Pooling Modalities and Pointwise Intersection: Semantics, Expressivity, and Applications

Abstract

We study classical modal logics with *pooling modalities*, i.e. unary modal operators that allow one to express properties of sets obtained by the *pointwise intersection* of neighbourhoods. We discuss salient properties of these modalities, situate the logics in the broader area of modal logics (with a particular focus on relational semantics), establish key properties concerning their expressive power, and discuss their application to epistemic/doxastic logic, the logic of evidence-based belief, deontic logic, and logics of agency and ability.

keywords: pointwise intersection, pooling modalities, distributed belief, coalition logic, classical modal logics, multi-agent systems

1 Introduction

Neighbourhood models are a well-established tool to study generalizations and variants of relational semantics and non-normal modal logics.[1] They have been successfully applied to i.a. the dynamics of evidence and beliefs [66], the logic of ability [14, 51], conflict-tolerant deontic logic [28], and the analysis of (descriptive or normative) conditionals [15, 42]. The generalization from relational semantics to neighbourhood semantics allows one to invalidate certain schemata that are problematic for a given interpretation of the modal operator $\Box$, but also to include other schemata that would trivialize any normal modal logic.[2] Apart from giving us more logical options, neighbourhood models can also be used as a purely technical tool in order to prove completeness or incompleteness with respect to other semantics.[3] Finally, neighbourhood models bear very close links to topological models [44, 64] and to subset spaces [47].[4]

A neighbourhood collection $\mathcal{N}(w)$ for a unary modality $\Box$ generalizes the idea of a set X of accessible worlds (from a given world w, by a relation R) that is familiar

[1] Scott [58] and Montague [46] are often seen as the inventors of neighbourhood models; Chellas [16] and Segerberg [59] are usually cited as the main figures in their development.

[2] Examples are the axioms $\Box\bot$ and $\Box(\varphi \vee \psi) \to (\Box\varphi \wedge \Box\psi)$, which have been studied in the context of deontic reasoning, cf. [67], and the axiom $(\Box(\varphi \vee \psi) \wedge \neg\Box\varphi) \to \Box\psi$ that is considered for logics of agency, cf. [45].

[3] One prototypical example of a completeness proof via neighbourhood semantics is [42]. In [32], neighbourhood semantics are used to prove the incompleteness of Elgesem's modal logic of agency [20].

[4] We refer to [50] for a critical introduction to the many forms, uses and advantages of neighbourhood semantics.

from relational semantics to a *set* of accessible sets $X_1, X_2, \ldots$ of worlds, called the *neighbourhoods of* w. On one specific interpretation of the modal operator, $\Box\varphi$ is true iff some neighbourhood X of w is identical to the truth set of φ.[5] Extensions of propositional classical logic with a modal operator $\Box$ of this type are standardly called *classical modal logics* following [16] and include normal modal logics as a special case.

Just as for relational semantics, one can study classical modal logics that are the result of imposing certain relations on the neighbourhood functions, or of treating some neighbourhood functions as defined from others. In particular, the standard operations of non-deterministic choice, sequential composition, and iteration from **PDL** [36] have been generalized to various types of neighbourhood semantics and formal languages, cf. [54, 63]. In [33, 34], operations on monotonic neighbourhood models are studied from an abstract, algebraic viewpoint, giving rise to highly generic completeness results.[6] However, notwithstanding these important achievements, the counterpart of *intersections* of accessibility relations for neighbourhood semantics is largely unknown.

The current paper fills this gap by introducing and studying the notion of *pointwise intersection.* Let us explain this concept by means of a simple example – exact details will be provided in subsequent sections. Suppose that $\mathcal{N}_1(w)$ and $\mathcal{N}_2(w)$ are two neighbourhood collections, representing the beliefs of agent 1, resp. agent 2 at the world w. Let $\mathcal{N}_1(w) = \{X, Y\}$ and $\mathcal{N}_2(w) = \{Z\}$, where each of X, Y, and Z are sets of possible worlds. Then the pointwise intersection of $\mathcal{N}_1(w)$ and $\mathcal{N}_2(w)$ is defined as the set $\mathcal{X}$ of all sets $V \cap U$, where $V \in \mathcal{N}_1(w)$ and $U \in \mathcal{N}_2(w)$. In particular,

$$\mathcal{N}_1(w) \Cap \mathcal{N}_2(w) = \{X \cap Z, Y \cap Z\}$$

Pointwise intersection is however not just limited to binary (or finite) combinations of distinct sets of neighbourhoods: one may also intersect the members of one neighbourhood collection, or use several members of one neighbourhood collection in combination with members of other neighbourhood collections, in forming a new neighbourhood. Continuing with our example, we have:

$$\mathcal{N}_1(w) \Cap \mathcal{N}_1(w) = \{X \cap X, Y \cap Y, X \cap Y\} = \{X, Y, X \cap Y\}$$

and

$$\mathcal{N}_1(w) \Cap \mathcal{N}_1(w) \Cap \mathcal{N}_2(w) = \{X \cap X \cap Z, Y \cap Y \cap Z, X \cap Y \cap Z\} = \{X \cap Z, Y \cap Z, X \cap Y \cap Z\}$$

More generally, given any function M that specifies, for each neighbourhood function $\mathcal{N}_i$ in the original model, how many members of $\mathcal{N}_i(w)$ should go in the intersection for the world w, we can define a unique new neighbourhood function $\mathcal{N}_M$. This new neighbourhood function can then be used to interpret a corresponding classical modal operator $\Box_M$.

[5] We refer to Section 2 for all definitions of these concepts. We discuss the alternative, "monotonic" interpretation of $\Box$ in Section 4.3.

[6] A neighbourhood model is monotonic if and only if at every world, the set of neighbourhoods is closed under supersets. See also Section 2.

We call operators of the type $\Box_M$ *pooling modalities*, as they allow us to express information that *would* be obtained if certain pieces of information or attributes – whether doxastic, evidential, agentive, deontic, or other – are pooled, i.e. combined by means of intersection.

After introducing pooling modalities and their semantics in general and exact terms, we will establish three claims in this paper. First, the operation of pointwise intersection generalizes the intersection of accessibility relations in the exact same way that neighbourhood models generalize relational models. Second and relatedly, pooling modalities add considerable expressive power to the basic (multi)modal language, allowing us to express various properties of neighbourhood models. Third, pooling modalities have natural interpretations and applications, ranging from "implicit belief of an agent", over "distributed evidence among a group of agents" and "norms entailed by a code or legal system", to "coalitional ability".

In the companion paper [4], we prove that notwithstanding their expressive power, many of the resulting logics are well-behaved, i.e. they have an elegant and unified sound and strongly complete axiomatization, they enjoy the finite model property, and they are decidable. Taken as a whole, these two papers are meant to provide a solid foundation for further research on applications of pooling modalities, their metatheory, and related formal languages.[7]

This paper is structured as follows. In Section 2 we recall some basics of neighbourhood semantics and classical modal logics. Section 3 spells out the exact definition of pointwise intersection as an operation on neighbourhood models, and introduces various formal languages that feature pooling modalities. In Section 4 we investigate the relation between pooling modalities and intersection in relational semantics in detail, and situate our formalism in the landscape of classical modal logics more generally. The expressive power of formal languages with pooling modalities is discussed in Section 5. In Section 6, finally, we argue for the philosophical use of pooling modalities, discussing a range of (potential) applications and relating them to the formal work of earlier sections. Readers who are predominantly interested in those applications may well skip Sections 4 and 5 on a first reading, but return to them after reading Section 6. Section 7 concludes with a list of open problems.

2 Neighbourhood Models: a Quick Rehearsal

To set the stage for our contribution, we recall the basics of neighbourhood semantics, fixing notation and terminology along the way. Readers who are familiar with [50]

[7] In [50, Section 3.3], the idea of pointwise intersection and pooling modalities is introduced (using different terms) as a generalization of distributed belief, but no formal results about the ensuing logics are given. In [68], Van De Putte and Klein established completeness results for fragments of some of the logics that are studied in the current paper. The main differences between [68] and the present paper are: (a) [68] only concerns pointwise intersections of *distinct* neighbourhoods, hence, not of a single neighbourhood with itself or more complex combinations with certain neighbourhoods being used more than once; as a result, (b) [68] does not discuss operations of "arbitrary intersection". Also, (c) in the present paper, we include the universal modality, and (d) consider the monotonic semantic clause (cf. Section 4.3). Finally, (e) [68] is only concerned with axiomatizations and completeness, whereas here we include results on expressivity, give a more elaborate discussion of applications, and (in [4]) establish the finite model property and decidability of the logics in question.

can skip this section.

Basic Languages We start by introducing two formal languages. Fix a countable set $I = \{1, 2, \ldots\}$ of ***indexes***[8] and a countable set $\mathfrak{P} = \{p_1, p_2, \ldots\}$ of propositional variables. Where p ranges over $\mathfrak{P}$ and i over I, the language $\mathfrak{L}$ is given by the following Backus-Naur form (BNF):

$$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_i \varphi.$$

The classical connectives $\wedge, \rightarrow, \leftrightarrow$ and the constant $\top$ are defined according to the well-known classical logic schemata. In the remainder of this paper, $\varphi, \psi, \ldots$ are used as metavariables for formulas, and $\Gamma, \Delta, \ldots$ as metavariables for sets of such formulas. Conjunction ($\bigwedge$) and disjunction ($\bigvee$) over finite sets Δ are defined in the standard way.[9]

For the sake of expressive power and in order to characterize specific frame conditions it is common to extend $\mathfrak{L}$ with a universal modality $[\forall]$, cf. [31]. The resulting language $\mathfrak{L}^{[\forall]}$ is given by the BNF:

$$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_i \varphi \mid [\forall]\varphi.$$

We use $[\exists]$ to denote the dual of $[\forall]$. In the remainder, we refer to $\mathfrak{L}$ and $\mathfrak{L}^{[\forall]}$ as the two ***basic languages*** that will be enriched with pooling modalities in Section 3.

Frames and Models A ***frame*** – is a tuple that specifies a non-empty set of possible worlds W and, for each index $i \in I$ and $w \in W$, a corresponding set $\mathcal{N}_i(w)$ of subsets of W, called the ***neighbourhoods of*** i ***at*** w. We will use the term ***neighbourhood set*** (for i) to refer to the set $\mathcal{N}_i(w)$. Depending on the application, $\mathcal{N}_i(w)$ may e.g. denote the beliefs that agent i holds at w, the permissions that are granted in view of normative system i at w, or the propositions one can guarantee given the resource i at w.

To interpret $\mathfrak{L}^{[\forall]}$, we extend frames to models (Definition 1) and give semantic clauses for each of the components of $\mathfrak{L}^{[\forall]}$ (Definition 2):

Definition 1 *A **model** $\mathfrak{M}$ is a triple $\langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$, where (i) $W \neq \emptyset$ is the **domain** of $\mathfrak{M}$, (ii) for every $i \in I$, $\mathcal{N}_i : W \to \wp(\wp(W))$ is a **neighbourhood function for** i, and (iii) $V : \mathfrak{P} \to \wp(W)$ is a **valuation function**.*

Definition 2 *Where $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ is a model and $w \in W$:*

0. $\mathfrak{M}, w \not\models \bot$

1. $\mathfrak{M}, w \models \varphi$ *iff* $w \in V(\varphi)$ *for all* $\varphi \in \mathfrak{P}$

2. $\mathfrak{M}, w \models \neg\varphi$ *iff* $\mathfrak{M}, w \not\models \varphi$

[8] Depending on the specific application, indexes may be interpreted in various ways: they may refer to agents whose beliefs, evidence, or abilities we want to model; to legal or moral codes implying certain obligations or permissions; or to resources enabling a given agent to guarantee certain states of affairs.

[9] If $\Delta = \emptyset$, then $\bigvee \Delta =_{\mathsf{df}} \top$ and $\bigwedge \Delta =_{\mathsf{df}} \bot$.

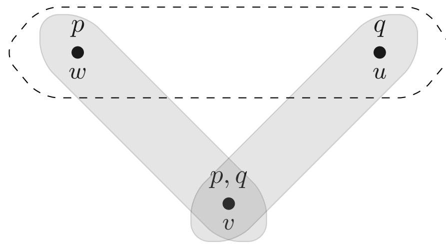

Figure 1: A representation of the model in Example 1. $\mathcal{N}_1(w)$ consists of the grey sets, $\mathcal{N}_2(w)$ of the dashed set.

3. $\mathfrak{M}, w \models \varphi \vee \psi$ *iff* $\mathfrak{M}, w \models \varphi$ *or* $\mathfrak{M}, w \models \psi$
4. $\mathfrak{M}, w \models \Box_i \varphi$ *iff* $\|\varphi\|^{\mathfrak{M}} \in \mathcal{N}_i(w)$
5. $\mathfrak{M}, w \models [\forall]\varphi$ *iff* $\mathfrak{M}, w' \models \varphi$ *for all* $w' \in W$.

where $\|\varphi\|^{\mathfrak{M}} = \{w \in W \mid \mathfrak{M}, w \models \varphi\}$.

Validity($\Vdash \varphi$) and semantic consequence ($\Gamma \Vdash \varphi$) are defined in the standard way, viz. as truth, resp. truth-preservation at all worlds in all models.

The semantics just presented captures the so-called ***exact reading*** of the modal operators $\Box_i$: for $\Box_i \varphi$ to be true at w, the exact truth set of φ has to be in the neighbourhood of i at w. This is often contrasted with a weaker reading where $\Box_i \varphi$ is true iff some neighbourhood of i at w is a (perhaps proper) subset of $\|\varphi\|^{\mathfrak{M}}$. We will discuss this weaker reading in more detail in Section 4.3.

Example 1 *Consider a model* $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ *with* $W = \{w, v, u\}$, $\mathcal{N}_1(w) = \{w, v\}, \{v, u\}$, *and* $\mathcal{N}_2(w) = \{w, u\}$, *and* $V(p) = \{w, v\}, V(q) = \{v, u\}$. *In this model,* $\Box_1 p$, $\Box_1 q$, *and* $\Box_2(\neg p \vee \neg q)$ *are all true at* w, $[\forall](p \vee q)$ *is true at each world. Figure 1 gives a graphic representation of this model.*

Frame Conditions As the above example illustrates, the semantics and logic characterized by the class of all models is very weak. Note that $\Box_1(p \wedge q)$, $\Box_1(p \vee q)$, and $\Box_1 \top$ are false at w in this model. These facts illustrate the failure of three well-known properties of logics characterized by relational semantics, viz. aggregation, monotony, and necessitation.[10] For $\mathfrak{L}$, the class of all neighbourhood models gives us the weakest modal logic that satisfies the principle of ***replacement of equivalents*** (RE):

$$\text{if } \Vdash \varphi \leftrightarrow \psi, \text{ then } \Box_i \varphi \to \Box_i \psi$$

To characterize the weakest logic in $\mathfrak{L}^{[\forall]}$, we need to add all axioms of **S5** for $[\forall]$ and replace (RE) with the axiom of ***replacement of global equivalents*** (RGE):[11]

[10] See Table 1 where those properties are specified.

[11] Note that, by necessitation for $[\forall]$, $\Vdash \varphi \leftrightarrow \psi$ entails $\Vdash [\forall](\varphi \leftrightarrow \psi)$. So from $\Vdash \varphi \leftrightarrow \psi$ we can derive $\Vdash \Box_i \varphi \leftrightarrow \Box_i \psi$ using (RGE).

(N_i)	$\Box_i \top$	$W \in \mathcal{N}_i(w)$
(P_i)	$\neg\Box_i \bot$	$\emptyset \notin \mathcal{N}_i(w)$
(T_i)	$\Box_i\varphi \to \varphi$	for all $X \in \mathcal{N}_i(w)$, $w \in X$
(M_i)	$\Box_i(\varphi \wedge \psi) \to (\Box_i\varphi \wedge \Box_i\psi)$	$\mathcal{N}_i(w) = (\mathcal{N}_i(w))^{\uparrow}$
(C_i)	$(\Box_i\varphi \wedge \Box_i\psi) \to \Box_i(\varphi \wedge \psi)$	$\mathcal{N}_i(w)$ is closed under finite intersections
(U_i)	$\Box_i\varphi \to [\forall]\Box_i\varphi$	$\mathcal{N}_i(w) = \mathcal{N}_i(w')$ for all $w, w' \in W$

Table 1: Modal axioms and corresponding frame conditions.

$$[\forall](\varphi \leftrightarrow \psi) \to (\Box_i\varphi \to \Box_i\psi)$$

Having applications to doxastic logic, deontic logic, or the logic of ability in mind, one may require various additional properties of the logic. Such properties are to be had if we impose additional *frame conditions*, i.e., conditions on the frames underlying our models. Frame conditions can also be motivated in terms of the interpretation of the models. For instance, if we interpret the members of a neighbourhood as "those states of affairs the agent can guarantee, independently of any other agent", then it is common to require the neighbourhood sets to be closed under supersets.

