NEGATIVE NUMBERS AS AN EPISTEMIC DIFFICULT CONCEPT: SOME LESSONS FROM HISTORY

Albrecht HEEFFER[1]
Center for Logic and Philosophy of Science
Ghent University, Blandijnberg 2,
B-9000 Ghent, Belgium
albrecht.heeffer@ugent.be

ABSTRACT

Historical studies on the development of mathematical concepts will serve mathematics teachers to relate their students' difficulties in understanding to conceptual problems in the history of mathematics. We argue that one popular tool for teaching about numbers, the number line, may not be fit for early teaching of operations involving negative numbers. Our arguments are drawn from the many discussions on negative numbers during the seventeenth and eighteenth centuries from philosophers and mathematicians as Arnauld, Leibniz, Wallis, Euler and d'Alembert. Not only the division by negative numbers poses problems for the number line, but also the very idea of quantities smaller than nothing has been challenged. Drawing lessons from the history of mathematics we argue for the introduction of negative numbers in education within the context of symbolic operations.

> Mon enthousiasme pour les mathématiques avaient peut-être eu pour base principale mon horreur pour l'hypocrisie; l'hypocrisie à mes yeux, c'était ma tante Séraphie, Mme Vignon, et leurs prêtres. Suivant moi, l'hypocrisie était impossible en mathématiques, et, dans ma simplicité juvénile, je pensais qu'il en était ainsi dans toutes les sciences où j'avais ouï dire qu'elles s'appliquaient. Que devins-je quand je m'aperçus que personne ne pouvait m'expliquer comment il se faisait que: moins par moins donne plus?
>
> (From *The Life of Henry Brulard* by Stendhal, 1890)

1. Against the number line

The number is line currently one of the important tools for teaching basic arithmetical concepts such as natural and real numbers in primary and secondary education. Hans Freudenthal (1983, 101) calls this mental object a "device beyond praise" and considers it a preferred vehicle to teach negative numbers (ibid, 437). In many countries the ordering of negative numbers by means of the number line is taught by the fifth grade (Howson, Harries and Sutherland, 1999). Despite its wide acceptance, the use of the number line in mathematics education is rather new. It seems to have originated in the 1950's.[2] Max Beberman, credited for many innovations in math teaching, used the earlier term 'number scale': "In teaching subtraction of signed numbers, I first draw a number scale" (Beberman and Meserve, 1956). However, not everyone is convinced of the benefits of using the number line for teaching negative numbers in primary education. In fact, the

[1] Fellow of the Research Foundation Flanders (FWO) Belgium.
[2] Several web sites reconstructed its history: Jeff Miller, http://members.aol.com/jeff570/. and Patt Ballew http://www.pballew.net/mathbooks.html are the most useful.

very teaching of operations on negative numbers is no longer allowed in basic education in Belgium. Negative numbers can only be used in “concrete situations”. The examples provided are the floors of a building and the temperature scale. The concept of an isolated negative number is an intrinsic difficult concept. Negative numbers emerged in history within the context of symbolic algebra. We share the view of the Belgian education program that the concept is best taught in secondary education and more specifically within an algebraic context. We will develop some arguments from the history of mathematics against the use of the number line for teaching about negative numbers.

But even the question of the historical acceptance of negative numbers is problematic. We have argued elsewhere against the interpretation of negative solutions in two accounts on the history of the subject by Sesiano (1985) and Gericke (1996) (Heeffer 2007). Algebraic practice of solving linear problems has lead repeatedly to situations in which one arrives at “a negative value”. Before the sixteenth century, such solutions were consistently called ‘absurd’ or ‘impossible’. The abbacus master, convinced of the correctness of his algebraic derivations, could interpret the negative value in some contexts as a debt. This does not imply that he accepted the solution as a negative value. On the contrary, by interpreting the solution as a debt, he removed the negative. Only from the beginning of the sixteenth century onwards, we see the first step towards negative values, in the form of algebraic terms affected by a negative sign. The fact that negative solutions were considered absurd for several centuries of algebraic practice is of significance to the teaching of mathematics. When teachers are aware that isolated negative quantities formed a conceptual barrier for the Renaissance habit of mind, it prepares them for potential difficulties in the student’s understanding of the concept. The arguments we will expound below were advanced and discussed by philosophers and mathematicians of the seventeenth and eighteenth century. The continuous struggle during that period to get a grip on such an elusive concept as that of a negative quantity exemplifies the intrinsic epistemic difficulty of some elementary concepts, now taught to ten year olds. The historical arguments and discussions are not trivial. They should be taken seriously by anyone teaching the subject. Questions arising in classroom practice may reflect historical concerns and taken positions which have since then been abandoned.

