Logique & Analyse **183–184** (2003), 289–313

A STRENGTHENING OF THE RESCHER–MANOR CONSEQUENCE RELATIONS*

DIDERIK BATENS

Abstract
The flat Rescher–Manor consequence relations — the Free, Strong, Weak, C-Based, and Argued consequence relation — are defined in terms of the classical consequences of the maximal consistent subsets of (possibly) inconsistent sets of premises. If the premises are inconsistent, the Free, Strong and C-Based consequence sets are consistent and the Argued consequence set avoids explicit inconsistencies (such as A and $\sim A$).

The five consequence relations may be applied to discussive situations as intended by Jaśkowski — the comparison with Jaśkowski's D2 is instructive. The method followed in [12] to extend D2 to an adaptive logic, may also be applied to the Rescher–Manor consequence relations. It leads to an extension of the Free, Strong, Weak, and C-Based consequence relations. The extended consequence sets are consistent and closed under Classical Logic. Applying the method to the Argued consequence relation leads to a different consequence relation, not an extension. Neither the Argued consequence relation nor its extension appear very interesting in the present application context.

1. Praeludium

Jaśkowski connected inconsistency to discussions. This leads to a very specific approach to paraconsistency, suitable to some application contexts only. Nevertheless, it is a landmark in the history of logic.

*The research for this paper was financed by the Fund for Scientific Research – Flanders, and indirectly by the Flemish Minister responsible for Science and Technology (contracts BIL98/37 and BIL01/80). I am indebted to Timothy Vermeir and to Liza Verhoeven for comments on a former draft of this paper. I am especially indebted to Joke Meheus for help with the research; some of her comments and counterexamples saved me several days of work.

Actually, I think it is more sensible to read Jaśkowski as about meetings in which decisions are taken — departmental or faculty meetings provide a case in point that Jaśkowski certainly was acquainted with.

In [12], Joke Meheus extends Jaśkowski's approach. By building an adaptive logic AJ from Jaśkowski's D2, she obtains a richer consequence set that has an appealing interpretation with respect to discussions. In view of earlier work it became clear to me that Meheus' result may be applied to extend the Rescher–Manor consequence relations — see Section 2 — and that these consequence relations as well as their extensions may sensibly sophisticate Jaśkowski's analysis of discussions.

In good order, the upshot is this. First, the Rescher–Manor consequence relations may be characterized as adaptive logics extending D2. This enlightens our understanding of discussions. Here is a simple example: if someone who does not contradict you in the discussion, makes a statement that you agree to, there is no need for you to repeat that statement. This is easily understood in terms of Rescher–Manor consequence relations, but goes beyond Jaśkowski's analysis. In other words, a discussion cannot be understood in terms of the statements made by the separate participants. It has to be described in terms of the interactions between the participants, most importantly in terms of the consequences of the statements made by groups of participants that do not contradict each other. Next, Jaśkowski's D2 as well as the Rescher–Manor consequence relations may be further extended in terms of Meheus' 'consistent core'. Even if two participants contradict each other at some point, they need not contradict each other at all points. Think about a departmental or faculty meeting. Notwithstanding fights about many points on the agenda, everyone may agree about other points and even about aspects of (or arguments relevant to) issues on which there is disagreement.

Even combining the approaches of Jaśkowski, Rescher–Manor, and Meheus, one does not obtain an adequate analysis of discussions. This is why the present paper ends with some open problems. At least one of these problems seems to be adequately resolved by Liza Verhoeven's [19]. This concerns the point that participants in a discussion may change their position. On that analysis, the position of some participant cannot always be reduced to the conjunction of statements made by him or her, but needs to be calculated again whenever the participant's stand is changed. Again, a realistic analysis requires that inconsistency is taken to indicate a change in position.[1] Some participants do not like the Canossa road, others simply fail to realize

[1] Unfortunately, Liza Verhoeven's approach is restricted to consistent interventions. This may be repaired, but a complication is involved. Any participant may be confused, but dialetheists may be serious in affirming an inconsistent position.

they changed their minds. That the latter are not very smart will not prevent other participants to rely on their support — meetings are fights about interests and (sometimes) about justice, not courses in logic.

