

# The Rule of Quantity by Chuquet and de la Roche and its Influence on German Cossic Algebra

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## 1. Abstract

The importance of *Larismethique* of de La Roche, published in 1520, has been seriously underestimated. One reason for the neglect is related to the inscrutable way he is referred to. Buteo and Wallis called him Stephanus à Rupe de Lyon. Other obscure references, such as Gosselin calling him Villafrancus Gallus have been overlooked by many commentators. His influence can be determined in several works that do not credit him but use problems or definitions from the *Larismethique*. However, most damaging for its historical assessment was Aristide Marre's misrepresentation of the *Larismethique* as a grave case of plagiarism. Marre discovered that the printed work of 1520 by Estienne de la Roche contained large fragments that were literally copied from Chuquet's manuscript of the *Triparty*. Especially on the *Appendice*, which contains the solution to a large number of problems, Marre writes repeatedly that it is a literal copy of Chuquet. However, he fails to mention that the structure of the text of de la Roche, his solution methods and symbolism differs significantly from Chuquet. De la Roche introduces several improvements, especially with regards to the use of the second unknown. We provide an in-depth comparison of some problems solved by the so-called *regle de la quantite* by Chuquet with those of de la Roche. We further report on the surprising finding that Christoff Rudolff's solution to linear problems by means of the second unknown in his *Behend vnnd Hubsch Rechnung* of 1525 depends on Chuquet and de la Roche. As it is generally considered that algebra was introduced in Germany through Italy this provides a new light on the transmission of algebraic knowledge from France to the rest of Europe.

## 2. More than plagiarism: Estienne de la Roche

The *Larismethique nouvellement composee par maistre Estienne de La Roche*, published in 1520 is after Pacioli's *Summa* and the work of Grammateus, the third printed book dealing with algebra and the first one in French.<sup>1</sup> We do not know much about de la Roche. Tax registers from Lyon reveal that his father lived in the Rue Neuve in the 1480s and that

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<sup>1</sup> Estienne de La Roche, *Larismethique nouvellement composee par maistre Estienne de La Roche dict Villefra[n]che natif de Lyo[n] sou le Rosne*. Imprimee par maistre Guillaume Huyon. Pour Constantin Fradin marchand & libraire, Lyon, 1520. Lucca Pacioli, *Summa de arithmetica geometria proportioni: et proportionalita. Continetia de tutta lopera*, Paganino de Paganini, Venice, 1494. Henricus Grammateus, *Ayn new kunstlich Buech welches gar gewiss vnd behend lernet nach der gemainen Regel Detre, welschen practic, Reglen falsi vñ etlichäe Regeln Cosse*, Johannem Stüchs, Nürnberg, 1518.

Estienne owned more than one property in Villefranche, from which he derived his nickname.<sup>2</sup> De la Roche is described as a “master of argorisme” as he taught merchant arithmetic for 25 years at Lyon. He owned the manuscript of the *Triparty* after the death of Chuquet. It is therefore considered that de la Roche was on friendly terms with Chuquet and possibly learned mathematics from him.

The importance of the *Larismethique* has been seriously underestimated. There are several reasons for this. Probably the most important one is Aristide Marre’s misrepresentation of the *Larismethique* as a grave case of plagiarism. Marre discovered that the printed work by Estienne de la Roche, contained large fragments that were literally copied from Chuquet’s manuscript.<sup>3</sup> Indeed, especially on the *Appendice*, which contains the solution to many problems, Marre writes repeatedly in footnotes “This part is reproduced word by word in the *Larismethique*” with due references.<sup>4</sup> However, giving a transcription of the problem text only, Marre withholds that for many of the solutions to Chuquet’s problems de la Roche uses different methods and an improved symbolism.<sup>5</sup> In general, the *Larismethique* is a much better structured text than the *Triparty* and one intended for a specific audience. Chuquet was a bachelor in medicine educated in Paris within the scholarly tradition and well acquainted with Boethius and Euclid.<sup>6</sup> On the other hand, de la Roche was a reckoning master operating within the abbaco tradition. It becomes clear from the structure of the book that de la Roche had his own didactic program in mind. He produced a book for teaching and learning arithmetic and geometry which met the needs of the mercantile class. He rearranges Chuquet’s manuscript using Pacioli’s *Summa* as a model. He even adopts Pacioli’s classification in books, distinctions and chapters. He moves problems from Chuquet’s *Appendice* to relevant sections within the new structure as did Pacioli with the many abbaco manuscripts from which he borrowed problems and solutions. He adds introductory explanations to each section of the book, such as for the second unknown, discussed below. With the judgment of an experienced teacher, he omits sections and problems from the *Triparty* which are of less use to merchants and craftsmen and adds others which were not treated by Chuquet such

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<sup>2</sup> Graham Flegg, Cynthia Hay and Barbara Moss (eds.), *Nicolas Chuquet, Renaissance Mathematician*, Kluwer, Dordrecht, 1985, p. 61.

<sup>3</sup> Aristide Marre, “Notice sur Nicolas Chuquet et son *Triparty* en la science des nombres”, *Bulletino di Bibliografia e di Storia della Scienze Matematiche e Fisiche*, 13, (1880) pp. 555-92, see introduction.

<sup>4</sup> Aristide Marre, “Appendice au *Triparty* en la science des nombres de Nicolas Chuquet Parisien”, *Bulletino di Bibliografia e di Storia della Scienze Matematiche e Fisiche*, 14, (1881) pp. 413-59.

<sup>5</sup> For example, problem 70 from Chuquet’s *Appendice* is solved by two unknowns by de la Roche (see below) while Chuquet solves it by false position. His improvements in the symbols for the second unknown are discussed below.

<sup>6</sup> Hervé L’Huillier, “Les mathématiques à Lyon à travers l’arithmétique commerciale d’Etienne de La Roche (1480-1520)”, in *Lyon: Cite de savants. Actes du 112e Congres National des Societes Savantes, Section d’histoire des sciences et des techniques, tome 1, Lyon, 1987, pp. 31-41.*

as problems on exchange and barter. This didactical program of de la Roche was apparently not understood by Marre whose conclusion on the book is very harsh:<sup>7</sup>

One can state, pure and simple that, [de la Roche] copied a mass of excerpts from the *Triparty*, that he omitted several important passages, especially on algebra, that he abridged and extended others for producing the *Arismetique*, a work much inferior to the *Triparty*.

Comparing the problem texts only, the denunciation of de la Roche would also apply to numerous others, including Chuquet's use of various problems from Barthelemy of Romans.<sup>8</sup> As yet, there has not been a published transcription of Chuquet's solution to problems of his *Appendice*. The partial English translation by Flegg, Hayes and Moss<sup>9</sup> only added more to the confusion than that it resolved the issue. We will further try to give a clear picture of the improvements made by de la Roche with regards to the second unknown.

