

Propositional Logic Extended With A Pedagogically Useful Relevant Implication*

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Abstract

First and foremost, this paper concerns the combination of classical propositional logic with a relevant implication. The proposed combination is simple and transparent from a proof theoretic point of view and at the same time extremely useful for relating formal logic to natural language sentences. A specific system will be presented and studied, also from a semantic point of view.

The last sections of the paper contain more general considerations on combining classical propositional logic with a relevant logic that has all classical theorems as theorems.

1 Some Motivation

In this paper I present an extension of **PC** (Classical Propositional Logic) with a relevant implication. The resulting system is called **PCR**. It figured in my introductory logic course for many years, even before this was published as [3]. Thousands of students applied the system for natural deduction (Fitch-style) proofs and for formalizing Dutch sentences. Still, **PCR** appears here for the first time in English.

The logic **PCR** was developed especially for elementary logic classes. Teaching elementary logic, one cannot avoid presenting the paradoxes of Classical Logic, henceforth **CL**.¹ Becoming familiar with inferences deemed correct by **PC**, even the slowest students start complaining after a while.

Having described the paradoxes, textbooks and logic teachers sometimes try to reason them away. Two types of moves are invoked in this connection. The first move is legitimate but insufficient: one shows that it is correct that the paradoxes are apparent only. Indeed, there is a discrepancy between **PC** and the logical constants from natural languages, but there is a systematic

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¹The paradoxes all show at the propositional level. The paradoxes of material implication are even typical for propositional logic—see below in the text.

and coherent² idea behind **PC**. So **PC** is all right as a logical system. This move is insufficient because of the discrepancy. It does not explain in which way a correct reasoning in **PC** can have a bearing on a reasoning in natural language. The second move is an attack on the discrepancy. It often proceeds in terms of Grice’s conversational maxims. The maxims are extremely useful, for example for explaining that $A \vee B$ indeed follows from A , but that it may be inappropriate for a person knowing A to affirm $A \vee B$ rather than A —and similarly for quantifiers and for modalities. This, however, does not have any bearing on the paradoxes of **CL**. If my grandson weeps because his doll is unrecognizably dirty, I shall tell him that he left the doll in the garden, that the overnight rain turned the sand into mud, and if a doll lies all night in the mud, it is unrecognizably dirty. It is just nonsense to say that the implication is inappropriate because the child knows that the consequent is true. Similarly, “It is not so that it rains if I want it to rain” and, more idiomatically, “It does not rain if I want it to” are decent English sentences. Their logical form is obviously not $W \rightarrow \neg R$ but $\neg(W \rightarrow R)$. Neither “I want it to rain” nor “it does not rain” is derivable from the sentences.

Textbook writers and teachers who don’t try to reason away the paradoxes may refer to relevant logics as a way out of the paradoxes. One may spell out some properties of one or more relevant logics. However, it seems beyond the reach of an elementary logic course to make students familiar with proofs within a relevant logic. The popular relevant logics are just too complex.

The resulting situation is frustrating for both teacher and students. The aim is to improve the students’ reasoning in the context of natural language. However, they become at best fluent in **CL**. It is this situation that urged me to devise **PCR**.

The logic **PCR** was not intended for solving all paradoxes of **CL**. As I see it, there are paradoxes of three kinds. (i) Some paradoxes derive from the fact that the consequence relation is not relevant. If there is a proof of A , then it is said that A is provable or derivable *from* any premise set Γ . An obvious example is $p \vdash_{\mathbf{PC}} q \supset q$. This is matched by the semantic consequence relation: what holds true in all models is a semantic consequence of every premise set. (ii) Even given (i), further paradoxes derive from the fact that contradictory theories have no models. So they are all trivial, cannot be distinguished from each other and reasoning from them is pointless. (iii) The meaning of the implication in **CL** differs drastically from that in natural language; the standard examples are $p \vdash_{\mathbf{PC}} q \supset p$, $p \vdash_{\mathbf{PC}} \neg p \supset q$, $\neg(p \supset q) \vdash_{\mathbf{PC}} p \wedge \neg q$.

The official relevance tradition, Anderson and Belnap and their many students and followers, has chosen to remove the paradoxes of all three kinds. This is by no means necessary. One may study ways to remove one of the kinds of paradoxes. Some such ways may have effects on other paradoxes, but not all of them.

The logic **PCR** was devised with the aim of removing only the paradoxes from (iii). In [3], paraconsistency is presented as a means to remove the paradoxes from (ii). Also, a different means is presented to remove the paradoxes from (i) and (ii) together. **PCR** takes care of the remaining paradoxes, those

²It is sometimes claimed that **PC** is incoherent, for example because classical negation would be a tonk-like operator, as is claimed in [5]. I think this claim is mistaken because coherence depends on possibility, not on actuality—but this is not the appropriate place to discuss the matter.

from (iii). It introduces a relevant implication within the context of **CL**. This is not an obvious matter, as we shall see in Section 2.

The discussion will be kept at the propositional level. The paradoxes from (ii) originate at the propositional level. The discrepancy between formal and natural language mainly plays at that level.³

2 Fitch-Style Rules for PC

Fitch-style proofs⁴ consist of a main proof and of zero or more subproofs, some of which may occur within other subproofs. There are three ‘structural’ rules: PREM to introduce premises, HYP to introduce hypotheses, thus starting a new subproof at the current point (in the main proof or in a subproof), and REIT to reiterate formulas from the main proof or from a subproof into one of its subproofs. There are ten rules of inference, two for each connective:

MP	$A, A \supset B / B$
CP	From a subproof starting with the hypothesis A and ending with B , to infer $A \supset B$.
ADJ	$A, B / A \wedge B$
SIM	$A \wedge B / A$ and $A \wedge B / B$
ADD	$A / A \vee B$ and $B / A \vee B$
DIL	$A \vee B, A \supset C, B \supset C / C$
EI	$A \supset B, B \supset A / A \equiv B$
EE	$A \equiv B / A \supset B$ and $A \equiv B / B \supset A$
DN	$\neg\neg A / A$
RAA	$A \supset B, A \supset \neg B / \neg A$

A subproof is said to be *closed* iff a formula was derived from it by CP.

Rather than further specifying the precise format, I present an example, a proof of $p \supset \neg q, (r \supset r) \supset q \vdash_{\mathbf{PCR}} \neg p$. Although there is nothing paradoxical about this inference, a paradox of kind (iii) is invoked at line 8.

1	$p \supset \neg q$	PREM
2	$(r \supset r) \supset q$	PREM
3	p	HYP
4	r	HYP
5	$r \supset r$	4, 4; CP
6	$(r \supset r) \supset q$	2; REIT
7	q	5, 6; MP
8	$p \supset q$	3, 7; CP
9	$\neg p$	1, 8; RAA

Definition 1 A **PC**-proof of A from Γ is a list of formulas L such that A is the last member of L , all members of L are justified by a **PC**-rule, only members of Γ are justified by PREM, and all subproofs in L are closed.

Definition 2 $\Gamma \vdash_{\mathbf{PC}} A$ (A is **PC**-derivable from Γ) iff there is a **PC**-proof of A from Γ .

³At the predicative level, implications take on a further function. The English sentence “all humans are mortal” does not even contain an implication and there is nothing paradoxical in deriving it from “all mammals are mortal” and “humans are mammals.”

⁴The reader is supposed to be familiar with the basics of Fitch-style proofs.

Definition 3 $\vdash_{\mathbf{PC}} A$ (A is a \mathbf{PC} -theorem) iff $\emptyset \vdash_{\mathbf{PC}} A$.

In a pedagogical context, a set of derivable rules of inference will be introduced, looking for an equilibrium between heuristic facility and the number of rules.

3 Relevant Consequence Relation

The official relevance tradition does not provide an obvious way to combine a relevant implication with \mathbf{PC} . Actually, there are two reasons why it doesn't. The first reason is that the tradition aims at removing all paradoxes together. So all its relevant logics are paraconsistent and their implication as well as their consequence relation is relevant. The second reason is that the consequence relation (which consequences are assigned to a premise set) is defined in a somewhat unusual way by the tradition and that this holds for the syntactic consequence relation as well as for the semantic one.⁵

That the official relevance tradition aims at removing all paradoxes together is easily understandable from its position, which is that 'the true logic' is relevant and that the mistaken conception of \mathbf{CL} is responsible for all its paradoxes.⁶ Obviously, this situation does not help us to realise the aim of the present paper.

The Fitch-style proofs presented in [1] establish that certain formulas are \mathbf{E} -theorems but do not enable one to derive consequences from a premise set. In other words, the proofs define theoremhood but not the syntactic consequence relation. The latter is defined in [1, §23.6] for \mathbf{E} and is there called "a proof in \mathbf{E} that A_1, \dots, A_n entail(s) B ". The definition refers to \mathbf{E} -theorems, thus presupposing that these are provided independently. Incidentally, it is easily seen that there is such a proof iff $(A_1 \wedge \dots \wedge A_n) \rightarrow B$ is a theorem of \mathbf{E} .⁷ The relation with \mathbf{E} -theoremhood reveals at once two important properties of "a proof in \mathbf{E} that A_1, \dots, A_n entail(s) B ". (i) No formula is entailed by the empty set. Indeed, this syntactic consequence relation presupposes that $n > 0$. (ii) The consequence relation is Tarski: reflexive, transitive, and monotonic.

Although the syntactic consequence relation is not defined for any other logic in [1], relevant logicians clearly have a similar entailment notion in mind for those logics.⁸

The Routley-Meyer semantics for relevant logics—see for example [7] and other publications including [6]—may be interpreted as defining the semantic consequence relation in a similar way. The models define the valid formulas

⁵It is perhaps more accurate to say that the role and status of theorems (respectively valid formulas) is unusual, both in the proof theory and in the semantics. See also below in the text.

⁶A multiplicity of relevant logics has been devised. It was never very clear whether relevance logicians see these as candidates for the title 'true logic' or as sensible logics for specific purposes. Thus in [1] the logic \mathbf{E} seems to be advocated as the 'true logic' but in §28.1 it is argued that \mathbf{R} is required for some purposes—see especially the paragraph on applicability.

⁷The widespread habit to define the set of theorems as the set of formulas derivable from the empty set—compare Definition 3—cannot be upheld for relevant logics as the two sets are not identical. See also below in the present paragraph of the text.

