Kepler's near discovery of the sine law: A qualitative computational model.

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Abstract. Computational models offer an excellent tool for the study and analysis of scientific discovery processes. The study of failures provides an insight into the history and philosophy of science as valuable as the study of successful discoveries. Using a computational model I analyzed Kepler's approach in formulating a quantitative law for refraction. Although Kepler ultimately failed in discovering the sine law, the model shows that his basic hypothesis as well as his approach by geometrical reasoning was a correct one. This went largely unnoticed by commentators on the history of optics. Based on this analysis I provide new evidence that Descartes and Snell found in Kepler's main hypothesis everything needed to deduce the sine law by pure geometrical reasoning. Our computational model is based on geometrical knowledge as contrasted with previous quantitative approaches. It has been implemented as a Prolog program.

1. Introduction

Recent studies on discovery, in the history and philosophy of science, show a shift of focus from the product (the discovery) to the process. Instead of asking "Who was the first to discover the sine law of refraction?" and "How did he come to the idea?", philosophy of science becomes more interested in the necessary conditions for, and the process of scientific discovery. One tends to move away from the romantic idea, popular in the nineteenth century and the beginning of the twentieth century, of the isolated bright mind individually bringing about giant steps in the progress of science. The scientific practice of a community of scientists, often with opposing and conflicting views and interpretations of basic concepts, becomes the new object of study. Within this new view on scientific discovery, failures as well as successes of scientific practice need to be studied. Given a common context of scientific knowledge, the failure to discover something by one scientist may learn us as much as the success of another. In this study we have attempted to model the historic process and ultimate failure of Johannes Kepler in formulating a quantitative law for the relation between the angles of incidence and refraction of light rays in different media. We have formalized Kepler's approach of generating and testing hypotheses and implemented his method in a computational model. Using this model we have been able to infer two important results: 1) Despite his failure, Kepler's approach was a correct one. It did generate promising hypotheses which could have led to the discovery of the sine law. 2) The model provides strong arguments for the claim that Kepler's method was Snell's as well as Descartes's path to discovery.

In this paper we cannot go into the history of optics[1], but will give a detailed account of Kepler's study of refraction based on his *Ad Vitellionem paralipomena* of 1604. Thanks to his diligent reporting style Kepler's writings are grateful objects of study and allow us to reconstruct his path of reasoning in detail. We will first sketch the historical context and then critically review a previous model of the discovery of the sine law of refraction. Our model of Kepler's discovery process is based on his central hypothesis that the optical densities of two media are proportional to some ratio of line segments in a geometrical representation of refraction. We will show how a computational model is able to derive the correct ratio from this central hypothesis, corresponding to the sine law.

2. Historical context

The discovery of the sine law is only a small fragment of a theory on refraction. The fact that the sine law was discovered independently by Thomas Harriot (around 1600), René Descartes (around 1620), Willebrord Snell (between 1621 and 1625) and Pierre de Fermat (1662) in a period of a few decades is significant.[2] It shows that the context of optical knowledge in natural philosophy at the beginning of the seventeenth century was the right one. Therefore, in order to understand Kepler's endeavours it is important to briefly sketch the historical context in which he operated. He could rely on a vast body of knowledge on optics which was founded centuries before, by Euclid (*Optics*, ca. 300 B.C.) and Hero of Alexandria (*Catoptrics*, ca. 60 A.D.). Euclid established the rectilinear propagation of the visual ray which was a prerequisite for the geometrical treatment of vision.[3] Visual rays reflect and refract in a single plane, which allows us to apply the laws of plane geometry. Refraction was considered to be caused by a change of media traveled through by the visual ray. Ptolemy (*Optics*, ca. 150 A.D.) conducted careful experiments to measure and quantify refraction between air and water, air and glass, and glass and water. He observed that, when traveling from a rare to a denser medium, rays are refracted towards the normal. Using his measurements he could predict the angle of refraction for a given angle of incidence within given media. Despite Ptolemy's sophisticated analysis of refraction and geometrical treatment of image location on plane and spherical surfaces, he did not come at a physical

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[1] Lindberg, 1976, is the most authorative study on the development of optics since the Arabs. Sabra, 1981, is a trustworthy source for seventeenth century optics. For a short overview specific to the sine law see Heeffer, 2003.