Table 1 lists some standard axiom schemata and frame conditions that characterize them. Here and in the remainder, $\mathcal{X}^{\uparrow}$ denotes the closure of $\mathcal{X}$ under supersets.[12] The quantification over w in the conditions on the neighbourhood functions is universal; e.g. the condition for (N) reads that *for all* $w \in W$, $W \in \mathcal{N}_i(w)$. Note that, for the last condition in this table, we need the universal modality in order to obtain a characteristic axiom. When some operator $\Box_i$ satisfies (M_i), we say it is ***monotonic***.

Augmented models, supplementation As mentioned in the introduction, neighbourhood models are a generalization of relational models for modal logic. More specifically, if we use the standard semantic clause for $\Box$ for relational models, and the exact semantic clause for neighbourhood models, then the class of relational models is modally equivalent to the class of neighbourhood models that are "augmented".[13] In the present context, the latter notion can be defined as follows:

Definition 3 *Let D be a set and $\mathcal{X} \subseteq \wp(D)$. $\mathcal{X}$ is **augmented** iff (i) $\mathcal{X} = \mathcal{X}^{\uparrow}$ (i.e., $\mathcal{X}$ is closed under supersets), (ii) $W \in \mathcal{X}$, and (iii) $\mathcal{X} = \Cap^{\infty}\mathcal{X}$ (i.e., $\mathcal{X}$ is closed under arbitrary intersections). A model $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ is **augmented** iff for all $w \in W$ and all $i \in I$, $\mathcal{N}_i(w)$ is augmented.*

A model is called *monotonic* iff for all $i \in I$, for all $w \in W$, $\mathcal{N}_i(w)$ is closed under supersets. The following well-known notion is useful in the study of monotonic models:

[12] Formally, where $\mathcal{X} \subseteq \wp(D)$ for some domain D: $\mathcal{X}^{\uparrow} = \{Y \subseteq D \mid X \subseteq Y$ for some $X \in \mathcal{X}\}$.

[13] This result dates back to [16]; see [50, Theorem 2.21] for a recent proof and discussion with the notation we use.

Definition 4 *Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a model. The* ***supplementation of*** $\mathfrak{M}$, *$\mathfrak{M}^{\uparrow}$, is the triple $\langle W, \langle \mathcal{N}_i^{\uparrow} \rangle_{i \in I}, V \rangle$, where, for all $w \in W$, $\mathcal{N}_i^{\uparrow}(w)$ is the closure of $\mathcal{N}_i(w)$ under supersets.*

3 Pointwise Intersection and Pooling Modalities

In this section, we enrich the basic languages with pooling modalities. In order to interpret them, we need to introduce the notion of pointwise intersection in full generality. After that, we will discuss some key validities and invalidities of the resulting logics.

Semantics of Pooling Modalities Consider again Example 1 as depicted in Figure 1. Here, the neighbourhood for index 1 contains $\|p\|$ and $\|q\|$, and the neighbourhood for index 2 contains $\|\neg(p \wedge q)\|$. Suppose we interpret $\Box_i \varphi$ as "i has some piece of evidence that indicates that φ is the case". Under this reading, aggregation is not valid: having pieces of evidence, each for various propositions, does not entail having a single piece of evidence for the conjunction of those propositions. However, in the example, there is some sense in which agent 1 has information that contradicts the information of agent 2. That is, if we *would* combine the various propositions for which 1 has evidence, then we *would* obtain information that contradicts 2's evidence. Claims like these can be made exact by introducing the notion of pointwise intersection, and expressed in a formal language by pooling modalities.

Let us start with the former:

Definition 5 *Let D be a set, let $\mathcal{X}, \mathcal{Y} \subseteq \wp(\wp(D))$, and let $k \in \mathbb{N}$.*

1. *$\mathcal{X} \Cap \mathcal{Y} = \{X \cap Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$ is the* ***pointwise intersection of $\mathcal{X}$ and $\mathcal{Y}$.***
2. *$\Cap^k \mathcal{X} = \{X_1 \cap \ldots \cap X_k \mid X_1, \ldots, X_k \in \mathcal{X}\}$ is the* ***pointwise k-intersection of $\mathcal{X}$ with itself.***
3. *$\Cap^\infty \mathcal{X} = \{\bigcap \mathcal{Y} \mid \mathcal{Y} \subseteq \mathcal{X}\}$ is the* ***pointwise arbitrary intersection of $\mathcal{X}$ with itself.***

In order to talk about pointwise intersections of neighbourhood sets, we first need a means to denote the indexes $i \in I$ that are being combined, and the number of neighbourhoods in each such $\mathcal{N}_i(w)$ that go into the intersection. This requires some additional notation.

Definition 6 ***Pooling profiles*** *are functions of the type $M : I \to \mathbb{N} \cup \{\infty\}$, where (i) for only finitely many $i \in I$, $M(i) \neq 0$ and (ii) for at least one $i \in I$, $M(i) > 0$. $\mathbb{M}_\infty$ denotes the set of all pooling profiles. $\mathbb{M}_f \subset \mathbb{M}_\infty$ is the set of all* ***finitary*** *pooling profiles, i.e. pooling profiles of the type $M : I \to \mathbb{N}$.*
Where $M \in \mathbb{M}_\infty$, $\mathbf{I(M)} =_{\mathsf{df}} \{i \in I \mid M(i) \neq 0\}$.

When writing about pooling profiles, we will often switch from functional to a simplified relational notation, writing every pooling profile as a finite set of pairs (i, k) for $k \in \mathbb{N}^+ \cup \{\infty\}$, thus omitting all pairs $(j, 0)$. Note that with such notation, every pooling profile is a finite set.

$\mathfrak{L}_\infty$	$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M\varphi$	where $p \in \mathfrak{P}$ and $M \in \mathbb{M}_\infty$
$\mathfrak{L}_\infty^{[\forall]}$	$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M\varphi \mid [\forall]\varphi$	where $p \in \mathfrak{P}$ and $M \in \mathbb{M}_\infty$
$\mathfrak{L}_f$	$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M\varphi$	where $p \in \mathfrak{P}$ and $M \in \mathbb{M}_f$
$\mathfrak{L}_f^{[\forall]}$	$\varphi := p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Box_M\varphi \mid [\forall]\varphi$	where $p \in \mathfrak{P}$ and $M \in \mathbb{M}_f$

Table 2: Languages with pooling modalities.

Intuitively, a pooling profile M indicates, for each $i \in I$, the number of sets $X \in \mathcal{N}_i(w)$ that we can use in order to obtain a member Y of the neighbourhood set $\mathcal{N}_M(w)$. The symbol ∞ can be read as "arbitrarily many". So e.g., that $X \in \mathcal{N}_{\{(1,\infty)\}}(w)$ means that X is the result of intersecting an arbitrary number of members of $\mathcal{N}_1(w)$; that $Y \in \mathcal{N}_{\{(1,2),(2,\infty)\}}(w)$ means that Y can be obtained by intersecting 2 members of $\mathcal{N}_1(w)$ with an arbitrary number of members of $\mathcal{N}_2(w)$.[14]

Definition 7 *Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i\in I}, V\rangle$ be a neighbourhood model and let $M \in \mathbb{M}_\infty$, with $I(M) = \{i_1, \ldots, i_n\}$. The **neighbourhood function** $\mathcal{N}_M$ is defined as follows: for every $w \in W$,*

$$\mathcal{N}_M(w) = (\Cap^{M(i_1)}\mathcal{N}_{i_1}(w)) \Cap \ldots \Cap (\Cap^{M(i_n)}\mathcal{N}_{i_n}(w))$$

Since we will often have to refer to specific (relatively small) pooling profiles in examples, we will use a simplified "multiset" notation for them: e.g. $\{1, 1, 2^\infty, 3\}$ denotes the pooling profile M for which $M(1) = 2$, $M(2) = \infty$, $M(3) = 1$, and $M(i) = 0$ for all $i \in I \setminus \{1, 2, 3\}$. When such sets occur in subscripts we will also omit set brackets whenever this causes no confusion.

Example 2 *Consider again Example 1 from the previous section (Figure 1). In this model, $\{w\} \in \mathcal{N}_{1,2}(w), \{u\} \in \mathcal{N}_{1,1,}(w)$, and $\emptyset \in \mathcal{N}_{1,1,2}(w)$.*

We are now in a position to define the logics that play central stage in this paper. The formal languages $\mathfrak{L}_\infty$ and $\mathfrak{L}_\infty^{[\forall]}$ are obtained by replacing $\Box_i$ ($i \in I$) with $\Box_M$ ($M \in \mathbb{M}_\infty$) in $\mathfrak{L}$, resp. $\mathfrak{L}^{[\forall]}$. In a similar fashion we obtain $\mathfrak{L}_f$ and $\mathfrak{L}_f^{[\forall]}$, where instead of $\mathbb{M}_\infty$ we use $\mathbb{M}_f$. Table 2 gives the BNFs of each of these languages.

These languages are interpreted in the exact same way as before (cf. Definition 2), except that we use the following semantic clause for the pooling modalities $\Box_M$:

Definition 8 *Where $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i\in I}, V\rangle$ is a neighbourhood model, $M \in \mathbb{M}_\infty$, and $w \in W$,*

$\mathfrak{M}, w \models \Box_M\varphi$ *iff* $\|\varphi\|^{\mathfrak{M}} \in \mathcal{N}_M(w)$.

Returning to Example 1, we have that $\mathfrak{M}, w \models \Box_{1,2}(p \wedge \neg q)$, $\mathfrak{M}, w \models \Box_{1,1}(p \wedge q)$, $\mathfrak{M}, w \models \Box_{1,1,2}\bot$, and $\mathfrak{M}, w \models \Box_{1^\infty,2^\infty}\bot$. The latter formula expresses that there is

[14] "Arbitrary" should be interpreted here in the strongest possible sense, corresponding to the third item of Definition 5.

conflicting information among the group of agents $\{1,2\}$. In contrast, the evidence of agent 1 alone is consistent, and $\mathfrak{M}, w \models \neg\Box_{1^\infty}\bot$.

It can be easily observed that for the border case where a pooling profile M coincides with a single index i, i.e. $M = (i, 1)$, this semantic clause is conservative with respect to the original semantic clause for $\Box_i$:

Fact 1 *Let $M \in \mathbb{M}_\infty$ and $i \in I$ be such that $I(M) = i$ and $M(i) = 1$. Then, for all models $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ and all $w \in W$: $\mathfrak{M}, w \models \Box_M \varphi$ iff $\mathfrak{M}, w \models \Box_i \varphi$.*

In view of this fact, the languages with pooling modalities are a conservative extension of the corresponding languages with only single-indexed operators.

Logical Properties of Pooling Modalities To illustrate the above notions and give the reader a feeling of the behavior of pooling modalities, let us go over a few validities and invalidities of this semantics. First, given the well-known correspondence between intersecting sets of worlds and classical conjunction, pooling distinct propositions – whether across different indexes, or within a single index – gives us the conjunction of those propositions for the associated pooling profile. The following three validities illustrate this:

$$\Vdash (\Box_1 p \wedge \Box_2 q) \rightarrow \Box_{1,2}(p \wedge q) \tag{1}$$

$$\Vdash (\Box_1 p \wedge \Box_1 r) \rightarrow \Box_{1,1}(p \wedge r) \tag{2}$$

$$\Vdash (\Box_1 p \wedge \Box_2 q \wedge \Box_1 r) \rightarrow \Box_{1,1,2}(p \wedge q \wedge r) \tag{3}$$

Note that in the consequents of (2) and (3), the multiset notation allows us to represent the sources of pooled information in a very perspicuous way. Conversely, one may ask whether the pooled information that φ given a pooling profile M always entails that φ is already attributable to some index $i \in I(M)$. This is not the case: it may be that *only* by using information that is distributed among the various indexes, or by combining distinct neighbourhoods for a single index, we can arrive at φ. So e.g.

$$\nVdash \Box_{1,2} p \rightarrow (\Box_1 p \vee \Box_2 p) \tag{4}$$

$$\nVdash \Box_{1,1} p \rightarrow \Box_1 p \tag{5}$$

The invalidity of the implication in (4) is well-known from the literature on intersection modalities in relational semantics, cf. [22]. (5) shows that, in the context of neighbourhood models (since neighbourhoods need not be closed under intersection), the implication also fails in the special case with only one index .

One may also consider the converse of (4). According to the standard account of distributed belief on relational models, whenever φ is believed by some G, then φ is also distributed belief for every supergroup $H \supset G$. Crucially, the analogue of this implication fails for the weaker, non-normal notions of pooled information that we study here:

$$\not\Vdash \Box_1 p \to \Box_{1,2} p \tag{6}$$

To see why the implication in (6) is invalid, consider a model $\mathfrak{M}$ with world w in which the neighbourhood for index 2 is empty, i.e., $\mathcal{N}_2 = \emptyset$, yet $\|p\| \neq \emptyset$. Then by Definition 7, also $\mathcal{N}_{1,2}(w) = \emptyset$ and hence there is no φ such that $\Box_{1,2}\varphi$ holds at w in $\mathfrak{M}$. So here already we see that, by abandoning the assumption that every agent must have at least some belief, the associated modalities for distributed belief behave differently. But even if we assume non-emptiness of the neighbourhoods, the implication in (6) may still fail. That is, suppose that $\mathcal{N}_1(w) = \{\|p\|^{\mathfrak{M}}\}$ and $\mathcal{N}_2(w) = \{\|q\|^{\mathfrak{M}}\}$. Then the only element of $\mathcal{N}_{1,2}(w)$ is $\|p \wedge q\|^{\mathfrak{M}}$, which may be distinct from both $\|p\|$ and $\|q\|$. So in such a case, $\Box_1 p$ holds whereas $\Box_{1,2} p$ fails.

More restricted versions of the implication in (6) do however obtain. First, if we just "enlarge" a pooling profile by adding more occurrences of a given index $i \in I(M)$, then we obtain at least as much pooled information:

$$\Vdash \Box_{1,2}(p \vee r) \to \Box_{1,1,2}(p \vee r) \tag{7}$$

This follows immediately from the fact that, in our definition of pointwise intersection, we do not require the intersected sets to be distinct. So e.g. where $X \in \mathcal{N}_1(w)$ and $Y \in \mathcal{N}_2(w)$, we have $X \cap X \cap Y = X \cap Y \in \mathcal{N}_{1,1,2}(w)$.

Second, in case we add a new index i, for which we know that $\mathcal{N}_i(w)$ contains some proposition φ that includes the original proposition ψ, then ψ is preserved when moving to the larger pooling profile. In other words, it suffices for that i to have some (possibly redundant) information that includes all φ-worlds, in order to ensure that the group including i also has the pooled information that ψ. Examples of associated validities are:

$$\Vdash (\Box_1 p \wedge \Box_2 \top) \to \Box_{1,2} p \tag{8}$$

and

$$\Vdash (\Box_{1,1,2}(\neg p \wedge q) \wedge \Box_3 q) \to \Box_{1,1,2,3}(\neg p \wedge q) \tag{9}$$

Finally, note that the total set W can only be obtained by intersecting some sets $X_1, \ldots, X_n$ when those X_i are themselves equal to W. As an immediate result, we obtain the following restricted variants of (4), resp. (5):

$$\Vdash \Box_{1,2}\top \to \Box_1 \top \tag{10}$$

$$\Vdash \Box_{1,1}\top \to \Box_1 \top \tag{11}$$

This completes our brief survey of some salient properties of pooling modalities. A thorough investigation of the set of all such validities – by means of a sound and strongly complete axiomatization – is provided in the companion paper [4].