Another reason for the neglect of de la Roche is related to the inscrutable way he is referred to. Buteo for example, calls him Stephanus à Rupe de Lyon,<sup>10</sup> and so does John Wallis.<sup>11</sup> Together with other obscure references, such as Gosselin's naming of Villafrancus Gallus, instances mentioning de la Roche have been overlooked by his peers as well as commentators.<sup>12</sup> Therefore his name and the title of his book are not very well represented in sixteenth-century works on algebra. However, he did have an influence which can be determined in several works that do not credit him but which use problems and definitions from the *Larismethique*.

### 3. The Rule of Quantity

That de la Roche made an important contribution to the emergence of symbolic algebra during the sixteenth century can best be argued by his treatment of the second unknown, sometimes called *Regula quantitatis* or Rule of Quantity. The importance of the use of multiple unknowns in the process leading to the concept of an equation cannot be overestimated. We have traced the use and the development of the second unknown in

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<sup>7</sup> A. Marre, *Notice*, art. cit. p. 28; this and subsequent translations are mine.

<sup>8</sup> As demonstrated in: Maryvonne Spiesser, *Une Arithmétique commerciale du XVe Siècle*. 'Le Compendy de la Praticque des nombres' de Barthélemy de Romans, Brepols, Turnhout, 2003.

<sup>9</sup> G. Flegg, C. Hay and B. Moss, *Nicolas Chuquet*, op. cit.

<sup>10</sup> Johannes Buteo, *Logistica*. Gulielmum Rovillum, Lyon, 1559, cited on pages 118, 183, 184, and 216.

<sup>11</sup> John Wallis, *A Treatise of Algebra Both Theoretical and Practical*, Printed by John Playford for Richard Davis, London, 1685, on p. 63.

<sup>12</sup> Guillaume Gosselin, *De arte magna, seu de occulta parte numerorum, quae & algebra, & almucabala vulgo dicitur, libri qvatvor*. Aegidium Beys, Paris, 1577.

algebraic problem solving from the introduction of Arabic algebra in Europe until the systematic treatment of a system of linear equations by Gosselin.<sup>13</sup> The first important step can be attributed to the Florentine abacus master Antonio de' Mazzinghi, who wrote an algebraic treatise around 1380.<sup>14</sup> Luca Pacioli copied almost literally the solution method in his *Summa* of 1494, and Cardano used the second unknown both in his *Arithmetica* and the *Ars Magna*.<sup>15</sup> A second thread of influence is to be distinguished through the *Triparty* by Chuquet and the printed works of de la Roche and Christoff Rudolff.<sup>16</sup> The Rule of Quantity finally culminates in the full recognition of a system of linear equation by Buteo and Gosselin. The importance of the use of letters to represent several unknowns goes much further than the introduction of a useful system of notation. It contributed to the development of the modern concept of unknown and that of a symbolic equation. These developments formed the basis on which Viète could build his theory of equations.<sup>17</sup>

Before discussing the examples by Chuquet, de la Roche and Rudolff, it is appropriate to emphasize the difference between the rhetorical unknown and unknowns used in modern transliterations. Firstly, the method of using a second unknown is an exception in algebraic practice before 1560. In general, algebraic problem solving before the seventeenth century uses a single unknown. This unknown is easily identified in Latin text by its name *res* (or sometimes *radix*), *cosa* in Italian and *coss* or *ding* in German. The unknown should be interpreted as a single hypothetical value used within the analytic method. Modern interpretations such as an indeterminate value or a variable, referring to eighteenth century notions of function and continuity, do not fit the historical context. In solving problems by means of algebra, abacus masters often use the term 'quantity' or 'share' or 'value' apart from the *cosa*. The rhetoric of abacus algebra requires that the quantities given in the problem text are formulated in terms of the hypothetical unknown. The problem solving process typically starts with 'suppose that the first value sought is one *cosa*'. These values or unknown quantities cannot be considered algebraic unknowns by themselves. The solution depends on the expression of all unknown quantities in terms

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<sup>13</sup> Albrecht Heffer, "The *Regula Quantitatis*: From the Second Unknown to the Symbolic Equation" Monograph, forthcoming.

<sup>14</sup> Gino Arrighi, (ed.) *Antonio de' Mazzinghi. Trattato di Fioretti, secondo la lezione del codice L.IV.21 (sec. XV) della Biblioteca degli Intronati di Siena..* Domus Galilaeana, Pisa, 1967.

<sup>15</sup> Girolamo Cardano, *Practica arithmetice, & mensurandi singularis. In qua que preter alias cōtinentur, versa pagina demnstrabit.* Mediolani : Io. Antonins Castellioneus medidani imprimebat, impensis Bernardini calusci, 1539. Girolamo Cardano, *Artis magna; sive, De regylis algebraicis, Lib. unus. Qui & totius operis de Arithmetica, quod opvs perfectvm inscripsit, est in ordine decimus.* Johann Petreius, Nürnberg, 1545.

<sup>16</sup> Christoff Rudolff, *Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeinlicklich die Coss genent werden . Darinnen alles so treülich an tag gegeben , das auch allein auss vleissigem lesen on allen mündtlichē vnterricht mag begriffen werden , etc.* Vuolfius Cephaleus Joanni Jung, Strassburg, 1520.

<sup>17</sup> François Viète, *Francisci Vietae In artem analyticem isagoge. Seorsim excussa ab Opere restituae mathematicae analyseos, seu algebra nova,* apud Iametium Mettayer typographum regium, Turonis, 1591.

of the *cosa*. Once a value has been determined for the *cosa*, the unknown quantities can then easily be determined.

However, several authors, even in recent publications, confuse the unknown quantities of a problem, with algebraic unknowns. As a result, they consider the rhetorical unknown as an auxiliary one. For example, in his commentary of Leonardo of Pisa's *Flos*, Ettore Picutti consistently uses the unknowns  $x, y, z$  for the sought quantities and regards the *cosa* in the linear problems solved by Leonardo to be an auxiliary unknown.<sup>18</sup> The "method of auxiliary variable" as a characterization by Barnabas Hughes for a problem-solving method by ben-Ezra also follows that interpretation.<sup>19</sup> We believe this to be a misrepresentation of the original text and problem-solving method.

The more sophisticated problems sometimes require a division into sub problems or subsequent reasoning steps. These derived problems are also formulated using an unknown but one which is different from the unknown in the main problem. For example, in the anonymous manuscript 2263 of the Biblioteca Riccardiana in Florence (c. 1365), the author solves the classic problem of finding three numbers in geometric proportion given their sum and the sum of their squares.<sup>20</sup> He first uses the middle term as unknown, arriving at the value of 3. Then the problem of finding the two extremes is treated as a new problem, for which he selects the lower extreme as unknown. We will not consider such cases as the use of two unknowns, but the use of a single one at two subsequent occasions.

