Regiomontanus and Chinese mathematics

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Introduction

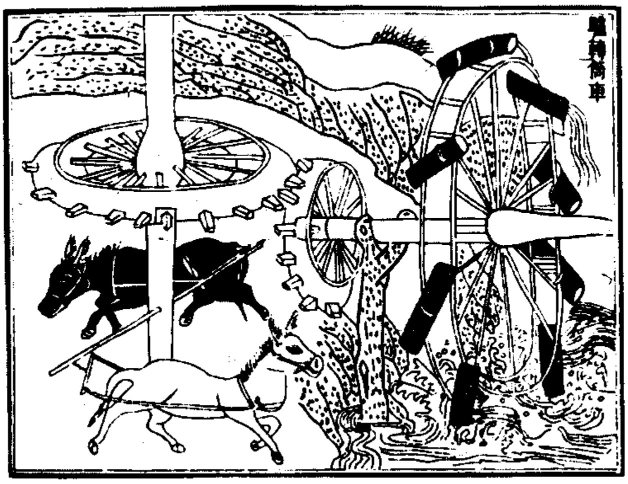
The title of this paper is borrowed from a section heading of Gavin Menzies's latest book titled *1434* (Menzies 2008, 147). Menzies already stirred considerable controversy with his first work titled *1421* (Menzies 2002) in which he claims that the Chinese fleet circumnavigated the world during the Ming and travelled to parts of the world yet undiscovered by the Europeans, such as the Americas. Menzies is a retired naval commander with no command of the Chinese language. His methods were criticized and his conclusions dismissed by several historians and sinologists but his hypothesis also attracted many followers, named "friends of the 1421 website", supporting the theory with "additional evidence". In a critical assessment of Menzies' "evidence" that has escaped "distinguished academics in the field", Bill Richardson concluded that "'[i]maginography' and uninformed, wildly speculative 'translations' of toponyms are not conducive to a credible rewriting of history" (Richardson 2004, 10).

His latest book is no less controversial. The new hypothesis in *1434* becomes apparent from the subtitle "The Year a Magnificent Chinese Fleet Sailed to Italy and Sparked the Renaissance". According to Menzies a large fleet of Chinese vessels visited Italy in the first half of the 15th century. As a consequence, many great men from the Renaissance such as Paolo Toscanelli, Leone Battista Alberti, Nicolas of Cusa, Regiomontanus, Giovanni di Fontana and Mariano Taccola found direct inspiration for their knowledge from the Chinese. He claims for example, that many of the inventions that Leonardo da Vinci is credited for actually depended on knowledge of Chinese contrivances through Taccola and Francesco di Giorgio. The evidence is presented demagogically by showing the similarity of Renaissance and Chinese illustrations side by side (see figure 1). Alberti's knowledge of perspective would depend on the *Shù shū Jiŭ zhāng* (數書九章, *Mathematical Treatise in Nine Sections*, hence *SSJZ*, Libbrecht 1973) by *Qín Jiŭshào* (秦九韶) written in 1247.[1] And so it goes on.

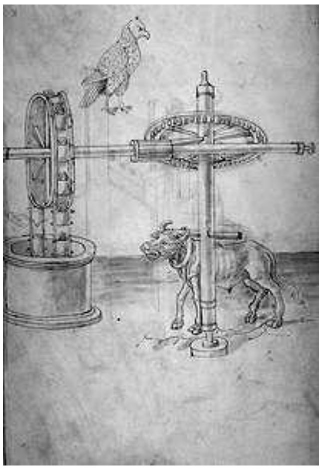
It is all too easy to dismiss the claims by Menzies by reasons of a lack of historical scrutiny, knowledge of Chinese language or scientific method. His claims are challenging, the parallels *are* surprising, and some evidence cries out for an explanation. As it is our belief that scholars

[1] We will use the pinyin transcription for the names of Chinese books and authors.

should not shy away from the claims presented in the book we will take up the challenge. We will focus here on one of Menzies's arguments dealing with mathematics: Regiomontanus had access to Chinese mathematical works because of his knowledge of the Chinese Remainder Theorem (*CRT*). We will demonstrate that the argument is false and the criticism applies to many similar claims in his book. However, there has been a transfer of mathematical knowledge from East to West. The *CRT* method or *Da-yan shu* (大衍術 Great Extension Mathematics) *is* a Chinese invention. Therefore we will also address the question of how then to account for dissemination of knowledge through distant cultures without the availability of written evidence.



From the *Nung shu*, 1313 (Needham, 1965)



Late 15th cent. copy of Taccola (Ms. Palatino 767, f. 32r, BNC Florence)

Figure 1: A typical example of Menzies's evidence for Chinese influences.