4 Relational Semantics as a Special Case

In this section, we show that pointwise intersection of neighbourhood functions is a generalization of the intersection of accessibility relations. In fact, there are two ways in which normal modal logics with intersection modalities can be captured in terms of pooling modalities:

(a) Over augmented neighbourhood models, pointwise intersection corresponds exactly to the intersection of the accessibility relations in the corresponding relational (Section 4.1).

(b) Normal modal logics with intersection modalities are a fragment of the pooling logics characterized by the class of monotonic models (Section 4.2).

Claim (a) was already mentioned in [50] for languages without a universal modality; there, the proof is left to the reader.[15] Claim (b) is new. Whereas neither of these claims come totally unexpected, it is useful to spell them out in exact detail, also for subsequent sections.

In Section 4.3 we consider the monotonic semantic clause for the modal operators $\Box_M$ and show that, using this clause, we obtain a logic that is equivalent to the logic of monotonic models – thus generalizing a well-known result for mono-modal neighbourhood semantics.

4.1 Augmented Neighbourhood Models

Let $\mathbb{G}$ denote the set of all finite, non-empty subsets of I. In the remainder we will slightly abuse notation, using G as a variable for both, sets of agents, and pooling profiles, where in the latter case we stipulate that $G = \{(i,1) \mid i \in G\} \cup \{(j,0) \mid j \notin G\}$. We focus here on the formal language $\mathfrak{L}_G^{[\forall]}$ obtained by adding $[\forall]$ and all operators $\Box_G$ for $G \in \mathbb{G}$ to classical propositional logic. Analogous observations can be made for the language without these operators.

A first thing to note is that, when interpreted over augmented neighbourhood models, $\mathfrak{L}_G^{[\forall]}$ is as expressive as the full language $\mathfrak{L}_\infty^{[\forall]}$. That is, since the neighbourhoods of an augmented model are already closed under arbitrary intersection, all distinctions between a given pooling profile M and the underlying set of indexes $I(M)$ are lost:

Fact 2 *Let $\mathfrak{M}$ be an augmented neighbourhood model and let w be a member of the domain of $\mathfrak{M}$. Then, for all $M \in \mathbb{M}_\infty$ and $\varphi \in \mathfrak{L}_\infty^{[\forall]}$, the following holds: $\mathfrak{M}, w \models \Box_M \varphi$ iff $\mathfrak{M}, w \models \Box_{I(M)} \varphi$.*

Recall that a relational model $\mathfrak{S}$ is just like a neighbourhood model, with each $\mathcal{N}_i$ replaced by a relation $\mathcal{R}_i \subseteq W \times W$. The semantic clauses for $\mathfrak{L}_G^{[\forall]}$, interpreted over relational models, are the same as for neighbourhood models, except for the modal operators of the type $\Box_G$:

Definition 9 *Let $\mathfrak{S} = \langle W, \langle \mathcal{R}_i \rangle_{i \in I}, V \rangle$ be a relational model. Where $G \in \mathbb{G}$, let $\mathcal{R}_G = \bigcap_{i \in G} \mathcal{R}_i$. Where $w \in W$ and $G \in \mathbb{G}$:*

[15] See [50, Section 3.3, Exercise 73].

$\mathfrak{S}, w \models \Box_G \varphi$ *iff* $\mathfrak{S}, w' \models \varphi$ *for all* $w' \in \mathcal{R}_G(w)$.

In order to establish a correspondence between augmented neighbourhood models and relational models, we define transformations from one to the other.[16] Where $\mathcal{N} : W \to \wp(\wp(W))$ is a neighbourhood function, let $\mathcal{R}_\mathcal{N} \subseteq W \times W$ be defined by putting $\mathcal{R}_\mathcal{N}(w) = \bigcap \mathcal{N}(w)$ for all $w \in W$. Conversely, for any accessibility relation $\mathcal{R} \subseteq W \times W$, let $\mathcal{N}_\mathcal{R}(w) = \{X \subseteq W \mid \mathcal{R}(w) \subseteq X\}$. In the remainder, we will use the term ***standard transformations*** to refer to both transformations. Where $\mathfrak{M}$ is a neighbourhood model, we use $\mathfrak{S}_\mathfrak{M}$ to denote the relational model that is obtained from $\mathfrak{M}$ using the standard transformation; conversely, $\mathfrak{M}_\mathfrak{S}$ denotes the neighbourhood model obtained from $\mathfrak{S}$ by the standard transformation. We now show that the standard transformations preserve validity in the richer languages that feature pooling modalities.

Lemma 3 *Let* $\mathfrak{S} = \langle W, \langle \mathcal{R}_i \rangle_{i \in I}, V \rangle$ *be a relational model. Then, for all* $w \in W$,

$$\Cap_{i \in G} \mathcal{N}_{\mathcal{R}_i}(w) = \{X \subseteq W \mid \mathcal{R}_G(w) \subseteq X\}$$

Proof. ($\subseteq$) Suppose $X \in \Cap_{i \in G} \mathcal{N}_{\mathcal{R}_i}(w)$. So there is a set $\{Y_i \mid i \in G, Y_i \in \mathcal{N}_{\mathcal{R}_i}(w)\}$ such that $\bigcap_{i \in G} Y_i = X$. Since each $\mathcal{N}_{\mathcal{R}_i}$ is obtained from $\mathcal{R}_i$ by the standard transformation, it follows that for all $i \in G$, $Y_i \supseteq \mathcal{R}_i(w)$. Hence, $X = \bigcap_{i \in G} Y_i \supseteq \bigcap \mathcal{R}_i(w) = \mathcal{R}_G(w)$.

($\supseteq$) Suppose that $\mathcal{R}_G(w) \subseteq X$. So $\bigcap_{i \in G} \mathcal{R}_i(w) \subseteq X$. For all $i \in G$, let $Y_i = \mathcal{R}_i(w) \cup X$. Note that $\bigcap_{i \in G} Y_i = (\bigcap_{i \in G} \mathcal{R}_i(w)) \cup X = X$. Since each $\mathcal{N}_{\mathcal{R}_i}$ is obtained from $\mathcal{R}_i$ by the standard transformation, it follows that each $Y_i \in \mathcal{N}_{\mathcal{R}_i}(w)$. Hence, $X = \bigcap_{i \in G} Y_i \in \Cap_{i \in G} \mathcal{N}_{\mathcal{R}_i}(w)$. ■

Theorem 1 *Let* $\mathfrak{S}$ *be a relational model with domain* W. *Then* $\mathfrak{M}_\mathfrak{S}$ *is an augmented neighbourhood model. Moreover,* $\mathfrak{S}$ *and* $\mathfrak{M}_\mathfrak{S}$ *are* $\mathfrak{L}_G^{[\forall]}$*-equivalent: for all* $w \in W$ *and all* $\varphi \in \mathfrak{L}_G^{[\forall]}$, $\mathfrak{S}, w \models \varphi$ *iff* $\mathfrak{M}_\mathfrak{S}, w \models \varphi$.

Proof. By an induction on the complexity of φ. We only consider the case where $\varphi = \Box_G \psi$. Here we have: $\mathfrak{S}, w \models \varphi$ iff [by Definition 9] $\mathcal{R}_G(w) \subseteq \|\psi\|^\mathfrak{S}$ iff [by Lemma 3] $\|\psi\|^\mathfrak{S} \in \Cap_{i \in G} \mathcal{N}_{\mathcal{R}_i}(w)$ iff [by the induction hypothesis] $\|\psi\|^{\mathfrak{M}_\mathfrak{S}} \in \Cap_{i \in G} \mathcal{N}_{\mathcal{R}_i}(w)$ iff [by Definition 2] $\mathfrak{M}_\mathfrak{S}, w \models \varphi$. ■

Lemma 4 *Let* $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ *be an augmented neighbourhood model. Then, for all* $w \in W$,

$$\mathcal{N}_G(w) = \{X \subseteq W \mid \bigcap_{i \in G} \mathcal{R}_{\mathcal{N}_i}(w) \subseteq X\}$$

Proof. ($\subseteq$) Suppose $X \in \mathcal{N}_G(w)$. Hence there is a set $\{Y_i \mid i \in G, Y_i \in \mathcal{N}_i(w)\}$ such that $\bigcap Y_i = X$. Note that by the standard transformation, for every $i \in G$, $\mathcal{R}_{\mathcal{N}_i}(w) \subseteq Y_i$. Consequently, $\bigcap_{i \in G} \mathcal{R}_{\mathcal{N}_i}(w) \subseteq X$.

($\supseteq$) Suppose that $\bigcap_{i \in G} \mathcal{R}_{\mathcal{N}_i}(w) \subseteq X$. For every $i \in G$, let $Y_i = \mathcal{R}_{\mathcal{N}_i}(w) \cup X$. Note that, since each $\mathcal{N}_i$ is augmented, each $Y_i \in \mathcal{N}_i(w)$. Moreover, by the supposition, $X = \bigcap_{i \in G} Y_i$. Hence, $X \in \Cap_{i \in G} \mathcal{N}_i(w) = \mathcal{N}_G(w)$. ■

[16] This transformation dates back at least to [16, Theorem 7.9].

Theorem 2 *Let $\mathfrak{M}$ be an augmented neighbourhood model with domain W. Then $\mathfrak{S}_{\mathfrak{M}}$ is a relational model. Moreover, $\mathfrak{M}$ and $\mathfrak{S}_{\mathfrak{M}}$ are $\mathfrak{L}_G^{[\forall]}$-equivalent: for all $w \in W$ and all $\varphi \in \mathfrak{L}_G^{[\forall]}$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{S}_{\mathfrak{M}}, w \models \varphi$.*

Proof. Analogous to the proof of Theorem 1, but relying on Lemma 4. ■

Taken together, Theorems 1 and 2 imply:

Corollary 1 *For all $\varphi \in \mathfrak{L}_G^{[\forall]}$, φ is valid on the class of all relational models iff φ is valid on the class of all augmented neighbourhood models.*

4.2 Normal Modal Logics as Fragments of Pooling Logics

In view of the preceding, one can characterize normal modal logics with intersection modalities by means of neighbourhood semantics with pooling modalities, by restricting the class of models. In this section, we show that the same can already be done within the broader class of monotonic models, by using the full expressive power of the language $\mathfrak{L}_\infty^{[\forall]}$.[17]

This requires some preparation. Where $G \in \mathbb{G}$, let $G^\infty = \{(i,\infty) \mid i \in G\} \cup \{(i,0) \mid i \notin G\}$. Note that $G^\infty \in \mathbb{M}_\infty$. We define the translation function $tr : \mathfrak{L}_G^{[\forall]} \to \mathfrak{L}_\infty^{[\forall]}$ as follows:

$tr(\varphi) = \varphi$ for all $\varphi \in \mathfrak{P}$
$tr(\neg\varphi) = \neg tr(\varphi)$
$tr(\varphi \vee \psi) = tr(\varphi) \vee tr(\psi)$
$tr([\forall]\varphi) = [\forall] tr(\varphi)$
$tr(\Box_G \varphi) = \bigvee_{\emptyset \subset H \subseteq G} \Box_{H^\infty} tr(\varphi) \vee [\forall] tr(\varphi)$

With every monotonic (multi-neighbourhood) model, we can associate a (multi-agent) relational model, by putting $\mathcal{R}_i(w) = \bigcap(\mathcal{N}_i(w) \cup \{W\})$ for all worlds w. With infinitary pooling modalities at our disposal, we can express all properties of that relational model that could be expressed by the corresponding normal modal operators, as we now show.

Definition 10 *Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a monotonic model. The* ***unit-augmentation of*** $\mathfrak{M}$ *is the model $\mathfrak{M}^\cap = \langle W, \langle \mathcal{N}_i^\cap \rangle_{i \in I}, V \rangle$, where for every $i \in I$ and every $w \in W$, $\mathcal{N}_i^\cap(w)$ is the closure of $\mathcal{N}_i(w) \cup \{W\}$ under arbitrary intersections, i.e., $\mathcal{N}_i^\cap(w) = \Cap^\infty(\mathcal{N}_i(w) \cup \{W\})$.*

Note that the unit-augmentation of a given monotonic model $\mathfrak{M}$ is an augmented model. The following is an immediate consequence of the definition of the neighbourhoods $\mathcal{N}_i^\cap$:[18]

[17]Without $[\forall]$, one can obtain similar results for the intermediate class of models that are monotonic and where, for every $w \in W$ and all $i \in I$, $W \in \mathcal{N}_i(w)$. Here the translation consists merely in replacing each G with G^∞, and hence $[\forall]$ is not required. The lemmas and theorems of this section can be rephrased in terms of such unit-monotonic models, for this simpler translation.

[18]Note the similarity between Fact 2 and Fact 5: both basically state that if neighbourhoods are already closed under arbitrary intersections, there is no difference between multisets and the sets of indexes from which they are built up.

Fact 5 *Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a monotonic model and let $\mathfrak{M}^{\cap} = \langle W, \langle \mathcal{N}_i^{\cap} \rangle_{i \in I}, V \rangle$ be its unit-augmentation. Let $G \subseteq I$ be non-empty and finite and let $w \in W$. If for all $i \in G$, $\mathcal{N}_i(w) \neq \emptyset$, then $\Cap_{i \in G} \mathcal{N}_i^{\cap}(w) = \mathcal{N}_{G^{\infty}}(w)$.*

Theorem 3 *Let $\mathfrak{M}$ be a monotonic model. For all w in the domain of $\mathfrak{M}$ and for all $\varphi \in \mathfrak{L}_G^{[\forall]}$: $\mathfrak{M}^{\cap}, w \models \varphi$ iff $\mathfrak{M}, w \models tr(\varphi)$.*

Proof. By an induction on the complexity of φ. The inductive base ($\varphi \in \mathfrak{P}$) and the induction step for the connectives and $[\forall]$ are safely left to the reader. So let $\varphi = \Box_G \psi$ and let $w \in W$, where W is the domain of $\mathfrak{M}$. Let $H = \{i \in G \mid \mathcal{N}_i(w) \neq \emptyset\}$. Note that for all τ, whenever $\mathfrak{M}, w \models [\forall]\tau$, then $\mathfrak{M}, w \models \Box_H \tau$ since $\mathfrak{M}$ is monotonic ($\star$).

Case 1: $H \neq \emptyset$. Note that $\Cap_{i \in H} \mathcal{N}_i^{\cap}(w) = \Cap_{i \in G} \mathcal{N}_i^{\cap}(w)$ (†). Hence, $\mathfrak{M}^{\cap}, w \models \varphi$ iff [by the semantic clause] $\|\psi\|^{\mathfrak{M}^{\cap}} \in \Cap_{i \in G} \mathcal{N}_i^{\cap}(w)$ iff [by (†)] $\|\psi\|^{\mathfrak{M}^{\cap}} \in \Cap_{i \in H} \mathcal{N}_i^{\cap}(w)$ iff [by the induction hypothesis] $\|tr(\psi)\|^{\mathfrak{M}} \in \Cap_{i \in H} \mathcal{N}_i^{\cap}(w)$ iff [by Fact 5] $\|tr(\psi)\|^{\mathfrak{M}} \in \mathcal{N}_{H^{\infty}}(w)$ iff [by the semantic clause and our choice of H, and by ($\star$)] $\mathfrak{M}, w \models \bigvee_{\emptyset \subset H' \subseteq G} \Box_{H'} tr(\psi) \vee [\forall] tr(\psi)$ iff $\mathfrak{M}, w \models tr(\varphi)$.

