Formalism.
The success(es) of a failure

1 Introduction

I had my first encounter with logic when I was still in high-school. My Dutch teacher chose to skip certain chapters in the course book including the one on logic. Fascinated as I was by things not taught in school I had a look at the chapter. Today I know that at that point I had no clue whatsoever what logic is. The chapter gave something in the spirit of the following syllogism as an example, adding that this was a correct derivation:

> All elephants are ruby red
> Bambi is an elephant
> ______________________
> Bambi is ruby red

I simply could not get this: how could this conclusion be correct? As is evident, the mistake I made was that I was focusing on the meaning of what is said rather than on the form of the deduction. I did not understand at that point in time that one can speak about truth in relation to form without reference to what *is* true, *empirically* speaking. Since that first encounter with logic, I have long disliked, if not opposed, the idea of formalization.

Nowadays I feel no longer appalled by logic-as-formalism, on the contrary. However, my interest is *not* in the formalization of something "empirical" like human reasoning but in the form *as* form, form without content. This paper is an account of why "form" matters and why the quest for "meaningfulness" sometimes obscures and even slows down certain developments or ideas.

In the introduction to *Proofs and refutations* [6], an influential book in the philosophy of mathematics and science in general, Lakatos makes clear that the motivation behind his book is to criticize and ultimately reject formalism as being the *latest link in the long chain of* dogmatist *philosophies of mathematics*" [6, p. 4]. By identifying formalism as the "bulwark" of logical positivism, he concludes that formalism somehow excludes the informal aspects of mathematics and that it denies mathematics its history. In this paper I will apply a Lakatos-inspired method on formalism itself viz. study *formalism-as* and embedded *in-a-practice* of (informal) mathematics. By doing so, I will argue that formalism as practiced is far removed from the kind of picture one gets from formalism upon reading Lakatos and that his view on formalism should at least be nuanced if one turns to actual formalism in action.

By way of a historically-inspired study of the work of a well-known mathematical logician, Emil Leon Post, I will make a stand for the concreteness, historicity and practicability of "form". It is shown that at least in the case of Post the formalist approach was a necessary prerequisite to unveil the fundamental limitations of formalism itself. Ultimately, this paper aims at showing how Post's formalism is relevant even today. By putting it into the perspective of computer science, it is suggested that it might be considered as an interesting philosophical and practical alternative to the "art of simulation" as a means to explore the limits and possibilities of computation.

2 In search of the ultimate form

2.1 Lewis' influence on Post's early work

Perhaps one of the most extreme formalist convictions can be found in Chapter 6 *Symbolic Logic, Logistic and mathematical method* of Lewis' *Survey of Symbolic Logic*, a chapter that was removed from the later editions of [8, 355-56]:

> A mathematical system is any set of strings of recognizable marks in which some of the strings are taken initially and the remainder derived from these operations performed according to rules which are independent of any meaning assigned to the marks. [The] distinctive feature of this definiton lies in the fact that it regards mathematics as dealing, not with certain denoted things – numbers, triangles, etc – nor with certain symbolized "concepts" or "meanings", but solely with recognizable marks, and dealing with them in such wise that it is wholly independent of any question as to what the marks represent. This might be called the "external view of mathematics" or "mathematics without meaning". [W]hatever the mathematician *has in his mind* when he develops a system, what he *does* is to set down certain marks and proceed to manipulate them [...]

With this lengthy quote, the stage is set for "pure" form, viz. form without meaning. As is clear, for Lewis, such form is in fact the ideal of mathematics *as an activity*.[1] Going a step further, if mathematics is the foundation of all of science then "*Logistik is the universal method for presenting exact science in ideographic symbols. It is the "universal mathematics" of Leibniz*" [8, 372].

[1]In the long footnote 17, p. 360, Lewis explains that this does not exclude creativity. The mathematician as the "manipulator" of the marks needs to be intelligent and ingenious for the derivation of required, interesting or valuable results. Lewis makes the analogy here with Gulliver "*who found the people of Brobdingnag (?) feeding letters into a machine and waiting for it to turn out a masterpiece. Well, masterpieces are combinations achieved by placing letters in a certain order! However mechanical the single operation, it will take a mathematician to produce masterpieces of mathematics.*" (Clearly, Lewis has made a mistake here: it is not in Brobdingnag but in the Academy of Lagado – where useless projects are undertaken – that Gulliver saw the machine for producing sentences and books.)