2. Numbers smaller than nothing

Although not devised as a dispute against the number line, when Arnauld raised the discussion about proportions involving negative numbers he provided a strong argument against its use. Take any point n on the line of natural numbers, the proportion of its neighbors $n + 1$ to $n - 1$ is always larger than the proportion of $n - 1$ to $n + 1$. This property disappears when you add negative numbers to the number line.

2.1. Antoine Arnauld (1612–1694)

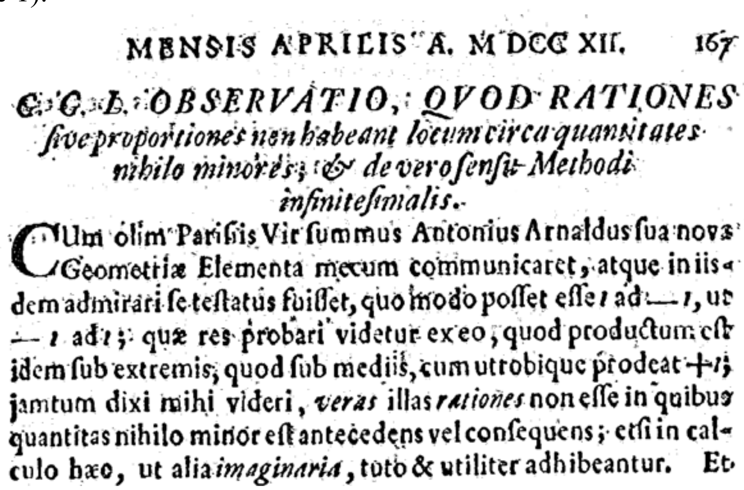
Antoine Arnauld, who wrote an important philosophical work known as *The Logic of Port-Royal* (Arnauld, 1662) published also his *Geometry* (Arnauld, 1667). In the book he includes an example of symbolic rules that he considers to be against our basic intuitions on magnitudes and proportions. His reasoning goes as follows: Suppose we have two numbers, a larger and a smaller one. The proportion of the larger to the smaller one should evidently be larger than the proportion of the smaller to the larger one. But if we use 1 as the larger number and – 1 as the smaller one this would lead to

$$\frac{1}{-1} > \frac{-1}{1} \qquad (1.1)$$

which is against the rules of algebra. Witnessing the multiple instances in which this discussion turns up during the seventeenth century, the clash between symbolic reasoning and classic proportion theory, taught within the *quadrivium*, was experienced as problematic. Schrecker (1935) was the first to describe the controversy on the topic initiated by Prestet in his *Elemens des mathematiques* of 1675. Both Mancosu (1996, 88-91) and Schubring (2005, 52-61) describe the positions taken. Prestet's response was basically that quantities can only be positive and that the signs refer to operations. So it is perfectly possible to subtract a larger quantity from a smaller one. The negative result means just that: a larger quantity subtracted from a smaller one. And when dealing with geometrical ratios one should neglect the signs all together.

2.2. Gottfried Wilhelm Leibniz (1646 - 1716)

Also Leibniz found it important enough to respond to Arnauld in an article (Leibniz, 1712, 167) (see Figure 1).


MENSIS APRILIS A. MDCCXII. 167

G. G. L. OBSERVATIO, QVOD RATIONES
sive proportiones non habeant locum circa quantitates nihilo minores; & de vero sensu Methodi infinitesimalis.

Cum olim Parisiis Vir summus Antonius Arnaldus sua nova Geometriæ Elementa mecum communicaret, atque in iisdem admirari se testatus fuisset, quo modo posset esse *1* ad *−1*, ut *−1* ad *1*; quæ res probari videtur ex eo, quod productum est idem sub extremis, quod sub mediis, cum utrobique prodeat *+1*; jamtum dixi mihi videri, *veras* illas *rationes* non esse in quibus quantitas nihilo minor est antecedens vel consequens; etsi in calculo hæc, ut alia *imaginaria*, tuto & utiliter adhibeantur. Et