⁸For example [6] is in line with this—see §3.1 and see the notion of a \mathbf{L} -theory in §4.6. Where a theory T is a set of formulas, T should obviously contain all formulas entailed by its members. Given the relation between entailment and theoremhood, this means that T should be closed under adjunction (if $A, B \in T$, then $A \wedge B \in T$) and ' \mathbf{L} -implication' (if $A \in T$ and $A \rightarrow B$ is a theorem of \mathbf{L} , then $B \in T$).

as those true at a specific world 0 of every model or as those true at every member of a specific non-empty set of worlds Z of every model.⁹ The set of valid formulas is provably identical to the set of theorems. Already in [7, §4], Richard Routley and Bob Meyer also define “ A **R**-entails B ” in a direct way and the definition is referred to approvingly in the postscript to the appendices of [6]. The definition comes to this: for every world of every model from the Routley-Meyer **R**-semantics, if A holds true in the world, then so does B . To see what this comes to, take into account that worlds may be inconsistent (verifying both A and $\neg A$ for some A) or negation incomplete (falsifying both A and $\neg A$ for some A) or both. However, consistency and negation completeness are required for world 0 (alternatively, for the members of Z).

It is not difficult to see the reason for the construction. Every **E**-theorem is verified at world 0 (at every member of Z in the other formulation) of every model. All theorems of **CL** are theorems of **E** and most other relevant logics. So if one were to define “ A is an **E**-semantic consequence of Γ ” as

$$A \text{ is verified by all } \mathbf{E}\text{-models of } \Gamma, \quad (1)$$

the semantic consequence relation would be non-relevant. Indeed, $p \vee \neg p$ is verified by every **E**-model of $\{q\}$ (because it is verified by every **E**-model) and q is verified by every **E**-model of $\{p \wedge \neg p\}$ (because $\{p \wedge \neg p\}$ has no **E**-models). Moreover, the formulas verified by world 0, respectively by all members of Z , form a *consistent* set. So if the **E**-semantic consequence would be defined by (1), it would validate Ackermann’s rule (γ), more widely known as Disjunctive Syllogism.¹⁰

4 Eliminating Nested Arrows

We shall need three sets of formulas: \mathcal{W} will be the set of formulas of the language of **PC**, compounded in the usual way from sentential letters, the unary connective \neg , and the binary connectives \vee , \wedge , \supset , and \equiv ; \mathcal{W}^\rightarrow will be like \mathcal{W} , except that there also is a binary connective \rightarrow ; \mathcal{W}^1 will be like \mathcal{W}^\rightarrow , except that it has no formulas containing nested arrows. A formula $A \in \mathcal{W}^\rightarrow$ contains a nested arrow if it has a (proper or improper) subformula $B \rightarrow C$ in which B or C contains an arrow.

The language of **PCR** will comprise the formulas of \mathcal{W}^1 . Choosing \mathcal{W}^1 instead of \mathcal{W}^\rightarrow leads to a drastic simplification of the Fitch-style proofs. Indeed, the subproofs that lead to the introduction of an arrow do not require sets of relevance indices, but a single symbol, in the present paper an asterisk. However,

⁹In [6], Z is called 0. Truth at world 0, respectively at every world in Z is defined by the general clauses that pertain to every world but also by specific clauses. Thus 0, respectively every member of Z , is negation consistent as well as negation complete—this warrants that all **PC**-theorems are valid—and relates to other worlds by the ‘accessibility relation’ R in such a way that all implicative theorems of the relevant logic come out true.

¹⁰On p. 497 of [2] we read: “Relevance logicians have so far invariably used a classical metalanguage [...] “preaching to the heathen in his own language.”” The point is that a truth-value like $\{t\}$ does not express, within a relevant metalanguage, what one usually means by “true only” because “only” here refers to classical negation, which is not available within the relevant metalanguage. One wonders, however, where this leaves the proof that rule (γ) is admissible in **E**, by which is meant that B is a theorem of **E** whenever $A \vee B$ and $\neg A$ are theorems of **E**. Indeed, in order for an **E**-theory to be closed under (γ) it is not sufficient that it is consistent; it should be consistent *only*.

the restriction to non-nested arrows does not render the logic useless with respect to formalizing sentences from natural language. It hardly is any hindrance at all as I show below. So the restriction makes the logic suitable for an introductory logic course.

It is hardly an exaggeration to claim that nested implications do not occur in natural language sentences. Most exceptions are sentences produced by logicians trying to read or instantiate formulas from formal languages in such a way that they sound minimally ‘natural’. When another natural language sentence seems to have the logical form $A \rightarrow (B \rightarrow C)$, the sentence is usually equivalent to one that has the logical form $(A \wedge B) \rightarrow C$ or else to a metalinguistic statement that has the form $A \vdash B \rightarrow C$. A similar equivalence seems to obtain for other forms.

Of course, **PCR** is too weak a logic to formalize all sentences from natural language—think about modalities, times and tenses, commands, and so on. Even if what is said in the previous paragraph would be false, it would still hold that many sentences from natural languages can be formalized by means of formulas that are members of \mathcal{W}^1 and that some practice with **PCR** improves the students’ logical competence in connection to implication.

5 Stars and Carrying them Over

To obtain the specific properties of the arrow as opposed to the horseshoe a special kind of subproof is required. The rule RHYP says that any member of \mathcal{W} may be introduced with a star attached to it. By applying RHYP one starts (what will be called) a *starred subproof*.

Certain rules will carry over stars from one formula to the other. If the starred subproof that starts with A^* ends with a starred formula, say B^* , the rule RCP (relevant conditional proof) allows one to infer $A \rightarrow B$ from the starred subproof. Here is a schematic representation:

\vdots	\vdots	
i	A^*	RHYP
\vdots	\vdots	\vdots
j	B^*	\dots
$j+1$	$A \rightarrow B$	i, j ; RCP
\vdots	\vdots	

All this will sound familiar to people acquainted with the work of Anderson and Belnap. The language being \mathcal{W}^1 , there is no need for index sets; the star will be sufficient to recall whether the hypothesis of the subproof is or is not relevant to the conclusion of the subproof. If it is, an arrow can be introduced.

An important *restriction* is that no subproof can be started within a starred subproof. Half of the restriction is necessary: if a starred subproof were started within a starred subproof, nested arrows would result. The other half of the restriction is conventional: starting a non-starred subproof within a starred subproof will never lead to a useful result (given the other rules that will be introduced).

The absence of nested arrows in the members of \mathcal{W}^1 entails that no formula containing an arrow should ever occur starred. If such formula occurred starred

in a proof, it would occur in a starred subproof. But then RCP would lead to nested arrows.

Let us now turn to the central point: which rules carry over stars? Consider a subproof that begins with the starred hypothesis A^* and in which occurs C^* . If we stop the subproof right there, applying RCP gives us $A \rightarrow C$. If we continue the subproof by applying a rule X to C^* and this results in D^* , we can derive $A \rightarrow D$ from the subproof. So it is justified that rule X carries over de star from C to D just in case $A \rightarrow C$ provides a warrant for $A \rightarrow D$. If rule X requires two starred formulas, say C_1^* and C_2^* in order to derive D^* , then it is justified that rule X carries over the star just in case $A \rightarrow C_1$ and $A \rightarrow C_2$ jointly provide a warrant for $A \rightarrow D$. And so on.

As carrying over the star is parasitic on the justification of the transition between arrow formulas, it is advisable to have an intended intuitive interpretation of the arrow. A simple and general, and rather unquestionable, such interpretation is the following:

(M \rightarrow) $A \rightarrow B$ means that reasons to accept A constitute reasons to accept B and that reasons to reject B constitute reasons to reject A .

In order for this to do its job, we better also consider the other connectives, skipping material implication and equivalence, which are definable anyway.

(M \neg) One has reasons to accept $\neg A$ iff one has reasons to reject A . One has reasons to reject $\neg A$ iff one has reasons to accept A .

(M \wedge) One has reasons to accept $A \wedge B$ iff one has reasons to accept A as well as reasons to accept B . If one has reasons to reject A or reasons to reject B , then one has reasons to reject $A \wedge B$.

(M \vee) If one has reasons to accept A or reasons to accept B , then one has reasons to accept $A \vee B$. One has reasons to reject $A \vee B$ iff one has reasons to reject A as well as reasons to reject B .

Let us call (M \rightarrow)–(M \vee) *meaning postulates*. Note that no exhaustive condition pertains to reasons for accepting a disjunction or to reasons for rejecting a conjunction. This is as it should be, but it follows that one cannot fully justify de Morgan properties in terms of these meaning postulates. One of the possible illustrations is in Figure 1. Figure 2 illustrates the situation where two of the

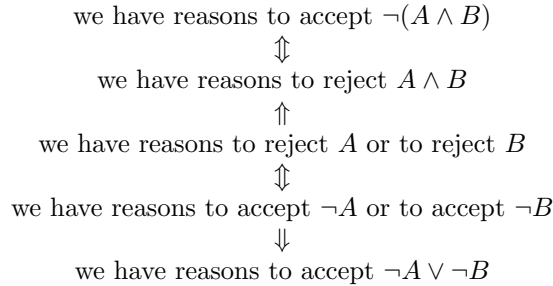


Figure 1: Present Meaning Postulates

statements are modified. The result is that the two unidirectional arrows become bidirectional but that another bidirectional arrow becomes questionable, as the

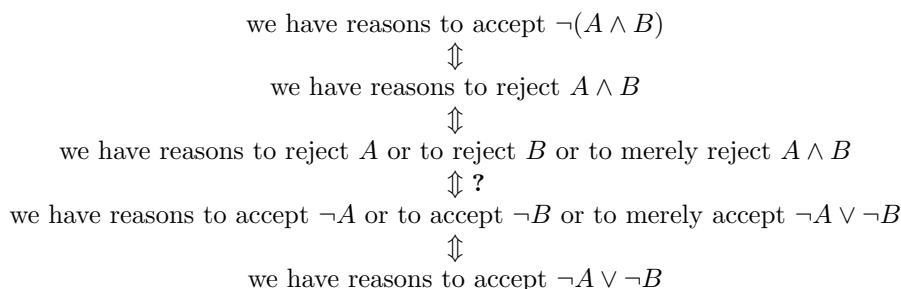


Figure 2: Closing Two Gaps and Creating Another?

question mark indicates. The question mark can be removed iff reasons to merely reject $A \wedge B$ can be identified with reasons to merely accept $\neg A \vee \neg B$.