2. *Aim of this Paper*

There are many approaches to handling inconsistency. Among the oldest ones are the Rescher–Manor consequence relations. The underlying idea is that inconsistent sets of sentences are divided into maximal consistent subsets — henceforth *MCS* — and that what 'follows' from the inconsistent set is defined in terms of the classical consequences of (a selection of) the *MCS*. Some consequence relations were implicitly present already in [14], and were articulated in [17]. Extensions and applications appeared in [15], [16], and elsewhere. Later, further consequence relations were defined within the same approach. Some of these are called "prioritized" because they depend on non-logical preferences. The others are called "flat". A survey and comparative study is presented in [8] and [9]. The present paper concentrates on the flat consequence relations: the Free, Strong, Argued, C-Based, and Weak consequence relation. The prioritized cases are studied in [20]; a generalization of them is presented in [18].

Unlike the usual (monotonic) paraconsistent logics — see for example [13] and [6] — and like inconsistency-adaptive logics — see for example [2] or [4] — the Rescher–Manor consequence relations do not invalidate rules of inference of classical logic (CL), but restrict their applications. They interpret a set of premises 'as consistently as possible.' This phrase is not unambiguous — the different Rescher–Manor consequence relations lead to different consequence sets (see Section 3) and most inconsistency-adaptive logics lead to still other consequence sets (see Section 8).

Even if the premise set Γ is inconsistent, its Free, Strong, and C-based consequence sets are consistent and closed under CL, and its Weak consequence set is in general inconsistent but never trivial. C-Based and Argued consequence relations (that are of more recent vintage than the others) are the outcome of attempts to extend the Strong consequence set in an intuitively justified way. The C-based consequence set of an inconsistent Γ is consistent and closed under CL, but intuitively justified applications seem to require multisets — see Section 3. The Argued consequence set of an inconsistent Γ avoids explicit inconsistencies (such as A and $\sim A$), but does not warrant consistency and is not closed under CL.

This raises the question whether the Strong consequence relation may be further extended in a sensible way, which warrants that the result is consistent and closed under CL. I shall show that it can be so extended. A further question is whether the other consequence relations may be extended along similar lines and with a sensible outcome as a result. I shall show that the answer to this question is positive as well, except for the Argued consequence relation.

The Rescher–Manor consequence relations are presented in Section 3. I concentrate on one specific application context: the statements made by participants in a discussion. Relying on ideas of Jaśkowski, as presented in [11] and earlier publications in Polish, the application context is approached in modal terms. This leads to a characterization of the Rescher–Manor consequence relations as adaptive logics based on S5 — see Section 4. The Rescher–Manor approach leads to stronger consequence sets than Jaśkowski's approach. However, in [12] an adaptive logic is studied that extends Jaśkowski's D2 in an intuitively attractive way. The same method, and a similar intuitive justification, extends all but one of the Rescher–Manor consequence relations; the results are presented in Section 5. The outcome for the Argued consequence relation is studied in Section 6. Dynamic proof theories for the extended consequence relations are presented in Section 7. Section 8 contains some comments and open problems.

The results of the present paper are the outcome of research on adaptive logics (see [2]) and on the characterization of the flat Rescher–Manor consequence relations in terms of adaptive logics (see [3], [7], and [5]). The present paper is self-contained in that it does not presuppose any knowledge of adaptive logics. Still, it is useful for the reader to know that, semantically, adaptive logics select a subset of models in terms of the abnormalities verified by them — several examples follow.[2] The most fascinating aspect of adaptive logics is their dynamic proof theory. These too proceed in terms of abnormalities, as we shall see.