We have given some examples of what should not be comprehended as a second unknown, but let us turn to a positive definition. The best characterization of the use of several unknowns is operational. We will consider a problem solved by several unknowns if all of the following conditions apply in algebraic problem solving:

- 1) The reasoning process should involve more than one rhetorical unknown which is named or symbolized consistently throughout the text. One of the unknowns is usually the traditional *cosa*. The other can be named *quantità*, but can also be a name of an abstract entity representing a share or value of the problem.
- 2) The named entities should be used as unknowns in the sense that they are operated upon algebraically by arithmetical operators, by squaring or root extraction. If no operation is performed on the entity, it has no operational function as unknown in solving the problem.
- 3) The determination of the value of the unknowns should lead to the solution or partial solution of the problem. In some cases the value of the second unknown is not determined but its elimination contributes to the solution of the problem. This will also be considered as an instance of multiple unknowns.

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<sup>18</sup> Ettore Picutti, "Il 'Flos' di Leonardo Pisano dal Codice E.75P. sup. Traduzione e Commenti", *Physis*, 25 (1983), pp. 293-387.

<sup>19</sup> Barnabas B Hughes, "A Treatise on Problem Solving from Early Medieval Latin Europe", *Mediaeval Studies*, 63, (2001) pp. 107-41; p. 126.

<sup>20</sup> Annalisa Simi, (ed.) *Anonimo (sec. XIV). Trattato dell'Alcibra Amuchabile (dal Codice Ricc. 2263 della Biblioteca Riccardiana di Firenze)*, Quaderni del Centro Studi della Matematica Medioevale, 22, Università di Siena, Siena, 1994, pp. 39-40.

- 4) The entities should be used together at some point of the reasoning process and connected by operators or by a substitution step. If the unknowns are not connected in this way the problem is considered to be solved by a single unknown.

In all the examples discussed below, these four conditions apply.

#### 4. Chuquet's use of the second unknown

Without any previous introduction or explanation, Chuquet uses a second unknown in problem 71 of the *Appendice*:<sup>21</sup>

There are three who each have a sum of denari in such proportion that when the first gets 7 from the others, he would have the fivefold of their rest plus one. The seconds says that when he gets 9 from the first and third, he would have the sixfold of their rest plus two. And the third says that when he gets 11 from the two others, he would have the sevenfold of their rest plus 3. There is to know how much denari each has of his own.

Chuquet's direct source is most likely the very similar problem from the *Le Compendy De la pratique des nombres*, by Barthelemy of Romans. Barthelemy's problem originates from Fibonacci.<sup>22</sup> In modern notation:

$$\begin{aligned} a+7 &= 5(b+c-7)+1 \\ b+9 &= 6(a+c-9)+2 \\ c+11 &= 7(a+b-11)+3 \end{aligned} \quad (1.1)$$

The solution method by Fibonacci is purely arithmetical. Barthelemy gives two solutions, the first corresponding with Fibonacci, the second using the rule of false position. The arithmetical solution depends on the reformulation of the problem using  $s$ , as the sum of the three shares, as follows:

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<sup>21</sup> Paris, Fonds Français 1346, 1484, f. 168<sup>r</sup>-169<sup>r</sup>: "Encore plus. Ilz sont troys qui ont chascun tant de deniers et en telle proporcion que le premier avoit 7 ds. des aultres il auroit le quintuple de ce qui leur reste plus 1. Le second dit que sil avoit 9 ds. du premier et du tiers il aurait le sextuple de leur reste et 2 plus. Et le tiers dit que sil avoit 11 ds. des deux aultres Il auroit le septuple de leur reste plus 3. Assavoir moult quantz deniers a ung chascun deulx", transcription and translation mine.

<sup>22</sup> M. Spiesser, *Une Arithmétique commerciale*, op. cit., pp. 90-100 and pp. 317-320. For Fibonacci's version see Baldessare Boncompagni, *Scritti di Leonardo Pisano, matematico del secolo decimoterzo*, pubblicati da Baldassarre Boncompagni, *Scritti di Leonardo Pisano, matematico del secolo decimoterzo* Vol. I, *Liber Abaci*, Rome, 1862, p. 200 for the Latin original and Laurence Sigler, *Fibonacci's Liber Abaci. A Translation into Modern English of Leonardo Pisano's Book of Calculation*, Springer, Heidelberg, 2002, p. 301, for the English translation.

$$a + 6 = \frac{5}{6}(s - 1)$$

$$b + 7 = \frac{6}{7}(s - 2)$$

$$c + 8 = \frac{7}{8}(s - 3)$$

From this, one can determine the value of the sum and hence the values of the three shares. Chuquet's solves the problem in two ways, each method uses algebra. The first solution is based on a combination of the rule of double false position with algebra ("par la Regle de deux positions et par la Regle des premiers tout ensemble").<sup>23</sup>

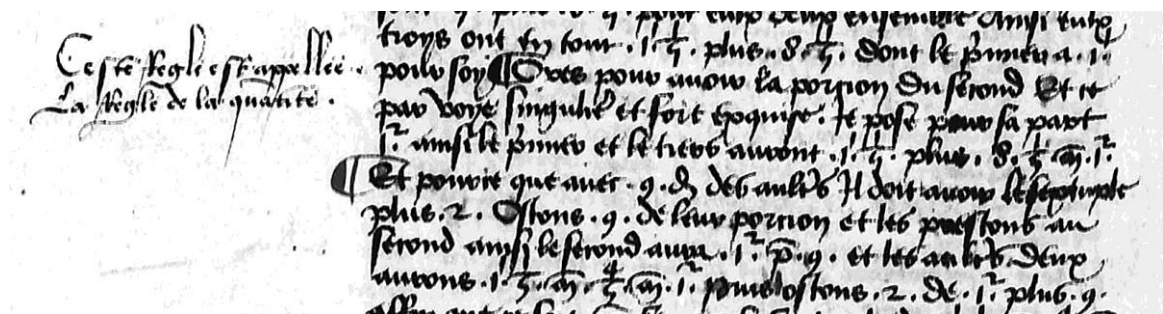


Figure 1: The first use of the second unknown in the *Triparty* by Chuquet (1484, f. 169v) (reproduced from the BNF microfilm)

Chuquet introduces  $1^2$  for the second unknown without any introduction or previous explanation. At this point, a marginal annotation reads "This rule is called the rule of quantity" ("Ceste Regle est appellée La Regle de la quantité"). This comment is most likely from the hand of de la Roche who explains the method in his arithmetic published in 1520.

Chuquet then turns to a method using "la Regle des premiers tant seulement" (f. 169v), which is his expression for the algebraic method. Here, he introduces the second unknown, without even using a name for the new entity. To differentiate the second unknown from the first  $1^1$ , he uses the notation  $1^2$ , which could be read as  $1y$ . The base number acts as the coefficient, so  $6^2$  signifies  $6y$ . This is really puzzling as he previously used that notation for the second power of the unknown. Chuquet most certainly was aware of the possible confusion, but apparently had no other means to his disposal. Chuquet's method can be summarized as follows:

- use the first unknown for the first share (1)
- use the second unknown for the second share
- express the value of the second share in the first unknown (2)
- use the second unknown for the third share
- express the value of the third share in the first unknown (3)
- add (1), (2) and (3) together and determine the value of the unknown

<sup>23</sup> A marginal notation, probably by de la Roche, gives "par la regle de deux positions et par la regle de la chose ensemble", f. 168f.