And indeed, to do so seems justified. A reason to merely reject $A \wedge B$ is a reason to reject $A \wedge B$ that is neither a reason to reject A nor a reason to reject B . Yet, if our knowledge became total *and would still constitute a reason to reject $A \wedge B$* , then it would constitute a reason to reject A or to reject B or to reject both A and B . Similarly for the reason to merely accept $\neg A \vee \neg B$. If our knowledge became total *and would still constitute a reason to accept $\neg A \vee \neg B$* , then it would constitute a reason to accept $\neg A$ or to accept $\neg B$ or to accept both $\neg A$ and $\neg B$. These considerations justify that we introduce a further meaning postulate:

(M $\wedge\vee$) One has reasons to merely reject $\neg A \wedge \neg B$ iff one has reasons to merely accept $A \vee B$. One has reasons to merely reject $A \wedge B$ iff one has reasons to merely accept $\neg A \vee \neg B$.

Digression The paragraph preceding (M $\wedge\vee$) contains two occurrences of a phrase that is emphasized because it is crucial. Indeed, the situation that obtains in the case of total knowledge cannot be taken as the norm.

This obtains notwithstanding the fact that we underwrite all presuppositions of **PC**—the game we are playing *is PC* extended with a relevant implication. So we take it for granted that total knowledge is consistent as well as negation-complete. But even if (the world as well as) total knowledge is as Classical Logic claims it to be, reasoning about our knowledge requires a relevant implication because our knowledge is not total. A good illustration of this is provided by $A \rightarrow (B \vee C) \not\vdash_{\mathbf{PCR}} (A \rightarrow B) \vee (A \rightarrow C)$. This is directly connected to the fact that our reasons to accept A may be reasons to merely accept $B \vee C$. This would be different if our knowledge became total, but it is not total right now.

So it is important to realize that (M $\wedge\vee$) does not presume that our knowledge is or will ever be total. The reasoning that leads to (M $\wedge\vee$) merely refers to total knowledge to identify, for example, reasons to merely accept $\neg A \vee \neg B$ with reasons to merely reject $A \wedge B$. This ends the digression.

With the five meaning postulates in mind, the carrying over of stars may be organized by the four conventions that follow. The reader is prayed to verify that each of the conventions is exhaustively justified by the meaning postulates.

(C1) Some formulas have the same meaning in view of the meaning postulates. Here are some examples: $A \supset B$ and $\neg A \vee B$, $A \equiv B$ and $(A \supset B) \wedge (B \supset$

A), $A \wedge (B \vee C)$ and $(A \wedge B) \vee (A \wedge C)$, $\neg\neg A$ and A , $\neg(A \vee B)$ and $\neg A \wedge \neg B$, etc. Rules that lead from a formula to a formula that has the same meaning carry over the star; reasons to accept a statement are obviously reasons to accept every statement that has the same meaning. Incidentally, the involved couples of formulas are actually *mutual* tautological entailments from [1, §15] and all mutual tautological entailments correspond to primitive or derivable **PCR**-rules that carry over the star in both directions.

(C2) ‘Simple weakenings’ like SIM and ADD carry over the star; reasons to accept $A \wedge B$ constitute reasons to accept A , and so on. Incidentally, the simple weakenings correspond to tautological entailments and all tautological entailments correspond to primitive or derivable **PCR**-rules that carry over the star.

(C3) Consider rules with a major and a minor (local) premise. Modus Ponens for the arrow is the easiest case. If A^* and $A \rightarrow B$ occur in the subproof¹¹ then B^* is derivable. Indeed, A^* tells us that reasons to accept the hypothesis constitute reasons to accept A and $A \rightarrow B$ tells us that reasons to accept A constitute reasons to accept B ; so it follows that reasons to accept the hypothesis constitute reasons to accept B . Similarly for reasons to reject.

The matter is wholly different for Disjunctive Syllogism or for material Detachment (Modus Ponens for material implication). If $\neg A^*$ and $A \vee B$ occur in the proof, reasons to accept the hypothesis of the subproof constitute reasons to reject A . However, $A \vee B$ is merely a truth function; its truth does not warrant that reasons to reject A constitute reasons to accept B . The situation remains the same if $\neg A^*$ and $A \vee B^*$ occur in the proof—see the digression below. The case is actually crucial as the non-theoremhood of $(A \wedge \neg A) \rightarrow B$ depends on it.

(C4) The result of an application of ADJ will only be starred if both conjuncts were starred. This is a direct consequence of $M\wedge$: a reason to accept the hypothesis is a reason to accept $A \wedge B$ iff it constitutes a reason to accept A as well as a reason to accept B . Incidentally, if $A \wedge B^*$ were derivable from A^* and B , then, as SIM carries over the star, B^* would be derivable from B . So the arrow would lose its relevant character.

The specific rules in Section 6 rely on conventions (C1)–(C4). Note that (C3) requires the possibility that one has reasons to accept a statement as well as reasons to accept its negation. Put differently, one may have reasons to accept a statement as well as reasons to reject it.

Digression Some people will consider it impossible that one has reasons to accept A as well as reasons to reject it. This is because they are thinking about reasons that are both final and exclusive. Some reasons, however, do not have these qualities. One may think of reasons provided by reliable witnesses that contradict each other, or provided by empirical data on the one hand and theoretical considerations on the other.

The *possibility* that someone has reasons to accept A as well as reasons to reject A is amply sufficient for not having $(A \wedge \neg A) \rightarrow B$ as a theorem of logic. Recall that the reason why $(A \wedge \neg A) \supset B$ is a theorem of **PC** is that **PC** considers it impossible that A and $\neg A$ are both *true*. Expressed in similar terms, **PCR** considers it impossible that A and $\neg A$ are both *true* but considers

¹¹As we have seen, $A \rightarrow B$ cannot possibly occur starred.

it possible that one has reasons to accept A as well as reasons to reject A . So $(A \wedge \neg A) \supset B$ is and $(A \wedge \neg A) \rightarrow B$ is not a theorem of **PCR**.

6 Fitch-Style for PCR

As **PCR** is an extension of **PC**, all Fitch-style rules of **PC** are retained in **PCR**. We have to add specific rules in order to handle the arrow and in order to specify which rules carry over stars.

The rule RHYP was already mentioned in the previous section. The rule REIT will be *restricted* to non-starred subproofs and a specific reiteration rule for starred subproofs, RREIT, is added. RREIT states that only formulas of the form $A \rightarrow B$ may be reiterated into a starred subproof.¹²

This is a matter of elegance and efficiency, not of principle. One might allow for starred subproofs that start with the starred hypothesis A^* and end with a non-starred formula B . From such a starred subproof, one might then derive $A \supset B$. It is obvious, however, in view of the rules of **PCR** that whatever is derivable from *such* a starred subproof, is also derivable from a non-starred subproof. A non-starred subproof is obtained for example by removing all stars and replacing RHYP by HYP.

Below follows a list of rules of inference. The list is redundant—some rules are easily derivable from others. To save some space, I write $A \parallel B$ to abbreviate A/B and B/A . The rules in the first list are primitive or derivable rules of **PC** for which the behaviour with respect to stars is here specified.

ADJ $A^*, B^*/A \wedge B^*$
SIM $A \wedge B^*/A^*$ and $A \wedge B^*/B^*$
ADD $A^*/A \vee B^*$ and $B^*/A \vee B^*$
MI $A \supset B^* \parallel \neg A \vee B^*$
ME $A \equiv B^* \parallel (A \supset B) \wedge (B \supset A)^*$
DN $\neg\neg A^* \parallel A^*$
ND $\neg(A \vee B)^* \parallel \neg A \wedge \neg B^*$
NC $\neg(A \wedge B)^* \parallel \neg A \vee \neg B^*$
DIST $A \wedge (B \vee C)^* \parallel (A \wedge B) \vee (A \wedge C)^*$

The second list contains specific properties of the arrow. The understanding is that RMP, RDIL, and RMT remain valid if the stars are removed from them.¹³

RMP $A^*, A \rightarrow B/B^*$
RDIL $A \vee B^*, A \rightarrow C, B \rightarrow C/C^*$
RMT $\neg B^*, A \rightarrow B/\neg A^*$
RCP From a subproof starting with the hypothesis A^* and ending with B^* , to infer $A \rightarrow B$.
REQ $A \leftrightarrow B \parallel (A \rightarrow B) \wedge (B \rightarrow A)$

A **PCR**-proof of A from Γ , $\Gamma \vdash_{\mathbf{PCR}} A$, and $\vdash_{\mathbf{PCR}} A$ are defined as for **PC**, viz. by replacing **PC** by **PCR** in Definitions 1–3.

¹²There are presumably means to retain REIT for all subproofs and nevertheless to avoid trouble. An example of trouble is presented at the end of the present section.

¹³The variants without stars are required in the main proof and in non-starred subproofs.

Many properties of the arrow are derivable. It is easy enough to produce proofs for $A \rightarrow B, B \rightarrow C \vdash_{\mathbf{PCR}} A \rightarrow C$ and $A \rightarrow B \vdash_{\mathbf{PCR}} \neg B \rightarrow \neg A$. The latter enables one to show that $\neg A \rightarrow \neg(A \wedge B)$ follows from $(A \wedge B) \rightarrow A$. So both are **PCR**-theorems and the right-left direction of NC is redundant.

It seems wise to mention a few properties of **PCR**. I add at best a hint to the proof because we are in propositional logic.