Figure 1: Leibniz's response to Arnauld in *Acta eruditorum* of 1712

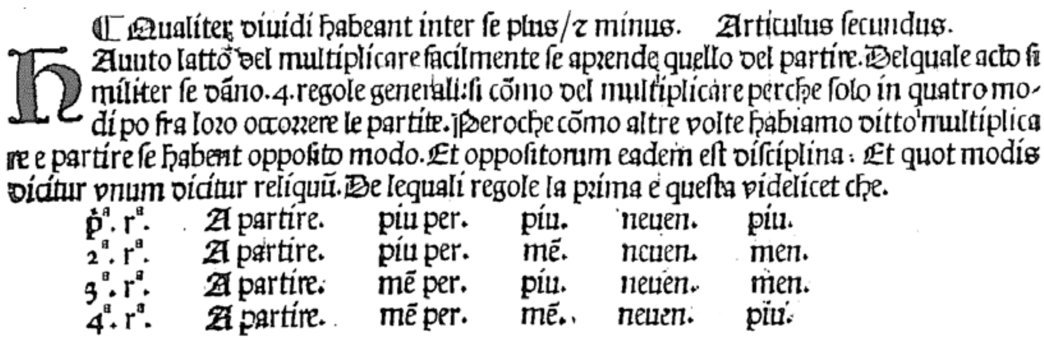
Leibniz acknowledges the problem as a genuine one, but states that the division should be performed as a symbolic calculation, the same way as we do with imaginary numbers. Indeed, when blindly applying the rules of signs there is no problem at all. When dividing a positive number by a negative one, the result is negative, and dividing a negative number by a positive one, the result is also negative. Therefore

$$\frac{1}{-1} = -\frac{1}{1} \qquad (1.2)$$

2.3. The abbaco tradition (1200-1500)

It may be surprising when you follow the seventeenth-century discussions but the kind of symbolic reasoning proposed by Leibniz was well accepted within the abbaco tradition in Italy by the end of the fifteenth century. Although symbols were not used in any consistent way within this tradition we find that the *maestri d'abbaco* had no problems with such kind of symbolic operations. We have argued elsewhere that the introduction of symbols such as the minus and plus signs is the result of a process toward symbolic thinking (Heeffer 2008). The epistemic validity of operations on negative numbers within the abbaco tradition was derived from a believe in the correctness of generally accepted practices. Very early within the algebraic practice of the *maestri d'abbaco* we see 'proofs' for the rules of sign. That was the kind of explanation Henry Brulard, or more precisely Marie-Henri Beyle, who is better known under his pseudonym Stendhal, longed for (see quote at the beginning). The first of such proofs in European mathematics appears in a treatise of c.1380 by Maestro Dardi titled *Aliabraa argibra* (f. 5v, Franci 2001, 44). It explains why a negative multiplied by a negative makes a positive. It is repeated in various other manuscripts dealing with algebra during the fifteenth century. The proof is derived from the well-know operations on binomials which often appear in the introductions of such manuscripts. The reasoning goes as follows: we know that 8 times 8 makes 64. Therefore (10 – 2) times (10 – 2) should also result in 64. One well known multiplication procedure is called *per casella*, (literally by the pigeonhole) meaning cross-wise multiplication in which you add all the sub-products (see Swetz 1987, 201-205). You multiply 10 by 10, this makes 100, then 10 times – 2 which is – 20 and again 10 times – 2 or – 20 leaves us with 60. The last product is – 2 times – 2 but as we have to arrive at 64, this must necessarily be + 4. Therefore a negative multiplied by a negative always makes a positive. In modern terminology we would say that the proof is based on distributive law of arithmetic.

By the end of the fifteenth century we see the rules of signs expressed in a more formal way. Luca Pacioli is the first to formulate them in an abstract form without referring to specific types of quantities (see Figure 2).


Qualiter diuidi habeant inter se plus/e minus. Articulus secundus.

Hauuto latto del multiplicare facilmente se aprende quello del partire. Del quale acto similiter se dano .4. regole generali: si como del multiplicare perche solo in quatro modi po fra loro occorrere le partite. Peroche como altre volte habiamo ditto multiplicare e partire se habent opposito modo. Et oppositorum eadem est disciplina. Et quot modis dicitur vnum dicitur reliquum. De lequali regole la prima è questa videlicet che.

pª. rª.	A partire.	piu per.	piu.	neuen.	piu.
2ª. rª.	A partire.	piu per.	mē.	neuen.	men.
3ª. rª.	A partire.	mē per.	piu.	neuen.	men.
4ª. rª.	A partire.	mē per.	mē.	neuen.	piu.