- the value of the other shares can be determined from (2) and (3)

In the *Appendice*, Chuquet solves five problems by this method.<sup>24</sup>

## 5. de la Roche and La Regle de la Quantite

De la Roche first mentions *la regle de la quantite* in the beginning of distinction six together with *la regle de la chose*. He properly introduces the second unknown in a separate chapter titled *Le neufiesme chapitre de la regle de la quantite annexee avec le dict premier canon, et de leur application*, in the sixth distinction of the first part :<sup>25</sup>

This rule of quantity is added and follows from the first canon of the rule of the thing as a realization and perfection of this rule. The motivation is that it happens many times, for several reasons of this canon, that one has to pose two, three or more times that the first position is 1  $\rho$ . After that, one has to pose a second or third time, etc. It is therefore necessary that the second, third or fourth position should be a number different from  $\rho$ . Because when the numbers for the second, third and fourth positions are the same and indistinguishable from the numbers for  $\rho$ , or the other positions, this would lead to confusion. In that case we are unable to determine which positions stand for the numbers or for  $\rho$ . Therefore it is necessary to make another distinction between the numbers to distinguish and differentiate the first position from the others. And the same for the second, third and fourth positions, etc.

In the *Rule of Quantity*, de la Roche sees a perfection of algebra itself. The use of several unknowns allows for an easy solution to several problems which might otherwise be more difficult or even impossible to solve. Interestingly, he insists on the use of clearly distinguishable notation for the other unknowns, obviously referring to the ambiguities in Chuquet's use of 1<sup>2</sup>. After this introduction, de la Roche gives six examples of the rule of quantity applied to the typical linear problems, though he removes the practical context. Then he presents five indeterminate problems under the heading "questions which have multiple responses", without use of the second unknown. Finally he solves five problems

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<sup>24</sup> We have only been able to inspect the complete manuscript during two days at the BNF on site, but problems 71, 75, 77, 78 and 79 seem to be the only ones solved by use of the second unknown.

<sup>25</sup> de la Roche 1538, f. 42<sup>v</sup> : "Ceste regle de la quantite est annexee et inferee au premier canon de la regle de la chose comme accomplissement et perfection dicelluy pource que souventes foys advient en plusieurs raisons de ce canon quil fault poser deux, trois ou plusieurs fois desquelles tousiours la premiere position est 1  $\rho$ . Sil vient apres quil faille poser une aultre fois ou deux ou troys etc. Il est besoing que la seconde, tierce ou quatre positions soyent aultres differences de nombre que  $\rho$ . Car qui seroit la seconde tierce ou quatre positions de 1  $\rho$ ., le nombres et  $\rho$  de la premiere positions seroyent mesmes et intrinques indifferenment avec les nombres et  $\rho$  des autres positions qui seroit confusion. Car on ne scavroit distinguer de quelle positions seroyent lesdictz nombres et  $\rho$ . Et pour ce est necessaire de trouver aultre difference de nombre pour distincter et differer la premiere position des aultres. Et par ainsi en la seconde, tierce ou quatre positions etc".



using the second unknown under the heading “other inventions on numbers”. At the end of the book there is a chapter on applications in which four more problems are given (f. 149<sup>v</sup>-150<sup>r</sup>). In total, there are twenty problems solved by the *regle de la quantite*.

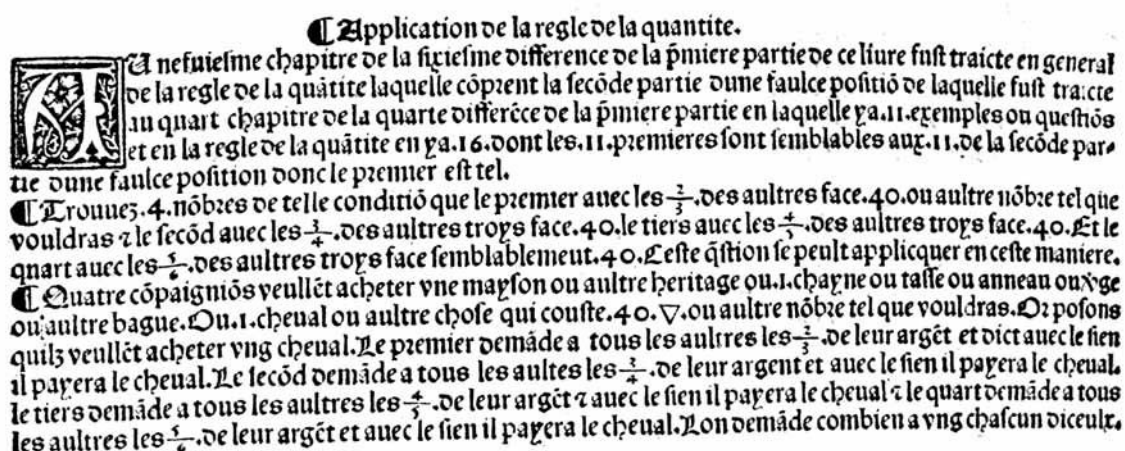


Figure 2: de la Roche (1538, f. 149v) applies the second unknown.

In a section on the application of the Rule of Quantity, de la Roche solves some difficult linear problems with the second unknown. The first problem is formulated in two ways. Once as properties of four numbers and once in a practical context of four companions buying a house (reproduced from a copy at the Université de Liège, Belgium, Réserve 194C).

We will look at a typical solution to problem 70 also treated by Chuquet.<sup>26</sup> Marre points out that de la Roche uses the text from Chuquet “word by word”, but does not mention that Chuquet solves the problem in a different way. As we have discussed above, Chuquet uses a combination of double false position and algebra. In meta-description the problem can be formulated as follows:

$$\begin{aligned} a+12 &= 2(b+c-12)+6 \\ b+13 &= 4(a+c-13)+2 \\ c+11 &= 3(a+b-11)+3 \end{aligned} \quad (1.2)$$

Estienne de La Roche solves the problem twice.<sup>27</sup> He proceeds in a way very similar to Chuquet. We discuss here the first solution.

Nr	Symbolic representation	Meta description	Original text
1	$a+12 = x+12$	choice of first	pose sue le premier soit 1 ρ

<sup>26</sup> Marre, *Appendice*, op. cit. f. 168<sup>r</sup>, pp. 432-3.