3. *Definitions of the Consequence Relations*

As expected, Γ_1 is a *MCS* of the set of formulas Γ iff (i) $\Gamma_1 \subseteq \Gamma$, (ii) $\Gamma_1 \nvdash_{\mathsf{CL}} \bot$, and (iii) for all $A \in \Gamma - \Gamma_1$, $\Gamma_1 \cup \{A\} \vdash_{\mathsf{CL}} \bot$. If each member of Γ is self-inconsistent (for all $A \in \Gamma$, $A \vdash_{\mathsf{CL}} \bot$), the set of *MCS* of Γ is empty. If Γ is consistent, the set of *MCS* of Γ is the singleton $\{\Gamma\}$. In all other cases, Γ has more than one *MCS*. Members of Γ that belong to all *MCS* of Γ are called *free* members of Γ. The *largest MCS* of Γ are those the cardinality of which is not smaller than the cardinality of any other *MCS* of Γ.

The flat Rescher–Manor consequence relations are easily defined:

Definition 1: $\Gamma \vdash_{Free} A$ *iff* A *is a* CL*-consequence of the free members of* Γ.

[2] The semantic selection mechanism was first introduced in [1].

Definition 2: $\Gamma \vdash_{Strong} A$ *iff A is a* CL*-consequence of all* MCS *of* Γ.

Definition 3: $\Gamma \vdash_{Weak} A$ *iff A is a* CL*-consequence of some* MCS *of* Γ.

Definition 4: $\Gamma \vdash_{C\text{-}Based} A$ *iff A is a* CL*-consequence of all largest* MCS *of* Γ.

Definition 5: $\Gamma \vdash_{Argued} A$ *iff A is a* CL*-consequence of some* MCS *of* Γ *and* $\sim A$ *is not a* CL*-consequence of any* MCS *of* Γ.

The set $\{p \wedge q, \sim q, (p \vee \sim q) \supset s\}$ has two *MCS*, $(p \vee \sim q) \supset s$ is a free member of it, and s is a Strong consequence of it. The set $\{p \wedge q, \sim q \wedge (p \vee \sim q) \supset s\}$ has also two *MCS*, but has no free members, and s is not a Strong consequence of it. This illustrates that the consequence relations are very dependent on the formulation of the premises. In general, their suitable application contexts are those in which each premise has a different source — an *MCS* then represents a set of jointly consistent sources. An example of a suitable application is where each premise is the conjunction of the statements made by one participant in a discussion. I shall concentrate on this application in the sequel of this paper.

Where each member of Γ represents a participant, a consistent subset of Γ represents a *consistent subgroup* (a group of participants whose statements form a consistent set). An *MCS* represents a maximal consistent subgroup. Inconsistent members of Γ represent participants that contradict themselves, and free members of Γ represent 'safe' participants: those that do not contradict any consistent subgroup. There is nothing wrong with the fact that some participants belong to several (or even all) maximal consistent subgroups. In discussions (or meetings) where the participants are rather autonomous, it is customary that some participants do not contradict any of several parties that contradict each other. Maybe the participant does not hold any view on the subject of dispute, or considers it unimportant.

Given this setting, it seems sensible to say that something is stated *by* a consistent subgroup: the statement is made by a participant that belongs to the consistent subgroup, and hence is not contradicted by any other member of that subgroup. In this sense, a *MCS* is stated by a maximal consistent subgroup.

Let us now turn to the consequence relations. The Free consequences follow from statements made by the safe participants, and hence from statements made by all maximal consistent subgroups. Anybody may be taken to agree about these. There is also agreement on the Strong consequences: each maximal consistent subgroup makes statements from which they follow. Two different forms of agreement surface here. The Free consequences follow from *statements* made by any maximal consistent subgroup and hence not contradicted by any consistent subgroup. The Strong consequences are agreed upon by all consistent subgroups, but each consistent subgroup may have a different *argument* for them.

The C-based consequences are agreed upon by the maximal consistent subgroups that are not outnumbered by any other maximal consistent subgroup. A few remarks are in place here. First, the agreement is at the level of the consequences, not at the level of the arguments — compare the two forms of agreement mentioned in the previous paragraph. Next, the C-based consequence relation is more naturally applied to multisets than to sets. Multisets are like sets except in that it matters how many times a member belongs to the multiset. Thus the multisets $\{p, \sim p\}$, $\{p, p, \sim p\}$ and $\{p, \sim p, \sim p\}$ are all different and have different C-based consequences. For the other consequence relations, it does not matter whether one considers sets or multisets. Finally, even in terms of multisets, C-based consequences do not necessarily represent a basis for democratic decision. If ninety-eight participants are isolated, a two-member consistent subgroup determines the C-based consequences. In the sequel, I shall disregard multisets, leaving their rather obvious incorporation to the reader.