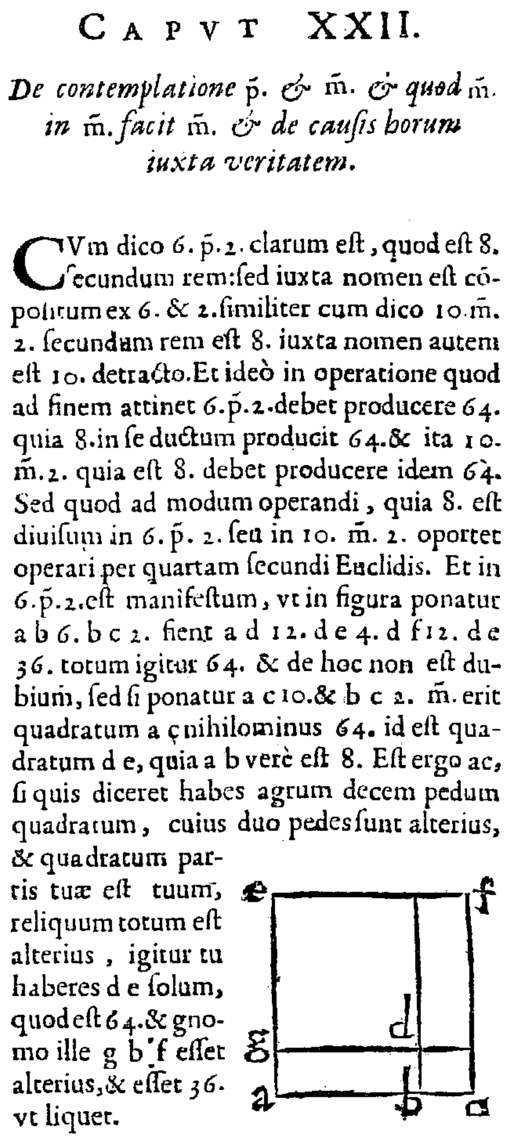
Figure 2: Pacioli's rules of sign for division from the *Summa* (1494, f. 113r)

Importantly, Pacioli introduced these rules in distinction 8, as a preparation to his treatment of algebra. In contrast with the discussion of the basic operations of arithmetic, the rules of signs are formulated in the most general way. Except for an illustrating example with numbers, the formulation of the rules does not refer to any sort of quantities, integers, irrational binomials or cossic numbers. The rules only refer to 'the negative' and 'the positive'. Despite the absence of any symbolism, we consider this an early instance of symbolic reasoning. And as Leibniz said this is the way to deal with the apparent anomaly. Forget about the values but apply the rules correctly and you necessarily arrive at (1.2).

2.4. Girolamo Cardano (1501 – 1576)

Cardano made profound contributions to the acceptance of negative numbers that are not that well recognized. He was the first to give a satisfactory argumentation for negative solutions to linear problems and the first to accept square roots of negative numbers. As we have discussed these both before (Heeffer 2007) we will here only sketch how he got doubts about the rules of signs during his later writing career.

Some years before his death Cardano wrote two treatises in which he reflects on his earlier treatment of negative and imaginary numbers in the *Ars Magna* (1545). The first one is titled *De Aliza Regulae* published in 1570 as part of the larger *De Propotionibus* and the second *Sermo de plus and minus* which was published posthumously in his collected works (Cardano 1663, IV, 435-439). Of particular interest is his 'refutation' of the proof for the rules of signs as it was generally accepted by the abbacists. Using the same example as Maestro Dardi he concludes the contrary, that minus time minus makes plus has the same truth as telling that plus times plus makes minus.


CAPVT XXII.

De contemplatione p̃. & m̃. & quod m̃. in m̃. facit m̃. & de cauſis horum iuxta veritatem.

CVm dico 6. p̃. 2. clarum eſt, quod eſt 8. ſecundum rem: ſed iuxta nomen eſt cõpoſitum ex 6. & 2. ſimiliter cum dico 10. m̃. 2. ſecundum rem eſt 8. iuxta nomen autem eſt 10. detracto. Et ideò in operatione quod ad finem attinet 6. p̃. 2. debet producere 64. quia 8. in ſe ductum producit 64. & ita 10. m̃. 2. quia eſt 8. debet producere idem 64. Sed quod ad modum operandi, quia 8. eſt diuiſum in 6. p̃. 2. ſeu in 10. m̃. 2. oportet operari per quartam ſecundi Euclidis. Et in 6. p̃. 2. eſt manifeſtum, vt in figura ponatur a b 6. b c 2. fient a d 12. d e 4. d f 12. d e 36. totum igitur 64. & de hoc non eſt dubium, ſed ſi ponatur a c 10. & b c 2. m̃. erit quadratum a c nihilominus 64. id eſt quadratum d e, quia a b verè eſt 8. Eſt ergo ac, ſi quis diceret habes agrum decem pedum quadratum, cuius duo pedes ſunt alterius, & quadratum partis tuæ eſt tuum, reliquum totum eſt alterius, igitur tu haberes d e ſolum, quod eſt 64. & gnomo ille g b f eſſet alterius, & eſſet 36. vt liquet.

Figure 3: Cardano's refutation of the rules of sign

The *Aliza* in the title of his treatise is derived from the Latin word *a'izzā* meaning 'risky' or 'doubtful' (Tanner 1980, 162). Therefore the conclusion should not be taken too literality. As the same argument was repeated in his *Sermo* we will only discuss the first.