<sup>27</sup> Once under the heading *Other Inventions on Numbers* in *L’arismethique*, 1538, f. 43<sup>v</sup> and once in the final chapter of *L’arismethique*, 1520, f. 217<sup>r</sup>; 1538, f. 150<sup>r</sup>. Marre only mentions the second solution.

		unknown	adiouster avec 12 fait 1 p p 12
2	$a+12-6=x-6$	subtract 6 from (1)	dont fault oster 6 r reste 1 p p 6
3	$\frac{a+12-6}{2}=\frac{x}{2}+3$	divide (2) by 2	dont la ½ gest ½ p p 3
4	$b+c-12=\frac{x}{2}+3$	substitute (3) in (1.2)(a)	sont les autres deux nombres
5	$a+b+c=\frac{3}{2}x+15$	add $x+12$ to (4)	ainsi tous troys sont 1 p ½ p 15.
6	$b=y$	choice of second unknown	Après pose le second nombre 1 quantite
7	$b+13=y+13$	add 13 to (7)	adiouste a 13 font 13 p 1 quantite.
8	$b+11=y+11$	subtract 2 from (7)	Dont fa[u]lt lever 2 et reste 11 p 1 quantite
9	$b+11=4(a+c-13)$	from (1.2)(b)	
10	$y+11=4(a+c-13)$	substitute (8) in (9)	
11	$\frac{1}{4}y+2\frac{3}{4}=a+c-13$	divide (10) by 4	dont le ¼ qui est 2 ¾ p ¼ quantite
12	$\frac{1}{4}y+2\frac{3}{4}=\frac{3}{2}x+2-y$	substitute (5) and (6) in (11)	sont egaulz a 1 p ½ p 2 m 1 quantite qui sont les aultres 2 nombres quantite les 13
13	$1\frac{1}{4}y=1\frac{1}{2}x-\frac{3}{4}$	add $y-2\frac{3}{4}$ to (12)	en sont levez ores egaliz
14	$y=1\frac{1}{5}x-\frac{3}{5}$	divide (10) by 5/4	et partiz par les quantitez et trouveras 1 p 1/5 m 3/5 pour le second nombre.
15	$c+11=z+11$	reuse of the second unknown	puis pose le tiers nombre 1 quantite adiouste avec 11 font 11 p 1 quantite.
16	$c+8=z+8$	subtract 3 from (15)	Dont fault lever 3 et restent 8 p 1 quantite
17	$z+8=3(a+b-11)$	substitute (16) in (c)	qui sont les aultres deux nombres quant les 11 en sont leves.
18	$\frac{1}{3}z+2\frac{2}{3}=a+b-11$	substitutes (15) in (16)	dont le 1/3 ques est 2 2/3 p 1/3 quantite
19	$\frac{1}{3}z+2\frac{2}{3}=1\frac{1}{2}x+4-z$	substitute (5) and (16) in (18)	sont egault a 1 p ½ p 4 - 1 quantite
20	$\frac{4}{3}z=1\frac{1}{2}x+\frac{4}{3}$	add $z-2\frac{2}{3}$ to (20)	ores egaliz

21	$z = 1\frac{1}{8}x + 1$	divide (20) by 4/3	et partis comme dessus trouveras 1 p 1/8 p 1 pour le tiers nombre.
22	$x + y + z = 3\frac{13}{40}x + \frac{2}{5}$	substitute (14) and (21) in (5)	Ores adioste tous les troys nombre ensemble
23	$3\frac{13}{40}x + \frac{2}{5} = 1\frac{1}{2}x + 15$	equal (20) and (5)	et auras 3 p 13/40 p 2/5 egaulz a 1 p 1/2 p 15
24	$\frac{73}{40}x = \frac{73}{5}$	from (23)	puis egaliz
25	$a = 8$	divide (24) by 40/73	et partiz le nombre par la p et auras 8 pour le premier nombre.
23	$b = 9, c = 10$		Et par consequent 9 pour le second et 10 pour le tiers.

By using a new symbol for the second unknown, de la Roche resolves one of the ambiguities from Chuquet's manuscript, the use of the same symbol as for the square of the unknown. The second ambiguity, the use of the same unknown for the second and third unknown quantities, is not eliminated. Only because the second and third unknowns are not related during the derivation, he can use the entity *quantite* twice.

## 6. Christoff Rudolff's Coss

The first printed book entirely devoted to algebra was published in 1525 in Strassburg under the title *Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeincklich die Coss genennt werden*. We do not know much about its author Christoff Rudolff. As he mentions that he used the examples from the book for teaching his students algebra, he may have been a professor in Vienna. Rudolff gives a short introduction on arithmetic, followed by the basic operations on polynomials. The rest of the book is filled with the algebraic solution to over four hundred problems, divided into eight sections corresponding with the eight equation types. Rudolff explicitly uses the word *equation*. The *Regula Quantitatis* is not mentioned in the introduction but used in the examples of the first rule (on linear equations). The rule is introduced as follows:<sup>28</sup>

<sup>28</sup> For a long time I have not been able to consult the original edition of the book. Some parts have been published in secondary sources. This quote is from Rudolff (1525, f. βvi<sup>v</sup>, and cited by Peter Treutlein, "Die Deutsche Coss", *Abhandlungen zur Geschichte der Mathematik*, Heft 2, (1879), pp. 1-124, p. 84: "Dise regl lernt wie man sich halten sol bey etlichen exempeln, so über den gesetzten radix (wie dann der brauch ist) auch andere position oder satzungen erfordern. Dann so 1 x einem ding gesetzt oder zugeben ist mag er in dem selbigen procesz (confusion oder irsal zu vermeiden) keinem andern ding zugestellt werden. Laut also. Wann nach setzung 1 x ein ding vorhanden ist welchem du (ausz vorgethaner underweysung) mit der position nit magst zukommen. Setz dasselbig ding sei 1 quantitet, und procedir nach laut der auffgab, so lang bisz zwo ordnung der zalen einander gleich werden ... Ist weiter etwas vorhanden. Nim war der vorigen satzungen, gib dem selbigen ding 1 quantitet, und procedir nach vorgemelter instruction" (translation mine). Treutlein discusses problems 188 and 191 from Rudolff and point at Pacioli's *Summa* as an earlier source for the *Regula quantitatis*. The quoted part has been ommitted in Stifel's edition: Michael Stifel, *Die Coss Christoffe Ludolffs mit schönen*

This rule teaches us how to approach numerous examples, by using also other positions or statements in addition to the root (as is usual). So, when you pose or allow 1  $x$  for one thing, then you cannot do the same for another thing within the same process (to avoid confusion or misunderstanding). It goes as follows. When, after using 1  $x$ , you find a thing for which you cannot pose or allow the position (as explained before). Suppose that this thing is 1 quantity, and proceed as the problem asks for, to the point that you find a value for it ... Is there yet another thing. Take the previous position, give this thing 1 quantity, and proceed as instructed above.

The procedure explains the way problems are solved with the second unknown by Chuquet and de la Roche, including the reuse of the second unknown for other unknown quantities in the problem. Example 9 goes as follows:<sup>29</sup>

Give two numbers, which together make 20. Dividing the smaller one by 8 and the larger one by 3. Adding the quotients of both gives 5.