- (1) **PCR** is a conservative extension of **PC**: if $A \in \mathcal{W}$, then $\vdash_{\mathbf{PCR}} A$ iff $\vdash_{\mathbf{PC}} A$.¹⁴
- (2) For all $A \in \mathcal{W}$, $\vdash_{\mathbf{PCR}} A$ iff $\vdash_{\mathbf{PCR}} \neg A \rightarrow A$.¹⁵
- (3) $A \rightarrow B$ is a **PCR**-theorem iff it is a tautological entailment as defined in [1, §15].
- (4) If $A \leftrightarrow B$ is a **PCR**-theorem and D is obtained by replacing the subformula A in C by B , then $C \leftrightarrow D$ is a **PCR**-theorem. This derivable rule may be called Replacement of Relevant Equivalents.
- (5) Replacement of (Material) Equivalents holds in **PC** but not in **PCR**. For example $\vdash_{\mathbf{PC}} p \equiv (p \vee (r \wedge \neg r))$ but $\not\vdash_{\mathbf{PCR}} (p \rightarrow q) \equiv ((p \vee (r \wedge \neg r)) \rightarrow q)$.¹⁶
- (6) Derivable rules: $A \rightarrow (B \wedge C) \parallel (A \rightarrow B) \wedge (A \rightarrow C)$
 $(A \vee B) \rightarrow C \parallel (A \rightarrow C) \wedge (B \rightarrow C)$
- (7) Negative results: For some A and B , $B \not\vdash_{\mathbf{PCR}} A \rightarrow B$, $\neg A \not\vdash_{\mathbf{PCR}} A \rightarrow B$, $\neg(A \rightarrow B) \not\vdash_{\mathbf{PCR}} A$, and $\neg(A \rightarrow B) \not\vdash_{\mathbf{PCR}} \neg B$.
 In general no *implication* paradox holds for the arrow of **PCR**.¹⁷
- (8) Other negative results: For some A, B, C , and D : $(A \wedge B) \rightarrow C \not\vdash_{\mathbf{PCR}} (A \wedge \neg C) \rightarrow \neg B$, $A \rightarrow (B \vee C) \not\vdash_{\mathbf{PCR}} \neg B \rightarrow (\neg A \vee C)$, and $(A \wedge B) \rightarrow (C \vee D) \not\vdash_{\mathbf{PCR}} (A \wedge \neg C) \rightarrow (\neg B \vee D)$. These are obviously important to prevent the theoremhood of irrelevant implications such as $(p \wedge \neg p) \rightarrow q$, $p \rightarrow (q \vee \neg q)$, and $(p \wedge \neg p) \rightarrow (q \vee \neg q)$.

As was announced in footnote 12, I present an example of the kind of trouble that would occur if REIT were retained for all subproofs. Disjunctive Syllogism (DS is obviously a derivable rule of **PC** and hence of **PCR**).

1	$\neg p$	PREM	
2	$\neg p \vee (p \rightarrow q)$	1; ADD	
3	p^*	RHYP	
4	$\neg p \vee (p \rightarrow q)$	2; REIT	wrong
5	$p \rightarrow q$	3, 4; DS	
6	q^*	3, 5; RMP	
7	$p \rightarrow q$	3, 6; RCP	

The proof illustrates that, if it were allowed to apply within starred subproofs the rule REIT (jointly with other rules that do not carry over stars), then it

¹⁴By an induction over Fitch-style proofs; or by an induction over the semantics (see Section 7 and Section 8); or by the fact that **PC** is Post-complete whereas **PCR** has non-trivial models.

¹⁵ \Rightarrow : if $A \in \mathcal{W}$ and $\vdash_{\mathbf{PCR}} A$, then $\vdash_{\mathbf{PC}} A$ by (1); so if $B_1 \vee \dots \vee B_n$ is a conjunct of the Conjunctive Normal form of A , then (i) some B_i is $\neg B_j$ for $i, j \in \{1, \dots, n\}$ and (ii) $\neg B_1 \wedge \dots \wedge \neg B_n$ is a disjunct of the Disjunctive Normal form of $\neg A$; so $\vdash_{\mathbf{PCR}} \neg A \rightarrow A$.
 \Leftarrow : $\neg A \rightarrow A \vdash_{\mathbf{PCR}} \neg \neg A \vee A$, and $\neg \neg A \vee A \vdash_{\mathbf{PCR}} A$.

The property is not merely a result of the absence of nested implications in \mathcal{W}^1 . Even $\neg(A \rightarrow A) \rightarrow (A \rightarrow A)$ is not a theorem of any of the popular relevant logics.

¹⁶For those in doubt: if $(p \rightarrow q) \equiv ((p \vee (r \wedge \neg r)) \rightarrow q)$ were a theorem of **PCR**, then so would be $(p \rightarrow p) \equiv ((p \vee (r \wedge \neg r)) \rightarrow p)$; but $\vdash_{\mathbf{PCR}} p \rightarrow p$ and $\not\vdash_{\mathbf{PCR}} (p \vee (r \wedge \neg r)) \rightarrow p$.

¹⁷The negative properties are easily proven from one of the semantic systems in the subsequent sections.

would be possible to derive $p \rightarrow q$ from $\neg p$. So the arrow would lose its relevance properties.

7 Worlds Semantics and Tableaux

It is easily seen that all connectives can be defined from $\{\neg, \vee, \rightarrow\}$. The worlds semantics will take this into account, leaving it to the reader to eliminate the definitions if desired.

A **PCR**-model is a triple $\langle W, w_0, v \rangle$ in which W is a set, $w_0 \in W$ and $v: \mathcal{W}^1 \times W \rightarrow \{0, 1\}$ is a valuation fulfilling:

- SPCR1 $v(\neg A, w_0) = 1$ iff $v(A, w_0) = 0$
- SPCR2 $v(A \rightarrow B, w_0) = 1$ iff, for all $w_i \in W$, $v(A, w_i) \leq v(B, w_i)$ and $v(\neg B, w_i) \leq v(\neg A, w_i)$
- SPCR3 $v(A \vee B, w_i) = \max(v(A, w_i), v(B, w_i))$
- SPCR4 $v(\neg\neg A, w_i) = v(A, w_i)$
- SPCR5 $v(\neg(A \vee B), w_i) = \min(v(\neg A, w_i), v(\neg B, w_i))$

$M \models A$ (a **PCR**-model $M = \langle W, w_0, v \rangle$ verifies A) iff $v(A, w_0) = 1$. A model of Γ is a model that verifies all members of Γ . $\Gamma \models A$ (A is a semantic consequence of Γ) iff all models of Γ verify A ; $\models A$ (A is valid) iff every model verifies A .

The set of formulas verified at world w_0 is consistent and negation complete (for each A , exactly one of A and $\neg A$ is verified at w_0). The set of formulas verified at a different world w_i may be inconsistent as well as negation-incomplete (for each A , the set contains any subset of $\{A, \neg A\}$). Note that formulas of the form $A \rightarrow B$ have an arbitrary truth-value at worlds different from w_0 . Incidentally, as $A \rightarrow B \in \mathcal{W}^1$, the A and B in SPC2 are arrow-free.

With the semantics we associate a tableau method. Being a great fan of Beth tableaux, especially for pedagogical use, I shall nevertheless present signed Smullyan tableaux to make the majority happy. I shall consider the logical constants $\{\neg, \vee, \wedge, \rightarrow\}$, leaving the others to the reader.

A tableau construction results from executing a procedure, which is defined by a set of instructions and by the order in which they are applied. The construction will consist of a main tableau, called 0, and of zero or more side tableaux, called 1, 2, ... Each tableau has one or more branches; for example tableau 0 may have branches $0_1, \dots, 0_4$. Each side tableau is *associated* to a unique branch of the main tableau. For each branch 0_i of the main tableau, the set S_i comprises 0_i as well as all branches of the side tableaux associated with 0_i . While the tableau construction is carried out, side tableaux may be added, and branches of a tableau may 'split'. A branch contains labelled formulas and the labels are T and F . As the label is not part of the formula, I shall write $TA \rightarrow B$ rather than $T(A \rightarrow B)$.

The tableau construction for $A_1, \dots, A_n \models B$ is *closed* iff its main tableau is closed. A tableau is *closed* iff all its branches are closed. A branch is *closed directly* iff there is a formula A for which TA as well as FA occur in the branch. A branch 0_i of the main tableau is *closed indirectly* iff a side tableau associated with 0_i is closed. A branch is *finished* iff it is either closed or no application of an instruction results in a new (labelled) formula being added to the branch. A branch is *open* iff it is not closed. A tableau construction is *open* iff it is not

closed and all its branches are finished. No rule is applied to extend a closed branch—to do would be harmless but useless.

A tableau construction for $A_1, \dots, A_n \models B$ is *started* by writing the list $\langle TA_1, \dots, TA_n, FB \rangle$, which at this point forms the single branch of the main tableau 0. First one applies the instructions that do not lead to the introduction of side tableaux:

$$\begin{array}{ll}
0_i T\neg A / 0_i FA & mT\neg \\
0_i F\neg A / 0_i TA & mF\neg \\
0_i TA \rightarrow B / j_k FA \mid j_{k'} TB & \text{for all } j_k \in S_i \quad mT\rightarrow 1 \\
0_i TA \rightarrow B / j_k F\neg B \mid j_{k'} T\neg A & \text{for all } j_k \in S_i \quad mT\rightarrow 2 \\
j_k TA \vee B / j_k TA \mid j_{k'} TB & aT\vee \\
j_k FA \vee B / j_k FA, j_k FB & aF\vee \\
j_k TA \wedge B / j_k TA, j_k TB & aT\wedge \\
j_k FA \wedge B / j_k FA \mid j_{k'} FB & aF\wedge
\end{array}$$

The first four instructions are only applied to the branches of the main tableau 0 whereas the last four instructions are applied to the branches of all tableaux—the first letter of the names refers to this. The instruction $mT\neg$, for example, says: if $T\neg A$ occurs in a branch of 0, then FA is added to the same branch. The instruction $aF\vee$ says: if $FA \vee B$ occurs in branch k of tableau j , then both FA and FB are added to branch j_k . The instruction $aT\vee$ says: if $TA \vee B$ occurs in branch k of tableau j , then the branch is split into k and k' , TA is added to j_k and TB to $j_{k'}$. The instruction $mT\rightarrow 1$ says: if $TA \rightarrow B$ occurs in branch i of the main tableau, then every branch $j_k \in S_i$ is split into j_k and $j_{k'}$, FA is added to j_k and TB to $j_{k'}$. Note that $S_i = \{0_i\}$ as long as only the above instructions are applied. So $mT\rightarrow 1$ has only effect on the main tableau for the time being. Note also that there are two instructions for implicative formulas that have label T .