The reasoning goes as follows (see his diagram in Figure 3). If we multiply 10 by 10 we get a square with side *ac* equal to 10 and area 100. Given that *bc* and *ag* equal 2, the square *egd* thus has an area of 64. To get from 100 (square *acef*) to 64 (square egd) we therefore have to subtract the two rectangles *cg* and *bf*. But by doing so we have subtracted the square *cd* twice, so we have to add it one more time. Thus arithmetically we have

$$100-(10\times 2)-(10\times 2)+(2\times 2)=64\,.$$

And this is the expansion of (10 – 2)(10 – 2) as we have seen it in the proof of Maestro Dardi. But Cardano argues that the + 4 is not the result of the multiplication of – 2 by – 2 but an area we must add again because we had subtracted the square twice. He refers to proposition 7 from book II of Euclid's *Elements* and concludes: "And therefore lies open the error commonly asserted that minus times minus produces plus, lest indeed it be more correct that minus times minus produces plus than plus times plus would produce minus".[3]

While Cardano doubts the rules of signs for multiplication (and also for division, see Tanner 1980, 167) on this occasion he does not err on their application to algebraic problems. So why his fulmination against it? While Tanner believes that Cardano "appears unique only in putting into print something of a rebel trend of thought, entertained in private by the majority, but soon to be disavowed by silent suppression", Schubring (2005, 45) believes in epistemological motives as Cardano experienced the mixture of operations of subtraction with those of multiplication as problematic. For us, the curious use of the same example from the well known proof of Dardi suggest that Cardano's motivation stems from a lack of epistemic validity for the generally accepted rules of sign. Cardano wanted to show he can challenge these all too easily accepted rules and doing so he reacted against what Stendhal called hypocrisy in mathematics.

3. Larger than infinity

Several historians of mathematics have pointed out that some major mathematicians of the sixteenth and seventeenth century believed that negative numbers were larger than infinity. From our reading of the original sources we believe that such interpretations are unwarranted. Statements about numbers greater than infinity occur in early mathematical treatments of infinitesimals and summation of series. They do not refer to negative numbers as such but expressions in involving division by negative quantities. We will here discuss the two most important ones.

3.1. John Wallis (1616–1703)

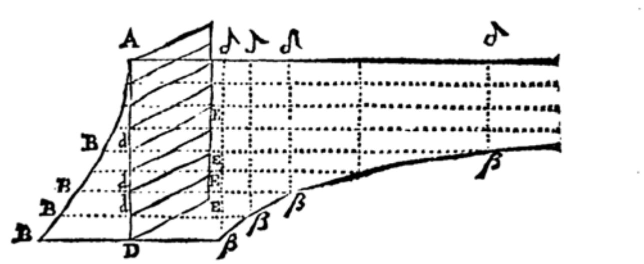
While he was Savilian professor of geometry at Oxford, John Wallis wrote two of his most important works, one on conic sections and the other on infinitesimal quantities. In the latter one, titled *Arithmetica Infinitorum* of 1656 he advanced the idea that dividing a positive number by a negative one the result is larger than infinity. The reasoning leading to this conclusion is found in proposition 104 (see Figure 4). A series of reciprocals $1/a$ grows to zero when a becomes very large and to infinity when a is becomes zero. Now, as you cross zero you get to negative denominators. Extending the reasoning, numbers divided by zero become infinite and if you go beyond that, you arrive at values larger than infinity. Although we have to understand this surprising conclusion within the context in which it appears, the quadrature of curves with equations having negative indices, the statement is nevertheless unambiguous: the ratio of a positive number to a negative one is a *rationem plusquam infinitam*, "a ratio greater than infinity".

[3] Cardano 1663, IV, 399: "Et ideo patet communis error dicentium, quod m . in m. producit p. neque enim magis m. in m. producit p. quam p. in p. producat m.", (this and subsequent translations are mine). Later Christian Wolff (1732, I, 240) uses the same diagram to prove the rules of sign for multiplication.

For some unknown reason several historians of mathematics misunderstood Wallis as if he claimed that negative numbers in itself were greater than infinity. William Rouse Ball (1912, 293) writes “It is curious to note that Wallis rejected as absurd the now usual idea of a negative number as being less than nothing, but accepted the view that it is something greater than infinity”. We will see below that Wallis did not reject at all numbers less zero. In fact, Wallis can be considered as the inventor of the number line for negative quantities. Morris Kline (1972; 1990, 253) possibly inspired by Ball also completely misses the point: “Though Wallis was advanced for his times and accepted negative numbers, he thought they were larger than infinity but not less than zero”. Some years later in his *Loss of Certainty* he writes (1983, 116): “Though Wallis was advanced for his times and accepted negative numbers, he thought they were larger than ∞ as well as less than 0”. Such incongruous misrepresentations caution us to always check the original sources.

