ON THE NECESSITY OF (SOMETIMES) BEING SYNTHETIC
COMMENT ON POGGIOLESI

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1. Introduction

The debate concerning the nature and philosophical value of logical systems is as long as the discipline itself. Modern logic, as it originated from the foundational debate of the XX^{th} century and throughout its evolution in the XXI^{th} century, sides with and (for some of us) grounds its siblings: mathematics and computer science. It is a difficult task to assess the value of a logical system with respect to its mathematical and computational interpretations. In her contribution, Poggiolesi suggests that it is preferable to use a formal criterion rather than an applicative one for such assessment and she identifies analyticity—in its formal counterpart, the subformula property—as *the* criterion to be chosen for evaluating proof systems (i.e. at least for one aspect of the multi-faceted nature of logical systems as we know them today). I mostly agree on the general point that formal properties should have priority in evaluating a logical system. Nonetheless, it is my aim in this brief commentary to highlight a viewpoint which, mostly inspired by the relevance of logical methods in knowledge representation and the modeling of defeasible reasoning, questions Poggiolesi's conclusions that (only) an analytical proof system is a good system. I shall start in section 2 by briefly reformulating Poggiolesi's argument; in section 3 I will present a thesis on which basis I formulate a premise; in section 4 I will draw the related conclusions.

2. Short Rehearsal

Let me very briefly restate the core of Poggiolesi's argument. Analyticity for proofs, intended as rational methods to establish the truth of a proposition, can be seen as a bottom-up procedure that provides a reduction from complex to simpler concepts and such that only concepts that occur in the conclusion are essential to its achievement. An appropriate formal translation of this property is reflected in a proof system enjoying the subformula property, as the latter generates analytic derivations. A calculus that enjoys the subformula property will admit the cut-rule and for every other rule of the system all formulas occurring in the premises are included in the conclusion. The main argument put forward by Poggiolesi is of a pragmatic nature: If a proof system enjoys the subformula property, useful and important properties can be discovered.

To support this thesis, she presents as an example Artemov's logic of proofs *Lp*, notoriously considered an adequate provability interpretation of modal logic *S*4, and its intuitionistic counterpart *Ilp*. In order to formulate analyticity for *Ilp*, Poggiolesi proposes to modify of the original language so that the constants of the logic of proofs are shown to be nothing but a shortcut to express closed λ-terms in normal form, typed by the corresponding axioms. By this step, one discovers that the rule $\odot$ violates the requirement on the subformula property, as its proof-polynomial corresponds to a proof containing two cuts. The altered calculus where proof-polynomials are extracted from the corresponding λ-terms, easily solves this issue in view of reductions for the λ-term corresponding to $\odot$. The search for an analytic calculus has led us to an important and relevant feature, an additional pragmatic argument in favor of analyticity.

3. A Thesis and its Premise

In the following, I will suggest that not every *useful* proof system *can* be analytic. Even though this does not directly imply that 'synthetic' proof systems are better off than analytic ones; nor that we should not prefer the latter over the former for some formal reason; nor, finally, that a useful proof system is better than a good proof system; nonetheless, I believe that this remark should be taken into account by logicians (or at least by proof theorists) in the appropriate measure and context. Moreover, I will support this argument with an example that is as pragmatic as Poggiolesi's. Let me start with a thesis:

Thesis 1. *Useful and realistic proof systems for knowledge representation and communications cannot always be analytic (in the sense of enjoying the subformula property).*

The conclusion I draw from this thesis is that analyticity is *not always* an appropriate feature to evaluate *good* proof systems. Obviously, this argument will have no relevance for those who believe that logical systems should *never* be evaluated in view of some application; and it will sound even less interesting to those who believe that neither knowledge representation nor communications systems should be treated in the form of a logical system (or of a proof system, more specifically).

Our thesis relies on a premise of a formal nature. Let us agree, for the sake of the argument, that our aim is in fact that of using a proof system for the purpose of representing knowledge acts by human rational agents and that we want such a system to be able to characterize essential features of this reasoning form. To reduce any sensible gap with the argument put forward by Poggiolesi, let us use the same kind of formal machinery: In particular, we shall use proof terms as the formal counterparts of *justifications*, and a epistemic system that endorses the intuitionistic notion of truth as existence of a justification. In such a system, therefore, a judgement of the form 'proposition A is true' would be based on a formula whose meaning is 'there is a justification t of A'. For this task, Artemov's calculus *Ilp* from Artemov [2002], as well as Martin-Löf's Type Theory from Martin-Löf [1984], Martin-Löf [1998], could be used. My formal premise is then the following:

Premise 1. *λ-terms are equivalent to functional abstracts, and they implement hypothetical reasoning in our system.*

Let me articulate a bit its meaning. The standard intuitionistic interpretation of abstraction is that of a proof-term whose computational content has been 'hidden' (substituted by an open variable), with the additional restriction that such content needs to be preserved in order to be restored at will. This is known as the "Forget-Restore Principle" by Valentini and Sambin, see

Sambin and Valentini [1998] and Valentini [1998]. In other words, any abstraction we perform, hence any *hypothesis* we use in a conditional reasoning form in a constructive setting, would be one of which we are actually able to compute the content; this in turn corresponds formally to a standard request on λ-terms, namely their closure, or reduction, as to ensure that no "empty" term is used.