Imagine the impression this must have left on the young mathematician Emil Post. It is important to keep in mind that at that point in time, mathematical logic as a discipline hardly existed in the United States [5]. In fact, Emil Post would become one of the few U.S. mathematical logicians who made fundamental contributions to the field in the early 20s. Lewis' book, was one of the few available English textbooks in circulation at that time, so it is logical that Post studied it.

In the same year as Lewis' book was published, Post was a postgraduate student at Columbia University. It was during that time that he was familiarized with the formal austerity of *Principia Mathematica* by Russel and Whitehead [18] through the teachings of Cassius J. Keyser. Together with Lewis' [8] this would become the main influence on Post's PhD *Introduction to a general theory of elementary propositions* [11].[2] Post however was not fully satisfied with the formal apparatus of *Principia* because [11, 163–164]:

> [...] owing to the particular purpose the authors had in view they decided not to burden their work with more than was absolutely necessary for its achievements, and so gave up the generality of outlook which characterized symbolic logic. [W]e might take cognizance of the fact that the system of 'Principia' is but one particular development of the theory [and] so [one] might construct a general theory of such developments.

Hence, instead of working with *Principia* Post decided to develop his own formal apparatus, one, as Post would later write, which *eschews all interpretation* [15].

2.2 The method of combinatory iteration

But why exactly did Post regard *Principia* as being too particular and why did he develop his own formal apparatus? The motivation behind this is what one could call (a kind of) methodological formalism: the development of the most general form of symbolic logic and ultimately mathematics as *instruments of generalization* which make possible a study of the *general* properties of the whole of mathematics. Post's idea was that if one wants to study the general properties of logic and mathematics then one needs not one particular system of symbolic logic or mathematics, but a general form that comprises all such possible systems. In this sense, Post's formalism can be regarded as a *method* to study mathematics.

In an unpublished note from the *Emil Post Papers* held at the *American Philosophical Society* titled *Note on a Fundamental Problem in Postulate Theory* and dated June 4, 1921 Post makes explicit that formalism can be used "*to obtain theorems about all [possible] assertions*" of mathematics but that such a "*complete specification of the logic that is employed [in a mathematical system]is not made in the usual mathematical developments, and indeed is not necessary.*" In other words, Post did not aim nor expect to effectively replace the usual

[2] See [4, 9, 20].

style of mathematics by formal logic; it was not his ambition to cleanse or cure mathematics from non-rigorousness or to get rid of its informality.

Even though Post's formalism can be called methodological, it is rooted in the believe that, ultimately, mathematics can be captured by form. This indeed does not necessarily mean that he expected that real-life mathematics would be replaced by formal proofs. What he did expect was that mathematics can be captured by (a) *general* form and that (b) by studying (particular instances of that) form it would become possible to prove theorems that say something about the whole of mathematics, about mathematics *in general*. An example of such a problem, which would in fact become his main focus in the period 1920-21 when he was a Procter fellow in Princeton, was what Post called *the finiteness problem* for first-order logic, viz. the famous Entscheidungsproblem proven undecidable by Church and Turing in 1936.

In its concrete realization, Post's formalism is completely in the spirit of Lewis and any hardcore formalist philosophy. Indeed, in the same unpublished note just mentioned, Post identifies this "method" as the method of combinatory iteration and describes it as follows:

> [T]he method of combinatory iteration completely neglects [...] meaning, and considers the entire system purely from the symbolic standpoint as one in which both the enunciations and assertions are groups of symbols or symbol-complexes [....] and where these symbol assertions are obtained by starting with certain initial assertions and repeatedly applying certain rules for obtaining new symbol-assertions from old.

How far this method of combinatory iteration would lead Post becomes clear if one relates Post's Ph.D. to his research during the period 1920-21 when he was a Procter fellow.

In his PhD Post made a start with his method of generalization: he introduced the truth-table method for propositional logic (isolated from *Principia*) and proved that this logic is complete and consistent with respect to this method. He also emphasized that the truth table method provides a method that allows to decide the decision problem for the propositional calculus. He also generalized the two-valued truth table method to an arbitrary finite number of truth values hence laying the foundations for multi-valued logic. Finally and most importantly here, he proposed a first form intended as a *general* framework to reason about all systems of symbolic logic and hence, ultimately, mathematics. He referred to this form as *generalization by postulation* [11, 176] and later called it the *canonical form A* [13, 15]. It is a form that captures an infinite number of formal systems understood as finitary symbol manipulation systems. This form resulted from a generalization of Post's formulation of propositional logic based on that from *Principia Mathematica*. Table 1 compares Post's formulation of propositional logic, using only the two logical functions $\sim, \vee$ with the canonical form A.[3,4]

[3] Note that I have not used the dots as brackets notation of Principia.