The da yan rule[2]

Let us first recall that the *da yan* rule describes that for a set of congruences $x \equiv a_i (\bmod m_i)$ in which the m_i are pair wise relative primes, the solution is given by

$$x = \sum_{i=1}^{n} a_i b_i \frac{M}{m_1}$$

in which *M* is the product of all the m_i and the b_i are derived by the congruence relation $b_i \frac{M}{m_i} \equiv 1 (\bmod m_i)$.

The Chinese version of the rule depends on a specific procedure which we will illustrate by a numerical example. See Needham (1959, 119-120), Libbrecht (1973, 333-354), Katz (1992,

[2] The name is in Western publications better known under its Wade-Giles transliteration *ta-yen*.

188) or Martzloff (2006, 310-323) for other examples. We will abbreviate congruence sets by the compact notation $x \equiv a_1(m_1), a_2(m_2), \ldots, a_n(m_n)$. Our example is $x \equiv 2\ (3),\ 3\ (5),\ 4\ (7)$.

1) Reduce the moduli (*ding mu* or fixed denominators) to a multiplication or a power of prime numbers unless they are prime or a power of a prime already (which is the case in our example). The relative prime moduli are called *ding shu*. This procedure is called *da yan qiu yi shu* (great expansion procedure for finding the unity) from which the *da yan* name is derived.
2) Find the least common multiple of the moduli, called the *yan mu* (multiple denominators). In our example the product of 3, 5 and 7 is 105.
3) Divide the *yen mu* by all the *ding shu*. The result is called *yan shu* (multiple numbers or operation numbers), in our example 35, 21 and 15 respectively.
4) Subtract from the *yen shu* the corresponding *ding shu* as many times as is possible. The remainders are called *qi shu* (surplus). Thus 35 – 3(11) = 2, 21 – 5(4) = 1, 15 – 7(2) = 1.
5) Calculate the *chêng lü* (multiplying terms) as $b_i \frac{M}{m_i} \equiv 1 (\bmod m_i)$, being $2b_1 = 1(\bmod 3), b_1 = 2$, $1b_2 = 1(\bmod 5), b_2 = 1$ and $1b_3 = 1(\bmod 7), b_3 = 1$.
6) Multiply the *chhêng lü* with the corresponding *yan mu* and the remainders. These are called *yong shu* (useful numbers). Thus 2.35 = 70, 3.21 = 63 and 4.15 = 60.
7) Multiply the *yong shu* with the remainders. Thus 70.2 = 140, 63.1 = 63 and 60.1 = 60.
8) Add these products together and you will get the *zong shu* (sum) thus 140 + 63 + 60 = 263.
9) Subtract the *yan mu* from this sum as many times as possible to get the solution, *x* = 263 – 105 – 105 = 53.

The terminology is taken from *Qín Jiǔshào*. An English translation of the complete text of the procedure is given by Libbrecht (1973, 328-332) and Dauben (2007, 314-15).

An epistemic argument

In this section we provide a fair representation of the epistemic argumentation developed by Menzies. The premise consists of the fact that Regiomontanus had knowledge of the *da yan* rule. The evidence presented stems from his correspondence with the astronomer Francesco Bianchini in 1463. The Latin correspondence was published by Curtze (1902). In a letter to Bianchini (not dated but late 1463 or early 1464) Regiomontanus poses eight questions, the last one being "Quero numerum, qui si dividatur per 17, manent in residuo 15, eo autem diviso per 13, manent 11 residua, At ipso diviso per 10 manent tria residua: quero, quis sit numerus ille" (Curtze 1902, 219). Within the context of the *quatrocento* such questions should be understood as challenges rather than genuine enquiries. Regiomontanus probably had the solution before posing the question. But Bianchini met the challenge and produced a correct answer in his letter of 5 Feb 1464 (Curtze 1902, 237). Bianchini answers the above problem with the solutions 1103 and 3313 and adds that there are many more solutions, but that he does not wish to spend the labor for finding more ("Sed in hoc non euro laborem expendere, in aliis numeris invenire"). This comment leads Curtze to conclude that Bianchini did not know the general method. In his answering letter (not dated, Curtze 1902, 254) Regiomontanus reveals that there is an infinite number of solutions and that the solutions are easily generated, annotated by a figure in the margin ("Huic si addiderimus numerum numeratum ab ipsis tribus divisoribus, scilicet 17, 13 et 10, habebitur secundus, item eodem addito resultat tertius etc."). Every time one adds the least common multiple of the three divisors, additional solutions are generated, not much labor required at all. But concluding from this, as Curtze does, that Regiomontanus knew the general solution for the remainder problem, is one step too far. As we shall demonstrate below, the solution can be obtained by tables or trial and error and the rule for generating additional solutions was known within the abbaco tradition. It even had its own name. It is impossible from the correspondence to establish the specific procedure he used for finding the first solution. The second premise, that a general procedure for the CRT is explained in the *Shù shū Jiǔ zhāng*, is of course justified. From this Menzies concludes that "it follows that Regiomontanus must have been aware of this Chinese book of 1247 unless he had quite independently thought up the *Da-yan* rule, which he never claimed to have done".[3] He adds: "The implications of Regiomontanus knowing of this massive book, which was the fruit of the work of thirty Chinese schools of mathematics, could be of great importance. (..) It may lead to a major revision of Ernst Zinner's majestic work on Regiomontanus" (ibid.).[4] In summary the argument is as follows:

Premisses: 1) Regiomontanus knew the general solution to the CRT in 1463

2) The CRT is explained in the *SSJZ* of 1247

Conclusion: Regiomontanus had knowledge of the *SSJZ*

We will show that the first premise cannot be justified and that the conclusion is not proven. But even if the premise would be true and if the conclusion would be valid, the reasoning still is flawed.

Other sources for the *da yan* rule

A typical fallacy of Menzies's argumentation is that he was looking for an explanation which necessarily involves Chinese knowledge, sources or artifacts which supposedly have been disseminated through a hypothetical visit of the Chinese in Italy in 1434. He therefore looks for Renaissance sources after 1434 and tries to match them with Chinese ones before that date ignoring all other contextual explanations. In our case he matches a very specific problem from the correspondence of Regiomontanus with one specific Chinese book. We will first show that the *CRT* was known long before *Shù shū Jiǔ zhāng* and secondly that the problem and alternative solutions were known in Europe before 1434.

[3] Menzies 2008, 149. Remark that this argument can be used to pose the opposite: "it follows that Regiomontanus must have independently thought up the *Ta-Yen* rule unless he was aware of this Chinese book of 1247, which he never claimed to have known".
[4] Ernst Zinner (1939) is the biographer of Regiomontanus.

Earlier sources in Asia

It is generally acknowledged by historians of Chinese mathematics that remainder problems grew out of calendar calculations in China during the third century. The problem of finding a common cycle for the lunar months and the tropical year can be formulated as finding a period of N days in which

$$N \equiv r_1 (\bmod D) \equiv r_2 (\bmod L) \equiv r_3 (\bmod Y)$$

with D as the length of a day, L the duration of a lunar month and Y the duration of a tropical year and the r_i as the remainders. The *zhāng* cycle (章) of 19 tropical years corresponding with 235 lunar months was determined in China already in the sixth century BC. Such a cycle was also known by the Babylonians around 500 BC and by the ancient Greeks (Meton at 432 BC).

Remainder problems appear in different forms but usually within a practical setting. The earliest extant source dealing with such problems is the *Sun Zi Suan Jing* (孙子算经, *Master Sun's Arithmetical Manual*. An English translation and discussion is given by Lam Lay Yong (2004). Sun Zi describes the problem $x \equiv 2\ (3),\ 3\ (5),\ 2\ (7)$ (Chap. 3, prob. 26, p. 10b, also translated and discussed by Needham (1959, 119), Mikami (1913, 32), Li and Du (93), Libbrecht (269)). The method spread to India as it appears already in the *Āryabhaṭīya* of Āryabhaṭa (of 499) where it becomes known as the *kuttaka* or pulverizer method (Clark 1930, 42-50, Datta and Sing 1934, 87-99, 131-133, Libbrecht 1973, 229). In the *Brāhmasphuṭasiddhānta* by Brāhmagupta of 628 the method is applied within the context of astronomy (Colebrooke 1817, 326-7). In the *Bījagaṇita* of 1150 Bhāskara II uses algebra to calculate the solution (Chap. VI, stanzas 160 & 162, Colebrooke 1817, 235-239). Although remainder problems are rare in Arabic treatises they were known by Ibn al-Haiṯam (c.1000), in particular the problem $x \equiv 1(2, 3, 4, 5, 6),\ 0(7),$ as discussed by Wiedemann (1892), Libbrecht (1973, 234-5) and Rashed (1994). Libbrecht points our attention to the similarity of the Arabic and Indian solutions.

Earlier European sources[5]

Already in 1202 Fibonacci discusses seven remainder problems in his *Liber abbaci*. His last problem is actually the one which we used as an example (Latin in Boncompagni 1857, 304, English by Sigler 2002, 428-29, also discussed and translated by Libbrecht 1973, 236-238). Fibonacci predates the *Shù shū Jiǔ zhāng* and while he gives a general method for the moduli 3, 5 and 7 and another one for the moduli 5, 7 and 9, his procedure is different from the one we listed above (Sigler 2002, 428-9):

He divides the chosen number by 3, and by 5, and by 7,

[5] We will limit the discussion of European sources to the period before 1464. Shortly after the correspondence of Regiomontanus and Bianchini the problem appears in Maestro Benedetto da Firenze's *Trattato* (c.1465, pp.68-69), in the Pseudo-dell'Abbaco of 1478 (ff. 69r-69v) and in Piero della Francesca's *Trattato d'abbaco* of c. 1480 (f.122). For its later history and influence on Gauss see Bullynck (2008).