Rudolff gives two solutions, one using a single unknown and the second using the *Regula quantitates*. Rudolff holds the rule in great admiration, calling it the perfection of the Coss (“ein vollkommenheit der Coss. Ja warlich ein sölche vollkommenheit, on welche sie nit vil mer gilt dan ein pfifferling”). Taking the first number as  $x$  and the second as 1  $q$  (“das ist ein quantitet und bedeutet 1  $q$ . auch ein ungezelete zal / als die noch ist verborgen / gleych so wol als 1  $x$ ”), he arrives at 1  $q = 20 - 1 x$ . Adding

$$\frac{x}{8} \text{ and } \frac{20-x}{3} \text{ gives } \frac{160-5x}{24} = 5 \text{ and } x \text{ is } 8.$$

He then adds that it is also possible to choose 1 $x$  for larger and 1  $q$ . for the smaller number. The value of  $x$  would then be 12.<sup>30</sup> Problems 188 to 217 in the section concerning

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*Exempeln der Coss*. Gedrückt durch Alexandrum Lutomyssensem. Zu Königsberg in Preussen, 1553, f. 186<sup>r</sup>. Wolfgang Kaunzner published an article which contain reproductions of some pages of the original edition in Wolfgang Kaunzner “Christoff Rudolff, ein bedeutender Cossist in Wien”, in Rainer Gebhardt and Helmuth Albrecht (eds.) *Rechenmeister und Cossisten der frühen Neuzeit*, Beiträge zum wissenschaftlichen Kolloquium am 21. September 1996 in Annaberg-Buchholz, Annaberg-Buchholz, (1996) pp. 113-38. Recently there is a new book by Kaunzner and Kart Röttel, *Christoff Rudolff aus Jauer in Schlesien. Zum 500. Geburtstag eines bedeutenden Cossisten und Arithmetikers, der aus diesem seinerzeit hoheitlich zur Krone von Böhmen gehörenden Landesteil stammt*. Polygon Verlag, 2006, which includes a complete facsimili of the original edition. The quote is listed on p. 87.

<sup>29</sup> Michael Stifel, *Die Coss Christoffe Ludolffs*, op.cit., f.185<sup>v</sup>: “Gib zwo zalen / welche zsamen 20 machen: wenn ich die kleynere dividir durch 8. Die grossere ducr 3. Thu die quotient zu samen / das 5 werden”. This book by Stifel is an annotated edition of the 1525 original, with several additions by Stifel, as on the cubic equation.

<sup>30</sup> This observation is important for the distinction of double solutions to the quadratic equation.

the first rule (linear problems) are all solved using the *regula quantitatis*. Problem 192 is a typical linear problem of four men buying a horse, formulated as follows:<sup>31</sup>

Four companions bought a horse for 11 fl. Each of them needed something from the other three companions. Namely, the first  $\frac{1}{2}$  of their money, the next  $\frac{2}{3}$ , the third  $\frac{3}{4}$ , and the fourth  $\frac{4}{5}$ . So that each of them could buy the horse. How much had each?

192. Vier gesellen haben ein roß laufft p 11 floren. Wegeret jeder zu dem das er hat / von seinen gesellen: nemlich der erst  $\frac{1}{2}$  ives gelts. der ander  $\frac{2}{3}$ . der dritt  $\frac{3}{4}$ . der vierd  $\frac{4}{5}$  alles gelts seiner geselschafft / so hab jeder das roß zu zalen: Setz dem ersten 12 flo / so manglen ime 11 9 - 12. Solcher mangel zeigt  $\frac{1}{2}$  des gelts seiner gesellen. volgt / das der ander / dritt / vierd / sammentlich haben 22 9 - 22. darzu addir 12 (des ersten gelt) kompt die summa aller flo: 22 9 - 12. procedier weiter / eben wie im negsten exempl. Setz dem andern 1 quantitet / so komen seiner geselschafft 22 9 - 12 - 1 quantitet / darauf  $\frac{2}{3}$  thun 14  $\frac{2}{3}$  9 -  $\frac{2}{3}$  12 -  $\frac{2}{3}$  quant. darzu addir 1 quantitet: (des andern gelt) summa fa: 14  $\frac{2}{3}$  9 -  $\frac{2}{3}$  12 -  $\frac{2}{3}$  12 +  $\frac{2}{3}$  quant / gleich 11 9. Thun im also. Addir  $\frac{3}{4}$  12 zu 11 9. Subtrahir 11 9 von 14  $\frac{2}{3}$  9 / pleibē 3  $\frac{2}{3}$  9 / die subtrahir von  $\frac{2}{3}$  12 (auff d; die quant. allein stet) Rest  $\frac{1}{4}$  12 - 3  $\frac{2}{3}$  9 gleich  $\frac{1}{4}$  quant. Diuidir 12 komen 22 - 11 9 / souil floren hat der ander. Weiter setz dem dritten 1 quantitet / so komen seiner geselschafft die abnigen floren / 22 9 - 12 - 1 quantitet. darauf  $\frac{3}{4}$

Exempl bet  
thun 16  $\frac{2}{3}$  9 -  $\frac{2}{3}$  12 -  $\frac{2}{3}$  quant: darzu addir des dritten gelt / summa fa: 16  $\frac{2}{3}$  9 -  $\frac{2}{3}$  12 +  $\frac{2}{3}$  quant / gleich 11 9. Procedier nach inhalt der cauteln / werden  $\frac{2}{3}$  12 - 5  $\frac{2}{3}$  9 gleich  $\frac{1}{3}$  quantitet. Diuidir 12 kompt des dritten gelt / 32 - 22 9. Entlich setz dem vierden 1 quantitet: vnd thun wie vorhin / komen dem vierden 42 - 33 9 Summa summarum alles gelts facit 102 - 66 9 gleich 22 9 - 12. Nachs; facit 12 8 9 / souil floren hat der erst. Der ander 5. Der dritt 2. Der vierd - 1 floren. volgt vnmügligheit. Proba. der ander / dritt / vierd haben 6 floren (-1 ist abzogen) darauf  $\frac{2}{3}$  12 / das gib ich dem ersten so hat er 11. Der erst / dritt / vierd haben 6 / dar auf  $\frac{2}{3}$  thun 6 / gib ich dem andern / werden 11. Der erst / ander / vierd / haben 12. drei viertheil auß 12 sein 9. gib ich dem dritten / werden 11. Der erst / andern / dritt / haben 15. vierfünftheil auß 15 / thun 12 / gib ich dem vierdest / werden auch 11. Hab ich wellen dar pvingen.