[4] Note that the description of propositional calculus in *Principia* is alsmot identical. How-

Table 1: Comparison between Post's formulation of propositional logic and his canonical form A.

	Propositional Logic	**Canonical form A**
I.	If p is an elementary proposition than so is $\sim p$ If p and q are elementary propositions than so is $p \vee q$	If $p_1, \dots, p_{m_1}$ are elementary propositions than so is $f_1(p_1, \dots, p_{m_1})$ $\vdots$ If $p_1, \dots, p_{m_\mu}$ are elementary propositions than so is $f_\mu(p_1, \dots, p_{m_\mu})$
II.	The assertion of a function involving a variable p produces the assertion of any function found from the given one by substituting for p any other variable q, or $\sim q$, or $(q \vee r)$[5]	The assertion of a function involving a variable p produces the assertion of any function found from the given one by subsituting for p any other variable q, or $f_1(q_1, \dots, q_{m_1})$, or $f_\mu(q_1, \dots, q_{m_\mu})$
III.	$\vdash P$ $\vdash \sim P \vee Q$ produce $\vdash Q$	$\vdash g_{11}(P_1, ..., P_{k_1}) \dots \vdash g_{r_1}(P_1, ..., P_{r_r})$ $\vdots$ $\vdash g_{1r_1}(P_1, ..., P_{r_1}) \dots g_{rr_r}(P_1, ..., P_{r_r})$ produce produce $\vdash g_1(P_1, ..., P_{k_1}) \dots \vdash g_r(P_1, ..., P_{r_r})$
IV.	Postulates: $\vdash \sim (p \vee p) \vee p$ $\vdash \sim (p \vee (q \vee r)) \vee (q \vee (p \vee r))$ $\vdash \sim q \vee (p \vee q)$ $\vdash \sim (\sim q \vee r) \vee (\sim (p \vee q). \vee (p \vee r))$ $\vdash (p \vee q) \vee (q \vee p)$	Postulates: $\vdash h_1(p_1, p_2, \dots, p_{l_1})$ $\vdash h_2(p_1, p_2, \dots, p_{l_2})$ $\dots$ $\dots$ $\vdash h_\lambda(p_1, p_2, \dots, p_{l_\lambda})$

As is clear from Table 1 what Post did was to extract the essential formal features from the postulational formulation of propositional logic and then generalized them. Instead of the two logical functions $\vee, \sim$, a system in form A can have an arbitrary but finite number of functions; instead of having one production rule it can have an arbitrary number of production rules and instead of five postulates it can have an arbitrary but finite number of them. Post's formulation of propositional logic clearly fits canonical form A: it is just one of an infinite number of symbol manipulation systems that can be expressed in form A. The result is that Post now has a means to study not one but an infinite number of formal devices and hence study properties of mathematical systems as symbol manipulation systems *in general.*

Shortly after finishing his PhD, Post became a Procter fellow in Princeton: it was during that time that Post developed and studied several other forms, originally with the aim of proving that there is a *general* method to decide for any formula in first-order logic and ultimately *Principia* whether or not it is

ever, it uses the three logical functions $\sim, \vee, \supset$. Remember that $p \supset q$ can be defined as $\sim p \vee q$ in propositional logic (See [18, 12]).

[5] This corresponds to substitution

derivable in that system, i.e., to prove the decidability of the finiteness problem for first-order logic, and, ultimately, for the whole of *Principia*. This was a very ambitious project. Indeed: "*Since Principia was intended to formalize all of existing mathematics Post was proposing no less than to find a single algorithm for all of mathematics*" [4]. Following the method of his PhD, his approach was to generalize and study meaningless form. In the introduction to *Account of an anticipation. Absolutely unsolvable problems and relatively undecidable propositions* [2], a manuscript which gives detailed descriptions on Post's research during 1920-21 and which was posthumously published in 1965 by Martin Davis,[6] Post explains that this was an important feature of his approach [15, 341–342]:

> Perhaps the chief difference in method between the present development and its more complete successors is its preoccupation with the outward forms of symbolic expressions, and possible operations thereon, rather than with logical concepts as clothed in, or reflected by, correspondingly particularized symbolic expressions, and operations thereon. [This] allows greater freedom of method and technique.