[2] The dating of Descartes's dicovery is based on his *fundamentum inventi mirabilis* (fundamental principles of a wonderful discovery) reported in the *Olympica*, part of his little notebook which only survives in a translation by Adrien Ballet of 1691. This early dating is controversial and is rejected by A. Mark Smith (1987) and William Shea (1991). However, Gaukroger (1995) defends the thesis that Descartes is refering to the sine law in the *Olympica*. Other writing of Descartes around this period show evidence of his study of Kepler's *Ad Vitellionem paralipomena*. Our model provides further support for a dating as early as 1620.

[3] For the Greeks, light emanating from the eye strikes the external objects that we observe. They therefore speak of visual rays instead of light rays. This extramission theory was replaced by intromission after Alhazen.

explanation of why light rays change their direction upon entering a medium with a different optical density. In the ten centuries following Ptolemy the most important contribution to the answer of this question came from the Arab scholar Abū Alī Al-Hasan Ibn Al-Hasan Ibn al-Haytham, more commonly referred to as Alhazan. Alhazan's *Optics* or *Kitāb al-manāzir* in Arabic had a profound influence in medieval Europe and on seventeenth-century natural philosophers. Alhazen was the first to decompose reflected and refracted light rays into normal (vertical) and tangential (horizontal) components, which was an important step in geometrical optics. Alhazen's theory of light explains refraction in terms of resistance and speed that depend on the resisting media. Light rays that enter the second medium, perpendicular to the surface, are not refracted and the perpendicular is therefore the path of the least resistance. Light rays passing obliquely from a rare to a dense medium are, because of this reason, refracted towards the normal. By the end of the sixteenth century it had become clear that what was to be found, was a quantitative law describing some proportionality between the angles of incidence and refraction as a constant ratio.

Kepler's interest in optics was aroused by the solar eclipse of 10 July 1600. He studied the works of Alhazen, Witelo and Pecham and conducted many experiments. His *Ad Vitellionem paralipomena*, published in 1604, was a *status questiones* of all current knowledge on human vision and optics correcting many of the anomalies and mistakes he had found with the perspectivists. He spent considerable efforts in finding a qualitative law for refraction because of its importance in providing an exact measure of atmospheric refraction. The major contribution of his book is the retinal image paradigm based on the analogy with the *camera obscura* and his analysis of threefold refraction of light in the eye. In 1611, the *Dioptrice* was published, in which his knowledge of optics is applied to the design and usage of lenses. In both works Kepler gave a quantitative relation between the angles of incidence and refraction, which is nearly correct for rays near the perpendicular, but fails for wider angles.

3. The failure of quantitative models of scientific discovery

The only previous attempt at a computational model for the discovery of the sine law, I know of, is from the BACON team from Carnegie Mellon University.[4] BACON is to be situated within a research programme on the modelling of human problem solving, originating from the 1950s. Herbert Simon and Alan Newell take merit for the important demystification of the romantic conception of discovery and a revival of interest in scientific discovery. Their working hypothesis is that scientific discovery is not a specific activity in itself but a special case of problem solving. Thomas Nickles calls their approach "the neo-enlightment counterpart of universal Reason, a faculty that could in principle solve any (solvable) problem in any domain".[5] The strength of reducing discovery to problem solving is also its weakness. On the one hand it allows us to use the instruments of cognitive science and artificial intelligence, mostly state-space search, and provide models for some

[4] The best summary of the methods and results of BACON is in Langley, Simon, Bradshaw and Zytkow, 1987. The "discovery" of the sine law by BACON.4, is treated on pp. 141-145 and of BACON.5 on pp. 174-178.