PROP. CIV. *Theorema.*

SI deniq; ejuſmodi Figura ADδβ, ſic continuo decreſcat juxta ſeriem quæ ſit reciproca directæ indicem habenti unitate majorem; habebit illa ad Parallelogrammum inſcriptum rationem plusquam infinitam: qualem nempe habere ſupponatur numerus poſitivus ad numerum negativum, ſive minorem nihilo. Nempe eam, quam habet 1 ad indicem unitate auctum.

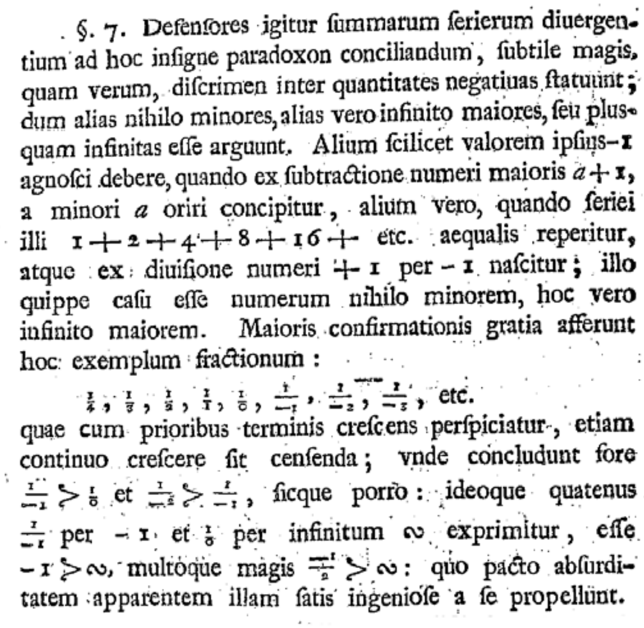


Puta cum indices ſeriei ſecundanorum, Tertianorum Quartanorum, &c. ſint 2, 3, 4, &c. (unitate majores,) indices ſerierum illis reciprocarum erunt −2, −3, −4, &c. qui quamvis unitate augeantur (juxta Prop. 64.) manebunt tamen negativi, puta −2 + 1 = −1, −3 + 1 = −2, −4 + 1 = −3, &c. & propterea ratio quam habet 1 ad indices illos ſic auctos, puta 1 ad −1, 1 ad −2, 1 ad −3, &c. major erit quam infinita, ſive 1 ad 0; quia nempe rationum conſequentes ſunt minores quam 0.

Figure 4: Wallis arguing that when a number is divided by a negative it becomes larger than infinity (from the *Arithmetica infinitorum*, **p. 78)**

3.2. Leonhard Euler (1707-1783)

About one century later Euler came to the same conclusion as Wallis through reasoning with divergent series. The Latin text *De seriebus divergentibus* [E247] was written in 1746, but not read to the Academy until 1754, and only published in 1760. Several English translations of the text have been published (Barbeau and Leah, 1976, Kline 1983, Sandiger 2006).



§. 7. Defenſores igitur ſummarum ſerierum diuergentium ad hoc inſigne paradoxon conciliandum, ſubtile magis, quam verum, diſcrimen inter quantitates negatiuas ſtatuunt; dum alias nihilo minores, alias vero infinito maiores, ſeu plusquam infinitas eſſe arguunt. Alium ſcilicet valorem ipſius -1 agnoſci debere, quando ex ſubtractione numeri maioris $a+1$, a minori a oriri concipitur, alium vero, quando ſeriei illi $1+2+4+8+16+$ etc. aequalis reperitur, atque ex diuiſione numeri $+1$ per -1 naſcitur; illo quippe caſu eſſe numerum nihilo minorem, hoc vero infinito maiorem. Maioris confirmationis gratia afferunt hoc exemplum fractionum:

$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{1}{0}, \frac{1}{-1}, \frac{1}{-2}, \frac{1}{-3}$, etc.

quae cum prioribus terminis creſcens perſpiciatur, etiam continuo creſcere ſit cenſenda; vnde concludunt fore $\frac{1}{-1} > \frac{1}{0}$ et $\frac{1}{-2} > \frac{1}{-1}$, ſicque porro: ideoque quatenus $\frac{1}{-1}$ per -1 et $\frac{1}{0}$ per infinitum ∞ exprimitur, eſſe $-1 > \infty$, multoque magis $\frac{1}{-2} > \infty$: quo pacto abſurditatem apparentem illam ſatis ingenioſe a ſe propellunt.