Hence, instead of starting from the logic of *Principia*, Post decided to focus on his canonical form A, convinced that if one can work with a generalized form, stripped of meaning, it might be more easy or straightforward to prove the decidability of decision problems. Post knew that if he would be able to prove the decidability of the finiteness problem for systems in canonical form A and if, on top of that, he would also be able to prove that first-order logic reduces to a system in canonical form A Post would have succeeded in his ambitious goal of proving that any mathematical problem can be decided in a finite number of steps. Post was indeed able to prove that first-order logic as described in *Principia* can be reduced to a system in canonical form A (by way of a second form, canonical form B).[7] All that remained to be done now was to demonstrate the decidability of the finiteness problem for systems in canonical form A. Post's approach here was to start from the simplest (classes of) cases, by studying systems in "*which the primitive functions are all functions of one variable, the resulting relative simplicity of the systems allowing a direct analysis of the formal processes involved*" [15, 346]. However, "*considerable further labor produced but minor dents in the problem for [systems in canonical form A] not so restricted*".

2.3 The frustrating problem of "tag"

So what to do next? Here things become a bit unclear, but it is known from [13, 15] that the next *important* step forward in the method of combinatory iteration are Post's tag systems [9].

[6]This manuscript was offered for publication to *American Journal for Mathematics* but rejected by Hermann Weyl. A significantly abbreviated version of it was finally published as Post's influential [13].

[7]See pp. 350–361 pf [15] for the details of the proof.

Definition 1 __*(v-tag system)*__ *A tag system T consists of a finite alphabet Σ of μ symbols, a deletion number $v \in \mathbb{N}$ and a finite set of μ words $w_0, w_1, \ldots, w_{\mu-1} \in \Sigma^*$ called the appendants, where any appendant w_i corresponds to $a_i \in \Sigma$. A v-tag system has a deletion number v.*

In a computation step of a tag system T on a word $A \in \Sigma^*$, T appends the appendant associated with the leftmost letter of A at the end of A, and deletes the first v symbols of A. This computational process is iterated until the tag system produces the empty word ϵ and hence halts. To give an example, let us consider the one tag system mentioned by Post [13, 15] with $v = 3$, $0 \to 00$, $1 \to 1101$ [13, 15]. If the initial word $A_0 = 110111010000$ we get the following productions:

~~110~~111010000
⊢ ~~111~~010000**1101**
⊢ ~~010~~00011011**1101**
⊢ ~~000~~11011101**00**
⊢ ~~110~~1110100**00**

The word A_0 is reproduced after 4 computation steps and is thus an example of a periodic word.

So how exactly did Post arrive at this form of "tag"? As explained by Post in his [15], he arrived at tag systems when working on a problem related to but different from the finiteness problem, which is now known as the unification problem [4]. This is the problem to determine for any two (logical) expressions what substitutions would make those two expressions identical.[8] Post furthermore found that tag systems and their decision problems are relevant for the finiteness problem the canonical form A. Hence, "*[tag systems] appeared as a vital stepping stone in any further progress to be made*" [15, 361]

If we compare the formal definition of tag systems with that of systems in canonical form A, it is clear that whereas the canonical form still bears a clear relation with propositional logic and its deductive nature, this is no longer the case for tag systems. These are mere string producing systems stripped of all meaning. These apparently simple forms do not care about expressing logical concepts. Having this meaningless and apparently simple form, Post could now fully focus on properties of the method of combinatory iteration at its purest. It was Post's hope that his study of tag systems would in fact be the first step towards a solution for the finiteness problem for systems in canonical form A. More specifically, Post hoped to tackle what he called the problem of

[8] Unification has played a central role in automated theorem proving and is also one of the central mechanism of Prolog, the well-known logic programming language. It was Robinson who proved that first-order unification is decidable by providing a uniform algorithm for unifying arbitrary first-order expressions [17] and it is exactly this algorithm that is incorporated into the resolution principle. To give the reader an idea of the problem of unification, consider the two terms $f(x, a)$ and $f(b, y)$. These two terms can be unified by setting $x = b$ and $a = y$. Note that Post had solved the unification problem for systems in canonical form A through what he called the L.C.M. process. As remarked by Davis, this is most probably nothing more than the famous unification algorithm developed by Robinson some 40 years later (See [4, footnote 6]).