[5] Nickles, 1994, p. 283.

simple quantitative laws. On the other hand, reducing scientific discovery to heuristic search is for most of the interesting cases an excessive over-simplification. An illustration of such simplification is their model for the discovery of the sine law. BACON.4 is provided with a table of 9 data lines which contains the sines of angles of incidence and refraction for three angles and combinations with three media. The program "discovers" that the ratio between the two sines is invariant for a given combination of media. In the case of BACON.5 the program is not given the sines but the projection of the ray to the normal, which is essentially the same. The problem with putting the notion of a quantitative invariance on par with scientific discovery is of course, that the data provided to BACON is giving away the discovery! Once you are looking for invariance in the ratio of sines, you have found the sine law. The real discovery is the hypothesis that ratio of optical densities corresponds to some geometrical proportion. The authors claim that the sine law is one of the "three instances where, in the actual history of the matter, the data – essentially the same data that were available to BACON – were interpreted erroneously before the "correct" law was discovered".[6] This is actually not the case. The data available to Kepler, Witelo and the mediaeval perspectivists writing on refraction were the tables of Ptolemy's *Optics*.[7] Ptolemy devised a bronze graduated disc constructed to measure refraction. A rotatable part, called *dioptron,* could be adjusted in steps of 10 degrees and the angle of the refracted ray could directly be measured by reading the marks in the quadrant. In this way Ptolemy compiled highly accurate tables with angles of incidence and measured angles of refraction for different combinations of media. The data that was used by Kepler and other philosophers of nature, should have been the input of BACON. It is easy to predict the result of the quantitative approach used by BACON. A polynomial curve fit using *Mathematica* on Ptolemy's data reveals the suspiciously nice quadratic relations between angles of incidence (i) and angles of refraction (r): $r = (33 - i^2) / 4$ for air/water, $r = (29 - i^2) / 4$ for air/glass and $r = (39 - i^2) / 4$ for water/glass. In tradition with astronomical tables, which did not reflect real observational data but which were generated by geometrical models, the refraction tables use constantly diminishing increments, which leads to this remarkable quadratic relation.[8] This formula would be the quantitative law that BACON would produce and not the sine law. The authors of BACON conclude their work with the words: "We would like to imagine that the great discoverers, the scientists whose behaviour we are trying to understand, would be pleased with this interpretation of their activity.."[9] I seriously doubt Kepler or Descartes would have been pleased with this unfair representation of their enquiries. A reconstruction of scientific discoveries is not possible without a serious study of the context and the history of scientific practice leading to the discovery. Such a study reveals the degree to which the discovery of the sine law is the result of a process that took several centuries. Let us look in some detail how Kepler conducted his experiments.

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[6] Langley, Simon, Bradshaw and Zytkow, 1987, p. 224.

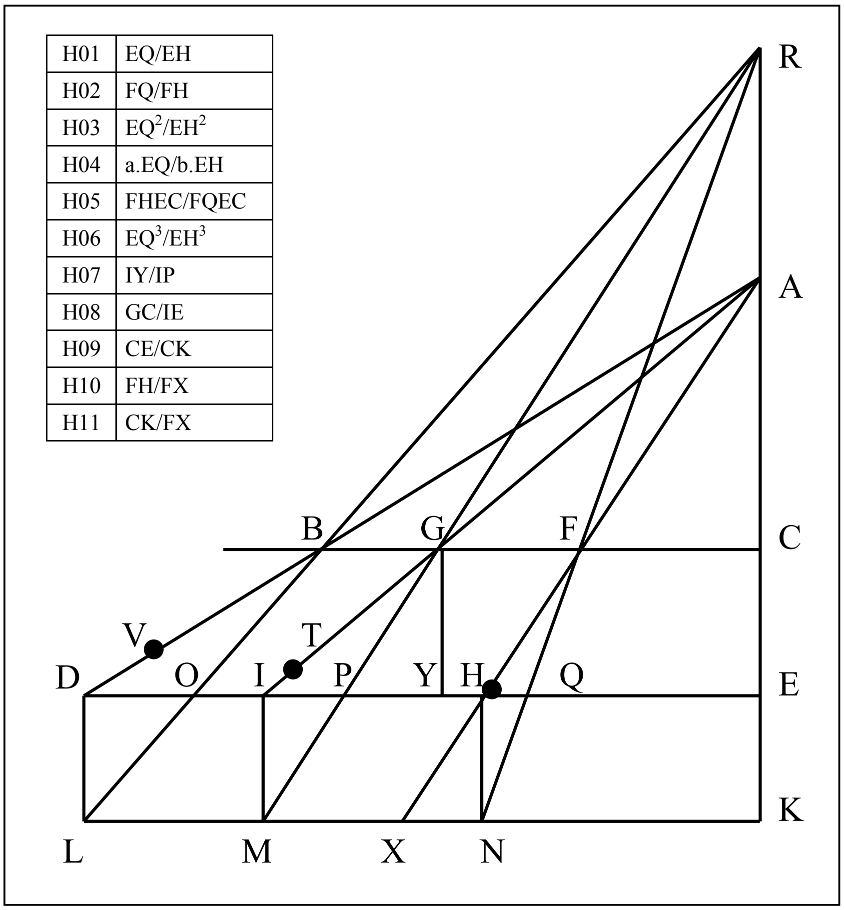
[7] Lejeune, 1956.