> and always you ask what are the remainders from each division. You truly for each unit of the remainder upon division by 3 keep 70, and for each unit of the remainder upon division by five you keep 21, and for each unit of the remainder upon division by seven you keep 15. And whenever the total exceeds 105, you throw away 105, and that which remains for you after all the 105 are thrown away will be the chosen number. For example, it is put that after division by 3 there remains 2 for which you twice seventy, that is 140; from this you take away the 105 leaving 35 for you. And after division by the five there remains 3 for which you keep three times 21, that is 63, which you add to the aforesaid 35; there will be 98. After the division by the seven there remains 4 for which you retain quadruple 15 that is 60 which you add to the aforesaid 98; there will be 158 from which you throwaway 105; there will remain for you 53 which is the chosen number.

In Fibonacci's procedure the *zong shu* are not calculated but given and the *yan mu* are subtracted from *zong shu* before they are summed together. In fact, Fibonacci's method corresponds closely with the text of the *Sun Zi Suan Ching* for the similar problem $x \equiv 2\ (3), 3\ (5), 2\ (7)$.[6] (Mikami 1913, 32):

> In general, take 70, when the remainder of the repeated divisions by 3 is 1; take 21, when the remainder of the repeated divisions by 5 is 1; and take 15, when the remainder of the repeated divisions by 7 is 1. When the sum of these numbers is above 106, subtract 105, before we get the answer. (…) The remainder divided by 3 is 2, and so take 140. The remainder divided by 5 is 3, and so take 63. The remainder divided by 7 is 2, and so take 30. Adding these together we get 293. There from subtract 210, and we obtain the answer.

Libbrecht (1973, 240) also discusses Fibonacci's solution and concludes that he "does not give the slightest theoretical or general explanation of his method of the remainder problem, and for this reason his whole treatment is on a level no higher than that of Sun Zi". Needham (I, 4, 34) and Libbrecht (1973, 241-2) mention a fourteenth-century Byzantine manuscript of the *Isagoge Arithmetica* by Nichomachus of Gerase which contains an appendix dealing with a problem finding a contrived number. The solution method as well as the recreational context is very similar to Fibonacci.

Two additional occurrences of the problem are documented in extant writings before Regiomontanus in Europe. The astronomer Giovanni Marliani was the teacher of Giorgio Valla and wrote a vernacular treatise on arithmetic. His *Arte giamata arismeticha* dates from c.1417 (codex A. II. 39, Biblioteca Universitaria de Genova) and is described and partly transcribed by Gino Arrighi (1965). The remainder problem is basically the same as the one by Fibonacci and the limited explanation provides no clues concerning the knowledge of the

[6] Martzloff (2006, 322) comes to the same conclusion: "In 1202, Fibonacci (..) proposed a solution to the remainder problem similar to that of Sun Zi, in other words incomparably less powerful than that due to Qin Jiushao".

CRT. Two other remainder problems appear in a pseudo-Paolo dell'abbaco of c.1440, with moduli 2 to 10 and the congruences one less than the moduli and 1:

> Truova uno numero che partjto per 2 ne rjmanghi uno, e partjto per 3 ne rjmanghi 2, e partjto per 4 ne rjmanghj 3, e partjto per 5 ne rimanghj 4, e choxj per insino in 10 (Arrighi 1964, 95).
>
> Truovamj uno numero che partjto per 2 ne rjmanghj uno, e partjto per 3 ne rjmanghi uno, e partjto per 4 ne rimanghj uno, e partjto per insino in 10 (Arrighi 1964, 96).

Now, these two problems are quite interesting as the author seems to have heard of these or similar remainder problems but he does not know the answer or the method of the *CRT*. For the first problems he gives the answer 3628799 and for the second 75601. Both solutions are valid, but of course not the smallest integral solutions as provided by the *CRT* method, 2521 and 2519 respectively. The author solves the first problem by multiplying all the moduli and subtracting one:

> Fa' choxj e di': in che si truova il 2 e 'l 3 e 'l 4 e 'l 5 e '16 e 'l 7 e 'l 8 e 'l 9 e 'l 10? E però multjpricha 2 via 3, fa 6, e 4 via 6, fa 24, e 5 via 24, fa 120, e 6 via 120, fa 720, e 7 via 720, fa 5040, e 8 via 5040, 40320, e 9 via 40320, fa 362880, e 10 via 362880, fa 3628800. E ora puoj dire: jo òe trovato uno numero che partjto per 2 no' rrimane alchuna choxa, e partjto per 3 no' rimane nulla e choxj per 4, per 5, per 6, per insino in 10. E ora di': traendo uno di questo numero, si nne verrà da xxezzo 1 meno che nel numero nel quale io ò partjto, uno di questo numero cioè di 3628800 che rresta 3628799 e questo numero è desso.

