup_{x \in X} Z_x \subseteq Y$. It suffices to prove two claims:

"Z is $\prec^{\mathsf{tr}}_X$-dense in X." Let $x \in X$; then $x' \in Z$ and $x' \preceq^{\mathsf{tr}}_X x$.

"Z is $\prec^{\mathsf{tr}}_X$-lower in X." Let $x \in Z$ and $y \in X$ with $y \prec^{\mathsf{tr}}_X x$. Note that $x \in Z_u$ for a $u \in X$. By the construction, also $y \in Z_u$ and hence $y \in Z$. ■

Theorem 6 $\Vdash_{\Pi^1} \; = \; \Vdash_{\Phi^1}$.

Proof. "$\subseteq$" Suppose $\Gamma \Vdash_{\Pi^1} A$. Let $\mathcal{M} \in \Pi^1(\mathcal{M}(\Gamma))$ be such that every $M \in \mathcal{M}$ verifies A. By Lemma 21.1, $\mathcal{M} \in \Phi^1(\mathcal{M}(\Gamma))$. The rest is immediate in view of Theorem 5.

"$\supseteq$" Suppose $\Gamma \Vdash_{\Phi^1} A$. By Theorem 5, there is a $\mathcal{M} \in \Phi^1(\mathcal{M}(\Gamma))$ such that every $M \in \mathcal{M}$ verifies A. By Lemma 21.2, there is a $\mathcal{M}' \subseteq \mathcal{M}$ such that $\mathcal{M}' \in \Pi^1(\mathcal{M}(\Gamma))$. It follows that $\Gamma \Vdash_{\Pi^1} A$. ■

In view of Theorem 6, it suffices to refer to Section 4.2 for the metatheoretic properties of $\Vdash_{\Phi^1}$. Another important result that follows from this theorem, together with Lemma 17.2 and Fact 13, is that the Limit Variant and the Boutilier-variant are equivalent in case $\prec$ is transitive:

Corollary 13 *If* $\prec$ *is transitive, then* $\Vdash_{\Phi^0} \; = \; \Vdash_{\Pi^1}$.

$\Vdash_{\Phi^0}$ versus $\Vdash_{\Phi^1}$ In contrast to the previous sections, this time it turns out that the local transitive closure-variant is not always at least as strong as the original variant. That is, for non-transitive $\prec$, if may be that $\Gamma \Vdash_{\Phi^0} A$ whereas $\Gamma \nVdash_{\Phi^1} A$.

This follows immediately from the fact that $\Vdash_{\Phi^0}$ does not have right satisfiability (see Section 5.1), whereas $\Vdash_{\Phi^1}$ does have this property in view of its equivalence to $\Vdash_{\Pi^1}$. So there may be unsatisfiable $\Gamma' \subseteq Cn_{\Phi^0}(\Gamma)$ for a satisfiable Γ, but it cannot be that those Γ' are included in $Cn_{\Phi^1}(\Gamma)$.

There are also simple cases where $\Vdash_{\Phi^1}$ is stronger than $\Vdash_{\Phi^0}$ – cases in which every sequence of models has a least element, yet $\prec$ is not transitive. For instance, let $\mathcal{M}(\Gamma) = \{M_1, M_2, M_3\}$ where $M_1 \prec M_2 \prec M_3$ but $M_1 \not\prec M_3$, and where only M_1 verifies p. In that case $p \in Cn_{\Phi^1}(\Gamma) - Cn_{\Phi^0}(\Gamma)$, in view of the model M_3.

[39] To see why $\Phi^1(X) \not\subseteq \Pi^1(X)$ for every X, let $X = \{x, y, z\}$ and $x \prec y \prec z$. Then the set $\{x, z\}$ is strongly $\prec^{\mathsf{tr}}_X$-dense in X, but not $\prec^{\mathsf{tr}}_X$-lower in X.

6 Comparing the Various Systems

In this section, we consider the various relations between each of the eight systems defined previously. Before doing so, we give an overview of their metatheoretic properties. We end with a brief comparison of the systems to **S**.

6.1 Overview of the Metatheoretic Properties

First, recall that all eight systems from this paper satisfy inclusion, left absorption, and singular right absorption. This is but a corollary of the fact that they all can be equivalently reformulated in terms of a multi-selection semantics, and Theorem 1.

On the negative side, none of the logics have the disjunction property in the general case – we return to this point in Section 7. However, as noted before, whenever $\prec$ is transitive, then each of the relations $\Vdash_{\Pi^0}, \Vdash_{\Pi^1}, \Vdash_{\Pi^2}, \Vdash_{\Phi^0}, \Vdash_{\Phi^1}$ are equivalent. So each of the corresponding consequence operations have the disjunction property whenever $\mathbf{CL} \subseteq \mathbf{L}$ and $\prec$ is transitive, as this was proven for Cn_{Π^0} in [17].