4. Some Consequences and Observations

I shall now draw some basic consequence from the premise formulated in the previous section. I will then conclude by putting forward two observations that I consider relevant against Poggiolesi's thesis that only an analytic proof system can be considered a *good* proof system. The strength of such conclusion is obviously based on the willingness of the reader to accept our thesis as formulated in the previous section.

A first consequence of our short notice on the use of abstract terms is the following: the explanation of the meaning of proof-theoretical acts involving abstractions affects in a relevant way our understanding of hypothetical reasoning; this, in particular, means that the meaning explanation of the implication connective is based on the presence of an abstract term or of its closure.

Consequence 1. *The meaning of an implication $A \to B$ where all terms are required to be closed (to reduce), is of the following form:*

> *"If A is proven, then B is proven".*

This apparently harmless, and surely consistent reading, holds also for the corresponding λ-term: $((x)b) : A \to B$. Its real meaning is given by the substitution of a redex for A and its application to the proof term for B, so that it corresponds to:

> "a construction transforming a proof a of A into a proof b of B".

Again, this reading is the consistent and standard translation with proof terms by the Brouwer-Heyting-Kolmogorov semantics. Though it behaves well for the semantics of mathematical proofs, it seems inappropriate for rendering some instances of implicational relations in knowledge processes, in particular for reasoning steps of the following form:

> "Assume a proof of A exists. Then one can infer there is a proof of B, *unless A is rejected*".

To put it plainly: There are reasoning processes that rely on refutable assumptions and not every useful implication proceeds from a known content. The explanation for the implication $A \to B$ as it is formulated in an analytic context cannot account for these cases.

This leads us to a few more observations. The problem that we are trying to formulate is crucially a problem of logical notions, in particular: How to make sense of the distinction between premises and assumptions? Despite the heavy use of the notion of *assumption* in Gentzen's style calculi, there is no obvious logical distinction in use with the notion of *premise*, where the former should recall a content that is only considered consistent, the latter a known content. The formal requirement is the usual one on discharging operations, but there is no obvious way to interpret a reasoning step to obtain a valid conclusion from an assumption that remains undischarged, typically a content that can be refuted at a later step. This is, in other words, the problem of providing a sensible interpretation of proof systems for defeasible reasoning and—even more strongly—to combine defeasible reasoning with a verificationist notion of truth.

Another way to approach the same issue is by considering the semantic interpretation, and the related notion of truth in use in our proof system: Is there a way to draw a distinction between truth by justification and truth by assumption? Can we seriously think of calculi that are able to combine these notions and to endorse this distinction? In **?** a modal type-theoretical system is formulated which combines two basic syntactic/semantic categories: One sort of expression *type* is justified by categorical constructions and induces standard constructive truth; another sort $type_{inf}$ is defined by syntactically admissible but not directly reduced terms.

$$\frac{a:A}{A\ type}\ \text{Type formation} \qquad \frac{a:A}{A\ true}\ \text{Truth Definition}$$

$$\frac{\neg(A \to \bot)}{A\ type_{inf}}\ \text{Informational Type formation}$$

$$\frac{A\ type_{inf} \qquad x:A}{A\ true^{*}}\ \text{Hypothetical Truth Definition}$$

$$\frac{a:A}{\neg A \to \bot}\ I\bot$$

$$\frac{A\ type_{inf} \qquad x:A \vdash B\ type_{inf}}{x:A \vdash B\ true^{*}}\ \text{Function Construction}$$

We cannot go into details on this occasion, and refer the reader to the full paper. What is crucial here is that admitting the Informational Type Formation and its related rules, this calculus cannot be considered analytic in the sense of enjoying the subformula property. Its main characteristic is to allow the combination of a justification-based notion of truth, with an appropriate counterpart for refutable contents. To the purpose of this commentary, any other similar system that does not satisfy the analyticity property could have been used.

To provide a logical treatment of knowledge acts based on the information transmission among rational agents in a proof system of the kind we are considering, we should be able to formalize epistemic processes involving falsifiable data. This leads us directly to the second consequence:

Consequence 2. *Not every epistemic process that we might want to analyze logically is information-preserving; some of these processes are information-destroying and finally some of them are information-extending or information-transforming processes.*

Our argument is of a different nature than the ones usually put forward, but it nonetheless recalls the long standing debate on the analytical nature of deduction and the well-known scandal of deduction, so well expressed by Cohen, Nagel's 'Paradox of Inference' (Cohen and Nagel [1934]). This problem has recently received renewed attention, see e.g. [Primiero, 2008, ch.2], Sequoiah-Grayson [2008], D'Agostino and Floridi [2009], Duži [2010].

Let us now return to the analyticity property to make our final point. For many epistemic processes, there is the possibility—and for some of us the evident necessity—to provide a logical treatment. These are processes that combine distinct epistemic states, that require dynamic structures and often involve non-standard properties, such as defeasibility, non-monotonicity or even paraconsistency. In this context, the ability of combining the strong value of justifications as truth-makers with weaker notions of truth, is a highly desirable task. We believe that for such purposes a proof system is a valuable tool. Nonetheless, for such a system, analyticity is less of a requirement, as it is the case with other good properties, such as polynomial computability. This does not reduce the importance of analyticity as a "good property" for those proof systems that advocate the standard role of logic, for example as a way to define the semantics of mathematical proofs. Nonetheless, it also gives rise to some interesting questions: What are the good principles for proof systems for refutable contents/weak truths/synthetic truths? And which kind of truth applies to computational contents that are context-dependent? These, and other related questions, are the core of large clusters of research in semantics and proof theory, despite their inner synthetic nature.

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