"tag" for which he formulated two variants nowadays known as the halting and reachability problem for tag systems [10]:

Definition 2 *The halting problem for tag systems is the problem to determine for a given tag system T and any initial word A_0 whether or not T will halt when started from A_0.*

Definition 3 *The reachability problem for tag systems is the problem to determine for a given tag system T, a fixed initial word A_0 and any arbitrary word $A \in \Sigma^*$, whether or not T will ever produce A when started from A_0.*

Post would spend nine months of research on his tag systems. His approach was to start from the simpler cases, and, if successful, try to "scale" the results for the simpler cases to the whole class of tag systems. He was able to prove that the class of tag systems with $v = \mu = 2$ has a decidable halting and reachability problem, a proof which *involved considerable labor*. He considered it as the *major success* of his Procter fellowship.[9] However, for cases that seemed to go but one step beyond the case $v = \mu = 2$, Post only found "*intractable*" cases and cases of a "*bewildering complexity*" [15, 382]. In the end, even though Post initially had been quite optimistic about the possibility of successfully proving the problem of "tag" decidable, it was his meeting and interaction with this form that ultimately led to the reversal of his entire program of proving the finiteness problem decidable [15, 363]:

> For a while the case $v = 2$, $\mu > 2$, seemed to be more promising, since it seemed to offer a greater chance of a finely graded series of problems. But when this possibility was explored in the early summer of 1921, it rather led to an overwhelming confusion of classes of cases, with the solution of the corresponding problem depending more and more on problems in ordinary number theory. Since it had been our hope that the known difficulties of number theory would, as it were, be dissolved in the particularities of this more primitive form of mathematics, the solution of the general problem of "tag" appeared hopeless, and with it our entire program of the solution of finiteness problems. This *frustration* [my emphasis], however, was largely based on the assumption that "tag" was but a minor, if essential, stepping stone in this wider program.

Post had clearly underestimated the complexity a simple form such as that of "tag" can give rise to. Instead of being convinced of the existence of an ultimate method to decide all of mathematics, he now first considered the possibility that this might be a hopeless ambition since even this "primitive form of mathematics" results in such difficulties.

[9] And I regard it as one of my own major successes to have reproven this result (See [10]). Note that (developing) this proof also involved considerable labor.

2.4 The actual reversal of Post's programme

After his frustrating experience with tag systems, Post developed two more formalizations during his Procter fellowship: systems in *canonical form* C, which are nowadays known as Post production systems in the context of formal language theory, and the normal form. I only describe the normal form here.

Systems in normal form, shortly, normal systems are a special class of systems in canonical form C. A system in normal form has only one initial word (postulate) and a finite set of production rules all of the following form:

$$g_i P_i$$
produces
$$P_i g_i'$$

Clearly, normal systems are very similar to tag systems. In fact, the production rules of a tag system are easily rewritten in normal form.

Knowing from his experience with tag systems that apparent formal simplicity does not necessarily imply real simplicity, Post started on a project of proving the "power" of systems in normal form, viz. their generality: he first proved that canonical form A and B can be reduced to a system in canonical form C and then, most importantly, proved that the canonical form C reduces to normal form. This fundamental result was later published as [13]. From this Post concluded that in fact the whole of *Principia* and hence mathematics could be reduced to the normal form:

> [F]or if the meager formal apparatus of our final normal systems can wipe out all of the additional vastly greater complexities of canonical form B, the more complicated machinery of [Principia] should clearly be able to handle formulations correspondingly more complicated than itself.

This insight resulted in the formulation of what Martin Davis has called Post's thesis I:

Post's Thesis 1 *Every generated set of sequences on a given set of letters* $a_1, a_2, ..., a_\mu$ *is a subset of the set of assertions of a system in normal form with primitive letters* $a_1, a_2, ..., a_\mu, a_1', a_2', ..., a_\nu'$*, i.e., the subset consisting of those assertions of the normal system involving the letters* $a_1, a_2, ..., a_\mu$.

Post's thesis identifies the vague notion of generated (set of) sequence(s) with generated by a normal system. Even though this thesis is quite technical in nature, it is logically equivalent to Turing's more famous thesis which identifies the intuitive notion of computability with computability by a Turing machine.