This leaves us with four properties: right satisfiability, right absorption, cumulative transitivity and cautious monotonicity. For these, the picture for unrestricted $\prec$ is more scattered; an overview is given in Table 2. Note that $\Vdash_{\Phi^1}$ is not represented in the table, as it is equivalent to $\Vdash_{\Pi^1}$.

	Cn_{Ψ^0}	Cn_{Ψ^1}	Cn_{Ψ^2}	Cn_{Π^0}	Cn_{Π^1}	Cn_{Π^2}	Cn_{Φ^0}
right satisfiability	+	+	+	+	+	+	-
right absorption	+	+	+	-	+	+	-
cumulative transitivity	-	+	+	-	f	f	-
cautious monotonicity	+	-	+	-	-	f	-

Table 2: Metatheoretic properties of each of the variants. "f" denotes the finitary version of the property.

As noted in Section 4, for the restricted case where $\prec$ is transitive, the Limit Variant satisfies finitary cumulativity (both directions), and right absorption. So as a corollary, we can infer that all the variants from Sections 4 and 5 have these properties for transitive $\prec$. The equivalence also implies that the Boutilier-variant does not warrant cumulativity in general, as this property fails for the Limit Variant.[40]

6.2 Completing the Puzzle

In Section 5, we discussed the relations between the consequence operations from that section and those from Section 4. We now prove similar theorems that relate the safe selections from Section 3 to the other consequence operations in this paper, after which we consider cases where they differ from one another.

First, $\Vdash_{\Psi^2}$ is included in each of the other variants except $\Vdash_{\Pi^2}$. That is, by Corollary 4, Fact 7 and Lemma 7, $\Psi^2(X) \in \Pi^0(X)$. Hence:

Corollary 14 $\Vdash_{\Psi^2} \subseteq \Vdash_{\Pi^0}$

[40]This point refutes an earlier claim by Makinson – see [12, p. 74].

One further corollary is that when $\prec$ is transitive, each of the safe selections variants are included in all the other consequence operations.

A second important result concerns the relation between $\Vdash_{\Psi^1}$ and $\Vdash_{\Pi^1}$. Note that by definition, $\Psi^1(X)$ is $\prec^{\mathsf{tr}}_X$-dense in X. By Lemma 12.2, it is also $\prec^{\mathsf{tr}}_X$-lower in X. Hence:

Fact 14 $\Psi^1(X) \in \Pi^1(X)$.

Corollary 15 $\Vdash_{\Psi^1} \subseteq \Vdash_{\Pi^1}$.

We end this survey with a number of examples that show where the various systems differ:

(i) In view of the example discussed in Section 3.4, Cn_{Π^2} is often stronger than Cn_{Ψ^1}, i.e. in cases where we have sequences of models with no least element.

This need not mean that $\mathcal{M}(\Gamma)$ contains *infinite* descending sequences – the same result can be obtained if we have cycles over the set of models. For instance, let $M_1 \prec M_2$, $M_2 \prec M_1$, $M_1 \prec M_3$ and $M_2 \prec M_3$; also, let M_3 be the only model that does *not* verify p. It can be easily verified that $\{M_1, M_2\} \in \Pi^2(\{M_1, M_2, M_3\})$, and hence p is an Π^2-consequence given this model-structure. However, $\Psi^1(\{M_1, M_2, M_3\}) = \{M_1, M_2, M_3\}$, and hence p is not an Ψ^1-consequence — note that M_3 cannot be excluded, since there are no $\prec$-minimal models at all.

As a corollary, there are cases in which each of the three safe selections-variants are weaker than all other consequence relations studied in this paper.

(ii) Let $\mathcal{M}(\Gamma) = \{M_1, M_2, M_3\}$ and $M_1 \prec M_2 \prec M_3$, but not $M_1 \prec M_3$. Suppose that $M_1, M_3 \models p$, $M_2 \not\models p$. Since $\Psi^0(\mathcal{M}(\Gamma)) = \{M_1, M_3\}$, $\Gamma \Vdash_{\Psi^0} p$. However, in view of M_3 and Definition 14, $p \notin Cn_{\Phi^0}(\Gamma)$.

By Fact 10.1 and Theorem 3 respectively, $\Vdash_{\Pi^2} \subseteq \Vdash_{\Pi^0} \subseteq \Vdash_{\Phi^0}$. Hence in view of the previous paragraph, $\Vdash_{\Psi^0}$ is sometimes stronger than $\Vdash_{\Pi^0}$ and $\Vdash_{\Pi^2}$ as well. Moreover, since $\Vdash_{\Psi^1}$ is always at least as strong as $\Vdash_{\Psi^0}$ (see Lemma 17), the same applies to $\Vdash_{\Psi^1}$.