The Weak consequences follow from the statements made by some consistent subgroup. They represent viewpoints that are defended in the discussion. These viewpoints are maintained by a consistent subgroup, but are possibly contradicted by another consistent subgroup.

Argued consequences seem the least interesting with respect to the present application context — and apparently with respect to all application contexts. They represent viewpoints upheld by a consistent subgroup and not contradicted by any other consistent subgroup. To see that this does not correspond to anything much fascinating, consider a 'discussion' between two participants, one claiming p and the other claiming $\sim p$. In this case, all of $p \vee A$, $p \vee \sim A$, $\sim p \vee A$, and $\sim p \vee \sim A$ are Argued consequences for any logically contingent A. So, the Argued consequence relation seems to be a half-hearted halfway house between Strong and Weak consequences. The concept behind the Argued consequence relation is obviously clear. Yet, the reasons for considering it as attractive seem to apply even more strongly to the extension of the Strong consequence relation presented in Section 5. Moreover, the latter provides a consequence set that is consistent and closed under **CL**, and has a straightforward intuitive interpretation.

4. *Enters Modal Logic*

Long before Nicholas Rescher started working on this problem, Jaśkowski had devised an approach to discussions. He interprets the set of premises Γ in modal terms, viz. as $\Gamma^{\Diamond} = \{\Diamond A \mid A \in \Gamma\}$. Today one would say that a possible world represents a viewpoint, the view of a participant in the discussion. Jaśkowski defines his D2 as follows:

$$\Gamma \vdash_{\mathsf{D2}} A =_{df} \Gamma^{\Diamond} \vdash_{\mathsf{S5}} \Diamond A.$$

As $\Diamond(p \wedge \sim p) \vdash_{\mathsf{S5}} \Diamond q$, we have $p \wedge \sim p \vdash_{\mathsf{D2}} q$. Nevertheless, D2 is a paraconsistent logic ($p, \sim p \nvdash_{\mathsf{D2}} q$) because it is non-adjunctive ($p, \sim p \nvdash_{\mathsf{D2}} p \wedge \sim p$).

As I need to choose a S5-semantics, let us settle on a usual worlds semantics: a model M is a quadruple $\langle W, D, Q, V \rangle$ in which W is a set of 'worlds', the domain D is a set, the function Q selects for each world $w \in W$ a subset of D as its domain, and the assignment V interprets the non-logical constants as usually (see [10, Ch. 10] for details). Such a model verifies A iff $V(A, w) = 1$ for all $w \in W$.

All Rescher-Manor consequence relations avoid $A \wedge \sim A \vdash B$ because self-inconsistent premises are simply disregarded: they do not belong to any *MCS* of the premises. The S5-models that verify all consistent members of $\Gamma^{\Diamond}$ will be called C-models of $\Gamma^{\Diamond}$. These may themselves be characterized as minimally abnormal models:[3]

Definition 6: $Ab^{c}_{\Gamma}(M) = \{A \mid A \in \Gamma; M \nvDash \Diamond A\}$

Definition 7: *A* S5-*model* M *is a* C-*model of* $\Gamma^{\Diamond}$ *iff there is no* S5-*model* M' *such that* $Ab^{c}_{\Gamma}(M') \subset Ab^{c}_{\Gamma}(M)$.

Obviously, each $\Gamma^{\Diamond}$ has C-models: there are S5-models such that, for every consistent $A \in \Gamma$, $V(A, w) = 1$ for some world w.

A central difference between Rescher's approach and Jaśkowski's approach is that Jaśkowski invalidates certain CL-rules (for example Adjunction and Disjunctive Syllogism), whereas Rescher restricts their applications.[4] For example, Rescher–Manor consequence relations allow for applications of Adjunction, Disjunctive Syllogism, etc., provided the rule is applied to formulas that follow from a specific set of *MCS* of the premises. To express this in terms of the S5-semantics, we need to make a selection of the C-models of $\Gamma^{\Diamond}$.