Figure 5: Fragment of Euler's text discussing division by a negative number

Euler's observations are based on the expansion posed by Leibniz in 1713 in which

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \cdots \qquad (1.3)$$

Euler begins the paragraph with stating that those who defend the idea of summing divergent series can resolve this paradox in a subtle way by discriminating between 1) quantities that become negative, 2) that stay less than zero, and 3) that become more than infinity. He then makes a distinction between two kinds of negatives: "Of the first sort is – 1, which is the difference between a its successor a + 1. Of the second sort is the – 1 that arises as 1 + 2 + 4 + 8 + 16 + etc., which is equal to the number one gets by dividing +1 by –1. In the first case, the number is less than zero, and in the second case it is greater than infinity". The first is a plain negative number as he later explains in his book on elementary algebra (Euler 1770). The second one is when you use $x = 2$ in (1.3). You then arrive at

$$\frac{1}{-1} = 1 + 2 + 4 + 8 + \cdots \qquad (1.4)$$

which according to Euler is greater than infinity. He then uses an argument analogous to Wallis: "This can be confirmed by the following example of a sequence of fractions:

$$\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}, \frac{1}{0}, \frac{1}{-1}, \frac{1}{-2}, \frac{1}{-3}, \ldots$$

where the first four terms are seen to grow, then grow to infinity, and beyond infinity they become negative. Thus the apparent absurdity is resolved in a most ingenious way."

Now again the idea that dividing a number by a negative one leads to something larger than infinity has been systematically been misunderstood. Kline writes "Euler, the greatest eighteenth-century mathematician believed that negative numbers are greater than infinity" (Kline 1981, 52) and later he later repeated "Euler concluded that – 1 is larger than infinity" (Kline 1983, 144). Sandiger (2006, 179) "Euler is claiming that numbers greater than infinity are the same as numbers smaller than zero" and recently William Dunham (2007, 138) Euler "is willing to accept that 'the same quantities which are less than zero can be considered to be greater than infinity'". Despite the last quote, Wallis or Euler never claimed that negative numbers are greater than infinity. The misunderstanding becomes apparent from an article by Kline (1983) on Euler. Instead of expression (1.4) Kline writes that Euler obtained:

$$-1 = 1 + 2 + 4 + 8 + \cdots$$

But that is taken already for granted that $\frac{1}{-1} = -1$, which is precisely the identity questioned by Wallis and Euler. In fact, Euler had no problems at all with negative numbers. In his book on elementary algebra he writes that "we may say that negative numbers are less than nothing" (Euler 1822, 5) and he explains so by enumerating the negative numbers from zero "in the opposite direction, by perpetually subtracting unity", *de facto* endorsing the number line. Concerning the product of two negatives, Euler gives an argumentation for its positive result. Based on the proof that minus times plus is minus, therefore minus times minus should be different and therefore be plus. A second argument using the distributive law of multiplication (as Maestro Dardi), is added by the translator in the English edition. The rules for division by a negative are derived from the multiplication rules and thus pose no problem for Euler in his *Algebra*.

3.3. Jean le Rond d'Alembert (1717-1783)

The discussion was not closed by Leibniz. Several eighteenth-century authors return to the question raised by Arnauld, such as Rolle (1690, 14-22). Maclaurin (1748, 6-7) does not refer to Arnauld but seems to be aware of the discussion. He considers – *a* and + *a* "equal as to quantity" but this does not mean that you can equate them in algebra as + *a* = – *a*. Their quality is the opposite of each other and "on account of this contrariety a negative quantity is said to be less than nothing".

Perhaps the most persistent in his struggle against the number line was d'Alembert. On several occasions both in the *Encyclopédie* as well as in his *Opuscules* he goes on at the idea of numbers smaller than nothing. d'Alembert was quite influential. Not only through his work as a mathematician but also in his contacts with Royal circles. His contributions on mathematics in the *Encyclopédie* were printed and read long after his death. Under 'négatif' he writes: "negative quantities are those which are affected by the minus sign and which are considered by several mathematicians as smaller than zero. This last idea is false, as will see in one moment".[4] His argumentation is somewhat based on the

[4] Diderot and d'Alembert, (1761-1790, 22, 289) : "quantités négatives sont celles qui sont affectuées du signe – et qui sont regardées par plusieurs mathématiciens comme plus petit que zero. Cette dernière idée n'est pas juste, comme le verra dans un moment".

considerations of Wallis and Euler. You cannot just claim that negative numbers are smaller than zero because the passage from positive to negative does not always goes over zero. In the simple case of $y = x - a$, y goes from positive to negative over zero. But in the case of $y = 1/(x - a)$ you will have $y = \infty$ when $x = a$.[5] So in contrast with Wallis and Euler, d'Alembert accepts that

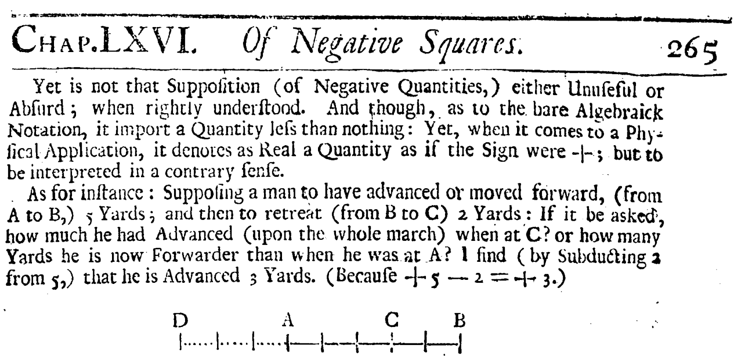
$$y = \frac{1}{-a}$$

will be negative, but it becomes negative while passing ∞. Therefore it is wrong to say that negatives numbers are always smaller than zero.