(iii) Sometimes $\Vdash_{\Psi^2}$ is stronger than $\Vdash_{\Pi^2}$. For instance, let $\mathcal{M}(\Gamma) = \{M_1, \ldots, M_5\}$, where $M_1 \prec M_2 \prec M_3 \prec M_4$ and $M_5 \prec M_4$. In that case, $\Pi^2(\mathcal{M}(\Gamma)) = \{\{M_1, \ldots, M_5\}\}$: we cannot ignore M_2 if we want to preserve $\prec$-density; hence we also cannot ignore M_3 if we want to ensure that our selected set is $\prec$-clear in $\mathcal{M}(\Gamma)$. Similarly, by $\prec$-clearness, M_4 cannot be ignored. However, since M_5 is $\prec$-minimal and $M_5 \prec M_4$, and since M_4 is not below any other model, we have $\Psi^2(\mathcal{M}(\Gamma)) = \{M_1, M_2, M_3, M_5\}$. Hence the weakest safe selection-variant still enables us to ignore M_4. In view of (i), this means that $\Vdash_{\Psi^2}$ and $\Vdash_{\Pi^2}$ are in general incomparable.

The above examples render our comparison of the eight systems defined in this paper complete. That is, for every two such systems, we have settled the question whether they are equivalent (in general or for transitive $\prec$), or one is stronger than the other, or they are mutually incomparable. An overview of these relations is given in Figure 6.

$$
\begin{tikzcd}
Cn_{\Psi^2} \arrow[r] \arrow[dr] & Cn_{\Psi^0} \arrow[r] \arrow[l, dotted, shift left] & Cn_{\Psi^1} \arrow[l, dotted, shift left] \arrow[d] \\
Cn_{\Pi^2} \arrow[r] & Cn_{\Pi^0} \arrow[r] \arrow[l, dotted, shift left] \arrow[d] & Cn_{\Pi^1} \arrow[l, dotted, shift left] \arrow[d, leftrightarrow] \\
 & Cn_{\Phi^0} \arrow[u, dotted, shift left] \arrow[r, dotted, leftrightarrow] \arrow[dr] & Cn_{\Phi^1} \arrow[d] \\
 & & Cn_{\mathbf{P}}
\end{tikzcd}
$$

Figure 6: Schematic overview of the relations between the various logics. Dotted arrows hold in the transitive case.

6.3 The Relation to S

Now that we have settled the relations between the various alternatives to **S**, it is a matter of routine to prove the following:

Theorem 7 *For every* $\mathbf{X}$ *defined in this paper, each of the following holds*

(i) $\Vdash_{\mathbf{X}} \subseteq \Vdash_{\mathbf{S}}$
(ii) If $\langle \mathcal{M}(\Gamma), \prec\rangle$ *is smooth, then* $Cn_{\mathbf{X}}(\Gamma) = Cn_{\mathbf{S}}(\Gamma)$

Proof. Ad (i). In view of the relations between the various logics (see Figure 6), it suffices to prove that both $\Vdash_{\Phi^0}$ and $\Vdash_{\Phi^1}$ are included in $\Vdash_{\mathbf{S}}$. As the reasoning is analogous in both cases, we only do this for $\Vdash_{\Phi^0}$. Suppose (†) $\Gamma \Vdash_{\Phi^0} A$. Let $M \in \mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$ – we have to show that $M \models A$. By (†), there is an $M' \in \mathcal{M}(\Gamma)$ such that (1) $M' \preceq M$ and (2) for all $M'' \in \mathcal{M}(\Gamma)$ with $M'' \preceq M'$, $M'' \models A$. Since M is $\prec$-minimal in $\mathcal{M}(\Gamma)$, we have $M' = M$ from (1), and hence by (2), $M \models A$.

Ad (ii). Suppose that $\prec$ is smooth. In view of the relations between the various logics and by item (i), it suffices to prove that (ii.a) $Cn_{\mathbf{S}}(\Gamma) \subseteq Cn_{\Psi^2}(\Gamma)$ and that (ii.b) $Cn_{\mathbf{S}}(\Gamma) \subseteq Cn_{\Pi^2}(\Gamma)$.

Ad (ii.a). By the supposition, $\Psi^0(\mathcal{M}(\Gamma)) = \mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$. Since by definition, no $M \in \mathcal{M}(\Gamma) - \mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$ can be below any $M' \in \mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$, also $\Psi^2(\mathcal{M}(\Gamma)) = \mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$. The rest is immediate in view of the definition of $\Vdash_{\Psi^2}$ and $\Vdash_{\mathbf{S}}$.