[8] Gilberto Govi, who published a Latin edition of Ptolemy's *Optics*, was the first to point out this relation: $r = ai - bi^2$ (Govi, 1885, pp XXII). A. Mark Smith, 1996, gives the relation $r = R-(n^2d_2-nd_2)/2$. Lejeune, 1956 warns that the figures may have been „adjusted" by previous translators "Il n'est pas absolument exclu que les tables aient été régularisées par un interpolateur grec our arabe", but Otto Neugebauer , 1957 gives the explanation of a constantly diminishing increments, typical for Babylonian astronomy.

[9] Langley, Simon, Bradshaw and Zytkow, 1987, p. 340.

4. Hypotheses formation by Kepler

Our analysis of Kepler's experiments is based on the fourth chapter of the *Paralipomena*. Crucial for his attempt at a quantitative law is one carefully designed drawing.[10]



H01	EQ/EH
H02	FQ/FH
H03	EQ^2/EH^2
H04	a.EQ/b.EH
H05	FHEC/FQEC
H06	EQ^3/EH^3
H07	IY/IP
H08	GC/IE
H09	CE/CK
H10	FH/FX
H11	CK/FX

The drawing is both a representation of empirical data as well as an ingenious analysis of refraction. Line *CB* represents the surface of a refracting medium (water). The three lines originating in point *A* are incidence rays. The lines originating in *R* represent the measured angles of refraction corresponding with the angles of the lines originating from *A*. The fact that the refraction lines all meet in point *R* is very unusual. A representation like this is the main step in discovering the sine law. Line *ED* is the bottom of the vessel with water. Line *KL* is a construction based on the crossing point of the perpendicular lines *HN*, *IM* and *DL* with the refracted rays. The perpendicular was also called *cathetus* and was used by the

[10] Figure 42 in Frish ,1859. Donahue, 2000 p. 101.

perspectivists for locating the reflected or refracted image.[11] This clever representation allows to reason about refraction using geometrical knowledge about rectangular triangles. Given this drawing, Kepler now formulates a central hypothesis:

> „Since density is obviously a cause of refraction, and refraction itself appears to be a kind of compression of light (i.e., towards the perpendicular), it comes to mind to ask whether the ratio of the media in the case of densities is the same as the ratio of the bottom of the spaces that light has entered into and strikes, first in an empty vessel, and then one filled with water."[12]

He then investigates several ratios between line segments in the drawing, which I have summarized in the table, included with the figure. His formulation only refers to H01, the ratio between *EQ* and *EH*, but then extends the hypothesis to other ratios (numbered H02 to H11). His belief is that one of these ratios corresponds to the "ratio of the media in the case of the densities", which we would now call the refraction index. After some pages of discussion he dismissed all eleven hypotheses as "refuted by experience, not to mention other procedures of enquiry".

The central hypothesis

Our model is based on Kepler's general hypothesis. In fact, the hypothesis he uses is more general than in the quotation above, because more ratios are investigated than those on the bottom of the vessel. A fair reformulation of Kepler's central hypothesis would be the following:

> "The ratio of the optical densities of two media is proportional to some ratio of two line segments in the geometrical representation of a light ray traversing the two media, the first line segment related to the angle of incidence, the second to the angle of refraction."