For the solution to the second problem he finds 7560 as a common multiple, though not the least common multiple, and adds one, and 7561 indeed is a solution. He then claims that 75601 is " a more secure solution".

While we can relate some remainder problems to the Arabic tradition and Fibonacci, it seems that such kind of problems were known, discussed and solved by methods which do not reflect any knowledge of the *da yan* rule. The occurrences of the problems as discussed here exhaust all our published sources of abbaco texts. Convinced that there had to be found something more we looked at a number of previously unpublished manuscripts. We will come back to the abbaco tradition in a further section. First we look at the problem in German cossic texts. Its appearance is unnoticed by Menzies but could be used as an argument in favour of his first premise.

The cossic tradition

In a classic overview of fifteenth-century algebra in Germany Maximilian Curtze points out that by the middle of the fifteenth century the CRT was a well known subject.[7] As evidence he refers to the *Deutsche Algebra,* the manuscript 14908 of the State library in Munich written

[7] Curtze (1895, p. 65 note): "Ich will jetzt hier den Beweis führen, dass um die Mitte des 15, .Jahrhunderts sie *mit ihrem Beweise* und ohne Benutzung des chinesischen Beispiels eine ganz bekannte Sache war".

in a mixture of German and Latin. Curtze names the author Gerhard, but it is since then established that it is the monk Fredericus Amann who authored the text in 1461 (Folkerts 1996, Gerl 1999, Gerl 2002 and Vogel 1981). While this treatise was written two years before the correspondence between Regiomontanus and Bianchini it still is difficult to establish who got it from whom. According to Folkerts (1996, 26) Fredericus based his text on the still unpublished Regiomontanus's manuscript on algebra written in 1456, Columbia University, Plimpton 188, (ff. 82v-96r). However, the algebra problems by Fredericus run from ff. 133v-157r and the method for solving the CRT is discussed on the preceding pages (124v-125r, Curtze 1895, 65). Though we do not have a transcription of Regiomontanus's problems we know that it includes one remainder problem namely $x \equiv 2\ (3),\ 4\ (5),\ 1\ (7)$ (problem 60, f. 93r). According to Folkerts (2002, 421) "it follows the *da yan* method". If at all Regiomontanus had a general method for solving remainder problems, which cannot be established from the correspondence or from what we know of the Plimpton 188, it is safe to assume that he and Amann used the same method. As we have a transcription available of Amann's solution let us therefore take this as a reference. In a curious mixture of old German and Latin, Amann provides the following solution to Regiomontanus's problem 60 with moduli (3, 5, 7) (Curtze 1895, 65; Vogel 1954, 120-1):

> Item ich wil wissen, wie vil pfenning in dem peutel odor im synn hast. Machs also. Hays yn dy dn, dy er hat, zelen mit 3, darnach *cum* 15, *postea cum* 7, vnd alz oft eins vber pleibt mit 3, so merck 70, vnd alz oft 1 vber pleibt mit 5, merk 21, vnd mit 7, merk 15. *Postea adde illos numeros in simul, et ab ista summa subtrahe radicem, hoc est multiplica* 3 *per 5 et 7 erit 105,* als oft du magst, vnd wz do pleibt, alz vil hat er ym sinn oder in peutel.

Amann further lists a table with the moduli (2, 3, 5), (3, 4, 5), (3, 4, 7), (2, 3, 7), (2, 7, 9), (5, 6, 7), (5, 8, 9) and (9, 11, 13) including the least common multiple of the moduli and but not what the Chinese called *yen shu*. So for the moduli (3, 5, 7), Amann gives 105 and then the numbers 70, 21, 15 instead of the *yen shu* 35, 21 and 15. We can thus conclude that Amann – and possibly Regiomontanus – followed the method of the *Sun Zi Suan Ching* also employed by Fibonacci instead of the one described in the *Shù shū Jiǔ zhāng*. As the method for solving remainder problems in fifteenth-century Europe is different from the superior one discussed in the *Shù shū Jiǔ zhāng* it can actually be interpreted as evidence *against* the hypothesis posed by Menzies:

Premisses: 1) If Regiomontanus knew the general solution to the CRT as explained in the *SSJZ* he would have used it

2) Regiomontanus (and Amann) did not use the method of the *SSJZ*

Conclusion: Therefore Regiomontanus had no knowledge of the *SSJZ*

The rule of numbers with no end

From a study of a number of unpublished manuscripts we discovered that remainder problems were treated already in the Italian abbaco tradition since the fourteenth century. We identified a family of at least six manuscripts which contain a table with remainders for moduli 3, 5 and