After the main tableau is finished, we construct the side tableaux according to a strict procedure.

Step 1 For every statement $FA \rightarrow B$ that occurs in branch i of 0, we start a new side tableau j , add both TA and FB in its sole branch 1 and stipulate that $j_1 \in S_i$:

$$0_i FA \rightarrow B / j_1 TA, j_1 FB \quad j \text{ new; stipulating } j_1 \in S_i \quad mF\rightarrow$$

Step 2 Next we apply to the side tableaux the above four instructions that pertain to all branches ($aT\vee$, $aF\vee$, $aT\wedge$, and $aF\wedge$) as well as the following specific instructions for branches of side tableaux:

$$\begin{array}{ll}
\text{for } j \neq 0 \text{ and } X \in \{T, F\}: & j_k X\neg\neg A / j_k XA & sX\neg\neg \\
& j_k X\neg(A \vee B) / j_k X(\neg A \wedge \neg B) & sX\neg\vee \\
& j_k X\neg(A \wedge B) / j_k X(\neg A \vee \neg B) & sX\neg\wedge
\end{array}$$

Step 3 When no further instruction from Step 2 can be applied to a branch of the side tableau, we apply one of $mT\rightarrow 1$ or $mT\rightarrow 2$ —this time the application will have an effect in branches of side tableaux. Whenever a member of S_i splits, both resulting branches are stipulated to be members of S_i . As soon as one of the branches of a side tableau is split and extended, we return to Step 2, going back and forth between Step 2 and Step 3 until no further instruction can be applied.

Consider the following three statements:

- (A) The construction for $A_1, \dots, A_n \models B$ is closed.
- (B) $A_1, \dots, A_n \models B$
- (C) $A_1, \dots, A_n \vdash B$

There are well-known standard means to prove the following three statements: If (A) then (B). If not (A) then not (B). If (C) then (B). These three show the equivalence of (A) and (B). The equivalence of (A) and (C) follows if we moreover are able to show: If (A) then (C). To show this is simple, viz. exactly as for **PC**, if all branches of the main tableau are closed directly. So I shall outline the longwinded but simple proof for the case in which some branches are closed indirectly.

Consider a closed tableau construction such that a branch of the main tableau is closed indirectly because side tableau i is associated with the branch and is closed directly. Let $A \rightarrow B$ occur in the branch with label F and let Δ be the set of implicative formulas that occur in the branch with label T . Let side tableau i result from applying $mF \rightarrow$ to $FA \rightarrow B$ (Step 1) and next by switching back and forth between Step 2 and Step 3. At the point where i is closed, a specific sequence of (Step 3, Step 2) pairs has been executed. Note that i will still be closed if these pairs are executed in a different order or if they are applied to one branch at some point in time, and to another at a later point in time.

Let us slightly reorganise Step 1 and the first Step 2. Executing half of Step 1, we write FB into tableau i and apply Step 2 to this. Let the result be branches i^1, \dots, i^n . These contain only FB and other F -labeled formulas obtained from FB . Let F^k be the *disjunction* of the formulas C for which FC occurs in branch i^k . Obviously

$$\vdash_{\mathbf{PCR}} \bigwedge \{F_k \mid k \in \{1, \dots, n\}\} \rightarrow B. \quad (2)$$

Next, the second half of Step 1 is executed, extending each of the branches with TA and Step 2 is applied to TA . For each branch i^k , this results in one or more branches: i_1^k, \dots, i_m^k . Apart from the F -labeled formulas that occur already in i^k , the so extended and possibly split branches contain TA and T -labeled formulas obtained from TA .

Consider the situation depicted in Figure 3. Step 1 and the first Step 2 were executed in the order specified in the previous paragraph. The first dashed dots indicate that the branch may have been split zero or more times before TA was introduced. The set of C for which FC occurs in branch i^k is represented by ϕ^k . After TA was introduced, the branch may have been split further before the split below τ_j^k . The set of C for which FC occurs in branch i_j^k is represented by τ_j^k .

Next, Step 3 is executed: $mT \rightarrow 1$ or $mT \rightarrow 2$ is applied to a member of Δ . The effect is that the considered branch i_j^k is split into i_j^k and $i_{j'}^k$, that FD is added to i_j^k , and that TE is added to $i_{j'}^k$.

After Step 3, another Step 2 is executed. The result is that F -labelled formulas are added to i_j^k —these are all obtained from FD —and T -labelled formulas to $i_{j'}^k$ —these are all obtained from TE . Branches i_j^k as well as $i_{j'}^k$ may split during this Step 2. In Figure 3, Σ_1 represents the present set of subbranches of i_j^k and these contain only F -labelled formulas that are obtained from FD ; Σ_2 represents the present set of subbranches of $i_{j'}^k$, and these contain

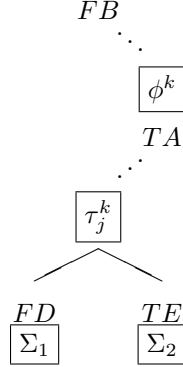


Figure 3: Left branch is closed without splitting

only T -labelled formulas that are obtained from TE . A member of Σ_1 is closed iff it contains a FC for which TC occurs in τ_j^k .

Remember that side tableau i is eventually closed. From this follows that, after Step 1 and the first Step 2 were executed (in the specified order), there is a member of Δ for which the application of Step 3 followed by Step 2 has the effect that all branches in Σ_1 are closed.¹⁸ By the same reasoning, after the execution of a (Step 3, Step 2) pair, another (Step 3, Step 2) pair can be executed for which all branches in the new Σ_1 close. This may be continued until the construction is closed because it contains a formula TC for which FC occurs in ϕ^k .

After this reordering of side tableau i , one can distinguish between left and right subbranches of a branch i^k . The *right* subbranches are those in which all formulas with label F occur in ϕ^k .¹⁹ Let R^k be the set of right subbranches of i^k at a given point. Let T_j be the *conjunction* of the formulas C for which TC occurs in branch i_j^k . After TA was introduced, the following holds:

$$\Delta \vdash_{\text{PCR}} A \rightarrow \bigvee \{T_j \mid j \in R^k\}. \quad (3)$$

Next, one executes the (Step 3, Step 2) pairs in an order that warrants that the pair starts with a $FC|TD$ move and ends in such a way that all branches that contain this FC are closed. The ‘remaining’ branches are members of R^k and contain an increasing number of T -labeled formulas. Eventually, all members of R^k contain a TC for which FC occurs in ϕ^k . It is easily seen that, after every execution of a (Step 3, Step 2) pair, (3) still holds.

As i is eventually closed, the right branches of each i^k close. This happens because there is a C such that TC and F both occur in i_j^k . Note that, in this case, (i) $C \in \phi^k$ and (ii) $A \rightarrow F_k$ for this specific $k \in \{1, \dots, n\}$. So, in (2), that specific F_k may be replaced by A . As the subbranches of all i^k close, by the same reasoning all F_k may be replaced by A in (2), whence $A \rightarrow B$.

¹⁸If this were not so, the application of $mT \rightarrow 1$ or $mT \rightarrow 2$ to the members of Δ , each time followed by Step 2, would eventually lead to a branch that contains no other T -labeled formulas than those in τ_j^k and this branch would not contain a FC for which TC occurs in τ_j^k . But then side tableau i cannot be closed at the end of the construction.

¹⁹Before any step 3 has been executed, all subbranches of i^k are right branches.

The upshot is that the so reordered side tableau can be turned into a subproof of a Fitch-style proof.²⁰ The subproof starts with introducing A^* by RHP. Applications of instructions are turned into application of rules in the standard way. As for **PC**, one first ‘translates’ T -labeled formulas until one reaches the TC that causes the branch to be closed; next one continues from the corresponding FC and ‘translates’ up to FB . After this, the subproof ends with B^* and justifies the derivation of $A \rightarrow B$ by RCP.

8 Algebraic Semantics

Consider an algebraic structure $\langle S, \leq, \bar{} \rangle$, called an *intensional lattice* by Michael Dunn—see [1, §28.2]. This structure has the following properties. (i) S is a set, \leq is a partial ordering relation (a reflexive, transitive, antisymmetric relation) over S , and $\bar{}$ is a function that maps every member of S to a member of S (if $a \in S$, then $\bar{a} \in S$). (ii) For all $a, b \in S$, $a \sqcup b, a \sqcap b \in S$, where $a \sqcup b$, the join of a and b , is the least upper bound of a and b in S with respect to \leq and $a \sqcap b$, the meet of a and b , is the greatest lower bound of a and b in S with respect to \leq .²¹ (iii) Join and meet distribute over each other: $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$ and $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$. (iv) De Morgan properties obtain: $\bar{\bar{a}} = a$; if $a \leq b$ then $\bar{b} \leq \bar{a}$.²² (v) The structure has a truth filter: for all a , $\bar{a} \neq a$.²³

An algebraic **PCR**-model is a structure $\langle S, \leq, \bar{}, R, v, T \rangle$ such that: $\langle S, \leq, \bar{} \rangle$ is an intensional lattice; R is a binary relation over S with the properties

- R1 if $a \leq b$, then Rab ,
- R2 if Rab , then $R\bar{b}\bar{a}$,
- R3 if Rab and Rbc , then Rac ,
- R4 if Rac and Rbc , then $R(a \sqcup b)c$, and
- R5 if Rab and Rac , then $Ra(b \sqcap c)$;

v is an assignment that maps the sentential letters to S ; T is a truth filter: (i) $T \subset S$, (ii) $\bar{a} \in T$ iff $a \notin T$, (iii) $a \sqcap b \in T$ iff $a, b \in T$, and (iv) if Rab , then $a \notin T$ or $b \in T$.²⁴

An interpretation of the members of \mathcal{W} is defined by:

- (i) if A is a sentential letter, $|A| = v(A)$
- (ii) $|\neg A| = \overline{|A|}$
- (iii) $|A \vee B| = |A| \sqcup |B|$
- (iv) $|A \wedge B| = |A| \sqcap |B|$

and verification by an algebraic model M is defined by:

²⁰In the reorganized construction, every execution of Step 3 corresponds to an application of RMP, RMT, or a derivable RDIL-variant (viz. $A \vee B, A \supset C / C \vee B$ and $A \vee B, B \supset C / A \vee C$). This requires some handling of cases. For example, if C occurs in the subproof and $(C \vee D) \rightarrow E$ is reiterated, one needs to derive $C \vee D$ before applying RMP. All this, however, falls into one’s hands when the meta-theoretic proof is written out.