Where $\mathcal{W}$ is the set of closed formulas (wffs) of the non-modal language, a world w of a model will be said to verify $\Delta \subseteq \mathcal{W}$ iff $V(A, w) = 1$ for

[3] I shall need several kinds of abnormal parts of models and sets of premises. They will be distinguished by subscripts and superscripts. The adaptive logics presented here are not in the standard from of [4]; I return to this in Section 8.

[4] Rescher never presented a proof theory. The claim in the text is obvious for simple and perspicuous examples, but also holds for the dynamic proof theories presented in [3] and [7].

all $A \in \Delta$. The minimally abnormal worlds with respect to Γ are defined as follows:

Definition 8: $Ab_\Gamma(w) = \{A \in \Gamma \mid w \text{ does not verify } A\}$.

Definition 9: *A world w of a* S5*-model M is minimally abnormal with respect to Γ iff no world w' of any* S5*-model M' is such that $Ab_\Gamma(w') \subset Ab_\Gamma(w)$.*

Definition 10: *A* C*-model M of $\Gamma^\Diamond$ is a* MA*-model of $\Gamma^\Diamond$ iff all worlds of M are minimally abnormal with respect to Γ.*

Lemma 1: *M is a* MA*-model of $\Gamma^\Diamond$ iff each world of M verifies a* MCS *of Γ.*

Proof. For the left–right direction, consider a world w of a S5-model M for which $\Delta \supset \{A \mid A \in \Gamma; w \text{ verifies } A\}$ is a *MCS* of Γ. There obviously is a S5-model M' of Γ in which some world w' verifies Δ. Hence, $Ab_\Gamma(w') \subset Ab_\Gamma(w)$, and M is not a MA-model of $\Gamma^\Diamond$. The right–left direction is obvious. □

As any *MCS* of Γ is consistent, any $\Gamma^\Diamond$ has MA-models.

From the MA-models of $\Gamma^\Diamond$ we define the RM-models of $\Gamma^\Diamond$ in terms of their abnormal parts with respect to the possibility of conjunctions of premises:

Definition 11: $Ab^\wedge_\Gamma(M) = \{\{A_1, \ldots, A_n\} \mid n > 1; A_1, \ldots, A_n \in \Gamma; M \not\models \Diamond(A_1 \wedge \ldots \wedge A_n)\}$.

Definition 12: *M is a* RM*-model of $\Gamma^\Diamond$ iff it is a* MA*-model of $\Gamma^\Diamond$ and there is no* MA*-model M' of $\Gamma^\Diamond$ such that $Ab^\wedge_\Gamma(M') \subset Ab^\wedge_\Gamma(M)$.*

In other words, a S5-model M is a RM-model of $\Gamma^\Diamond$ iff all its worlds are minimally abnormal with respect to Γ and the model itself is minimally abnormal with respect to the possibility of conjunctions of members of Γ.

Theorem 1: *M is a* RM*-model of $\Gamma^\Diamond$ iff each world of M verifies a* MCS *of Γ and each* MCS *of Γ is verified by some world of M.*

Proof. Let M and M' be MA-models of $\Gamma^\Diamond$, whence by Lemma 1 every world of both M and M' verifies some MCS of Γ. Moreover, let every *MCS* of Γ be verified by some world of M whereas some *MCS* of Γ are not verified by any world of M'. It is easily seen that $Ab^{\wedge}_{\Gamma}(M)$ is the set of all inconsistent $\{A_1, \ldots, A_n\} \subset \Gamma$, whereas $Ab^{\wedge}_{\Gamma}(M')$ comprises the same subsets of Γ as well as the *MCS* of Γ that are not verified by any world of M'. The right–left direction is again obvious. □

It is obvious in view of the theorem that each $\Gamma^{\Diamond}$ has RM-models. Moreover, as each world that verifies a *MCS* of Γ, verifies all its consequences, Theorem 1 together with Definitions 2, 3 and 5 give us:

Theorem 2: $\Gamma \vdash_{Weak} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{RM}} \Diamond A$.[5]

Theorem 3: $\Gamma \vdash_{Argued} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{RM}} \Diamond A$ *and* $\Gamma^{\Diamond} \not\models_{\mathsf{RM}} \Diamond{\sim}A$.