7. The table is listed in relation to one in a series of remainder problems. The purpose of that problem is to find numbers that satisfy $x \equiv 2\ (3),\ 3\ (5),\ 1\ (7)$. The unpublished manuscripts containing the table are the following (including sigla and folio location):

- **Z** (c. 1395) Florence, BNC, Conv.Soppr.G7.1137, Inc. "Queste sonno le fegurre nostre dello abocccho ceh le guagli tu pot inscrivere qualunque numero tu vogli.." (*corsiva mercantesca formata*) ff. 241r-241v
- **α** (c.1417), a lost archetype of which the following are derived copies; the table of contents in a later copy points to ff. 141
- **A** (c.1433) Florence, BNC, Magl. Cl. XI. 119, Inc: "Concio sia cosa che sono nove figure nell'abacho per le quali chi conoscie quelle agievolmente conosciera poi l'altre .." (neat *corsiva cancellaresca formata*) ff. 141(?)
- **B** (c1440) Florence, Biblioteca Mediceo-Laurenziana, Ash. 608, Inc: "Concio sia chosa che sono nove fighure nel abacho per le quali chi chonoscie quele agievolmente chonosciera poi l'altre .." (rapid *corsiva mercantesca*) ff. 101v-106r
- **C** (c1440) London, BL, Add. 10363, Inc: "Concio sie cosa che son 9 fighure nell'abbaco per le quali chi chonoscie quelle agevolmente conosciera poi l'altre …", (very neat humanistic bookhand) ff. 149r-152r
- **E** (1442) London, BL, Add. 8784, Inc: "E choncio sia chosa che sono 9 fighure nell'abacho per le quale chi conosce quelle agievolmente conoscera poi l'altre...", (fairly neat Italian Gothic bookhand, by Agostino di Bartolo) ff. 138v-140v

The manuscripts **α** and **A** to **E** are part of a related family of copies of an abbaco treatise which is described in Heeffer (2008). We will here use the same sigla. The table (see figure 2) in its earliest form is contained in a *zibaldone* (an author's notebook, hence **Z**) from the end of the fourteenth century. All the cited manuscripts contain the table and the main problem to find 11 numbers which satisfy the congruence relation. The solution is listed (8, 113, 218, 323, 428, 533, 638, 743, 848, 953 and 1058) and it is verified that they have the same remainders for the three moduli 3, 5 and 7 but apart from the table no method is presented. These texts therefore show no evidence of any knowledge of the *da yan* rule, not even the limited version from the *Liber abbaci*. The table starts from 8 and runs to 113, the first two numbers in the congruence set. The accompanying text explains how to generate additional solutions. First the property is observed that the remainders are the same when one is added to both solutions 8 and 113. The remainders are also the same when you add 2, 3 and more. When one adds the difference between 8 and 113 (which is 105) to 8 and to 113, the remainders will also be the same. This allows us to generate infinitely many numbers with the same remainders for a given set of moduli. In fact, as far as we know, the reasoning is the earliest instance of mathematical induction in European writings (Heeffer 2010). Because of the demonstrated infinity of possible solutions the rule is called "la reghola del numero sanza fini" or the rule of numbers with no end. We remark that the limited knowledge about remainder problems in these manuscripts is sufficient to explain the reasoning of both Regiomontanus and Bianchini in their correspondence. There are infinitely many solutions and these solutions are in an arithmetical progression. This undermines Menzies's argument since all what Regiomontanus needed to know was available already in fourteenth-century Italy.

8	2	3	1
9	0	4	3
10	1	0	3
11	2	1	4
12	0	2	5
13	1	3	6
14	2	4	0
15	0	0	1
16	1	1	2
17	2	2	3
18	0	3	4
19	1	4	5
20	2	0	6
21	0	1	0
22	1	2	1
23	2	3	2
24	0	4	3
25	0	0	4
26	2	1	5
27	0	2	6
28	1	3	0
29	2	4	1
30	0	0	2
31	1	1	2
32	2	2	4
33	0	3	5
34	1	4	6
35	2	0	0
36	0	1	1
37	1	2	2
38	2	3	3
39	0	4	4
40	1	0	5
41	2	1	6
42	0	2	0
43	1	3	1
44	2	4	2
45	0	0	3
46	1	1	4
47	2	2	1
48	0	3	6
49	1	4	0
50	2	0	1
51	0	1	2
52	1	2	3
53	2	3	4
54	0	4	5
55	1	0	6
56	2	1	0
57	0	2	1
58	1	3	2
59	2	4	3
60	0	0	4
61	1	1	5
62	2	2	6
63	0	3	0
64	1	4	1
65	2	0	2
66	0	1	3
67	1	2	4
68	2	3	5
69	0	4	6
70	1	0	0
71	2	1	1
72	0	2	2
73	1	3	3
74	2	4	4
75	0	0	5
76	1	1	6
77	2	2	0
78	0	3	1
79	1	4	2
80	2	0	3
81	0	1	4
82	1	2	5
83	2	3	6
84	0	4	0
85	1	0	1
86	2	1	2
87	0	2	3
88	1	3	4
89	2	4	5
90	0	0	6
91	1	1	0
92	2	2	1
93	0	3	2
94	1	4	3
95	2	0	4
96	0	1	5
97	1	2	6
98	2	3	0
99	0	4	1
100	1	0	2
101	2	1	3
102	0	2	4
103	1	3	5
104	2	4	6
105	0	0	0
106	1	1	1
107	2	2	2
108	0	3	3
109	1	4	4
110	2	0	5
111	0	1	6
112	1	2	0
113	2	3	1
0	0	0	0
0	0	0	0
X	X	X	X
X	X	X	X
X	X	X	X