²¹So $a \leq a \sqcup b$ and $b \leq a \sqcup b$ and, for all $x \in S$, if $a \leq x$ and $b \leq x$ then $a \sqcup b \leq x$. Analogously for $a \sqcap b$. Note that $a \sqcup a = a$ etc.

²²It follows that $a \sqcup b = \overline{\bar{a} \sqcap \bar{b}}$ and $a \sqcap b = \overline{\bar{a} \sqcup \bar{b}}$.

²³The phraseology is classical: the difference between a and \bar{a} warrants that the truth filter can pick exactly one of A and $\neg A$ as true.

²⁴In view of (iv), the left-right direction of (iii) is redundant. The following are derivable: (i) $a \sqcup b \in T$ iff $a \in T$ or $b \in T$, (ii) if $R\bar{a}a$ then $a \in T$, and (iii) if $a \in T$ and $a \leq b$, then $b \in T$.

- (a) Where $A \in \mathcal{W}$, $M \Vdash A$ iff $|A| \in T$.
- (b) Where $A, B \in \mathcal{W}$, $M \Vdash A \rightarrow B$ iff $R|A||B|$.
- (c) $M \Vdash \neg A$ iff $M \not\Vdash A$.
- (d) $M \Vdash A \wedge B$ iff $M \Vdash A$ and $M \Vdash B$.
- (e) $M \Vdash A \vee B$ iff $M \Vdash A$ or $M \Vdash B$.

Purists will restrict (c) to the case where $A \notin \mathcal{W}$ and will restrict (d) and (e) to the case where $A \notin \mathcal{W}$ or $B \notin \mathcal{W}$.

A model of Γ is a model that verifies all members of Γ . $\Gamma \vDash A$ (A is a semantic consequence of Γ) iff all models of Γ verify A ; $\vDash A$ (A is valid) iff every model verifies A .

The obvious *soundness* proof is left to the reader. *Completeness* is also easy. We start from a Γ and A for which $\Gamma \not\vDash_{\mathbf{PCR}} A$. We consider a sequence $L = \langle B_1, B_2, \dots \rangle$ of all formulas of \mathcal{L}^1 .²⁵ Next we define

$$\begin{aligned} \Delta_0 &= Cn_{\mathbf{PCR}}(\Gamma) \\ \Delta_{i+1} &= \begin{cases} Cn_{\mathbf{PCR}}(\Delta_i \cup \{B_{i+1}\}) & \text{if } A \notin Cn_{\mathbf{PCR}}(\Delta_i \cup \{B_{i+1}\}) \\ \Delta_i & \text{otherwise} \end{cases} \\ \Delta &= \Delta_0 \cup \Delta_1 \cup \dots \end{aligned}$$

The rest of the proof proceeds as for **PC**: from Δ one defines a structure and shows that this is a **PCR**-model.

One first defines relevant equivalence classes for arrow free formulas: $B \in [A]$ iff B is equivalent to A in view of the following equivalences: $A \leftrightarrow A \vee A$, $A \leftrightarrow A \wedge A$, $(A \vee B) \leftrightarrow (B \vee A)$, $(A \wedge B) \leftrightarrow (B \wedge A)$, $((A \vee B) \vee C) \leftrightarrow (A \vee (B \vee C))$, $((A \wedge B) \wedge C) \leftrightarrow (A \wedge (B \wedge C))$, $((A \vee B) \wedge C) \leftrightarrow ((A \wedge C) \vee (B \wedge C))$, $((A \wedge B) \vee C) \leftrightarrow ((A \vee C) \wedge (B \vee C))$, $\neg\neg A \leftrightarrow A$, $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$, and $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$, and the rule if $A \leftrightarrow B$ then $(A \wedge C) \leftrightarrow (B \wedge C)$ and $(A \vee C) \leftrightarrow (B \vee C)$. Next one defines:

- $S = \{[A] \mid A \in \mathcal{W}\}$
- \leq is the transitive closure of the set of ordered pairs $\{(x, y) \mid x, y \in S; \text{ for some } A \text{ and } B, A \wedge B \in x \text{ and } A \in y \text{ or } A \in x \text{ and } A \vee B \in y\}$
- for each $a \in S$, \bar{a} is the $x \in S$ such that, for some $A \in \mathcal{W}$, $A \in a$ and $\neg A \in x$
- $R = \{([A], [B]) \mid A \rightarrow B \in \Delta\}$
- $v(A) = [A]$
- $T = \Delta \cap \mathcal{W}$

and shows that $\langle S, \leq, \bar{}, R, v, T \rangle$ is an algebraic **PCR**-model. The demonstration is standard and left to the reader.

²⁵The proof becomes a trifle easier if it is required that, for all $A, B \in \mathcal{W}$, $\neg(A \rightarrow B)$ precedes in L every other formula that has $A \rightarrow B$ as a subformula. This guarantees that $A \rightarrow B \in \Delta$ iff $A \rightarrow B \in Cn_{\mathbf{PCR}}(\Gamma)$.

9 Embarrassing Strength?

The reader may have noticed that the relevant implication of **PCR** is unusually strong. Thus $A \rightarrow B \vdash_{\mathbf{PCR}} A \rightarrow (A \wedge B)$ and $A \rightarrow B \vdash_{\mathbf{PCR}} (A \wedge C) \rightarrow (B \wedge C)$. The reason for this strength is not difficult to understand. Given that theorems are seen as provable ‘from’ the empty set, we have $\emptyset \vdash_{\mathbf{PCR}} A \rightarrow A$ and hence $A \rightarrow B \vdash_{\mathbf{PCR}} A \rightarrow A$ by monotonicity. Moreover $A \rightarrow B \vdash_{\mathbf{PCR}} A \rightarrow B$ by reflexivity. But $A \rightarrow B, A \rightarrow A \vdash_{\mathbf{PCR}} A \rightarrow (A \wedge B)$ —the corresponding inference holds in nearly every relevant logic. So one obtains $A \rightarrow B \vdash_{\mathbf{PCR}} A \rightarrow (A \wedge B)$ by transitivity.

We knew all along that the inference relation is not relevant. In the present context, the absence of relevance derives from the fact that theorems are seen as derivable from the empty set and hence, by monotonicity, as derivable from every set. In view of this insight, that $A \rightarrow B \vdash_{\mathbf{PCR}} A \rightarrow (A \wedge B)$ and similar inference statements hold true is not a problem. This is especially so because **PCR** is obviously meant to be at most applicable in contexts in which **PC** is applicable.

Some may suspect that the non-relevant character of the inference relation makes the implication non-relevant. I think this is not so. For one thing it is easily seen that all **PCR**-theorems of the form $A \rightarrow B$ are tautological entailments and vice versa. In more technical terms,

$$\emptyset \vdash_{\mathbf{PCR}} A \rightarrow B \text{ iff } A \rightarrow B \text{ is a tautological entailment.}$$

Moreover, there is a set of statements that connect complex first-degree entailments to simpler ones. Their form is: for certain $A, B \in \mathcal{W}$, a formula of the form $A \rightarrow B$ holds true just in case a set of simpler implicative formulas hold true. Thus $\neg(A \wedge B) \rightarrow (C \wedge D)$ holds true just in case $\neg A \rightarrow C$, $\neg B \rightarrow C$, $\neg A \rightarrow D$, and $\neg B \rightarrow D$ hold true. The interesting point is that statements of the form

$$A \rightarrow B \text{ iff } (A_1 \rightarrow B_1) \wedge \dots \wedge (A_n \rightarrow B_n)$$

are valid in **PCR** iff they are valid for first-degree entailments—see [1, §19].

All this reveals that, whenever $A \rightarrow B$ is stronger than one would expect by standard relevance lights,²⁶ then $A \rightarrow B$ is relevantly equivalent to a conjunction of formulas $C \rightarrow D$ and each of these formulas is an unsuspect consequence of Γ or a theorem of **PCR**—all theorems of **PCR** are unsuspect (as theorems of **PCR**). In view of the above, it readily follows that: given a set Γ of formulas of the form $A \rightarrow B$ and where Δ is the set of tautological entailments, $\Gamma \vdash_{\mathbf{PCR}} A \rightarrow B$ iff $A \rightarrow B$ can be obtained from (zero or more members of) $\Gamma \cup \Delta$ by (primitive and derivable) rules of the formalization of tautological entailments. The upshot is that the oddities derive from the non-relevant inference relation but do not locate any problems with the relevant implication. Is this situation completely satisfactory? It seems wise to consider the question in the more general setting of the next section.

²⁶Not all relevantists agree about these lights, as may be seen from [6, §3.6], but the statement in the text holds even true if the ‘lights’ are identified with **E** or **R**.

10 A General Recipe and A Lesson

There is a general recipe for combining the **PC**-consequence relation with a relevant logic **L** of which all **PC**-valid formulas are theorems. The recipe is general, viz. it works fine even if nested relevant implications occur in **L**-theorems.²⁷ Let us denote the particular combination as **PC+L**.

First, there is a general recipe that proceed in terms of the semantics and which I shall briefly outline. The semantic consequence relation for the combination of **PC+L** may be defined as follows: $\Gamma \vDash_{\mathbf{PC+L}} A$ iff, in the Routley-Meyer semantics for **L**, A is verified by every model that verifies all members of Γ . To see that this is correct, note that the set of formulas verified by a **L**-model is closed under the **PC**-consequence relation.²⁸ From this follows (i) that **L**-models verify all **PC**-theorems and (ii) that the **PC+L**-semantic consequence relation assigns every theorem as a consequence to every set of formulas and assigns every formula as a consequence to every negation of a theorem.

The Fitch-style approach is at least as perspicuous. Anderson and Belnap provided Fitch-style rules for proving theorems of relevant logics. In these, each formula has an index set which is a (proper or improper) subset of $\{1, 2, 3, \dots\}$. In order to provide Fitch-style rules for the consequence relation that Anderson and Belnap have in mind—see Section 3—it is sufficient to extend the set of rules for proving theorems with a premise rule: Prem: a formula may be introduced with index set $\{0\}$. A Fitch-style proof in **E** that A_1, \dots, A_n entail(s) B is then defined as a proof written by application of the aforementioned rules, in which at most A_1, \dots, A_n are introduced by the rule Prem, and in which B occurs with index set $\{0\}$ in the main proof.²⁹ Similarly for **R** and other relevant logics.