Without naming Arnauld he also comments on the apparent anomaly where we started with: "Those who pretend that you cannot relate 1 to – 1 and think that the ratio of 1 to – 1 is different from – 1 to 1 are making a double error. Firstly, we perform such algebraic operations every day and secondly the equality of the product of – 1 with – 1, and of +1 by +1, shows that 1 is to – 1 as – 1 is to 1".[6]

4. Conclusion

Before we get to our conclusion we may ask the question where the idea of a number line showing negative numbers originated. We find the answer in Smith's source book (1956, 46-7): John Wallis (see Figure 6). Remarkably Wallis introduces the number line for the purpose of illustrating addition and subtraction involving negative numbers. When a man advances 5 yards from A and he returns 8, how far is he then from his starting point? Wallis gives the answer – 3, as it is taught now in primary education.


CHAP. LXVI. *Of Negative Squares.* 265

Yet is not that Suppoſition (of Negative Quantities,) either Unuſeful or Abſurd; when rightly underſtood. And though, as to the bare Algebraick Notation, it import a Quantity leſs than nothing: Yet, when it comes to a Phyſical Application, it denotes as Real a Quantity as if the Sign were +; but to be interpreted in a contrary ſenſe.

As for inſtance: Suppoſing a man to have advanced or moved forward, (from A to B,) 5 Yards; and then to retreat (from B to C) 2 Yards: If it be asked, how much he had Advanced (upon the whole march) when at C? or how many Yards he is now Forwarder than when he was at A? I find (by Subducting 2 from 5,) that he is Advanced 3 Yards. (Becauſe +5 − 2 = +3.)

Figure 6: John Wallis introducing the number line in his *Algebra*

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[5] ibid. p. 300 : "Le passage du positif au négatif se fait toujours par zéro ou par l'infini. Soit, par exemple, $y = x - a$ on aura y positif tant que $x > a$, y négatif lorsque $x < a$ et $y = 0$ lorsque $x = 0$; dans ce cas le passage se fait par zéro. Mais si $y = 1/(x - a)$ on aura y positif tant que x est $> a$, y négatif lorsque x est $< a$ et $y = \infty$ lorsque $x = a$; le passage se fait alors par infini".

[6] ibid. p. 299 : "Ceux qui prétendent que 1 n'est pas comparable à -1, et que le rapport entre 1 et -1 est différent du rapport entre -1 et 1 sont dans un double erreur, 1) parce qu'on divise tous les jours dans les opérations algébriques, 1 par – 1, 2) l'égalité du produit de – 1 et – 1, et de +1 par +1, fait voir que 1 est à – 1 comme – 1 a 1".

So the person responsible for the idea that dividing a number by a negative results is something larger than infinity, is also the first one to use the number line in a printed book. The idea was followed up by many others including Newton (1707, 3). Newton makes a distinction between affirmative quantities and negative quantities. A negative quantity arises when a large affirmative quantity is subtracted from a smaller one. And to illustrate the point Newton draws a line which amounts to a number line showing negative quantities.

As it was a source of controversy and discussion in the seventeenth and eighteenth century, it should come as no surprise that it raises questions and difficulties in the classroom. A conceptual history of mathematics can prepare teachers for such difficulties and show them that such questions must be taken seriously. It may show also that certain didactic approaches have their potential trap falls. The use of the number line in teaching negative numbers is in direct conflict with d'Alembert, who argues that most of the difficulties with Arnauld's identity arise from viewing negative numbers as smaller than zero. Then how should we teach the subject?

Barry Mazur in his highly enjoyable and philosophical book *Imagining numbers*, spends a lot of attention to the question why we accept that minus times minus equals plus. He concludes (Mazur 2003, 102-3) that "there is, in fact, *only one way* to extend the definition of multiplication to all whole numbers, negative as well as positive, if we wish (we do!) 1 times any number N equal N, and if we wish (we do!) the distributive law to hold". So this is an argument if favor of a proof as we know from the abbacists. However, d'Alembert's definition of negative numbers: "those which are affected by the minus sign" is for us the most appealing. It captures the way how negative numbers were introduced in history and how they fit within symbolic manipulations. We therefore believe that they should be taught within the context of early algebra.

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