Figure 2: Table of remainders for moduli 3, 5 and 7 (used with permission © The British Library Board, ms. Add.10363 ff. 152rv)

Transmission from China to Europe

If Regiomontanus did not get his knowledge of the *CRT* from the *Shù shū Jiǔ zhāng* how did he and other Europeans learn about the problem and ways to solve it? As with regard to transmission between cultures the Western historiography is deeply influenced by the humanist prejudice that all higher intellectual culture, in particular all science, had risen from Greek soil. A most typical example is the monumental and influential four-volume work on the history of mathematics by Moritz Cantor (1880-1908). Like many men of his era Cantor was entrenched with humanist ideas about the cultural superiority of Western knowledge. For example, when dealing with Hindu algebra Cantor takes every opportunity to point out the Greek influences on India. Some examples: the Indians learned algebra through traces of algebra within Greek geometry; Brāhmagupta's solution to quadratic equations has Greek origins; or the Indian method for solving linear problems in several unknowns depended on the Greek method of *Epanthema* and so on.[8] However, when he comes to the *CRT*, Cantor writes the following:[9]

> It is not impossibly that the Chinese problem and its solution could have become known somewhere by a Byzantine who took note of it, possibly through Arabic mediation. The reverse course, which is so often the case, were Western knowledge penetrates China, is here hardly possible, because it is only in Chinese texts that the explanation of the procedure is provided, quite difficult to understand, but comprehensible nevertheless, as experience has shown us.

This is one of the few instances were Cantor admits an influence from the East to the West although he is puzzled about how such a transmission could have taken place. The whole of his *Vorlesungen* solely depends on written sources. He knew about the Byzantine manuscript we cited before and did not know of any Arabic sources dealing with remainder problems. The quotation reflects perfectly the current knowledge about written resources at that time. So does Menzies also rely on written sources is his latest book.

However, remainder problems do not solely belong to the scholarly domain of mathematics, the kind of mathematics that supported the state and bureaucracy of ancient China. They also have a clear recreational value. These problems and their solutions belong to what Jens Høyrup has coined as sub-scientific mathematics. The narration of stories, riddles and recreational puzzles is the most important factor in the transition of arithmetical problems and their solution methods between generations, cultures and continents. They are passed on through oral traditions from master to apprentice and are mostly situated in the practices of merchants, lay surveyors and craftsmen. To get a grip on the oral tradition I proposed

[8] See Heeffer (2009) for a discussion on the reception of the first translations of Hindu classics on mathematics.
[9] Cantor (I, 644) "Es ist nicht unmöglich, dass die chinesische Aufgabe und ihre Auflösung etwa durch arabische Vermittlung irgend einem Byzantiner hekannt geworden sein kann, der sie sich aufnotirte. Ein umgelrehrter Gang, dass also hier wie so vielfach im Westen Bekanntes nach China drang, ist kaum anzunehmell, weil nur im chinesischen Texte die Begründung des Verfahrens angedeutet ist, freilich schwer zu verstehen, aber doch zu verstehen, wie die Erfahrung gezeigt hat." (my translation).

elsewhere a concrete implementation for sub-scientific knowledge in the form of proto-algebraic rules (Heeffer 2007). A proto-algebraic rule is a procedure or algorithm for solving one specific type of problem. Our main hypothesis is that many recipes or precepts for arithmetical problem solving, in abbacus texts and arithmetic books before the second half of the sixteenth century, are based on proto-algebraic rules. We call these rules proto-algebraic because they are or could be based originally on algebraic derivations. Yet their explanation, communication and application do not involve algebra at all. Proto-algebraic rules are disseminated together with the problems to which they can be applied. The problem functions as a vehicle for the transmission of this sub-scientific structure. Little attention has yet been given to sub-scientific mathematics or proto-algebraic rules. We provided a framework for identifying proto-algebraic rules which are common to Renaissance and Indian arithmetic and algebra. Some Chinese rules such as of *Yíng bù zú* 'excess and deficit'(盈不足) definitely fall under this category.[10]