Case 2: $H = \emptyset$. Then $\mathcal{N}_G^{\cap}(w) = \{W\}$ and, for all $G' \subseteq G$ with $G' \neq \emptyset$ and all τ, $\mathfrak{M}, w \not\models \Box_{G'} \tau$ (‡). Hence, $\mathfrak{M}^{\cap}, w \models \Box_G \psi$ iff $\|\psi\|^{\mathfrak{M}^{\cap}} = W$ iff [by the induction hypothesis] $\|tr(\psi)\|^{\mathfrak{M}} = W$ iff $\mathfrak{M}, w \models [\forall] tr(\psi)$ iff [by (‡)] $\mathfrak{M}, w \models \bigvee_{\emptyset \subset G' \subseteq G} \Box_{G'} tr(\psi) \vee [\forall] tr(\psi)$ iff $\mathfrak{M}, w \models tr(\varphi)$. ∎

Lemma 6 *If $\mathfrak{M}$ is augmented, then for all $\varphi \in \mathfrak{L}_G^{[\forall]}$: $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models tr(\varphi)$.*

Proof. By an induction on the complexity of φ. Again we should only worry about the case $\varphi = \Box_G \psi$. Here we can rely on the fact that, if $\mathfrak{M}$ is augmented, then it validates (i) $\Box_G \psi \leftrightarrow \Box_{G^{\infty}} \psi$ and (ii) $\Box_{G^{\infty}} \psi \leftrightarrow (\Box_{G^{\infty}} \psi \vee [\forall] \psi)$. ∎

Theorem 4 *For all $\varphi \in \mathfrak{L}_G^{[\forall]}$, φ is valid on the class of all relational models iff $tr(\varphi)$ is valid on the class of all monotonic models.*

Proof. ($\Rightarrow$) Suppose there is a monotonic $\mathfrak{M}$ and a w in its domain such that $\mathfrak{M}, w \not\models tr(\varphi)$. By Theorem 3, $\mathfrak{M}^{\cap}, w \not\models \varphi$. Note that $\mathfrak{M}^{\cap}$ is augmented. By Theorem 2, $\mathfrak{S}_{\mathfrak{M}^{\cap}}, w \not\models \varphi$. Hence, φ is not valid on the class of all relational models.

($\Leftarrow$) Suppose there is a relational model $\mathfrak{S}$ and a world w in its domain such that $\mathfrak{S}, w \not\models \varphi$. Note that $\mathfrak{M}_{\mathfrak{S}}$ is augmented and hence also monotonic. By Theorem 1, $\mathfrak{M}_{\mathfrak{S}}, w \not\models \varphi$. By Lemma 6, $\mathfrak{M}_{\mathfrak{S}}, w \not\models tr(\varphi)$. Hence $tr(\varphi)$ is not valid on the class of all monotonic models. ∎

4.3 The Monotonic Semantic Clause

As noted in the introduction, we focus in this paper on the exact semantic clause for the operators $\Box_M$: $\Box_M \varphi$ holds at w iff the exact truth set of φ is a member of the neighbourhood $\mathcal{N}_M(w)$. There is, however, a well-known connection between monotonic neighbourhood models and a different semantic clause, viz. the *monotonic semantic clause*. In this section we show how this connection generalizes to pointwise intersection and pooling modalities.

For the sake of convenience, let us introduce a new symbol $\models_m$ for truth at a world in a model according to the monotonic clause. This notion is defined in the same way as $\models$, except that we replace the semantic clause for $\Box$ with the following:

Definition 11 *Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be a model, $w \in W$, and $\varphi \in \mathfrak{L}_\infty^{[\forall]}$. Then*

$$\mathfrak{M}, w \models_m \Box_M \varphi \text{ iff there is an } X \in \mathcal{N}_M(w) \text{ such that } X \subseteq \|\varphi\|^{\mathfrak{M}}.$$

It has been argued that a focus on monotonic neighbourhood models – or equivalently, using the monotonic semantic clause – allows for a smoother model theory [50]. In contrast, at the level of the semantic consequence relation, the exact reading offers a perspective that is strictly more general, since one can characterize that semantic consequence relation simply by imposing the frame condition of monotony. We will now show how this generalizes to languages that feature pooling modalities.

In the remainder, we use $\mathcal{N}_M^{\uparrow}$ to refer to the neighbourhood function for the pooling profile M, in the supplemented model $\mathfrak{M}^{\uparrow}$ (cf. Definition 4). That is, $\mathcal{N}_M^{\uparrow} = \{\bigcap_{i \in I(M)} X_i \mid X_i \in \Cap^{M(i)} \mathcal{N}_i^{\uparrow}(w)\}$. The following lemma states that, essentially, it makes no difference whether we (a) first close all neighbourhood sets under supersets, and afterwards apply pointwise intersection, or (b) first apply pointwise intersection to a given tuple of neighbourhood sets, and only afterwards close the resulting neighbourhood set under supersets.

Lemma 7 *Let $M \in \mathbb{M}_\infty$ and let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ be given. Then $\mathcal{N}_M^{\uparrow}(w) = (\mathcal{N}_M(w))^{\uparrow}$.*

Proof. ($\subseteq$) Suppose that $Y \in \{\bigcap_{i \in I(M)} X_i \mid X_i \in \Cap^{M(i)} \mathcal{N}_i^{\uparrow}(w)\}$. So for every $i \in I(M)$, for every $k \in \{1, \ldots, M(i)\}$, there is an $X_i^k \in \mathcal{N}_i^{\uparrow}(w)$ such that $\bigcap_{k \in \{1, \ldots, M(i)\}} X_i^k = X_i$, and $\bigcap_{i \in I(M)} X_i = Y$. For every such k, i, let $Z_i^k \in \mathcal{N}_i(w)$ be such that $Z_i^k \subseteq X_i^k$. It follows that $Z_i = \bigcap_{k \in \{1, \ldots, M(i)\}} Z_i^k \in \Cap^{M(i)} \mathcal{N}_i(w)$ and hence $Z = \bigcap_{i \in I(M)} Z_i \in \{\bigcap_{i \in I(M)} X_i \mid X_i \in \Cap^{M(i)} \mathcal{N}_i(w)\}$. Moreover, $Z \subseteq Y$ and hence $Y \in \{\bigcap_{i \in I(M)} X_i \mid X_i \in \Cap^{M(i)} \mathcal{N}_i(w)\}^{\uparrow}$.

($\supseteq$) Suppose that $Y \in \{\bigcap_{i \in I(M)} X_i \mid X_i \in \Cap^{M(i)} \mathcal{N}_i(w)\}^{\uparrow}$. So there is an $X \in \{\bigcap_{i \in I(M)} X_i \mid X_i \in \Cap^{M(i)} \mathcal{N}_i(w)\}$ such that $X \subseteq Y$. So for all $i \in I(M)$, for all $k \in \{1, \ldots, M(i)\}$, there is an $X_i^k \in \mathcal{N}_i(w)$ such that each $X_i = \bigcap_{k \in \{1, \ldots, M(i)\}} X_i^k$ and $\bigcap_{i \in I(M)} X_i \subseteq Y$. For all $i \in I(M)$ and all $k \in \{1, \ldots, M(i)\}$, let $Y_i^k = Y \cup X_i^k$. Then in view of the preceding, each $Y_i^k \in \mathcal{N}_i^{\uparrow}(w)$. Also, $Y = \bigcap_{i \in I(M), k \in \{1, \ldots, M(i)\}} Y_i^k$. Hence, $Y \in \{\bigcap_{i \in I(M)} X_i \mid X_i \in \Cap^{M(i)} \mathcal{N}_i^{\uparrow}(w)\}$. ■

By a standard induction on the complexity of formulas, we obtain:

Corollary 2 *Where $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ is a model, $w \in W$, and $\varphi \in \mathfrak{L}_\infty^{[\forall]}$: $\mathfrak{M}, w \models_m \varphi$ iff $\mathfrak{M}^{\uparrow}, w \models \varphi$.*

Hence, the logic obtained by using the monotonic semantic clause for $\Box_M$ is just the same as the logic of the restricted class of models in which the neighbourhood sets are closed under supersets. In [4], this insight is put to work in the completeness proof for monotonic modal logics with pooling modalities.

Of course, this does not take away that there are other, independent reasons for working with the monotonic clause: e.g. when the neighbourhoods are also used to define other operators in a given formal language. A prominent example is Evidence Logic (cf. Section 6), in which the monotonic clause is used to interpret an operator expressing evidence, but the neighbourhoods are also used to interpret evidence-based belief (in terms of maximal non-empty intersections of *exact* pieces of evidence). Having the neighbourhood sets closed under supersets in this setting would ruin the logic of belief, in the sense that inconsistent evidence would always result in having only tautological beliefs.

It should be noted here that there is a yet different characterization of monotonic modal logics, viz. in terms of *multirelational semantics* [26,27]. Over multirelational models, intersection of the accessibility relations corresponds to pointwise intersection of neighbourhood sets.[19] In sum, monotonic models (using the exact semantic clause), arbitrary models (using the monotonic semantic clause), and multirelational models (using the multi-relational semantic clause) can all be used to interpret the same logic of pooling modalities.

5 Expressivity

In this section we map out the expressive power of the various formal languages introduced so far. We first show that, for the six formal languages considered in this paper, proper inclusion of the languages also entails a proper difference in expressive power at the level of models. We then consider one specific condition on models under which all pooled information is expressible in the languages without pooling modalities, linking this to existing work in epistemic logic on the so-called principle of "full communication". Finally, we move to the level of frames, showing that also there, pooling modalities yield a surplus in expressive power.

5.1 Expressivity at the Level of Models

Let a ***pointed model*** be a couple $\mathfrak{M}, w$, where w is a member of the domain W of $\mathfrak{M}$. Following a customary practice in modal logic, one may identify the expressive power of a language with the distinctions it can make among pointed models. In order to apply this criterion, we specify first what it means that two pointed models are indistinguishable from the viewpoint of a given formal language $\mathfrak{T}$.

Definition 12 *Let $\mathfrak{T} \in \{\mathfrak{L}, \mathfrak{L}^{[\forall]}, \mathfrak{L}_f, \mathfrak{L}_f^{[\forall]}, \mathfrak{L}_\infty, \mathfrak{L}_\infty^{[\forall]}\}$. Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V\rangle$ and $\mathfrak{M}' = \langle W', \langle \mathcal{N}'_i \rangle_{i \in I}, V'\rangle$ be models and let $w \in W$, $w' \in W'$. Then $(\mathfrak{M}, w) \equiv_{\mathfrak{T}} (\mathfrak{M}', w')$ iff for all $\varphi \in \mathfrak{T}$, $\mathfrak{M}, w \models \varphi$ iff $\mathfrak{M}', w' \models \varphi$.*

Where $\mathfrak{T}, \mathfrak{T}'$ are formal languages, we write $\mathfrak{T} \leq \mathfrak{T}'$ to indicate that $\mathfrak{T}'$ is at least as expressive as $\mathfrak{T}$. That is: $\mathfrak{T} \leq \mathfrak{T}'$ iff, for all $(\mathfrak{M}, w)$ and $(\mathfrak{M}, w)$: if $(M, w) \equiv_{\mathfrak{T}'} (M', w')$, then $(M, w) \equiv_{\mathfrak{T}} (M', w')$. If the converse fails, we say that $\mathfrak{T}'$ is strictly more expressive than $\mathfrak{T}$, formally: $\mathfrak{T} < \mathfrak{T}'$.

The expressivity relations between the six languages considered in this paper are summarized by the following theorem:

Theorem 5 *For all $\mathfrak{T}, \mathfrak{T}' \in \{\mathfrak{L}, \mathfrak{L}^{[\forall]}, \mathfrak{L}_f, \mathfrak{L}_f^{[\forall]}, \mathfrak{L}_\infty, \mathfrak{L}_\infty^{[\forall]}\}$: $\mathfrak{T} \leq \mathfrak{T}'$ iff $\mathfrak{T} \subseteq \mathfrak{T}'$.*

[19] Spelling out this claim in full generality and proving it is somewhat tedious but can be safely left to the reader.

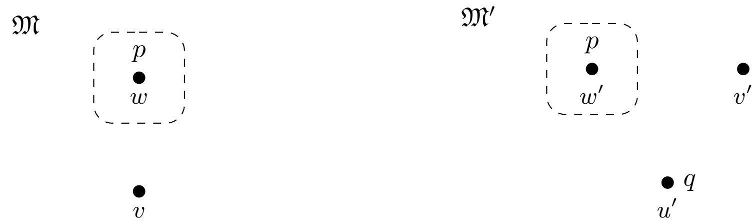

Figure 2: Two pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ that are $\mathfrak{L}_\infty$-equivalent but not $\mathfrak{L}^{[\forall]}$-equivalent. The dashed set denotes the agent's (only) neighborhood at all worlds.

Obviously, if $\mathfrak{T} \subseteq \mathfrak{T}'$, then $\mathfrak{T} \leq \mathfrak{T}'$. It order to show the converse, it suffices to observe three basic facts.

Fact 8 $\mathfrak{L}^{[\forall]} \not\leq \mathfrak{L}_\infty$.

Let us illustrate this fact by means of a simple example for $I = \{1\}$. Let $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ with $W = \{w, v\}$, $\mathcal{N}_1(w) = \mathcal{N}_1(v) = \{\{w\}\}$, $V(p) = \{w\}$, and $V(q) = \emptyset$. Let $\mathfrak{M}' = \langle W', \langle \mathcal{N}_i' \rangle_{i \in I}, V' \rangle$ with $W' = \{w', v', u'\}$, $\mathcal{N}_1'(w') = \mathcal{N}_1'(v') = \mathcal{N}_1'(u') = \{w'\}$, $V(p) = \{w'\}$, and $V(q) = \{u'\}$. These two models are depicted in Figure 2. It can easily be verified by an induction that $(\mathfrak{M}, w) \equiv_{\mathfrak{L}_\infty} (\mathfrak{M}', w')$. However, $\mathfrak{M}, w \models [\forall]\neg q$, whereas $\mathfrak{M}', w' \models [\exists]q$.

In the context of non-normal modal logics – which allow us to model i.a. agents having inconsistent information or normative systems that issue incompatible norms –, the universal modality is particularly useful, in that it allows us to express that two or more propositions are jointly compatible. For instance, using a non-normal modality $\Box_1$ we may express that agent 1 has evidence for p by $\Box_1 p$, and that this same agent has evidence for q ($\Box_1 q$). The existential modality then allows us to express that moreover, these pieces of evidence are jointly compatible: $[\exists](p \wedge q)$. In Section 6 we will consider other ways in which $[\exists]$ can be put to work when characterizing (multi-agent) logics for evidence and beliefs.

Fact 9 $\mathfrak{L}_f \not\leq \mathfrak{L}^{[\forall]}$.

As argued in Section 4, pointwise intersection provides a conservative generalization of the intersection of accessibility relations in relational semantics. It is a well-known fact that, for relational semantics, adding intersection modalities makes for a strict increase in expressive power. As a corollary, within the class of augmented models, one can already find models that are distinguishable only by means of (finitary) pooling modalities.[20]

In the general class of neighbourhood models, we can also illustrate the expressive power of finitary pooling modalities by two single-agent models. That is, let $\mathfrak{M} =$

[20]In fact, the mentioned pooling profiles M are even more specific, since they correspond to regular sets, i.e., $M(i) = 1$ for all $i \in I(M)$.

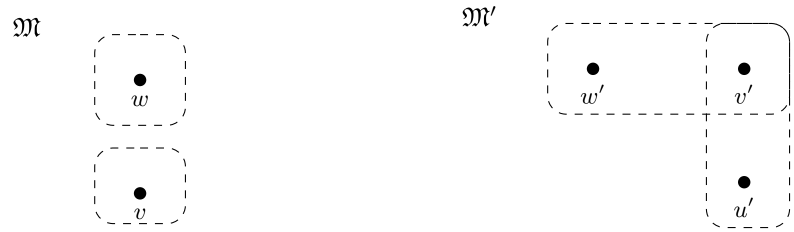

Figure 3: Two pointed models $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ that are $\mathfrak{L}^{[\forall]}$-equivalent but not $\mathfrak{L}_f$-equivalent.

$\langle W, \mathcal{N}_1, V\rangle$ with $W = \{w, v\}$, $\mathcal{N}_1(w) = \{w\}, \mathcal{N}_1(v) = \{v\}$, and $V(p) = \emptyset$ for all $p \in \mathfrak{P}$. Let $\mathfrak{M}' = \langle W', \mathcal{N}_1', V'\rangle$ with $W' = \{w', v', u'\}$, $\mathcal{N}_1'(w') = \mathcal{N}_1'(v') = \mathcal{N}_1'(u') = \{\{w', v'\}, \{v', u'\}\}$. These two pointed models are depicted in Figure 3. Note that none of the neighbourhoods in these models correspond to the truth set of some formula. Relying on this insight, it can easily be verified that $\mathfrak{M}, w \equiv_{\mathfrak{L}^{[\forall]}} \mathfrak{M}', w'$. In the first model, $\Box_{1,1}\bot$ holds, whereas in the second it fails.[21]

Fact 10 $\mathfrak{L}_\infty \not\leq \mathfrak{L}_f^{[\forall]}$.