Theorem 4: $\Gamma \vdash_{Strong} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{RM}} \Box A$.

As all Strong consequences of Γ are verified by all worlds that verify a *MCS* of Γ, the proof of the following theorem is obvious:

Theorem 5: $\Gamma \vdash_{Strong} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{MA}} \Box A$.

The two other consequence relations require special treatment. The C-based consequence relation is most easily incorporated by introducing a special (but simple) modality $\Box_{\Gamma}$ (where $\Gamma \subset \mathcal{W}$). Let $\#(w, \Gamma)$ be the cardinality of the set of members of Γ verified by w, and let $w \in m(\Gamma)$ iff there is no $w' \in W$ such that $\#(w', \Gamma) > \#(w, \Gamma)$. Extend the S5-semantics with the clause:

$$V(\Box_{\Gamma} A, w) = 1 \text{ iff } V(A, w') = 1 \text{ for all } w' \in m(\Gamma)$$

Theorem 6: $\Gamma \vdash_{C\text{-}based} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{RM}} \Box_{\Gamma} A$.

For the Free consequence relation, we need a different selection of the models. First we define the abnormal part of Γ — it obviously is the set of non-free members of Γ:

Definition 13: $Ab_F(\Gamma) = \{A \mid A \in \Gamma;$ *for some* $B_1, \ldots, B_n \in \Gamma : B_1, \ldots, B_n \nvdash_{\mathsf{CL}} \bot$ *and* $A, B_1, \ldots, B_n \vdash_{\mathsf{CL}} \bot\}$

Next, we define the abnormal part of the models with respect to the necessity of members of Γ:

[5] $\Gamma^{\Diamond} \models_{\mathsf{RM}} \Diamond A$ is obviously short for: all RM-models of $\Gamma^{\Diamond}$ verify $\Diamond A$.

Definition 14: $Ab^{\Box}_{\Gamma}(M) = \{A \mid A \in \Gamma; M \not\models \Box A\}$

Definition 15: *M is a* F*-model of* $\Gamma^{\Diamond}$ *iff M is a* C*-model of* $\Gamma^{\Diamond}$ *and* $Ab^{\Box}_{\Gamma}(M) = Ab_F(\Gamma)$.

The F-models of $\Gamma^{\Diamond}$ are all S5-models that verify $\Diamond A$ for all $A \in \Gamma$ and $\Box A$ for all A that are free members of Γ. So, it is obvious that any $\Gamma^{\Diamond}$ has F-models, and that:[6]

Theorem 7: $\Gamma \vdash_{Free} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{F}} \Box A$.

Having defined all Rescher–Manor consequence relations in terms of S5, I now move on to the promised extensions. I shall start with the motivation that leads to the extensions.

5. *The Extensions*

In this section I disregard Argued consequences altogether, postponing the discussion of their 'extension' to Section 6.

Suppose that a participant in a discussion states $p \wedge q$, that p is contradicted by another participant stating, for example, $\sim p \wedge r$, but that q is not contradicted by any participant. As some consistent participant stated $p \wedge q$, and no participant contradicted q, it seems sensible to conclude that all participants implicitly agree on q.

The underlying idea is that, in a discussion, one should contradict statements one does not agree about. The fact that one contradicts another participant at one point, does obviously not entail that one disagrees at all points. Of course, some participants may consider a statement as unimportant, and for this reason not contradict it. But this situation is similar to a case where some participants defend a viewpoint they do not subscribe to. The aim of the present paper does not (and cannot) relate to the convictions of the participants, but to the statements they make (or implicitly support) during the discussion.