Figure 3: A solution to the “men buy a horse” problem from Rudolff (1525 f. Qi<sup>r</sup>-Qi<sup>v</sup>) using the *regula quantitatis*. (facsimili from “Christoff Rudolff, ein bedeutender Cossist in Wien”, 128-9)

This translates andern into modern notation as follows:

$$\begin{aligned} a + \frac{1}{2}(b + c + d) &= 11 \\ b + \frac{2}{3}(a + c + d) &= 11 \\ c + \frac{3}{4}(a + b + d) &= 11 \\ d + \frac{4}{5}(a + b + c) &= 11 \end{aligned} \quad (1.3)$$

<sup>31</sup> Rudolff, *Behend vnnnd Hubsch Rechnung*, op. cit. f. Qi<sup>r</sup>-Qi<sup>v</sup>: “Vier kauften ein Roß um 11 flo; sie begehrten je der Reihe nach vom Geld der übrigen Gesellen, um kaufen zu können: Der erste  $\frac{1}{2}$ , der andere  $\frac{2}{3}$ , der dritte  $\frac{3}{4}$ , der vierte  $\frac{4}{5}$ ”.

Rudolff uses the first unknown  $x$  for the money of the first person. The other three thus have  $22 - 2x$ . Add to this  $x$  and you have the sum of all four as  $22 - x$ . Then he introduces the second unknown for the second person's money ("Setz dem andern 1 quantitet, so komen seiner geschelschafft 22 Ø - 1x - 1 quantitet"). The sum of the three others then is  $11 - 1x - 1y$ . Using this value in the second expression of (1.3) this gives

$$14\frac{2}{3} - \frac{2}{3}x + \frac{1}{3}y = 11 \text{ or } \frac{2}{3}x - 3\frac{2}{3} = \frac{1}{3}y$$

and the second persons money is  $y = 2x - 11$ .

Rudolff then uses the second unknown for the share of the third one ("Weiter setz dem dritten 1 quantitet"), which we also represent as  $y$ . Using the same reasoning, the share of the three others is  $22 - 1x - 1y$ . Using the third expression in (1.3) and simplifying this leads to

$$\frac{3}{4}x - 5\frac{1}{2} = \frac{1}{4}y \text{ or } y = 3x - 22.$$

Again for the fourth which gives  $4x - 33$ . Adding the four shares together gives  $10x - 66 = 22 - x$  leading to the interesting solution (8, 5, 2, -1). Rudolff explicitly denies the possibility of a negative value ("Der vierd - 1 floren, volgt unmugligkeit") but proceeds with a *proba* proving the validity of the solution. Stifel will later reproduce the problem, slightly adapting its solution.<sup>32</sup> He uses basically the same method but changes the second unknown to  $1B$ ,  $1C$  and  $1D$  respectively and uses  $1A$  for the sum of the three others. He gives an interpretation of the negative value as follows:<sup>33</sup>

The fourth has  $0 - 1$  fl., meaning that he has no money at all. He thus owes a debt of 1 fl. to the one who sells the horse. While the others have to pay 11 fl. for the horse, the fourth will have to pay 12 fl.

The interpretation by Stifel of the fourth man's share as a debt is denying the existence of negative solutions in the same spirit as Rudolff. It is therefore doubtful to consider this as an instance of the acceptance of negative solutions.<sup>34</sup>

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<sup>32</sup> Stifel, *Die Coss Christoffe Ludolffs*, op.cit., ff. 310<sup>v</sup> - 312<sup>f</sup>.

<sup>33</sup> Stifel, *Die Coss Christoffe Ludolffs*, op.cit.

<sup>34</sup> Similar linear problems by Fibonacci from the *Flos* and the *Liber Abaci* have recently been discussed both by Jacques Sesiano, "The Appearance of Negative Solutions in Mediaeval Mathematics", *Archive for History of Exact Sciences*, 32, (1985) pp. 105-50, and Helmut Gericke, "Zur Geschichte der negativen Zahlen", in J. Dauben, M. Folkerst, E. Knobloch and H. Wußing (eds.) *History of Mathematics. States of the Art*, Academic Press, New York, 1996, pp. 279-306. Fibonacci also gives the interpretation that the man started with a debt to the other

Allegedly, Rudolff borrowed heavily from the Latin manuscript Vienna 5277 as was known by his peers.<sup>35</sup> Stifel refers to these allegations in his introduction to the 1553 edition.<sup>36</sup> He does not really counter these claims, instead pitying those who demean oneself by such criticism. Even if it were true, he did not hurt anyone by doing so.<sup>37</sup> However, Rudolff must have kept some other sources secret. The Vienna codex contains practically no linear problems, while they are prominent in the *Coss*. No other known manuscripts from Vienna and München around 1500 use more than one unknown. Rudolff mentions that he was given the problem by Johann Seckerwitz from Breslau, and that he solved it using the *regula quantitatis*.<sup>38</sup>

Rudolff's solution compares very well with that of Chuquet and de la Roche as well as the example of Pacioli not discussed here.<sup>39</sup> Clearly, one of those must have been the source for the method and the name *regula quantitatis*. Treutlein and Tropfke assume that Rudolff learned the rule from Pacioli, but they do not cover the rule in relation to Chuquet.<sup>40</sup> While the exact influence will be hard to demonstrate there are strong

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persons of 3 besants. Sesiano and Gericke have no trouble interpreting the text as an instance of a solution with a negative number.

<sup>35</sup> Moritz Cantor, *Vorlesungen über Geschichte der Mathematik* (4 vols.), (1880-1908) Vol. I (1880): "Von den ältesten Zeiten bis zum Jahre 1200 n. Chr.," Vol. II (1892): "Von 1200-1668", 2nd ed. Teubner, Leipzig, 1894, 1900, vol II. p. 424. Wolfgang Kaunzner, "Über einige algebraische Abschnitte aus der Wiener Handschrift nr. 5277", *Österreichische Akademie der Wissenschaften. Mathematisch-Naturwissenschaftliche Klasse. Denkschriften*, 116 / 4 Vienna (1972), pp. 115-88, p. 115.

<sup>36</sup> Stifel, *Die Coss Christoffe Ludolffs*, op. cit.

<sup>37</sup> Stifel, *Die Coss Christoffe Ludolffs*, op.cit. f. A3: "Ob denn gleich Christoff Rudolff seine Exempla nicht alle selbs hatte gedichtet, sondern etzliche in der Librey zu Wien abgeschrieben, und uns die selbige durch den truck mitgeteylet, wem hat er damit schaden gethon? Niemand's."

<sup>38</sup> Rudolff, *Behend vnnd Hubsch Rechnung*, op. cit., Rii<sup>v</sup>, example 210: "Mit diesem exempl ward ich zu Breslaw von meinem guten freund Johansen Seckerwitz, Rechenmeister daselbs, durch die coss zu machen ersucht, war zur selbigen zeyt der regl quantitatis nit bericht". Johann Seckerwitz published several arithmetic books. In 1519 *Rechpuchleyn* at Breslaw; and *Regel oder Satzbüchlein... auff der Linien*, Rękopis, and *Rechbuchleyn*, in 1535 at Wrocław, republished in 1547. This passage is deleted in Stifel's edition, see Gustav Eneström, "Kleine Bemerkungen zur zweiten Auflage von Cantors *Vorlesungen über Geschichte der Mathematik*", *Bibliotheca Mathematica*, 3 /7 (1906-7), p. 204.