The step from finitary pooling modalities to arbitrary pooling modalities is easily made, conceptually speaking. However, it again results in a strict increase in expressive power. To see why, consider the single-index example given by Figure 4. Here, $\mathcal{N}_1(w)$ consists of only one set, i.e. the unit W. In contrast, for every $w_i' \in W'$, $\mathcal{N}_1'(w_i')$ contains W' and, in addition, an infinite series of ever smaller sets $X_n = W' \setminus \{w_1', \ldots, w_n'\}$. Since there is no proposition that distinguishes any of the worlds w_i', the X_i do not correspond to the truth set of any formula in $\mathfrak{L}_f^{[\forall]}$. The same holds for all sets $\bigcap_{i \leq k} X_i$ for $k \in \mathbb{N}^+$. Relying on this observation one can easily show that $(\mathfrak{M}, w) \equiv_{\mathfrak{L}_f^{[\forall]}} (\mathfrak{M}', w_1')$. However, $\mathfrak{M}, w \not\models \Box_{1^\infty}\bot$ but, since $\bigcap_{i \in \mathbb{N}} X_i = \emptyset$, $\mathfrak{M}', w_1' \models \Box_{1^\infty}\bot$.

It is no coincidence that the model $\mathfrak{M}'$ in Figure 4 has an infinite domain. That is, suppose $\mathfrak{M}$ has finite domain W. Then there can be at most $k = |\wp(W)|$ many different subsets of W. Hence any intersection of a set of subsets of W can be rewritten as an intersection of k subsets of W. So, letting $M_k = \{(i, k) \mid M(i) = \infty\} \cup \{(j, l) \mid M(j) = l \neq \infty\}$, it follows that $\mathfrak{M}, w \models \Box_M\varphi$ iff $\mathfrak{M}, w \models \Box_{M_k}\varphi$ for all $\varphi \in \mathfrak{L}_\infty^{[\forall]}$. So whatever can be expressed about finite models using formulas in $\mathfrak{L}_\infty$ ($\mathfrak{L}_\infty^{[\forall]}$) can also be expressed using formulas in $\mathfrak{L}_f$ ($\mathfrak{L}_f^{[\forall]}$). Consequently:

Fact 11 *Suppose that $\mathfrak{M}$ and $\mathfrak{M}'$ are finite. The each of the following hold:*

1. $(\mathfrak{M}, w) \equiv_{\mathfrak{L}_\infty} (\mathfrak{M}', w')$ *iff* $(\mathfrak{M}, w) \equiv_{\mathfrak{L}_f} (\mathfrak{M}', w')$
2. $(\mathfrak{M}, w) \equiv_{\mathfrak{L}_\infty^{[\forall]}} (\mathfrak{M}', w')$ *iff* $(\mathfrak{M}, w) \equiv_{\mathfrak{L}_f^{[\forall]}} (\mathfrak{M}', w')$.

[21] The same observations can be made about the supplementation of $\mathfrak{M}$ and $\mathfrak{M}$, obtained by closing their respective neighbourhood sets under supersets.

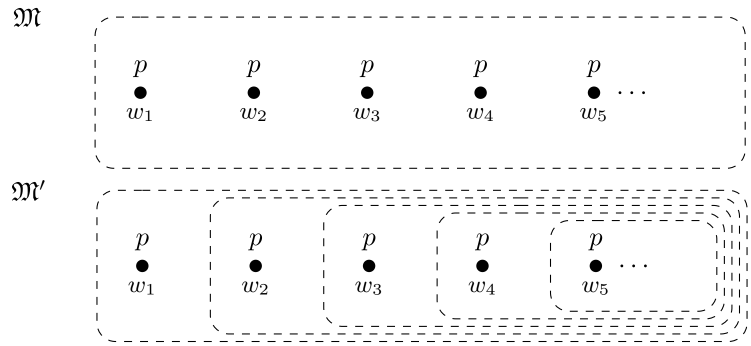

Figure 4: Two pointed models $(\mathfrak{M}, w_1)$ and $(\mathfrak{M}', w_1')$ that are $\mathfrak{L}_f^{[\forall]}$-equivalent, but not $\mathfrak{L}_\infty$-equivalent.

5.2 Differentiability and Full Communication

In the preceding we saw that pooling modalities add expressive power to $\mathfrak{L}$ and $\mathfrak{L}^{[\forall]}$. In metaphoric terms, the pooled information φ – say, the beliefs of different agents – may sometimes stay "under the radar", since it is not witnessed by any individual pieces $\psi_1, \psi_2, \ldots$ of information that, when combined, yield φ. In the present section, we delineate a specific class of models in which such witnesses are always available, and show that for this class, $\mathfrak{L}$ and $\mathfrak{L}^{[\forall]}$ are as expressive as their extensions with pooling modalities. Next, we show that such models also satisfy a principle that is well-known from the study of normal doxastic and epistemic logics.

Let us start by defining the notion of differentiability, cf. [3, 25, 56].[22]

Definition 13 *$\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$ is **$\mathfrak{T}$-differentiable** iff for every $w \in W$, for every $i \in I$, and for every $X \in \mathcal{N}_i(w)$, there is a $\varphi \in \mathfrak{T}$ such that $X = \|\varphi\|^{\mathfrak{M}}$.*

Intuitively, that $\mathfrak{M}$ is $\mathfrak{T}$-differentiable means that every neighbourhood (of every world w, for every index i) in the model can be "named" by some formula in $\mathfrak{T}$.

Theorem 6 *Suppose that $\mathfrak{M}$ and $\mathfrak{M}'$ are both finite and $\mathfrak{L}^{[\forall]}$-differentiable. If $(\mathfrak{M}, w) \equiv_{\mathfrak{L}^{[\forall]}} (\mathfrak{M}', w')$, then $(\mathfrak{M}, w) \equiv_{\mathfrak{L}_\infty^{[\forall]}} (\mathfrak{M}', w')$.*

Proof. Suppose that $\mathfrak{M}$ and $\mathfrak{M}'$, with respective finite domains W and W', are both $\mathfrak{L}^{[\forall]}$-differentiable and $(\mathfrak{M}, w) \equiv_{\mathfrak{L}^{[\forall]}} (\mathfrak{M}', w')$. We first prove the following:

($\dagger_1$) for all $v \in W$, there is a $v' \in W'$ such that $(\mathfrak{M}, v) \equiv_{\mathfrak{L}^{[\forall]}} (\mathfrak{M}', v')$

($\dagger_2$) for all $v' \in W'$, there is a $v \in W$ such that $(\mathfrak{M}, v) \equiv_{\mathfrak{L}^{[\forall]}} (\mathfrak{M}', v')$

To see why ($\dagger_1$) holds, assume for contradiction that $v \in W$ but, for no $v' \in W'$, $v \equiv_{\mathfrak{L}^{[\forall]}} v'$. So for all $v' \in W'$, some $\psi_{v'} \in \mathfrak{L}^{[\forall]}$ is true at v' in $\mathfrak{M}'$, but false

[22] We borrow terminology from [3], but relativize our definition to the underlying language. In other papers, differentiable models are called "locally distinguishing" (with respect to a multi-agent epistemic language), cf. [25, 56].

v in $\mathfrak{M}$. It follows that the conjunction $\bigwedge_{v' \in W'} \neg\psi_{v'}$ is true at v in $\mathfrak{M}$ and hence, $\mathfrak{M}, w \models [\exists] \bigwedge_{v' \in W'} \neg\psi_{v'}$. However, in view of the preceding, $\mathfrak{M}', w' \models [\forall] \bigvee_{v' \in W'} \psi_{v'}$. This contradicts the supposition that $(\mathfrak{M}, w) \equiv_{\mathfrak{L}^{[\forall]}} (\mathfrak{M}', w')$. The proof for $(\dagger_2)$ is entirely analogous. Let us denote the conjunction of $(\dagger_1)$ and $(\dagger_2)$ by $(\dagger)$.

We prove by induction that, for all $v, v' \in W \times W'$ with $v \equiv_{\mathfrak{L}^{[\forall]}} v'$, for all $\varphi \in \mathfrak{L}^{[\forall]}_{\infty}$, $\mathfrak{M}, v \models \varphi$ iff $\mathfrak{M}', v' \models \varphi$. The base case ($\varphi \in \mathfrak{P}$) and the induction step for the connectives are trivial. For $\varphi = [\forall]\psi$, we have that $\mathfrak{M}, v \models \varphi$ iff for all $u \in W$, $\mathfrak{M}, u \models \psi$ iff [by the induction hypothesis and $(\dagger)$] for all $u' \in W'$, $\mathfrak{M}', u' \models \psi$ iff $\mathfrak{M}', v' \models \varphi$.

For $\varphi = \Box_M \psi$, note first that by induction hypothesis (IH) and the induction step for the classical connectives and $[\forall]$:

$$\text{for all } \tau \in \mathfrak{L}^{[\forall]}, \mathfrak{M}, v \models [\forall](\tau \leftrightarrow \psi) \text{ iff } \mathfrak{M}', v' \models [\forall](\tau \leftrightarrow \psi)$$

Assume now that $\mathfrak{M}, v \models \Box_M \psi$ for some $v \in W$ and that $v \equiv_{\mathfrak{L}^{[\forall]}} v'$. By the semantic clause for $\Box_M$, for every $i \in I(M)$, there must be a (finite) set $\mathcal{X}_i \subseteq \mathcal{N}_i(v)$, such that $\bigcap_{i \in I(M)} \bigcap \mathcal{X}_i = \|\psi\|^{\mathfrak{M}}$. Since $\mathfrak{M}$ is $\mathfrak{L}^{[\forall]}$-differentiable, for every $i \in I(M)$, for every $X \in \mathcal{X}_i$, there is a $\tau_X \in \mathfrak{L}^{[\forall]}$ such that $X = \|\tau_X\|^{\mathfrak{M}}$. Hence, for every such X, $\mathfrak{M}, v \models \Box_i \tau_X$. Note that

$$\bigcap_{i \in I(M)} \bigcap \mathcal{X}_i = \bigcap_{i \in I(M)} \bigcap_{X \in \mathcal{X}_i} \|\tau_X\|^{\mathfrak{M}}$$

Let $\tau = \bigwedge_{i \in I(M)} \bigwedge_{X \in \mathcal{X}_i} \tau_X$. So we have:

$$\|\tau\|^{\mathfrak{M}} = \|\psi\|^{\mathfrak{M}}$$

It follows that $\mathfrak{M}, v \models [\forall](\tau \leftrightarrow \psi)$ and hence also $\mathfrak{M}', v' \models [\forall](\tau \leftrightarrow \psi)$ $(\star)$.

By the supposition that $v \equiv_{\mathfrak{L}^{[\forall]}} v'$, for every τ_X (with $X \in \mathcal{X}_i, i \in I(M)$), $\mathfrak{M}', v' \models \Box_i \tau_X$. Consequently, $\mathfrak{M}', v' \models \Box_M \tau$. By $(\star)$, $\mathfrak{M}', v' \models \Box_M \psi$. The proof for the other half of the equivalence is entirely analogous. ■

Let us briefly return to one of the potential applications of pooling modalities, i.e. the specification of non-normal logics for distributed information of agents and groups of agents. In this context, the principle of full communication [25, 56, 69] roughly says that φ can be a piece of group knowledge only if there is some way the agents in G would be able to arrive at the knowledge that φ by communicating each of their individual pieces of knowledge. One plausible formal explication of this principle reads:[23]

(FC) for all $\varphi \in \mathfrak{L}^{[\forall]}$: if $\mathfrak{M}, w \models \Box_{G^\infty} \varphi$, then $\{\psi \in \mathfrak{L}^{[\forall]} \mid \mathfrak{M}, w \models \Box_i \psi \text{ for some } i \in G\} \Vdash \varphi$

A first thing to note is that (FC) fails on the class of all models. That is because (FC) fails on relational semantics for distributed knowledge, and more generally, on any relational semantics that interprets $\Box_{G^\infty}$ by referring to the intersection of the relations $\mathcal{R}_i$ that correspond to $\Box_i$ [25, 56, 69]. The failure of (FC) for pooling

[23] Over monotonic models, the convere of (FC) is always valid; for arbitrary models it may fail. Obvious ways to vary on (FC) concern the language in which the ψ's are contained. We choose $\mathfrak{L}^{[\forall]}$ here for the sake of generality and simplicity.

modalities over (augmented) neighbourhood models follows from these known results as a corollary, in light of Section 4.1.

In [56], an overview of increasingly weak sufficient conditions for (FC) is given. Most of these conditions cannot easily be generalized to neighbourhood semantics, as they rely on the consistency of the agent-relative information sets $KS_i = \{\psi \in \mathfrak{L}^{[\forall]} \mid \mathfrak{M}, w \models \Box_i \psi\}$. One exception is the condition of differentiability relative to $\mathfrak{L}^{[\forall]}$.

Theorem 7 *If $\mathfrak{M}$ is finite and $\mathfrak{L}^{[\forall]}$-differentiable, then (FC) holds for all w in the domain of $\mathfrak{M}$.*

Proof. Suppose the antecedent holds and $\mathfrak{M}, w \models \Box_{G^\infty} \varphi$ for some $\varphi \in \mathfrak{L}^{[\forall]}$. Since W is finite, there are $X_1, \ldots, X_n$ and $i_1, \ldots, i_n \in G$ such that each $X_j \in \mathcal{N}_{i_j}(w)$ and $X_1 \cap \ldots \cap X_n = \|\varphi\|^{\mathfrak{M}}$. Since $\mathfrak{M}$ is differentiable, for each X_j there is a $\psi_j \in \mathfrak{L}^{[\forall]}$ such that $X_j = \|\psi_j\|^{\mathfrak{M}}$ and hence $\mathfrak{M}, w \models \Box_{i_j} \psi_j$. It follows that $\| \bigwedge_{1 \leq j \leq n} \psi_j \| = \|\varphi\|$ and hence $\| \bigwedge_{1 \leq j \leq n} \psi_j \to \varphi\|^{\mathfrak{M}} = W$. Hence, for all $j \in \{1, \ldots, n\}$, $\|\psi_j\|^{\mathfrak{M}} = \|\psi_j \wedge (\bigwedge_{1 \leq j \leq n} \psi_j \to \varphi)\|^{\mathfrak{M}}$. It follows that, for all $j \in \{1, \ldots, n\}$,

$$\mathfrak{M}, w \models \Box_{i_j} (\psi_j \wedge (\bigwedge_{1 \leq j \leq n} \psi_j \to \varphi))$$

For all $j \in \{1, \ldots, n\}$, let $\tau_j = \psi_j \wedge (\bigwedge_{1 \leq j \leq n} \psi_j \to \varphi)$. We have: $\{\tau_j \mid j \in \{1, \ldots, n\}\} \Vdash \varphi$ and for every $j \in \{1, \ldots, n\}$, $\mathfrak{M}, w \models \Box_{i_j} \tau_j$. Hence (FC) holds for w in $\mathfrak{M}$. ■

In conclusion, the class of finite, $\mathfrak{L}^{[\forall]}$-differentiable models constitutes a proper subclass of the class of all models, over which pooling modalities add no expressive power and full communication is valid.

5.3 Expressivity at the Level of Frames

Beside studying the expressive power of various formal languages at the level of models, one may also compare their expressive power in terms of frames. Let us first fix terminology. A ***pointed frame*** is a pair $(\mathfrak{F}, w)$, where $\mathfrak{F}$ is a frame and w is a member of the domain of $\mathfrak{F}$. Where W is the domain of $\mathfrak{F}$, $\mathfrak{F}, w \models \varphi$ iff for all valuations $V : W \to \wp(\mathfrak{P})$, $\langle \mathfrak{F}, V \rangle, w \models \varphi$.