In order to obtain Fitch-style proofs for the combination of **PC** with a relevant logic in which all **PC**-theorems (as well as other formulas) are theorems, two modifications are sufficient. First, one modifies the Prem rule, introducing premises with the empty index set. Next, one adds the material Modus Ponens rule for formulas with empty index set: to derive B_\emptyset from A_\emptyset and $A \supset B_\emptyset$. A Fitch-style proof of $A_1, \dots, A_n \vDash B$ is then defined as a proof written by application of the aforementioned rules, in which at most A_1, \dots, A_n are introduced by the rule Prem, and in which B occurs with index set \emptyset .

What this comes to is that premises and theorems of logic are put on the same foot *and* that the set of premises and theorems of logic is closed under material Modus Ponens (or Disjunctive Syllogism). Some may find it more transparent to retain the index set $\{0\}$ for premises and actually it is instructive to consider this version of the Fitch-style proofs. Let us call it *Version Two*. In Version Two, the original Prem rule is retained, but two rules are added:

$$\text{To derive } A_{\{0\}} \text{ from } A_\emptyset. \quad (4)$$

$$\text{To derive } B_{\{0\}} \text{ from } A_{\{0\}} \text{ and } A \supset B_{\{0\}}. \quad (5)$$

A Fitch-style proof of $A_1, \dots, A_n \vDash B$ is defined as a proof written by application

²⁷In terms of the recipe, **PCR** may be seen as the combination of **PC** and first-degree entailments.

²⁸It holds indeed that, for every **L**-model, world 0 (respectively every world in Z) is consistent and negation complete.

²⁹That this is correct is completely obvious if one introduces the restriction that formulas with index set $\{0\}$ are not reiterated within subproofs. Next one shows that no new consequences can be derived by removing the restriction.

of the aforementioned rules, in which at most A_1, \dots, A_n are introduced by the rule Prem, and in which B occurs with index set $\{0\}$. Note that the added rule (4) literally comes to: every theorem is a consequence of every premise set.

Both proof theories are equivalent and they are sound and complete with respect to the semantics outlined before. It is straightforward that, where \mathbf{L} is a relevant logic of which all \mathbf{PC} -theorems are theorems, $\mathbf{PC}+\mathbf{L}$ is the weakest logic that fulfils the following conditions: (i) the $\mathbf{PC}+\mathbf{L}$ -consequence relation is reflexive, transitive, and monotonic, (ii) if $\Gamma \vdash_{\mathbf{PC}} A$, then $\Gamma \vdash_{\mathbf{PC}+\mathbf{L}} A$, (iii) if A_1, \dots, A_n \mathbf{L} -entail B —see Section 3—then $A_1, \dots, A_n \vdash_{\mathbf{PC}+\mathbf{L}} B$, and (iv) if A is a theorem of \mathbf{L} , then $\emptyset \vdash_{\mathbf{PC}+\mathbf{L}} A$.

It is clear at once that the combination is stronger than one might have expected. The reason for this is that the consequence relation is not relevant. The unexpected strength of \mathbf{PCR} is shared by every combined logic $\mathbf{PC}+\mathbf{L}$ in which the relevant logic \mathbf{L} has all \mathbf{PC} -valid formulas as theorems; for example one will obtain $A \rightarrow B \vdash_{\mathbf{PC}+\mathbf{L}} A \rightarrow (A \wedge B)$ and $A \rightarrow B \vdash_{\mathbf{PC}+\mathbf{L}} (A \wedge C) \rightarrow (B \wedge C)$. These hold for exactly the same reason as the corresponding \mathbf{PCR} statements and this reason was explained in the previous section. One of the consequences is this: $\mathbf{PC}+\mathbf{L}$ is a conservative extension of \mathbf{PC} ; as the $\mathbf{PC}+\mathbf{L}$ -consequence relation contains the \mathbf{PC} -consequence relation, it cannot possibly be a conservative extension of \mathbf{L} ; however, even for premises and conclusions that belong to $\mathcal{W}^\rightarrow - \mathcal{W}$, the $\mathbf{PC}+\mathbf{L}$ -consequence relation is stronger than the \mathbf{L} -consequence relation.

11 Peter’s Complaint

A correspondence on the consequence relations was going on between Peter Verdée and me while I was writing up the present paper—see also the acknowledgment footnote. This led to several discussions. During one of them, Peter argued that $A \rightarrow B \vdash A \rightarrow (A \wedge B)$ and $A \rightarrow B \vdash (A \wedge C) \rightarrow (B \wedge C)$ are unacceptable because they don’t hold in the relevant logic.

It is possible to devise combinations that agree with Peter’s intuitions. As $\Gamma \vdash_{\mathbf{PC}} p \vee \neg p$, it is unavoidable that the consequence relation of the combined logic assigns $p \vee \neg p$ as a consequence to every premise set. However, nothing requires that the combined logic assign implicative \mathbf{L} -theorems as consequences to every premise set. The relevant logic \mathbf{L} does not require it because its consequence relation does assign them so and \mathbf{PC} does not require it—the arrow does not even belong to the language of \mathbf{PC} .

Let \mathbf{L} be a relevant logic and let all \mathbf{PC} -theorems be \mathbf{L} -theorems. I now introduce a combination $\mathbf{PC} \oplus \mathbf{L}$. It validates all \mathbf{PC} -inferences as well as all \mathbf{L} -inferences but does not assign specific theorems of the relevant logic \mathbf{L} as consequences to all premise sets—so $A \rightarrow B \not\vdash_{\mathbf{PC} \oplus \mathbf{L}} A \rightarrow (A \wedge B)$ and $A \rightarrow B \not\vdash_{\mathbf{PC} \oplus \mathbf{L}} (A \wedge C) \rightarrow (B \wedge C)$. The logic $\mathbf{PC} \oplus \mathbf{L}$ may be defined from the Version Two Fitch-style formulation of $\mathbf{PC}+\mathbf{L}$ by replacing rule (4) by the rule EM: “to introduce $A \vee \neg A_{\{0\}}$ ” (for any A).

Every \mathbf{PC} -theorem A is a $\mathbf{PC} \oplus \mathbf{L}$ -consequence of every premise set. If all \mathbf{PC} -theorems are theorems of the relevant logic \mathbf{L} , this is so because every \mathbf{PC} -theorem is \mathbf{L} -equivalent to a conjunction of one or more formulas of the form $C \vee \neg C \vee D$. In other words, for every \mathbf{PC} -theorem A , there are formulas B_1, \dots, B_n such that $((B_1 \vee \neg B_1) \wedge \dots \wedge (B_n \vee \neg B_n)) \rightarrow A$ is a theorem of

L. So if A is a **PC**-theorem, a proof of $\Gamma \vdash_{\mathbf{PC} \oplus \mathbf{L}} A$ is obviously obtained as follows: (i) introduce each of $(B_1 \vee \neg B_1), \dots, (B_n \vee \neg B_n)$ with index set $\{0\}$ by **EM**, (ii) obtain their conjunction, (iii) prove the **L**-theorem and (iv) apply \rightarrow E (Modus Ponens for the arrow) to obtain A with index set $\{0\}$.

It is obvious from the Fitch-Style system that not all specific theorems of the relevant logic **L** (those that are not also theorems of **PC**) are **PC** \oplus **L**-consequences of every premise set. Indeed, there is no way in which all specific theorems of **L** can obtain index set $\{0\}$. This is the reason why, for example, $q \not\vdash_{\mathbf{PC} \oplus \mathbf{L}} p \rightarrow p$ obtains in general.

Of course, one would like a more embracing claim, but it is difficult to phrase one for all relevant logics. So let me consider some logics separately. The most straightforward claim is possible about the relevant logic **E**: no formula $A \rightarrow (B \rightarrow C)$ is a theorem of **E** if A is a ‘factual formula’—a formula that contains no arrow (and no necessity, which contextually abbreviates an arrow anyway). So if no member of Γ contains an arrow, then $\Gamma \cup \{B \vee \neg B \mid B \in \mathcal{W}^\rightarrow\} \not\vdash_{\mathbf{E}} A \rightarrow A$ and actually $\Gamma \cup \{B \vee \neg B \mid B \in \mathcal{W}^\rightarrow\} \not\vdash_{\mathbf{E}} A \rightarrow C$ for all A and C .³⁰ It follows that, if $\Gamma \subseteq \mathcal{W}^\rightarrow - \mathcal{W}$, then $\Gamma \vdash_{\mathbf{PC} \oplus \mathbf{E}} A$ iff $\Gamma \vdash_{\mathbf{E}} A$.

The situation is more complex for **R**, which has theorems like $p \rightarrow ((p \rightarrow q) \rightarrow q)$. So it is unavoidable that $p \vdash_{\mathbf{PC} \oplus \mathbf{R}} (p \rightarrow q) \rightarrow q$ and hence also that $\Gamma \vdash_{\mathbf{PC} \oplus \mathbf{R}} ((p \vee \neg p) \rightarrow q) \rightarrow q$ for all Γ (including \emptyset).³¹ So $((p \vee \neg p) \rightarrow q) \rightarrow q$ and many other **R**-theorems are **PC** \oplus **R**-consequences of every premise set. The situation is even ‘worse’ in **RM**. This logic has $A \rightarrow (A \rightarrow A)$ as a theorem and so $\Gamma \vdash_{\mathbf{PC} \oplus \mathbf{RM}} (p \vee \neg p) \rightarrow (p \vee \neg p)$ holds for every Γ . Moreover, if $\Gamma \vdash_{\mathbf{PC} \oplus \mathbf{RM}} A$, then unavoidably also $\Gamma \vdash_{\mathbf{PC} \oplus \mathbf{RM}} A \rightarrow A$.