The recreational aspect of remainder problems is prominent from its earliest occurrence. Libbrecht (1973) provides a comprehensive overview of all Chinese sources and we note that several formulations of the problem are in verse and belong to the recreational domain. General *Hán Xìn*'s method of counting soldiers (韩信乱点兵) is a cryptical formulation of a remainder problem but we can recognize the multipliers 70, 21 and 15 and the least common multiple 105 of moduli 3, 5 and 7, which occurs in many of the problems already cited. Hán Xìn was a general of Emperor Liu Bang of the Han Dynasty and lived around 200 AD:

> Not in every three persons is there one aged three score and ten,
> On five plum trees only twenty-one boughs remain,
> Every fifteen days rendezvous the seven learned men,
> We get our answer by subtracting one hundred and five over and again.

The folk rhyme is still known in the oral tradition today by many Chinese and also Japanese. Li and Du (1987, 93-94) discuss several other versions over the next centuries. Libbrecht (1973, 286) quotes a stanza from the *Zhiyatang zachao*, written in 1290 by Zhou Mi:

> A child of three years old [when the father] is seventy, it is rare.
> At five to leave behind the things of 21, it is even more rare.
> At seven one celebrates the Lantern festival, again they meet together.
> The Cold meal holiday [on the 105th day] then you will get it.

[10] The problem appears already in the earliest extant Chinese treatise, the The *Suàn shù shū* (筭數書) or 'Writings on reckoning': "In sharing cash, if [each] person [gets] 2 then the surplus is 3. If [each] person [gets] 3 then the deficit is 2. Question: how many persons, and how many cash? Result: 5 persons and 13 cash". Christopher Cullen (2004, 81-88), who published the transcription of this treatise, identifies this rule with the rule of false position. However, the rule of false position, also known in Chinese arithmetic, is commonly considered as a general method for solving linear problems of the type $ax = b$. The *Yíng bù zú* however corresponds with the Indian rule *gulikā-antara* or the *regula augmenti* in European arithmetic, for problems of the form $ax + b = cx - d$ (Heeffer 2007, 15-21).

Again we recognize the numbers of Master Sun's original problem in a form which facilitates the recollection of the base numbers to solve problems for moduli 3, 5 and 7. Alexei Volkov (2002, 402) describes a Vietnamese mathematical work of the fifteenth century *Toan phap dai thanh* in which a similar verse appears. So, with these examples we find evidence that the problem travelled in South-East Asia in the form of folk rhymes. As to its influence on Hindu mathematics opinions vary, with Indian scholars favoring the precedence of India over China. Remainder problems in India are known as *kuttaka* or pulveriser problems and are very prominent in scholarly texts on mathematics and astronomy. Its earliest occurrence is in the *Āryabhaṭīya* of 499 (Clark 42-50). However, any sensible analysis, as Libbrecht (1973) does in his chapter 18, must come to the conclusion that the methods differ too widely to assume any influence from the one side to the other.

We finally come to the question of how the problem came to Europe. As we remarked already the treatment by Fibonacci very well reflects the Chinese method of Master Sun and bears little relation with methods from Hindu arithmetic. On the other hand, problems such as that of men finding a purse, where Fibonacci deals with many variants in the *Liber Abbaci* and the *Flos* seem to have originated in India (Heeffer 2007). If we look at the context in which the two remainder problems are discussed in the *Liber Abbaci* it comes as no coincidence that Fibonacci treats them within a series of recreational problems. The section called *de quibusdam divinationibus* concerns divination problems in which a number or object has to be guessed (Boncompagni 1857, I, 303; Sigler 2002, 427). Such number divination or combinatorial divination problems were quite popular in Medieval Europe. Menso Folkerts (1968) found many instances in a study of 32 fourteenth and fifteenth century Latin manuscripts. Also the German text by Fredericus Amann, cited earlier, sets the remainder problem as a divination problem: "I wish also to know how many coins he has in his purse or in his mind" (translation by Libbrecht 1972, 244-5). After Regiomontanus we find remainder problems treated as part of divination problems in Luca Pacioli's *Viribus Quantitatis* (c. 1500, problems 22 to 25, Peirani 1997), as well as in the seventeenth-century works *Problèmes Plaisantes* by Bachet (1612, problem V) and the *Recréations Mathématiques* ([Leurechon] 1624, problems 51, 52). The problem continues to appear in books on recreational mathematics during the following centuries. Being part of this oral tradition of challenging others with riddles and puzzles it comes as no surprise that the recreational version of the problem travelled on the trade routes from East to West. We do not have to assume a single visit of a Chinese delegation in Italy in 1434 to explain the fact that mathematical knowledge travels between cultures and continents. Mathematics, being a product of culture, has been exchanged between cultures for centuries, together with other artifacts and products of cultures.

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