We say that $\varphi \in \mathfrak{L}_\infty^{[\forall]}$ ***characterizes*** the frame condition (C) iff the following equivalence holds, for all pointed frames $(\mathfrak{F}, w)$:

$$\mathfrak{F}, w \models \varphi \text{ iff } (\mathfrak{F}, w) \text{ satisfies (C)}$$

In other words, $(\mathfrak{F}, w)$ does *not* satisfy the given frame condition if and only if one can find a valuation V over $\mathfrak{F}$ such that, given V, φ is false at w. For the sake of convenience we will state our results in terms of pointed frames, since the conditions we consider typically refer to neighbourhood sets of a given world. One can however easily derive corollaries that apply to frames more generally. In particular, if φ characterizes the frame condition (C), then it follows that φ is valid on a frame $\mathfrak{F}$ iff condition (C) holds at every world in $\mathfrak{F}$.[24]

[24] Following standard terminology, φ is ***valid*** on $\mathfrak{F}$ iff for all w in the domain of $\mathfrak{F}$ and all valuations V over $\mathfrak{F}$, $\langle \mathfrak{F}, V \rangle, w \models \varphi$.

We give two relatively simple examples to show that, also from the perspective of frames, pooling modalities give us more expressive power.

Theorem 8 *Let $\mathfrak{F} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I} \rangle$ be a frame and let $w \in W$. Then $\mathcal{N}_2(w) \subseteq \mathcal{N}_1(w) \Cap \mathcal{N}_1(w)$ iff $\mathfrak{F}, w \models \Box_2 p \to \Box_{1,1} p$. Moreover, this frame condition cannot be characterized within $\mathfrak{L}^{[\forall]}$.*

Proof. The proof of the first part of this theorem is safely left to the reader. For the second part, assume for contradiction that the frame condition can be characterized by a formula $\varphi \in \mathfrak{L}^{[\forall]}$. Let $I = \{1,2\}$, $\mathfrak{F} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I} \rangle$ with $W = \{w, v\}$, $\mathcal{N}_1(w) = \mathcal{N}_1(v) = \emptyset$, and $\mathcal{N}_2(w) = \mathcal{N}_2(v) = \{\{w\}\}$. Note that the pointed frame $(\mathfrak{F}, w)$ does not satisfy the mentioned frame condition. Let $V : W \to \wp(\mathfrak{P})$ be such that $\mathfrak{M}, w \not\models \neg\varphi$, where $\mathfrak{M} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I}, V \rangle$.

Let now $\mathfrak{M}' = \langle W', \langle \mathcal{N}'_i \rangle_{i \in I}, V' \rangle$, where $W' = \{w', v'_1, v'_2\}$, $\mathcal{N}'_1(w') = \mathcal{N}'_1(v'_1) = \mathcal{N}'_1(v'_2) = \{\{w', v'_1\}, \{w', v'_2\}\}$, $\mathcal{N}'_2(w') = \mathcal{N}'_2(v'_1) = \mathcal{N}'_2(v'_2) = \{\{w'\}\}$, and $V'(w') = V(w)$, $V'(v'_1) = V'(v'_2) = V(v)$. Then one can show by an induction that $(\mathfrak{M}, w) \equiv_{\mathfrak{L}^{[\forall]}} (\mathfrak{M}', w')$ and hence $\mathfrak{M}', w' \models \neg\varphi$. (For the induction step with $\varphi = \Box_1 \psi$, observe that neither $\{w', v'_1\}$, nor $\{w', v'_2\}$ are definable in the model $\mathfrak{M}'$.) Note however that $\mathcal{N}'_2(w') = \mathcal{N}'_1(w') \Cap \mathcal{N}'_1(w')$, contradicting the fact that φ expresses this frame condition. ■

Theorem 9 *Let $\mathfrak{F} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I} \rangle$ be a frame and let $w \in W$. Then $\mathcal{N}_2(w) \subseteq \Cap^{\infty} \mathcal{N}_1(w)$ iff $\mathfrak{F}, w \models \Box_2 p \to \Box_{1^\infty} p$. Moreover, this frame condition cannot be characterized within $\mathfrak{L}_f^{[\forall]}$.*

Proof. Again, proving that the frame condition corresponds to the mentioned formula is left to the reader. To show that it cannot be characterized within $\mathfrak{L}_f^{[\forall]}$, let I and $\mathfrak{M}$ be as in the proof of Theorem 8. Let $\mathfrak{M}' = \langle W', \langle \mathcal{N}'_i \rangle_{i \in I}, V' \rangle$ where $W' = \{w'_i \mid i \in \mathbb{N}\} \cup \{v'\}$, $\mathcal{N}'_1(x) = \{W' \setminus \{w'_1, \ldots, w'_n\} \mid n \in \{1, 2, \ldots\}\}$ for all $x \in W'$, and $\mathcal{N}'_2(x) = \{\{v'\}\}$ for all $x \in W'$. V' just treats each of the w'_i as copies of w and v' as a copy of v. It can be easily verified that (a) no $X \in \mathcal{N}'_1(w'_1)$ is definable and (b) $\{v'\} \in \Cap^{\infty} \mathcal{N}'_1(w'_1)$. From (b) it follows that $\mathfrak{F}' = \langle W', \langle \mathcal{N}'_i \rangle_{i \in I} \rangle$ satisfies the frame condition of concern. Relying on (a) we can prove that $(\mathfrak{M}', w'_1) \equiv_{\mathfrak{L}_f^{[\forall]}} (\mathfrak{M}, w)$ — a contradiction. ■

Theorems 8 and 9 concern frame conditions that are relatively uncommon, though perhaps useful from the viewpoint of certain applications. After these warm-ups, we turn to a frame condition that is often discussed, i.e. closure under intersections. First, it is well-known that

$$(\Box p \wedge \Box q) \to \Box(p \wedge q)$$

is characteristic for closure of the neighbourhood set under *finite* intersections — see e.g. [50, Lemma 2.20]. In contrast, there is no formula in $\mathfrak{L}^{[\forall]}$, or, for that matter, in $\mathfrak{L}_f^{[\forall]}$, that characterizes closure under arbitrary intersections. Here, infinitary pooling modalities turn out to yield a strict gain in expressive power, compared to the language with only finitary pooling modalities. Likewise, only with infinitary pooling modalities can one express that the intersection of all members of a neighbourhood is non-empty.

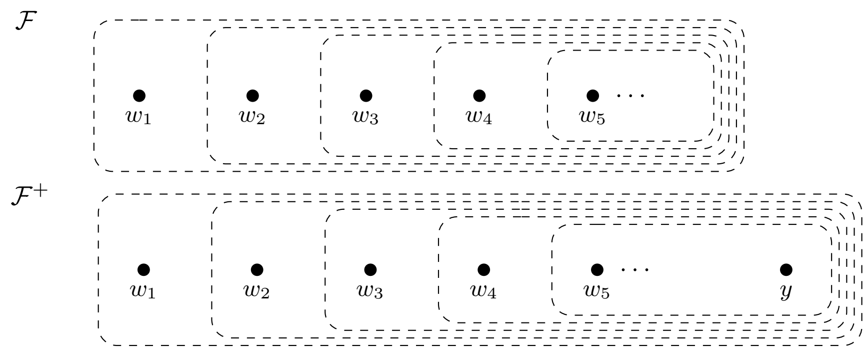

Figure 5: The two frames $\mathcal{F}$ and $\mathcal{F}^+$ in the proof of Theorem 10.

Theorem 10 *Let $\mathfrak{F} = \langle W, \langle \mathcal{N}_i \rangle_{i \in I} \rangle$ be a frame and let $w \in W$. Then each of the following hold:*

1. *$\mathcal{N}_i(w)$ is closed under arbitrary intersections iff $\mathfrak{F}, w \models \Box_{i^\infty} p \to \Box_i p$*
2. *$\bigcap \mathcal{N}_i(w) \neq \emptyset$ iff $\mathfrak{F}, w \models \Box_{i^\infty} p \to [\exists] p$.*

Moreover, neither of these frame conditions can be characterized by formulas within $\mathfrak{L}_f^{[\forall]}$.

Proof. Ad 1. Again, proving the positive claim is left to the reader. Although the negative claim for the language without $\Box_{i^\infty}$ modality is perhaps not too surprising, proving it requires some work. We provide a counterexample with $I = \{1\}$ and we omit the subscript when referring to the neighbourhoods for index 1. Our counterexample uses constant neighbourhood functions, and neighbourhood sets that are closed under finite intersections. As a result, for every $k \in \mathbb{N}$, $\Box_{1^k}\varphi$ is equivalent to $\Box_1$ in the models we construct below. Hence, it suffices to focus on the basic modal language $\mathfrak{L}^{[\forall]}$ with a single operator $\Box_1$.

Assume for contradiction that there is some $\varphi \in \mathfrak{L}^{[\forall]}$ that characterizes closure under arbitrary intersections. Let $W = \mathbb{N}$ and $W^+ = \mathbb{N} \cup \{y\}$. Let, for all $n \in \mathbb{N}$, $X_n = W \setminus \{0, \ldots, n\}$ and $X_n^+ = W^+ \setminus \{0, \ldots, n\}$. Let $\mathcal{N} = \{X_n \mid n \in \mathbb{N}\}$ and $\mathcal{N}^+ = \{X_n^+ \mid n \in \mathbb{N}\} \cup \{\{y\}\}$. Let $\mathfrak{F} = \langle W, \mathcal{N} \rangle$ and $\mathfrak{F}^+ = \langle W^+, \mathcal{N}^+ \rangle$. These two frames are depicted in Figure 5.

Note that $\mathcal{N}$ is not closed under arbitrary intersection, since it does not contain $\emptyset$, while $\bigcap \mathcal{N} = \emptyset$. By our assumption, there is a valuation V on W and a $v \in W$ such that $\langle \mathfrak{F}, V \rangle, v \models \neg\varphi$. In the remainder, let $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$. So we have:

Fact 12 *$\mathfrak{M}, v \models \neg\varphi$.*

We define the valuation $V^+ : W^+ \to \wp(\mathfrak{P})$ that extends V, as follows. First, for all $w \in W$, we let $V^+(w) = V(w)$. We define $V^+(y)$ inductively. Pick an enumeration $p_1, p_2, \ldots$ of $\mathfrak{P}$. As induction base let $\rho_0 = \top$ and $\Delta_0 = \emptyset$. Trivially, there are infinitely many $w \in W$ such that $\mathfrak{M}, w \models \rho_0$. For the induction step assume that ρ_{i-1} and Δ_{i-1} have already been defined, and there are infinitely many

$w \in W$ such that $\mathfrak{M}, w \models \rho_{i-1}$. If there are infinitely many worlds w such that $\mathfrak{M}, w \models \rho_{i-1} \wedge p_i$, set $\rho_i = \rho_{i-1} \wedge p_i$ and $\Delta_i = \Delta_{i-1} \cup \{p_i\}$. Note that if, on the other hand, there are only finitely many worlds $w \in W$ such that $\mathfrak{M}, w \models \rho_{i-1} \wedge p_i$, then there are infinitely many worlds $w \in W$ such that $\mathfrak{M}, w \models \rho_{i-1} \wedge \neg p_i$. In the latter case, set $\rho_i = \rho_{i-1} \wedge \neg p_i$ and $\Delta_i = \Delta_{i-1}$. Finally, let $V^+(y) = \Delta_0 \cup \Delta_1 \cup \ldots$.

Let $\mathfrak{M}^+ = \langle W^+, \mathcal{N}^+, V^+ \rangle$. Note that $\mathcal{N}^+$ is closed under arbitrary intersections. with $\bigcap_{X \in \mathcal{X}^+} = \{y\}$ for *any* infinite $\mathcal{X}^+ \subseteq \mathcal{N}^+$. To arrive at a contradiction we prove that, for all worlds $w \in W$, $(\mathfrak{M}, w) \equiv_{\mathfrak{L}^{[\forall]}} (\mathfrak{M}^+, w)$. This is Lemma 17 below; we first need a few other lemmata in preparation.

Lemma 13 *For every $n, k \in \mathbb{N}$, there is a $w \in X_n$ such that $\mathfrak{M}^+, w \models \rho_k$.*

Proof. Let $n, k \in \mathbb{N}$. By our construction, there are infinitely many $w \in W$ such that $\mathfrak{M}, w \models \rho_k$. Since V^+ agrees with V on all these w, and since ρ_k is just a truth-functional compound of propositional variables, there are also infinitely many $w \in W$ such that $\mathfrak{M}^+, w \models \rho_k$. Since there are only finitely many $w \in W \setminus X_n$, it follows that there is at least one (in fact, there are infinitely many) $w \in X_n$ such that $\mathfrak{M}^+, w \models \rho_k$. ■

Lemma 14 *For every finite set Δ of propositional variables, for every $n \in \mathbb{N}$: there is a $w \in X_n$ such that, $V^+(w) \cap \Delta = V^+(y) \cap \Delta$.*

Proof. Let Δ be a finite set of propositional variables. Pick a $k \in \mathbb{N}$ such that all variables from Δ occur in ρ_k. Let n be arbitrary. By the preceding lemma, there is a $w \in X_n$ such that $\mathfrak{M}^+, w \models \rho_k$. Note that also $\mathfrak{M}^+, y \models \rho_k$. In view of the construction of ρ_k, this means that $V^+(w) \cap \Delta = V^+(y) \cap \Delta$. ■

Lemma 15 *For every $n \in \mathbb{N}$ and $\psi \in \mathfrak{L}^{[\forall]}$: if, for all $w \in X_n$, $\mathfrak{M}^+, w \models \psi$, then $\mathfrak{M}^+, y \models \psi$.*

Proof. Suppose the antecedent holds. Since we work with constant neighbourhood functions, every formula of the form $\Box\tau$ is either false everywhere or true everywhere in $\mathfrak{M}^+$. Hence, every such formula is either equivalent to $p \vee \neg p$ or to $p \wedge \neg p$ in $\mathfrak{M}^+$. The same applies mutatis mutandis to formulas of the form $[\forall]\tau$. As a result, there must be a truth-functional combination of propositional variables, say ψ', for which $\|\psi'\|^{\mathfrak{M}^+} = \|\psi\|^{\mathfrak{M}^+}$. Let Δ be the (finite) set of variables that occur in ψ'. Note that the truth value of ψ' at any world $w \in W^+$ is a function of $V^+(w) \cap \Delta$. By Lemma 14, there is a $w \in X_n$ such that $V^+(w) \cap \Delta = V^+(y) \cap \Delta$. Consequently, $\mathfrak{M}^+, y \models \psi'$ and hence $\mathfrak{M}^+, y \models \psi$. ■

Corollary 3 *For every $n \in \mathbb{N}$ and $\psi \in \mathfrak{L}^{[\forall]}$: if $\mathfrak{M}^+, y \models \psi$, then there is a $w \in X_n$ such that $\mathfrak{M}^+, w \models \psi$.*

Corollary 4 *For every $\psi \in \mathfrak{L}^{[\forall]}$: if $\mathfrak{M}^+, y \models \psi$, then there is a $w \in W$ such that $\mathfrak{M}^+, w \models \psi$.*

Lemma 16 *For every $\psi \in \mathfrak{L}^{[\forall]}$, each of the following hold:*

1. $\mathfrak{M}, w \models \Box\psi$ *iff there is an* $n \in \mathbb{N}$ *such that* $X_n = \|\psi\|^{\mathfrak{M}}$
2. $\mathfrak{M}^+, w \models \Box\psi$ *iff there is an* $n \in \mathbb{N}$ *such that* $X_n^+ = \|\psi\|^{\mathfrak{M}^+}$

Proof. Item (1.) is immediate in view of the definition of $\mathcal{N}$. For item (2.) it suffices to prove that there is no $\psi \in \mathfrak{L}_f^{[\forall]}$ such that $\|\psi\|^{\mathfrak{M}^+} = \{y\}$. So assume there is one such ψ. Then $\mathfrak{M}^+, y \models \psi$. By Corollary 4, there is a $w \in W$ such that $\mathfrak{M}^+, w \models \psi$. But then $w \in \|\psi\|^{\mathfrak{M}^+}$ — a contradiction, since $w \neq y$. ■

Lemma 17 *For all $w \in W$ and all $\psi \in \mathfrak{L}_f^{[\forall]}$, each of the following hold:*

(i) $\mathfrak{M}, w \models \psi$ *iff* $\mathfrak{M}^+, w \models \psi$
(ii) $\|\psi\|^{\mathfrak{M}} = \|\psi\|^{\mathfrak{M}^+} \cap W$
(iii) for all $n \in \mathbb{N}$, $X_n = \|\psi\|^{\mathfrak{M}}$ iff $X_n^+ = \|\psi\|^{\mathfrak{M}^+}$

Proof. (i) and (ii) are easily seen to be equivalent. We prove (i)/(ii) and (iii) by a simultaneous induction on the complexity of ψ. The base case ($\psi \in \mathfrak{P}$) for (i)/(ii) is immediate in view of the construction. For (iii), we start with right to left. Suppose $X_n^+ = \|\psi\|^{\mathfrak{M}^+}$. Then $X_n^+ \cap W = \|\psi\|^{\mathfrak{M}^+} \cap W$. Since $X_n = X_n^+ \cap W$ and by (ii), $X_n = \|\psi\|^{\mathfrak{M}}$. For left to right, suppose that $X_n = \|\psi\|^{\mathfrak{M}}$. So for all $w \in X_n$, $\mathfrak{M}, w \models \psi$. By (i), for all $w \in X_n$, $\mathfrak{M}^+, w \models \psi$. By Lemma 15, $\mathfrak{M}^+, y \models \psi$. By (ii), $\|\psi\|^{\mathfrak{M}^+} \cap W = \|\psi\|^{\mathfrak{M}} = X_n$. Hence, $X_n^+ = X_n \cup \{y\} = \|\psi\|^{\mathfrak{M}^+}$.