Implementing this central hypothesis in our computational model, the latter generates all of Kepler's hypotheses H01 to H11, as well as many more. We notice from the table that, with exception of H08 and H11, in all ratios the two line segments have one point in common. If we add the restriction in the starting hypothesis, that the line segments used in the ratios, should start in the same point,

[11] The cathetus rule for locating images is discussed in Dupré 2002, 125-141. Dupré writes: "this path of discovery [of the sine law] was not open to Kepler, because he rejected the cathetus rule as a way of finding the image. Eventually, Kepler considered the search for a law of refraction from the determination of the location of images by the cathetus rule to be in vain" (p. 134). Our model precisely shows that Kepler's original hypothesis could have led to the sine law.

[12] Kepler 1604, Ch. 4, § 2, Frish, 1859 Bd II, p. 182: "Cum ergo densitas plane sit in causa refractionum, et refractio ipsa compressio quaedam videtur lucis, utpote ad perpendicularem, subiit animum inquirere, an quae proportio mediorum causa densitatis eadem sit proportio fundi spatiorum, quae lux primum in vacuum vas, dein aqua superfusa, introgressa feriat." Translation by Donahue 2000, p. 102.

the model generates only nine (linear) instances. These ratios cover all of Kepler's except the special case H07.

The next step is to either prove or disprove the generated hypotheses. Kepler's approach is to eliminate hypotheses by deduction or observation. Most hypotheses can be refuted deductively, by simple geometrical reasoning. For example, H11 states that the ratio *CK*/*FX* remains the same for varying angles of incidence. This is evidently not the case. As the angle of incidence increases, the length of *FX* will increase while *CK* remains the same. Therefore the ratio *CK*/*FX* decreases and hypothesis H11 is refuted. The tragedy of this case study is that Kepler succeeded in formulating both a suitable representation of the problem and the correct hypothesis that some geometrical proportion corresponds to the refraction index. He failed in identifying the correct ratio. Both *FR*/*FA* and *FN*/*FH* correspond to the ratio of optical densities of the two media. These ratios can be proved to be constant by geometrical reasoning. We can use line segment *FC* to establish a relation between the angles of incidence and refraction as it is the same side of the right-angled triangles *FRC* and *FAC*. This unfortunate oversight was Kepler's failure in discovering the sine law. Both Snell and Descartes have read the *Paralipomena* and undoubtedly found here their main inspiration for the sine law. Snell's formulation was based on a ratio of cosecants equaling *FH*/*FN*.[13] Descartes never mentioned his sources and took care not to reveal his path of discovery of the sine law. Several authors have formulated hypotheses on how Descartes came to his discovery. In an early study by Kramer, it was suggested that Descartes hit upon the sine law through his study of conic sections, in particular the problem of the anaclastic curve.[14] Others, such as William Shea believe that Descartes used the sine law to solve the anaclastic.[15] Shea argues that the demonstration of a refractometer, presented by Descartes in a letter to his lens cutter Ferrier, as the procedure leading to the sine law. Given Kepler's analysis as sketched above and the fact that Descartes called Kepler "my first teacher in optics"[16] I consider it most likely that Descartes found in Kepler's drawing and his main hypothesis everything needed to deduce the sine law by pure geometrical reasoning as early as 1620.

5. Computational model

Our computational model generates hypotheses in the form of candidate ratios and tests their invariance by applying trigonometry to the three angles of incidence and refraction. The model is implemented as a Prolog program. Our main intention is to demonstrate the validity of Kepler's main hypothesis rather than developing a model to perform rigorous geometry theorem proving. We therefore took a pragmatic approach and implemented only the functions necessary for the task. Our approach can best be compared with the theorem prover for Euclidean plane

[13] Volgraff, 1918, p. 21b.

[14] Kramer 1882, p 256-8 and note 39.

[15] Shea 1991: „Once Descartes had found the sine law, the next step was to use it to construct lenses that would bring all incoming parallel rays into focus, thereby yealding the anaclastic." p 157.

[16] Descartes to Mersenne, 31 March 1638 and also in Descartes to Mersenne, 13 mei 1638: „..ce qui n'empêche pas que je n'avoue que Kepler a été mon premier maître en optique, et qu'il est celui de tous hommes qui en a le plus su par ci-devant." Cousin, Bd VII, p 161.

geometry by Coelho and Cotta (1988, 88-97). Our Prolog program consists of three parts.