Post soon understood the implications of this thesis. He had already learned from tag systems that his programme of proving the whole of mathematics decidable might in fact be hopeless. He was now able to prove with the diagonal method that there is no finite method to decide for any normal system and some word whether or not that word can be generated by that normal system. Being convinced of the universality of normal systems in the sense that there is a normal system that can generate whatever one can generate with some other process, he was now fully convinced of the existence of absolutely unsolvable problems. He had proven that there are problems that cannot be solved by any finite process. Post even went one fundamental step further and concluded, on the basis of these results, that no logic is complete hence anticipating part of Gödel's results be it without formal proofs.

Hence, starting out from the idea of mechanizing the whole of mathematics and a strong believe in the power of formalization to tackle this kind of general problems, Post, through his experience with something that resulted from his formalist approach, viz. tag systems, came to exactly the opposite conclusion. The impact on one's mind of having not only to change but *reverse* ones views on the foundations of mathematics, because of ones experience with the very systems developed to support these original views cannot be underestimated. This was the case for Post: his "philosophical" thoughts that would follow relate back to this fundamentally *human* experience.

2.5 "I study mathematics as a product of the human mind"

Having established a thesis logically equivalent to Church's and Turing's 15 years before the facts, Post understood that even though he was now convinced of the universality of normal form [15, 387]:

> [for the thesis to obtain its full generality] an analysis should be made of all the possible ways the human mind can set up finite processes to generate sequences.

This view is very similar to what Turing would later write in his famous 1936 paper *On computable numbers* [19] where he states that regarding his thesis:

> The real question at issue is: "What are the possible processes which can be carried out in computing a number?"

Although hardly ever acknowledged in the literature, in 1921 Post was already aware of the significance of what Turing calls the "real question at issue". In fact it can be argued that Post made a start with such an analysis as early as 1921-22, an analysis which lay at the basis of Post's note from 1936 [12] which contains a formulation which is almost identical to Turing machines. How else does one explain that both Post and Turing found quasi-identical formalisms?[10]

[10] The main reason why Turing's thesis is considered in the literature as the better one is exactly because of this analysis of the processes involved when the mathematician is computing a number, an analysis which, by eliminating all the non-essential features of this process, resulted in the well-known Turing machine. Even though I value Turing's work very highly, it is my view that several recent historical and philosophical studies on the topic are too much biased in their high praise for Turing's work against Church's and especially Post's. I have found no satisfying reason in the literature to regard Turing's thesis as being superior to Post's second thesis (see below) especially since it uses a formalism which is almost identical to the Turing machine. It is not because Post's paper does not contain the (philosophical) analysis nor the major results of Turing's [19] that the thesis as such would be less.

In this 1936 note, Post proposed a second thesis which identifies the vague notion of *solvability* of a problem with solvability by his *formulation 1.* Although almost identical to Turing machines there is one important and philosophical difference between Post's and Turing's approaches: Turing's analysis is one of the mathematician in the process of computing a number, for Post it is about an analysis of the possible *mental processes* involved when generating a set and, later in 1936, when solving a decision problem. This is reflected in the fact that whereas Turing proposes a formalism in terms of idealized computing machines, Post's was in terms of sets of instructions in a formal language (see [4]).The fact that for Post his theses are related to human mental processes is reemphasized in his note [12, 105]:

> Its purpose [of formulation 1] is not only to present a system of a certain logical potency but also, [...] of psychological fidelity

It is exactly for this reason that Post could not agree with Church on the idea of regarding his thesis (or any other logically equivalent one) as being but a formal definition of a vague, intuitive concept.[11] It is also why for Post his thesis should be understood as a *working hypothesis* and, in case more and more support could be found for it, a *natural law.* Indeed, for Post his thesis is about the human mind and its mathematical capabilities, hence [12, 103]:

> [...] to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of the mathematicizing power of *Homo Sapiens* has been made and blinds us to the need of its continual verification.