<sup>39</sup> Pacioli, *Summa*, f. 192<sup>r</sup>: "Tre hano denari. Dici el primo a li 2 altri: dateme la ½ di vostri che haro 50. Dici el secondo alialtri 2, datime el 1/3 di vostri che haro 50. Dici el 3° alialtri 2, datime el ¼ di vostri che haro 50. Dimando che a per uno"

<sup>40</sup> Treutlein "Die Deutsche Coss", op. cit. Johannes Tropfke, *Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter*, vol. 1 (1930): *Rechnen*, vol. 2 (1933): *Algemeine Arithmetik*, vol. 3 (1937): *Proportionen, Gleichungen*, vol. 4 (1940): *Ebene Geometrie*, (third edition), Walter de Gruyter, Leipzig. From vol. II, p. 52: "Die Kenntnis der Neuerung Paciolis kam auch zu den deutschen Cossisten", and also vol. III, p. 46.

arguments in favor of Chuquet or de la Roche rather than Pacioli. This would throw a new light on the German cossist tradition because a line of transmission from France to Germany has not previously been established. Let us look at the arguments:

- 1) Rudolff's use of the word *quantitet* favors de la Roche rather than Pacioli's who cites *quantita sorda* as the original name "used by the ancients".
- 2) Rudolff explicitly uses the term *Regula Quantitatis*. The French term is used by de la Roche in the heading of chapter 9 and in "Application de la regle de la quantite". Pacioli, on the other hand, does not make any reference to a rule. He considers the *quantita sorda* more as a generalization of the *cosa*.
- 3) de la Roche's reference to 'perfection' in "Ceste regle de la quantite est annexee et inferee au premier canon de la regle de la chose comme acomplissement et perfection dicelluy", is echoed in Rudolff's description "ein volkommenheit der Coss". We do not find such a qualification in the *Summa*.
- 4) Both Rudolff ("confusion oder irsal zu vermeiden") and de la Roche ("le nombres et p de la premiere positions seroyent mesmes et intrinques indifferenment avec les nombres et p des autres positions qui seroit confusion") argue for the use of *quantite* or *q* for the sake of avoiding confusing. The use of the foreign word *confusion* in German is significant. Pacioli makes no reference to possible confusions over *cosa* and *quantita*.

Convinced by such arguments, we embarked on a detailed comparison of linear problems solved with the second unknown in Chuquet, de la Roche and Rudolff. The result, summarized in the Appendix, is quite surprising. The relationship between Rudolff and Chuquet is closer than between Rudolff and de la Roche. Furthermore, several consecutive problems from Rudolff's *Coss* can be found in the same order in Chuquet's manuscript. It seems as if Rudolff went through Chuquet's manuscript and copied the most interesting cases while changing the values. Such close correlation raises more questions than it answers. The provenance of the manuscript was traced by Marre. After de la Roche it came into the hands of the Italian Leonardo de Villa (sold for 80 solidis), only to return to France, in the library of Jean-Baptiste Colbert, long after the publication of Rudolff.<sup>41</sup> With additional solid evidence, such influence of the early French algebraist on Rudolff would require a revision of the history of the German cossist tradition.

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<sup>41</sup> See Marre, "Notice sur Nicolas Chuquet" and Flegg, Hay and Moss, *Nicolas Chuquet*, op. cit. p.17.



## 7. Appendix B: Linear problems from Chuquet and Rudolff

Roche	Chuquet (1484)		Rudolff (1525)	
107r	(3, 2; 4, 3; 5, 5)	CTA056		
	(7, 2; 9, 6).	CTA057	(3, 3; 3, 3)	RBH1134
	(20, 2; 30, 3).	CTA058	(2, 2; 3, 3)	RBH1135
	(1, 1; 1, 2).	CTA059		
	(2, 1; 3, 4).	CTA060		
	II-(3, 2; 4, 3; 5, 5).	CTA061		
	II-(2, 2; 2, 3; 2, 4).	CTA062	II-(2, 2; 3, 3; 4, 4).	RBH1136
	I-(7, 5; 9, 6; 11, 7).	CTA063	I-(200, 3; 300, 3; 376, 4)	RBH1137
			I-(2, 2; 3, 3; 4, 4)	RBH1138
	III-(7, 5, 1; 9, 6, 1; 11, 6, 1)	CTA064		
	IV-(7, 5, 1; 5, 6, 1)	CTA065		
	IV-(7, 3, 3; 9, 4, 1)	CTA066		
	IV-(7, 5, 1; 5, 7, 2)	CTA067		
	IV-(7, 5, 2; 9, 7, 5)	CTA068		
	IV-(7, 5, 1; 5, 6, 12)	CTA069		
150r	III-(12, 2, 6; 13, 4, 2; 11, 3, 3)	CTA070		
	III-(7, 5, 1; 9, 6, 2; 11, 7, 3)	<b>CTA071</b>	III-(14, 2, 1; 13, 3, 1; 12, 4, 1)	<b>RBH1197</b>
	IV-(7, 5, -1; 5, 7, -1)	CTA072		
	IV-(7, 5, -1; 5, 7, -3)	CTA073		
	IV-(7, 5, 4; 5, 7, -6)	CTA074		
	III-(7, 2, -1; 5, 9, -2; 3, 4, -3)	<b>CTA075</b>	III-(10, 1, -3; 10, 2, -3; 10, 3, -3)	<b>RBH1198</b>
	IV-(7, 5, 4; 5, 7, -6)	CTA076		
	(7, 5; 9, 6; 11, 6)	<b>CTA077</b>	(1, 2; 4, 3; 8, 4)	<b>RBH1199</b>
	(3, 100; 4, 106; 5, 145; 6, 170)	<b>CTA078</b>	(2, 20; 3, 13; 4, 8; 5, 22),	<b>RBH1200</b>
	(7, 2; 8, 3; 9, 4; 10, 5; 11, 6).	<b>CTA079</b>		
	(7/12, 14; 9/20, 12)	CTA080		
		CTA081		
	II-(1/2, 2/3) 30	CTA082		
	II-(1/3, 1/4, 1/5) 20	CTA083		

(a, b; c, d) the general form for two people “If I had had  $a$  from you, I would have  $b$  times ..”

I-(a, b; c, d; ...) denotes the case for more people where ‘you’ means all the others.

II-(a, b; c, d; ...) denotes the same where ‘you’ means just the next person, taken cyclically.

III-( $a_1, b_1, c_1; a_2, b_2, c_2; \dots$ )

$$x + a_1 = b_1(y + z - a_1) + c_1$$

IV-( $a_1, b_1, c_1; a_2, b_2, c_2; \dots$ )

$$y + a_2 = b_2(x + z - a_2) + c_2$$

same as III but cyclically

$$z + a_3 = b_3(x + y - a_3) + c_3$$