For the induction step for (i)/(ii), we only need to consider formulas of the form $\Box_1\psi$ (cf. our remark at the outset of the proof of Theorem 10). We have: $\mathfrak{M}, w \models \Box_1\psi$ iff [by Lemma 16.1] there is some $n \in \mathbb{N}$ such that $X_n = \|\psi\|^{\mathfrak{M}}$ iff [by the induction hypothesis, item (iii)] there is some $n \in \mathbb{N}$ such that $X_n^+ = \|\psi\|^{\mathfrak{M}^+}$ iff [by Lemma 16.2] $\mathfrak{M}^+, w \models \Box_1\psi$.

For (iii), we reason just as for the base case, relying on the fact that (i)/(ii) was already proven for ψ. ■

By Fact 12 and Lemma 17.(i), $\mathfrak{M}^+, v \models \neg\varphi$, contradicting the assumption that φ is characteristic for closure under arbitrary intersections. This completes our proof of item 1 of Theorem 10.

Ad 2. The positive claim is again safely left to the reader. The negative claim follows from our proof for item 1 of Theorem 10: in the above construction, $\bigcap \mathcal{N} = \emptyset$, whereas $\bigcap \mathcal{N}^+ = \{y\}$. ■

6 Applications

In this section, we outline a number of (potential) applications for pooling modalities. In doing so we pursue three aims. The first is to demonstrate that pointwise intersection occurs within a wide range of applications in philosophical logic. Second, we show that specific such applications call for certain conditions on models and the corresponding logics. These conditions then reoccur in the more technical companion paper [4], where we present a general technique for obtaining sound and complete axiomatizations that is modular with respect to those conditions. Our third aim, finally, is to identify a number of technical problems and open ends raised by the mentioned applications.

6.1 Distributed, Non-normal Informational Attitudes

Syntactically, the *distributed knowledge* of a group of agents G can be circumscribed as the knowledge that would be obtained if some third agent combined the individual knowledge of all group members G and closed the resulting information under logical consequence [1]. Semantically, distributed knowledge is usually interpreted over multi-agent epistemic models, in terms of the intersection of the epistemic equivalence relations $\sim_i$ ($i \in G$). Analogously, one can study *distributed beliefs* and other types of distributed information across groups following either a syntactic or semantic characterization.

A small warning is in place here though. As Gerbrandy [24] shows, the syntactic and semantic readings do not always coincide. This incidentally relates to our discussion of the principle of Full Communication in Section 5.2, where we identify a condition, finiteness in combination with $\mathfrak{L}^{[\forall]}$-differentiability, under which both readings are equivalent. One interesting alley for further research is to see whether there are other sufficient conditions for this equivalence, e.g. by generalizing the notion of *tightness* and *epistemic saturation* from [56] to non-normal modal logics and neighbourhood models.

In what follows, we adopt the semantically driven view, following the seminal [22]. As shown in Section 4.1, pointwise intersection generalizes the semantic view to cases where informational attitudes are non-normal (cf. Theorem 1). Let us point out a number of applications for which this is useful.

Knowability and Subset Spaces In an attempt to model the interaction between knowledge and (alethic) possibility, logics of knowability assume a picture of potential knowledge where either obtaining information or drawing logical inferences from it may come at a price. A prominent example of the former view are subset space logics [8, 10, 19, 49]. These represent informational situations with a neighbourhood, i.e. a family of subsets, each indicating information the agent could have, were she to invest the appropriate amount of effort.

With potential knowledge and belief cashed out in terms of neighbourhood systems, their aggregation in terms of group or distributed knowledge and belief naturally invokes pointwise intersections in neighbourhood systems, as for instance spelled out in [5], see also [71]. This in turn allows one to express what groups may come to know, through joint efforts of their members.

Non-normal Doxastic Logics Turning to doxastic logic, there are a number of distinct motivations for weakening the underlying logic of belief to a non-normal one.

First, abandoning the assumption that knowledge is negatively introspective, Stalnaker [61] proposes a combined epistemic-doxastic logic where belief is interpreted as the mental component of knowledge. While this weakening of the classical axiomatizaion preserves normality of knowledge and belief, further weakenings do not. Positive introspection, for instance, has been heavily criticised on philosophical grounds. Correspondingly, Klein et al. propose two logics that weaken Stalnaker's framework further by also omitting positive introspection [40]. It turns out that this renders belief a non-normal modality: belief is closed under weakening but not under intersection, i.e. the agent can believe φ and ψ without believing $\varphi \wedge \psi$.

Other philosophically motivated weakenings of classic epistemic logic bear an even closer connection to non-normal modal logics. The fragmentation theory of knowledge, for instance, holds that closure under conjunction or consistency requirements only apply within certain domains of knowledge, but not across those domains [37]. Correspondingly, knowledge and belief are represented by neighbourhood systems that need not satisfy (C_i) or $\neg\Box_{i^\infty}\bot$.

Another recent strand of literature makes a distinction between explicit beliefs of an agent and its potential beliefs and background knowledge, following [21]. On this picture, explicit beliefs are usually not closed under conjunction or under weakening (though they are closed under replacement of equivalents). They can hence be studied in terms of neighbourhood systems, as shown in [6].

Evidence Logics Tightly related to the subset space approach, *evidence logics* draw a fine grained picture that not only includes the agents' belief, but also the evidence this belief is based upon. First proposed by Van Benthem and Pacuit [66], the main component of evidence logic is a neighbourhood of the agent's available evidence. Only weak requirements are made on this evidence, as it is usually neither assumed truthful nor jointly consistent, let alone closed under intersection. Belief, in these logics, is a derived attitude that is obtained from the agent's available evidence by considering the maximal non-empty intersections of evidence sets. Exact details differ, see for instance [7,66] and also the discussion on monotonic vs. exact readings on evidence modalities in Section 4.3.

In this context, pooling modalities and pointwise intersection are again just around the corner. In fact, they literally show up in a proposal by van Benthem and Pacuit to model the fact that 'agents can also "pool" their evidence creating a new evidential states by combining their evidence' [66, pp. 88-89].

Note however that, *strictu sensu*, pooling modalities cannot express evidence-based belief as the latter specifically refers to *non-empty* intersections only, and also employs the $\subseteq$ relation on these non-emtpy intersections. There may however be a way around that difficulty. Baltag et al. [7] show that one can capture a specific Evidence Logic by combining three relatively simple ingredients: a universal modality $[\forall]$, an operator E^b for *basic factive evidence*, and an operator E for *combined factive evidence*, i.e. for any finite intersection of basic factive evidence. Baltag et al. show evidence for φ and evidence-based belief in φ to be definable in this language by $[\exists]\mathsf{E}^b\varphi$ and $[\forall]\neg\mathsf{E}\neg\mathsf{E}\varphi$ respectively. Analogously, distributed evidence and belief based on distributed evidence can then be defined by $[\exists]\mathsf{E}^b_G\varphi$ and $[\forall]\neg\mathsf{E}_G\neg\mathsf{E}_G\varphi$ respectively.

These are but some examples of non-normal logics for informational attitudes of agents. In all these logics, hence, one may investigate the logical behavior of distributed information within a group, generalizing the mentioned work on distributed knowledge and belief. Moreover, when an agent's informational attitude is not assumed to be closed under aggregation, pooling modalities may also be used to reason about implicit beliefs or knowability in the limit of a single agent, i.e. those propositions that follow from a (finite or arbitrary) aggregation of their information.

Issues and Open Questions: A first task is to provide axiomatizations of the resulting logics with pooling modalities. In line with standard properties of knowledge and belief, the modalities to be pooled will often satisfy axiom (M_i), also known as *weakening* or *inheritance*. In some of the cases outlined above, additional conditions will be in place such as axiom (T_i) also known as *factivity* for knowledge and knowability, or binary consistency for belief (cf. [40]). In [4], we provide axiomatizations for logics that include any combination of the mentioned axioms, leading to a counterpart of the axiomatization of normal distributed belief provided in [22] (cf. Section 4). For other axioms (e.g. those concerning positive and negative introspection), the question of a sound and complete characterization remains open.

A related open task concerns a multi-agent version of evidence logic. Going multi-agent triggers a number of conceptual and formal issues. In evidence based frameworks, different types of distributed beliefs become definable. For instance, one may first determine individual belief sets and then derive the corresponding distributed belief. Alternatively, one may first combine the agents' available evidence into the group's evidence and, from there, calculate the corresponding group belief. While these two notions of distributed belief will, in general, not coincide, their exact relationship remains to be determined.

6.2 Deontic Reasoning

Neighbourhood semantics have been used in deontic logic to model the interaction of various norms, for instance in (non-explosive) conflict-tolerant normative reasoning [28,29], or in agents forming plans to meet their various obligations [39,43]. Here, $\Box_i\varphi$ can, e.g., be taken to mean that there is at least one norm in the normative system S_i that makes φ obligatory; the presence of two conflicting norms can then account for the truth of a deontic conflict of the type $\Box_i\varphi \wedge \Box_i\neg\varphi$. Within our richer setting, we can express such things as S_i being conflict-free ($\neg\Box_{i^\infty}\bot$) or S_i being conflict-free (exactly) up to level n ($\neg\Box_{i^n}\bot$ and $\Box_{i^{n+1}}\bot \wedge \neg\Box_{i^n}\bot$ respectively). The former property plays an important role, for instance, in determining which obligations shall be translated into goals adopted by the agent [39].

In this context, pointwise intersection across distinct indexes can be interpreted as the piecemeal aggregation of norms from possibly different normative systems. A formula such as $\Box_{1,2}p$ then expresses that there are two norms, one in S_1, the other in S_2, such that obeying both requires the truth of p. With a universal modality in the language one can moreover express that these norms are mutually compatible, as witnessed by $[\exists]p$.

An altogether different application of pooling modalities consists in reading the indices as *reasons* for one's obligations. On this view, $\Box_r\varphi$ expresses that r is a reason for φ to be obligatory, and one can then aggregate reasons alongside with obligations: $\Box_r\varphi \wedge \Box_{r'}\psi$ yields $\Box_{r,r'}(\varphi \wedge \psi)$. As argued in [23, 48], reasons play an important, but often neglected role in our normative reasoning. A thorough logical investigation of their interaction and aggregation in deontic logic (at the level of the object language) is still lacking.

Issues and Open Questions: Also in deontic logic, a standard task is to identify sound and complete axiomatizations given various frame conditions. Notably, this can be quite different from the corresponding problem in doxastic logic, as the two validate very distinct logical principles. In some deontic approach, the neighbourhood set corresponding to a normative source S consists of all those $X \subseteq W$ that are exactly the set of all worlds satisfying some norm n issued by S. As S may not have issued any norm that is equivalent to $\top$, the set W of all worlds need not be contained in this neighbourhood, i.e. (N_i) may be violated. Conversely, a neighbourhood set may contain $\emptyset$, violating (P_i), e.g. if a general norms issued by S is not satisfiable in the present situation (cf. [39]).

Lastly, in a different but related approach, neighbourhood sets may be taken to collect all sets of legal or permitted worlds with respect to some set of norms [57]. In this interpretation, neighbourhood sets are *downward* closed (i.e. closed under taking subsets), as the property of not containing illegal worlds is.

6.3 Coalitional Agency, Ability, and Forcing Powers in Games

Modal logics of agency have a venerable history, dating back to the middle ages.[25] More recently, agency operators were proposed as a necessary complement of deontic operators in the analysis of rights and normative positions [60]. In these systems, $\Box_i\varphi$ can be read as "i brings it about that φ", or "i is the agent of φ". With the coming of age of **STIT** logic, the logic of "sees to it that"-locutions, it has been noticed that multiagent logics of agency bear strong links with various game-theoretic notions [55, 62] (see [65] for an overview) and in particular with coalition logic, which can express the powers of coalitions in game forms [30, 52].

Neighbourhood semantics provide a very general framework for interpreting logics of agency, cf. [45]. In such semantics, $X \in \mathcal{N}_i(w)$ is usually interpreted as: "X is a state of affairs brought about by i at w", or X is a state of affairs, that i could bring about at w. Various logics of agency differ in the exact viewpoint taken and, hence, in the additional assumption imposed on neighbourhood functions. Horty's famous "Chellas **STIT**" operator, for instance, assumes a border case perspective where actions are taken to produce instantaneous effects, different actions taken at world w are assumed compatible and the effect of actions is modelled as closed under weakening. At the formal level, this corresponds to the validity of (T_i), (M_i), and (C_i). This, however, is but one conceptualization of agency. Other perspectives such as Broersen's **XSTIT** [11, 12] depict actions as partially determining the transition from a current state to some next state, hence invalidating (T_i). Likewise, the "deliberative **STIT**" operator from [9] invalidates (N_i) and (M_i).

In logics of abilities or strategic powers, $X \in \mathcal{N}_i(w)$ can be interpreted as the agent having some available action that brings about X. Here, (C_i) need not hold, as an agent's option to choose between various possible actions does not imply that she can execute several of these simultaneously. Moreover, logics of *exact* strategic abilities invalidate (N_i) and (M_i). In [62] such logics are shown to be instrumental for defining notions of equivalence between games.

When working in multiagent settings, pooling modalities are a good candidate for talking about both, group agency and group abilities. For instance, the compatibility of actions postulate of game or coalition logics (cf. [53, 55]) is expressible as $\neg\Box_G\bot$. Hence, an adequate understanding of group abilities is tightly connected to the study

[25] See [9, Section 1D] for an overview of this history before 1990; [9] itself remains a key reference for the logic of agency.

of pooling modalities. Incidentally, this relates to logics of abilities and existing semantics for group **STIT**, which can be phrased in terms of the intersection of equivalence relations, cf. [18].

Issues and Open Questions: A first issue, again, is to provide sound and complete axiomatizations of pooling modalities the correspond to specific versions of agency and ability. *Independence of agency* and related independence conditions from game logic add additional complexity here, as philosophically motivated desiderata that may require the expressive power of pooling modalities.

A second issue is related to decidability: can we find a suitable, sufficiently strong logic of collective agency that captures some notion of free, independent agency, yet is still decidable, in contrast to group **STIT**? As shown in [13], Pauly's Coalition Logic [51] is a decidable fragment of a particular **STIT** logic for groups introduced in [38]. So there may well be other such decidable fragments that are complete with respect to certain pictures of group agency.

A third topic concerns enriched pictures of (group) agency and ability. First, Elgesem [20] and more recently McNamara [45] argued that we often distinguish between the states of affairs φ a given agent brings about, and a "basic proposition" ψ by means of which φ was brought about. The resulting dyadic neighbourhood functions raise a number of questions, for instance whether pointwise intersection should concern only the antecedents, only the consequents or both. Second, Alechina and co-workers [2] put forward a picture of group ability that not only depicts the agents involved, but also the resources these agents need to invest in order to bring about a certain state. Besides classic group actions, pooling modalities may then also be used to track the strategic abilities of groups that redistribute resources among their members to achieve certain outcomes.