This is a very strong philosophical point of view not only with respect to the thesis but also with respect to mathematics in general. It shows that even though Post can be considered as a formalist, this does not mean that he understood mathematics and its formalizations as something that can be isolated from humans, a point made even more explicit here [15, 403]:

> I study mathematics as a product of the human mind not as absolute

Does the conclusion of a fundamental limitation of the "method of combinatory iteration" mean that Post had turned his back to symbolic logic and "form"? No. On the contrary, the normal form would remain a fundamental form throughout his work. He even used it as the formal framework in his founding paper on recursive functions [14]. More important here, given the discovered limitations, it is symbolic logic itself that can be used as a method to explore and develop these limitations [15]:

> [...] the creativeness of human mathematics has a counterpart inescapable limitation thereof – witness the absolutely unsolvable (combinatory) problems. Indeed, with the bubble of symbolic logic as

[11] For instance, in Church's thesis, effective calculability is defined as λ-definability and general recursive functions

> universal logical machine finally burst, a new future dawns for it as the indispensable means for revealing and developing those limitations. For [...] Symbolic Logic may be said to be Mathematics become self-conscious.

In a letter to Church dated March 24, 1936 a similar point is made:[12]

> For if symbolic logic has failed to give wings to mathematicians this study of symbolic logic opens up a new field concerned with the fundamental limitations of mathematics, more precisely the mathematics of Homo Sapiens.

To be clear: this particular view does not imply that Post somehow supported computationalism, viz., the idea that the mind is like a Turing machine.[13] It only means that there are things we cannot do (at least if we indeed interpret the thesis as something that relates to human activity) and that it is symbolic logic than can be used to study the boundaries of these human limitations.[14]

3 Re: some high-speed logic. A discussion

The fact that Post emphasized that his thesis (and those that are logically equivalent to it) should be understood as a hypothesis because, if true, it implies a discovery of a fundamental human limitation, is most probably rooted in Post's explorations of "form" in the period 1920-21 and the reversal of his program that resulted from it. Indeed, quite unlike Turing who started out from the idea of formalizing the vague notion of computability, Post formulated his thesis on the basis of a profound analysis of systems of symbolic logic.[15] The insight that something as simple as tag systems cannot be controlled algorithmically, confronted Post with the limits of finite methods and, since these methods are human, also with his own limitations. Hence, Post's formalist approach ultimately resulted in a view on symbolic logic that seems far removed from the kind of picture one gets from Lakatos' reading of formalism. Here is an excerpt from his [6]:

[12] The letters from Post to Church can be found in the Alonzo Church papers, box 20, Folder 14; Department of Rare Books and Special Collections, Princeton University Library.

[13] It should be noted here that after his discoveries he became more and more convinced of the significance of mathematical creativity. On several occasions he pleaded for a mathematics that is more informal (!) and less axiomatic. In fact in the introduction of [15, 343] he even makes a plea for *a reversal of the entire axiomatic trend [...] with a return to meaning and truth*"

[14] Regretfully this is not what the debate on the Church-Turing thesis focuses on nowadays. On the contrary, if one takes some of the statements by people like Copeland and Wegner seriously then one cannot but conclude that they want to deny us these very limitations. In an attempt to go "beyond" Turing with, what seems to me, the underlying motivation to prevent the world of being like a Turing machine, they forget that the thesis is, above all, about these limitations instead of, what they seem to assume, a support of computationalism.

[15] This aspect of Post's early work is quite parallel to the way Church arrived at the first formulation of his thesis in which he identifies calculability with λ-definability. It was only by studying (properties of) λ-calculus and understanding its power that Church first came to the idea of defining (in his view) the vague notion of calculability.

> But what can one *discover* in a formalized theory? [...] *First*, one can discover the solution to problems which a suitably programmed Turing machine could solve in a finite time [...]No mathematician is interested in following out the dreary mechanical 'method' prescribed by such decision procedures. *Secondly*, one can discover the solutions to problems (such as: is a certain formula in a non-decidable theory a theorem or not?), where one can be guided only be the 'method' of unregimented insight and good fortune. Now this bleak alternative between the rationalism of a machine and the irrationalism of blind guessing does not hold for live mathematics

In the light of Post's struggle with "form" which resulted in a philosophical point of view that understands symbolic logic as the means to develop and explore the limits of mathematics and its formalizations, but also as "*mathematics become self-conscious*", this black-and-white picture of formalism vs. informal mathematics should at least be nuanced.[16] Furthermore, even at the time one Post was still a full-blood formalist he did not expect that *real-world* mathematics would be replaced by formalism (See Sec. 2.2, p. 4).