5.1 Representation

The first module contains a representation of Kepler's figure 42. We have chosen not to use coordinates as is done in many geometry theorem provers. The specific nature of refraction makes it impossible to place all point on integer coordinates. For the purpose of this application, we must be able to define points, line segments and triangles and relate them to the three angles of incidence and refraction. Prolog predicates are used to describe all points and their relation to one of the three rays. Points are named by a letter as in the figure, e.g. `point(1,f)` represents point F on ray 1. Line segments are described by a binary predicate; `cn(f,h)` designates the connection of points *F* and *H* by a straight line. The reformulation of Kepler's main hypothesis distinguishes points related to the angle of incidence and angle of refraction. This is represented by the predicate `rel`, thus `rel(a,h)` means that point *H* is related to incidence point *A*. Some points in the drawing, such as F are related to both incidence and refraction point. Also the points *C*, *E* and *K* on the cathetus are considered to be related to both incidence and refraction. Finally we have to define right-angled and similar triangles. The predicate `rt(X,Y,Z)` represents a right-angled triangle with *X* as the point of the right angle and *Y-X* as the opposite of *Z*. An example of similar triangles are *HFN* ~ *AFR*, represented as `similar(h,f,n,a,f,r).` We adopted a careful choice of canonical naming to avoid combinatorial problems in matching line segments, angles and triangles. Without canonical naming, the representation of the fact that *HFN* and *AFR* are similar, would require 72 clauses to cover all naming variations. The program takes care of generating the possible variations at the moment when a fact is retrieved from the database during the hypothesis generation or proof.

5.2 Hypotheses generation

A second module implements the hypotheses generation. Kepler looked for ratios of line segments related to the angle of incidence and angle of refraction. The top-level rule of the hypotheses generation in our program is as follows:

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ratio(X, Y1-Y2, Y1-Z2) :-
          line(X, a, Y1-Y2),
          line(X, r, Y1-Z2), Y2 \== Z2.
```

A candidate ratio for a given angle `X`, consists of two line segments, originating from the same point *Y1* and connecting to (different) points related to the angle of incidence and angle of refraction. Thus, for angle 1 and for example point *H*, the predicate `ratio` will generate the candidates *HN/HX* and *HE/HN*. The tracing of lines of connected points is implemented by the predicate `line`. In addition to Kepler's hypothesis H09 our computational model also generates the correct ratios *FH*/*FN* and *FA*/*FR*.

5.3 Hypotheses testing

A third module tests the nine generated hypotheses through geometrical reasoning. A hypothesis is considered correct if the corresponding ratios can be proved to be invariant for the three angles of incidence and refraction. It is considered faulty if this proof fails. Proving the invariance is based on geometrical reasoning, for which we have to introduce some knowledge about trigonometry. Our program has knowledge about the six trigonometric functions, although not all are used. The sine of angle *YZX* of the right-angled triangle *XYZ* with *X* as the right angle, is defined as `sin(X,Y,Z,Y-X,Y-Z).` The line segments *Y-X* and *Y-Z* represent the opposite and the hypotenuse. The sine of the angle at point Y is the ratio between these two line segments. The ratios *FA*/*FR* can be proved invariant in a direct way. The hypothesis *FH*/*FN* is found to be correct by proving its equivalence to the previous ratio. The direct proof uses the equivalence of the adjacent side *FC* in the sines of the angles *AFC* and *RFC* and thus establishes the invariance of *FA*/*FR*, *GA*/*GR* and *BA*/*BR*. Once the invariance is proved, the ratio is reformulated in terms of the sines of the angles of incidence *FAC* and refraction *FRC* (see table 1).

$$FA.\sin(FAC) = FR.\sin(FRC)$$

$$GA.\sin(GAC) = GR.\sin(GRC)$$

$$\frac{\sin(FAC)}{\sin(FRC)} = \frac{FR}{FA} = \frac{FN}{FH}$$

Table 1: ratios leading to the sine law

6. Conclusion

A computational model proves to be an excellent tool in reconstructing historical cases and provides an insight in the methods, successes and failures of scientific discovery. A detailed analysis of Kepler's method of investigation into refraction reveals that at some point he formulated the correct hypothesis for finding the sine law. This was unnoticed by commentators for centuries. I believe that his approach was not overlooked by Descartes and Snell and that Kepler's *Paralipomena* was their most important source of inspiration in formulating the law.

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