How satisfactory is the proposed approach? Let me start with the bad news. We considered $p \rightarrow q \vdash_{\mathbf{PC} + \mathbf{R}} p \rightarrow p$ as objectionable because $p \rightarrow q$ and $p \rightarrow p$ strictly belong to the language of **R**—they are not formulas of the language of **PC**—and $p \rightarrow p$ is not a **R**-consequence of $p \rightarrow q$. We tried to repair this by forging a logic **PC** \oplus **R**, in which **R**-theorems are means to pass from premises to conclusions but are not consequences of every premise set (but in which **PC**-theorems are still consequences of every premise set). However, the proposed solution does not and *cannot* answer the objection completely. Indeed, some specific **R**-theorems such as $((p \vee \neg p) \rightarrow q) \rightarrow q$ are **R**-consequences of **PC**-theorems. So they are **PC** \oplus **R**-consequences of every premise set, including subsets of $\mathcal{W}^\rightarrow - \mathcal{W}$. Needless to say, $((p \vee \neg p) \rightarrow q) \rightarrow q$ is not a **R**-consequence of \emptyset , of $\{r \rightarrow s\}$, and so on. Note also that more unwanted properties follow, for example $r \rightarrow s \vdash_{\mathbf{PC} \oplus \mathbf{R}} (r \wedge ((p \vee \neg p) \rightarrow q)) \rightarrow (s \wedge q)$ and also $r \rightarrow s, p \vdash_{\mathbf{PC} \oplus \mathbf{R}} (r \wedge (p \rightarrow q)) \rightarrow (s \wedge q)$. And things are worse for the combination involving **RM**. We have $r \rightarrow s \vdash_{\mathbf{PC} \oplus \mathbf{RM}} (r \wedge (p \vee \neg p)) \rightarrow (s \wedge (p \vee \neg p))$ as well as $r \rightarrow s, p \vdash_{\mathbf{PC} \oplus \mathbf{RM}} (r \wedge p) \rightarrow (s \wedge p)$.³²

As suggested before, there is also good news. According to the relevance tradition, theorems of logic are not consequences of the empty set. While the **PC**-consequence relation requires that **PC**-theorems are consequences of every

³⁰The formulation is correct: the claim holds true even if B contains arrows.

³¹Indeed, $\emptyset \vdash_{\mathbf{PC} \oplus \mathbf{R}} p \vee \neg p$ and $p \vee \neg p \vdash_{\mathbf{PC} \oplus \mathbf{R}} ((p \vee \neg p) \rightarrow q) \rightarrow q$. So the transitivity of $\vdash_{\mathbf{PC} \oplus \mathbf{R}}$ gives us $\emptyset \vdash_{\mathbf{PC} \oplus \mathbf{R}} ((p \vee \neg p) \rightarrow q) \rightarrow q$ and the monotonicity of $\vdash_{\mathbf{PC} \oplus \mathbf{R}}$ gives us $\Gamma \vdash_{\mathbf{PC} \oplus \mathbf{R}} ((p \vee \neg p) \rightarrow q) \rightarrow q$.

³²For relevant logics in which some ‘factual’ formulas entail relevant implications, the bad news is worsened by the fact that the premise set is closed under the non-relevant **PC**-consequence relation. Thus $p \vee q, \neg p \vdash_{\mathbf{PC} \oplus \mathbf{R}} q$ and hence also $p \vee q, \neg p \vdash_{\mathbf{PC} \oplus \mathbf{R}} (q \rightarrow r) \rightarrow r$. And so on.

premise set, including the empty set, $\mathbf{PC}\oplus\mathbf{L}$ enables one, in contradistinction to $\mathbf{PC}+\mathbf{L}$, to avoid that all *specific* theorems of the relevant logic \mathbf{L} are consequences of the empty set. So $\mathbf{PC}\oplus\mathbf{L}$ is half-hearted but at least it is so in a systematic way, siding with the classicists in that it extends the \mathbf{PC} -consequence relation, but siding with the relevantists in connection with the extension. However half-hearted, the systematicity of the approach has an immediate technical pay-off. Indeed, $p \vdash_{\mathbf{PC}\oplus\mathbf{L}} q \vee \neg q$ but $p \not\vdash_{\mathbf{PC}\oplus\mathbf{L}} q \rightarrow q$. Moreover, it is not difficult to prove that $Cn_{\mathbf{PC}\oplus\mathbf{L}}(\Gamma)$ is the smallest set Σ such that (i) $\Gamma \in \Sigma$, (ii) $Cn_{\mathbf{PC}}(\emptyset) \subseteq \Sigma$, (iii) if $A, A \supset B \in \Sigma$, then $B \in \Sigma$, and (iv) if $A \in \Sigma$ and $A \rightarrow B$ is a theorem of \mathbf{L} , then $B \in \Sigma$. From this, it is easily proved that $\mathbf{PC}\oplus\mathbf{L}$ is the weakest logic that fulfils the following conditions: (i) the $\mathbf{PC}\oplus\mathbf{L}$ -consequence relation is reflexive, transitive, and monotonic, (ii) if $\Gamma \vdash_{\mathbf{PC}} A$, then $\Gamma \vdash_{\mathbf{PC}\oplus\mathbf{L}} A$, and (iii) if A_1, \dots, A_n \mathbf{L} -entail B , then $A_1, \dots, A_n \vdash_{\mathbf{PC}\oplus\mathbf{L}} B$. So there is a natural sense in which $\mathbf{PC}\oplus\mathbf{L}$ is the weakest logic that contains both the classical \mathbf{PC} -consequence relation and the relevant consequence relation of \mathbf{L} . Note that $\emptyset \vdash ((p \vee \neg p) \rightarrow q) \rightarrow q$ is unavoidable in any logic that contains the \mathbf{PC} -consequence relation as well as the \mathbf{R} -consequence relation. Indeed, $\emptyset \vdash_{\mathbf{PC}} p \vee \neg p$ holds and $(p \vee \neg p) \rightarrow (((p \vee \neg p) \rightarrow q) \rightarrow q)$ is a theorem of \mathbf{R} .

To conclude this section, a bit of extremely good news. The ‘unwanted properties’ of $\mathbf{PC}\oplus\mathbf{R}$ concern nested implications.³³ So if we can formulate a logic, call it \mathbf{PCR} , that relates to \mathbf{PCR} in the same way as $\mathbf{PC}\oplus\mathbf{R}$ relates to $\mathbf{PC}+\mathbf{R}$, \mathbf{PCR} would not have those ‘unwanted properties’. The simplest Fitch-style proofs I found are longwinded (if natural) and the matter is presumably not worth the reader’s attention. However, a very simple algebraic semantics is available: replace R1 by the following clause: “If $a \leq b$ and Rbc , or Rab and $b \leq c$, then Rac .” This reveals the important difference between \mathbf{PCR} and \mathbf{PCR} . \mathbf{PCR} conflates logical implications and implications derived from (in the relevant sense) the premises while \mathbf{PCR} keeps them apart, but leaves room for the logical strengthening of the *implicans* and for the weakening of the *implicatum*. Do not underestimate the impact of the change. For example, R4 remains unchanged but $R(a \sqcup b)c$ does not follow from Rac and $b \leq c$. For the inference relation, the effect is that $A \rightarrow B \vdash_{\mathbf{PCR}} (A \wedge C) \rightarrow (B \vee D)$, but that $A \rightarrow B \not\vdash_{\mathbf{PCR}} (A \wedge C) \rightarrow (B \wedge C)$.

This finishes my comments on Peter’s complaint. Meanwhile, however, I understand that Peter is following an approach for combining \mathbf{PC} with a relevant implication and that this approach is very different from the one I proposed in this section.

12 Concluding Remarks

The logic \mathbf{PCR} has very simple Fitch-style proofs. Its semantic characterizations are a trifle more complicated. The tableau method is an extremely easy and perspicuous decision method. A conceptual matter is that the arrow of \mathbf{PCR} is obviously relevant, notwithstanding the fact that the inference relation is not. In view of all this, it is hard to see a reason for not making students familiar with the properties of this relevant implication. The proofs will give

³³The situation is different for $\mathbf{PC}\oplus\mathbf{RM}$.

them a feel for the distinction between material implication and relevant implication. As soon as relevant implication is available, negations of implicative sentences can be expressed without paradox.

The logic **PCR** is more interesting than **PCR** from a theoretical point of view. It gives the specific theorems of the relevant logic a different status than the theorems of **PC**. The specific relevant theorems are not seen as universal truths, but as means to derive the consequent from the antecedent and to derive the negation of the antecedent from the negation of the consequent. The price to pay is a more sophisticated proof theory and a more sophisticated semantics.

The two preceding sections were intended to offer general insights. In a sense, the results on **PCR** and **PCR** are just specific applications of those insights (to the case where **PC** is extended with the logic of tautological entailments). The difference between the two approaches turns on the status of theorems in relevant logics as opposed to classical logic, here **PC**. Note that there is no way to adjust the status of **PC**-theorems without adjusting **PC**-derivability—every set of rules that is sufficient to justify $\Gamma \vdash_{\mathbf{PC}} A$ whenever A and all members of Γ are **PC**-contingent, also justifies $p \vdash_{\mathbf{PC}} q \vee \neg q$.³⁴ So, for most relevant implications, combining **PC** with the relevant implication will not lead to a conservative extension of the relevant logic in any interesting fragment of the language.

For all the positive results presented in this paper, there is also a pessimistic message. Many consider relevant logics as too drastically remote from classical logic. At the same time, the properties of relevant implications seem so attractive that few sensible people would like to lose operators having those properties in exchange for retaining classical logics, in general or even for some specific contexts only. This naturally leads to the aim of combining classical logic with relevant implications. However, as the present paper shows, there are reasons for pessimism with respect to precisely this aim. So we need a new approach. The approach should retain **PC**, or rather **CL**, at least for specific contexts. At the same time the approach should allow one to retain the nice properties of relevant implications while avoiding the clutter arising from the approaches presented here. As we have seen, the outlook of the Anderson-and-Belnap relevance logics differs heavily from that of classical logic. The outlook required by the suggested new approach will presumably need to be very different from both classical logic and today's relevance logics. So be it. That is what makes the problem intriguing and fascinating. All this, however, does not diminish the present pedagogical and practical value of a system like **PCR**.

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³⁴The reference to *rules* is essential. If this is replaced by a reference to *instructions*, the situation becomes at once different, as is shown in [4].

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