Ad (ii.b). Note that the supposition holds iff $\mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$ is $\prec$-dense in $\mathcal{M}(\Gamma)$. By definition, $\mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$ is also $\prec$-lower in $\mathcal{M}(\Gamma)$. Finally, since there are no $M, M' \in \mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$ with $M \prec M'$, it follows that $\mathsf{min}_{\prec}(\mathcal{M}(\Gamma))$ is also $\prec$-clear in $\mathcal{M}(\Gamma)$. The rest is immediate in view of the definitions of $\Vdash_{\Pi^2}$ and $\Vdash_{\mathbf{S}}$. ■

For the local transitive closure-variants, a slightly stronger result can be shown. That is, suppose that $\langle \mathcal{M}(\Gamma), \prec^{\mathsf{tr}}_{\mathcal{M}(\Gamma)}\rangle$ is smooth – note that this does not imply that $\langle \mathcal{M}(\Gamma), \prec \rangle$ is smooth, unless we presuppose transitivity of $\prec$. In that case, it can easily be verified that $\Psi^1(\mathcal{M}(\Gamma)) = \mathsf{min}_\prec(\mathcal{M}(\Gamma))$ and $\mathsf{min}_\prec(\mathcal{M}(\Gamma)) \in \Pi^1(\mathcal{M}(\Gamma))$. So we have:

Theorem 8 *If $\langle \mathcal{M}(\Gamma), \prec^{\mathsf{tr}}_{\mathcal{M}(\Gamma)} \rangle$ is smooth, then $Cn_{\Psi^1}(\Gamma) = Cn_{\Pi^1}(\Gamma) = Cn_{\mathbf{S}}(\Gamma)$.*

In view of Theorem 8, one may motivate Cn_{Π^1} as a way to extend Cn_{Π^0} and obtain full **S** in case $\langle \mathcal{M}(\Gamma), \prec^{\mathsf{tr}}_{\mathcal{M}(\Gamma)} \rangle$ is smooth, yet nevertheless preserve the main intuition behind Cn_{Π^0} in the general case. On the one hand, by taking the local transitive closure of $\prec$, we can ensure that whenever there are no infinitely descending sequences of models, only the minimal models are selected.[41] On the other hand, if there are infinitely descending sequences, then Π^1 will take these into account, and hence not lead to the absurdities of **S** that were described in the introduction.

7 Concluding Remarks and Future Work

As shown in this paper, it is possible to obtain preferential semantics with a fairly strong metatheory, also in the absence of properties such as smoothness or transitivity. We showed how important properties do not hold for previous proposals from the literature in such cases, and proposed variants that fare better. In particular, Cn_{Ψ^2} and Cn_{Π^2} seem to preserve the main intuitions of their original counterparts, yet have a significantly stronger metatheory in non-transitive cases.

However, one small warning is in place: none of the variants discussed in this paper preserve the disjunction property in the non-transitive case. One aim of future research may thus be to develop still other variants that warrant the disjunction property in the general case. Also, one may ask if any representation theorems can be obtained for the variants studied in this paper.

We introduced the notion of multi-selection semantics and studied its properties. This allowed us to simplify several meta-proofs in the paper. So another topic that deserves further attention is the study of this broad type of semantics, and its application to problems other than non-smooth configurations of models. In particular, one may use this type of semantics to model so-called credulous approaches in AI, where properties such as right absorption and right satisfiability typically fail.

In the current paper, we focused on consequence relations, hence on meta-level expressions. However, the constructions we presented can just as well be used to model object-level expressions of the type "A normally implies B", or "if A is the case, then B should be the case". In fact, the research on preferential entailment often switches between the object-level and meta-level interpretation of **S**– see also our discussion of the Boutilier-variant in Section 5.

Let us end with a more personal remark, which points at a still different branch of future research. Our work on this subject was originally instrumental to the development of a generic format of adaptive logics in terms of selection functions – see [X]. However, when working out the main ideas for that paper, certain problems arose which apply to any preferential semantics in the sense

[41] Note that the requirement that $\Pi^0(X)$ is $\prec$-dense in X forces us to select some non-minimal models in cases where $\langle \mathcal{M}(\Gamma), \prec \rangle$ is not smooth.

of **S**. This urged us to work on the current subject, which in turn gave rise to the notion of multi-selection semantics (as a generalization of Schlechta's Limit Variant). With this concept in mind, we may now return to adaptive logics. Hence we may ask: can we further generalize the format from [X] in order to incorporate the idea of multi-selection semantics? And under what conditions may we then preserve certain metatheoretic properties, such as the ones from our current checklist?

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