7 Summary and Outlook

In this paper, we have argued that pooling modalities provide a highly expressive and well-behaved extension of the standard multi-index language interpreted over neighbourhood models. We showed that they moreover have a wide range of applications, gave a number of foundational results and introduced useful notation and terminology. Having already hinted at different applications and the specific issues they raise, we end this paper with an overview of more general open problems concerning pointwise intersection and pooling modalities.

1. Building on Section 4.2, can we further enrich the modal language with operators that talk about the supplementation of a given neighbourhood model? Note that such a language would allow us to describe the properties of the relational model that corresponds to the augmentation of an arbitrary neighbourhood model, and combine the expressive power of both the exact and the monotonic semantics for classical modal logics.

2. Bisimulation is a standard vehicle for analyzing modal equivalence and expressivity of the modal language. For classical modal logics, similar characterizations of modal equivalence have been investigated in recent work [35].

A natural question, hence, is for a suitable characterization of modal equivalence in languages with pooling modalities (cf. Section 5.1). Relatedly, one may investigate the possibility of a standard translation from $\mathfrak{L}_\infty^{[\forall]}$ or one of its sublanguages into first-order logic, again following [35]. Such translations are known to generate numerous important results such as compactness and enumerability.

3. It is well-known that classical modal logics can be embedded into normal modal logics. In particular, monotonic operators can be represented by a combination of two normal modal operators; for the embedding of arbitrary classical modal operators, three normal modalities are required [41]. Similarly, one may ask whether logics featuring pooling modalities can be embedded in normal modal logics, using intersection modalities. While this has been done for the specific case of coalition logic (which is a fragment of group **STIT**, cf. [13]), a more general recipe for such embeddings is still lacking.

4. Many of the applications presented call for a dynamic perspective. In the case of evidence and at the level of a single agent, dynamic operations in terms of adding or retracting pieces of evidence were already studied in [66]. A first question, here, is how such operations can be generalized to the multi-agent case. However, going multi-agent also opens the possibility of formalizing and axiomatizing interactive operations, where e.g. one agent shares its evidence with other agents, or two agents "merge" their evidence. Here pooling modalities may be particularly useful, in that they allow to reduce inter-agent forms of sharing information, in line with existing work on resolution operations and operators of distributed knowledge [1].

   Analogous observations can be made about sharing or exchanging norms (cf. Section 6.2). Finally, also logics of abilities may have use for dynamic operations. For instance, agents may *delegate* their powers or abilities to others, as happens frequently in institutional settings and in certain voting scenarios [17]. From a formal perspective, delegation from agent i to agent j may be understood as a dynamic operation, where the resulting powers of j coincide with the original powers of the group $\{i, j\}$.

   Importantly, dynamic operations such as the ones mentioned have formal connections to products of neighborhood systems in a similar way that product models of dynamic epistemic logic can be seen as a special type of product model [70]. Also here, the exact relationship remains to be worked out.

5. Finally, as mentioned in the introduction, pointwise intersections is just one prototypical example of combinatorics over neighbourhood sets. As such, it is a member of a larger family alongside, for instance, operations motivated by PDL's non-deterministic choice or sequential composition [54, 63]. It remains to be seen how the latter combine with pointwise intersection. Relatedly, one may also generalize pointwise intersection to other boolean operations of union and negation, and investigate the expressivity and logical properties of ensuing systems.

References

[1] Thomas. Ågotnes and Yi Nicholas Wáng. Resolving distributed knowledge. *Artificial Intelligence*, 252:1–21, 2017.

[2] Natasha Alechina, Brian Logan, Hoang Nga Nguyen, and Abdur Rakib. Logic for coalitions with bounded resources. *Journal of Logic and Computation*, 21(6):907–937, 2011.

[3] Carlos Areces and Diego Figueira. Which semantics for neighbourhood semantics? In *Proceedings of the 21st International Jont Conference on Artifical Intelligence*, IJCAI'09, pages 671–676, San Francisco, CA, USA, 2009. Morgan Kaufmann Publishers Inc.

[4] The Author. Pooling modalities and pointwise intersection: Axiomatization and decidability. Under review (accepted subject to major revisions). Companion paper to the current one.

[5] Can Başkent. Topics in subset space logic. *Institute for Logic, Language and Computation, Universiteit van Amsterdam (July 2007)*, 2007.

[6] Philippe Balbiani, David Fernández-Duque, and Emiliano Lorini. The dynamics of epistemic attitudes in resource-bounded agents. *Studia Logica*, 107(3):457–488, 2019.

[7] Alexandru Baltag, Nick Bezhanishvili, Aybüke Özgün, and Sonja Smets. Justified belief and the topology of evidence. In Jouko Väänänen, Åsa Hirvonen, and Ruy de Queiroz, editors, *Logic, Language, Information, and Computation*, pages 83–103, Berlin, Heidelberg, 2016. Springer Berlin Heidelberg.

[8] Alexandru Baltag, Nick Bezhanishvili, Aybüke Özgün, and Sonja Smets. A topological approach to full belief. *Journal of philosophical logic*, 48(2):205–244, 2019.

[9] Nuel D. Belnap, Michael. Perloff, and Ming Xu. *Facing the Future: Agents and Choice in Our Indeterminist World.* Oxford University Press, 2001.

[10] Adam Bjorndal and Aybüke Özgün. Logic and topology for knowledge, knowability, and belief. *The Review of Symbolic Logic*, pages 1–28, 2019.

[11] Jan Broersen. Deontic epistemic stit logic distinguishing modes of mens rea. *Journal of Applied Logic*, 9(2):137 – 152, 2011. Special Issue: Selected and revised papers from the Ninth International Conference on Deontic Logic in Computer Science (DEON 2008).

[12] Jan Broersen. Making a start with the stit logic analysis of intentional action. *Journal of Philosophical Logic*, 40:399–420, 2011.

[13] Jan Broersen, Andreas Herzig, and Nicolas Troquard. A normal simulation of coalition logic and an epistemic extension. In Dov Samet, editor, *Proceedings of the 11th Conference on Theoretical Aspects of Rationality and Knowledge*, TARK '07, pages 92–101, New York, NY, USA, 2007. ACM.

[14] Mark A. Brown. On the logic of ability. *Journal of Philosophical Logic*, 17(1):1–26, February 1988.

[15] José M. C. L. M. Carmo and Andrew J. I. Jones. Deontic logic and contrary-to-duties. In Dov M. Gabay and Franz Guenther, editors, *Handbook of Philosophical Logic*, volume 8, pages 147–264. Kluwer Academic Publishers, 2nd edition, 2002.

[16] Brian Chellas. *Modal Logic: an Introduction*. Cambridge: Cambridge university press, 1980.

[17] Zoé Christoff and Davide Grossi. Binary voting with delegable proxy: An analysis of liquid democracy. *arXiv preprint arXiv:1707.08741*, 2017.

[18] Roberto Ciuni and Emiliano Lorini. Comparing semantics for temporal STIT logic. *Logique et Analyse*, 2018.

[19] Andrew Dabrowski, Lawrence S. Moss, and Rohit Parikh. Topological reasoning and the logic of knowledge. *Annals of Pure and Applied Logic*, 78(1):73 – 110, 1996.

[20] Dag Elgesem. The modal logic of agency. *Nordic Journal of Philosophical Logic*, 2:1–46, 1997.

[21] Ronald Fagin and Joseph Y. Halpern. Belief, awareness, and limited reasoning: Preliminary report. In *Proceedings of the 9th International Joint Conference on Artificial Intelligence - Volume 1*, IJCAI85, page 491501, San Francisco, CA, USA, 1985. Morgan Kaufmann Publishers Inc.

[22] Ronald Fagin, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge*. MIT Press, Cambridge, Massachusetts, 2003.

[23] Federico L. G. Faroldi. *Hyperintensionality and Normativity*. Springer, Dordrecht, 2019.

[24] Jelle Gerbrandy. Distributed knowledge. In Joris Hulstijn and Anton Nijholt, editors, *Twendial 98: Formal Semantics and Pragmatics of Dialogue, TWLT 13*, pages 111–124, Enschede, 1998. Universiteit Twente.

[25] Jelle Gerbrandy. *Bisimulations on Planet Kripke*. PhD thesis, University of Amsterdam, 1999.

[26] Lou Goble. Multiplex semantics for deontic logic. *Nordic Journal of Philosophical Logic*, 5:113–134, 2000.

[27] Lou Goble. Preference semantics for deontic logic. Part II: Multiplex models. *Logique et Analyse*, 185–188:335–363, 2004.

[28] Lou Goble. A logic for deontic dilemmas. *Journal of Applied Logic*, 3:461–483, 2005.

[29] Lou Goble. Prima facie norms, normative conflicts, and dilemmas. In Dov Gabbay, Leon van der Torre, John Horty, and Xavier Parent, editors, *Handbook of Deontic Logic and Normative Systems*, volume 1, chapter 4, pages 241–351. College Publications, 2013.

[30] Valentin Goranko, Wojciech Jamroga, and Paolo Turrini. Strategic games and truly playable effectivity functions. *Autonomous Agents and Multi-Agent Systems*, 26(2):288–314, 2013.

[31] Valentin Goranko and Solomon Passy. Using the universal modality: Gains and questions. *Journal of Logic and Computation*, 2(1):5 – 30, 1992.

[32] Guido Governatori and Antonino Rotolo. On the axiomatisation of elgesem's logic of agency and ability. *Journal of Philosophical Logic*, 34(4):403–431, Aug 2005.

[33] Helle Hvid Hansen and Clemens Kupke. Weak completeness of coalgebraic dynamic logics. In Ralph Matthes and Matteo Mio, editors, Proceedings Tenth International Workshop on *Fixed Points in Computer Science,* Berlin, Germany, September 11-12, 2015, volume 191 of *Electronic Proceedings in Theoretical Computer Science*, pages 90–104. Open Publishing Association, 2015.

[34] Helle Hvid Hansen, Clemens Kupke, and Raul Andres Leal. Strong completeness for iteration-free coalgebraic dynamic logics. In Josep Diaz, Ivan Lanese, and Davide Sangiorgi, editors, *Theoretical Computer Science*, pages 281–295, Berlin, Heidelberg, 2014. Springer Berlin Heidelberg.

[35] Helle Hvid Hansen, Clemens Kupke, and Eric Pacuit. Neighbourhood structures: Bisimilarity and basic model theory. *Logical Methods in Computer Science*, 5(2), 2009.

[36] David Harel, Dexter Kozen, and Jerzy Tiuryn. *Dynamic Logic*. Cambridge, MA: MIT Press, 2000.

[37] Peter Hawke, Aybüke Özgün, and Francesco Berto. The fundamental problem of logical omniscience. *Journal of Philosophical Logic*, 2019.

[38] John F. Horty. *Agency and Deontic Logic*. Oxford University Press, New York, 2001.

[39] Dominik Klein and Alessandra Marra. From oughts to goals: A logic for enkrasia. *Studia Logica*, 2019.

[40] Dominik Klein, Olivier Roy, and Norbert Gratzl. Knowledge, belief, normality, and introspection. *Synthese*, 195(10):4343–4372, 2018.

[41] Marcus Kracht and Frank Wolter. Normal monomodal logics can simulate all others. *The Journal of Symbolic Logic*, 64(1):99–138, 1999.

[42] David Lewis. *Counterfactuals*. Harvard University Press, Cambridge, Mass., 1973.

[43] Alessandra Marra and Dominik Klein. Logic and ethics: An integrated model for norms, intentions and actions. In Wiebe van der Hoek, Wesley H. Holliday, and Wen-fang Wang, editors, *Logic, Rationality, and Interaction*, pages 268–281. Springer, 2015.

[44] J. C.C̃. McKinsey and Alfred Tarksi. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[45] Paul McNamara. Toward a systematization of logics for monadic and dyadic agency & ability (preliminary version). In *Proceedings for Deontic Logic and Normative Systems (Deon 2018), 3-6 July 2018, Utrecht, the Netherlands*, London, 2018. College Publications.

[46] Richard Montague. Universal grammar. *Theoria*, 36:373–398, 1970.

[47] Lawrence Moss and Rohit Parikh. Topological reasoning and the logic of knowledge. In Yoram Moses, editor, *Proceedings of TARK IV*. Morgan Kaufmann, 1992.

[48] Shyam Nair and John F. Horty. The logic of reasons. In Daniel Star, editor, *The Oxford Handbook of Reasons and Normativity*. Oxford University Press, in press.

[49] Aybüke Özgün. *Evidence in epistemic logic: a topological perspective*. PhD thesis, LORIA and ILLC Amsterdam, 2017.

[50] Eric Pacuit. *Neighbourhood Semantics for Modal Logic*. Springer, 2017.

[51] Marc Pauly. A modal logic for coalitional power in games. *Journal of Logic and Computation*, 1(1):149–166, 2002.

[52] Marc Pauly. A modal logic for coalitional power in games. *Journal of Logic and Computation*, 1(1):149–166, 2002.

[53] Marc Pauly. *Complexity in Interaction*. PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, 2011.

[54] Marc Pauly and Rohit Parikh. Game logic - an overview. *Studia Logica*, 75:165–182, 2003.

[55] Bezalel Peleg. Effectivity functions, game forms, games, and rights. *Social Choice and Welfare*, 15(1):67–80, 1998.

[56] Floris Roelofsen. Distributed knowledge. *Journal of Applied Non-Classical Logics*, 17(2):255–273, 2007.

[57] Olivier Roy, Albert Anglberger, and Norbert Gratzl. The logic of obligation as weakest permission. In *International Conference on Deontic Logic in Computer Science*, pages 139–150. Springer, 2012.

[58] Dana Scott. *Advice on Modal Logic*, pages 143–173. Springer Netherlands, Dordrecht, 1970.

[59] K. Segerberg. An essay in classical modal logic, 1971.

[60] Marek Sergot. Normative positions. In Dov Gabba, John Horty, Xavier Parent, Ron van der Meyden, and Leendert van der Torre, editors, *Handbook of Deontic Logic and Normative Systems*, pages 353–406. College Publications, 2013.

[61] Robert Stalnaker. On logics of knowledge and belief. *Philosophical Studies*, 128(1):169–199, 2006.

[62] Johan van Benthem, Nick Bezhanishvili, and Sebastian Enqvist. A new game equivalence, its logic and algebra. *Journal of Philosophical Logic*, 48(4):649–684, 2019.

[63] Johan van Benthem, Nick Bezhanishvili, and Sebastian Enqvist. A propositional dynamic logic for instantial neighborhood semantics. *Studia Logica*, 107:719–751, 2019.

[64] Johan van Benthem and Guram Bezhanisvilli. Modal logics of space. In *Handbook of Spatial Logic*, pages 217–298. Springer, 2007.

[65] Johan van Benthem and Dominik Klein. Logics for analyzing games. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2019 edition, 2019.

[66] Johan van Benthem and Eric Pacuit. Dynamic logics of evidence-based beliefs. *Studia Logica*, 99(1):61, Aug 2011.

[67] Frederik Van De Putte. That will do: Logics of deontic necessity and sufficiency. *Erkenntnis*, 82(3):473–511, 2017.

[68] Frederik Van De Putte and Dominik Klein. Pointwise intersection in neighbourhood modal logic. In Guram Bezhanishvili and Giovanna D'Agostino, editors, *Advances in Modal Logic (AiML 12)*, pages 591–610. College Publications, 2018.

[69] Wiebe van der Hoek, Bernd. van Linder, and John-Jules Meyer. Group knowledge is not always distributed (neither is it always implicit). *Mathematical social sciences*, 38:215240, 1999.

[70] Hans van Ditmarsch, Wiebe van Der Hoek, and Barteld Kooi. *Dynamic epistemic logic*, volume 337 of *Synthese Library*. Springer, 2007.

[71] Yi Nicholas Wang and Thomas Ågotnes. Multi-agent subset space logic. In *Twenty-Third International Joint Conference on Artificial Intelligence*, 2013.