If there is one thing one can learn from Post's formalism, it is that it is formalism *itself* and *in practice* that makes possible the study of its very own limitations. In this sense, Lakatos' question *But what can one discover in a formalized theory?* gets a very different answer then the one provided by Lakatos. It is Post's formalist method of simplifying through generalizations that led to his results and philosophical point of view and it is hard to imagine that Post would have called this method one of "unregimented insight and good fortune", formalist though he was at that time.

Post's story shows us that it was exactly the Lewisean heterodox view on mathematics, a mathematics stripped of all meaning, that resulted in the anticipation of the fundamental results of the 30s by Gödel, Church and Turing, even though it was not published at the time. Indeed, whereas [3]:

> Hilbert and his school went on to approach the decision problem for quantification theory semantically, Post evidently felt that was not a promising direction because the combinatorial intricacies of predicate logic were too great to penetrate into that manner, and what he proposed instead was to simplify through generalization.

Some 10-15 years after Post, the formalist school of thought would officially achieve the height of its (own) failure. The very limitations already discovered by Post in 1921 were now proven in detail and published. Gödel's incompleteness results are often seen as the death knell of the Hilbertian optimism so

[16] Even though I adept the philosophical view on mathematics which is historically-embedded and practice-based – a view which has been influenced by lakatos' [6], the way formalism is described by Lakatos is very much a-historical and at best caricatural. The fact that Tarski, Curry, Church et al are put on the same line as the logical positivists seems unfair. For instance Curry's formalism is much more delicate than the image one gets from Lakatos' Curry [1].

famously (and ironically) captured in Hilbert's epitaph "wir müssen wissen, wir werden wissen". It showed that no finite axiomatic system would ever be able to capture the whole of mathematics. Some five years later it would be up to Church and Turing to furthermore prove that no finite (formal) method will ever be found which is able to decide problems logically equivalent to the Entscheidingsproblem of first-order logic.

Despite the failure of the formalist program in the sense of Hilbert, it is not the case that formalism was death and buried after that. In fact, out of the ashes of the failure some of the foundations would be laid for a new discipline to be: computer science. Indeed, with the rise of the electronic and programmable computer it became clear that the formal devices developed by Church, Post, Turing et al were in fact very useful. Hence, the results of that which is often regarded as an abstract and old-fashioned philosophy of mathematics, attained a new and vigorous life in the context of the machine that we all use on an everyday basis.

Is this somehow surprising? In a sense it is not. One should not forget that the computer can in fact be understood as the physical realization of "calculability" and is hence the physical pedant of the forms developed by Church, Post and Turing. In this sense, the computer can also be understood as a machine without meaning, at least to some extent. That the computer is a machine without meaning, a machine that does not really understand in the way we are able to understand, is in fact one of the classical arguments of those who are against the idea of an intelligent machine, and, quite often, consciously or unconsciously, in favor of a pejorative and derogatory view on the machine. On the other side of the spectrum there are those who are trying to understand how the machine can be made (more) intelligent and/or (more) natural mostly by focusing on simulation.

If we look at what is done with computers nowadays, the least one can say is that these machines are quite influential in our everyday and professional lives. Understood as machines without meaning who can only "understand" form but not meaning, this would mean that it is mechanized "formal logic" that we all so much depend on. Of course, when we are interacting with the machine, we are hardly aware of this. This is due to the fact that it is the explicit purpose of software developers to create an illusion of meaningfulness made possible by adding many layers on top of the bare electrical pulses of the machine so that the user does not need to be bothered with the technicalities of the machine, all, of course, for the sake of "user-friendliness". In the meantime philosophers keep debating for or against the 'art of simulation'.

Few, however, are taking up the challenge posed by Derrick H. Lehmer, a number theorist and computer pioneer, in his paper *Some high-speed logic* [7]: instead of trying to let the machine excel in the art of simulation, or criticize it because it is poor at mimicking us, we should perhaps start to take serious the idea of having a fair contest/interaction, one in which the machine is allowed to do what it is good at. Taking such challenge philosophically serious, Post's formalism put into modern perspective could be one possible approach. It was argued here that Post dismissed meaning convinced that by focusing instead on the formal aspects/structure of mathematics it would be possible to understand some fundamental properties of the whole of mathematics. It was this approach which allowed Post to take form serious, to explore it and to uncover not only its possibilities but also its limitations. Similarly, it is perhaps by interacting with and studying the computer as a machine without meaning, stripped of its simulated "semantics", that we will be able to understand and explore the limitations and possibilities of computation in a context averse to the philosophically laden idea of the mimicking machine.

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