Handling implicit agreement is somewhat touchy. As any premise is a conjunction of the statements made by a single participant, the participant who stated $p \wedge q$ may have stated p and q separately, but may also have stated their conjunction. In the latter case, the participant can only be said to have stated q implicitly: q is derivable from the statement made. But then,

[6] A world of a F-model of $\Gamma^{\Diamond}$ need not verify a *MCS* of Γ. Hence, if Γ has no free members, only S5-valid formulas are necessary in some F-models of $\Gamma^{\Diamond}$.

the participant also implicitly stated $p \vee r$. I supposed that some participants contradicted p. If every participant implicitly agrees to $p \vee r$ (because no participant contradicted it), then the participants that contradict p implicitly agree to r. But by the same reasoning they implicitly agree to $\sim r$.

There is an obvious solution to this problem (and it is well known from adaptive logics). The point is (i) that one should not concentrate on implicit agreements, but on explicit disagreements, (ii) that disagreements may be 'connected', and (iii) that one should express these connections in terms of the simplest formulas that cause the disagreements. Let me explain.

There is disagreement about p if both $\Diamond p$ and $\Diamond\sim p$ are derivable from the consistent members of $\Gamma^\Diamond$. Suppose, however, that some participant states p, another q, and a third $\sim p \vee \sim q$. From $\Diamond p, \Diamond q, \Diamond(\sim p \vee \sim q)$ follows $(\Diamond p \wedge \Diamond\sim p) \vee (\Diamond q \wedge \Diamond\sim q)$, whereas neither disjunct follows. This is what I meant by a connected disagreement.

Consider the connected disagreement $(\Diamond(p \wedge q) \wedge \Diamond\sim(p \wedge q)) \vee (\Diamond(r \vee s) \wedge \Diamond\sim(r \vee s))$. It is easily seen that this entails a connected disagreement in terms of simpler formulas: $(\Diamond p \wedge \Diamond\sim p) \vee (\Diamond q \wedge \Diamond\sim q) \vee (\Diamond r \wedge \Diamond\sim r) \vee (\Diamond s \wedge \Diamond\sim s)$. This disjunction will be called a disjunction of abnormalities, and will be abbreviated by $Dab(p, q, r, s)$. To be more precise, a Dab-formula (disjunction of abnormalities) is a disjunction of formulas of the form $\exists(\Diamond A \wedge \Diamond\sim A)$ in which A is a *primitive formula* (a formula containing no logical symbol except for identity) and $\exists$ abbreviates an existential quantifier over every formula free in A. In the expression $Dab(A_1, \ldots, A_n)$, the A_i are called the *factors* of the Dab-formula.

For people not acquainted with adaptive logics, the following example will clarify the complication required by the predicative case.

Example 1. Let $\Gamma = \{(\forall x)(Px \vee Qx), (\exists x)(\sim Px \wedge \sim Qx)\}$. The *MCS* are $\{(\forall x)(Px \vee Qx)\}$ and $\{(\exists x)(\sim Px \wedge \sim Qx)\}$. There are no closed formulas $A_1, \ldots, A_n$ such that $\Gamma^\Diamond \vdash_{S5} Dab\{A_1, \ldots, A_n\}$. However, there are open such formulas. Indeed $\Gamma^\Diamond \vdash_{S5} Dab\{Px, Qx\}$, that is $\Gamma^\Diamond \vdash_{S5} (\exists x)(\Diamond Px \wedge \Diamond\sim Px) \vee (\exists x)(\Diamond Qx \wedge \Diamond\sim Qx)$.

As $Dab(\Delta) \vdash_{S5} Dab(\Delta \cup \Theta)$ (where Δ and Θ are finite sets), the derivability of a Dab-formula from the consistent members of $\Gamma^\Diamond$ does not warrant a connected disagreement about all factors of the Dab-formula. This is why one needs to concentrate on the *minimal Dab-consequences of* $\Gamma^\Diamond$.

Definition 16: *$Dab(\Delta)$ is a minimal Dab-consequence of $\Gamma^\Diamond$ iff $\Gamma^\Diamond \vdash_{S5} Dab(\Delta)$ and there is no $\Theta \subset \Delta$ such that $\Gamma^\Diamond \vdash_{S5} Dab(\Theta)$.*

The factors of the minimal Dab-consequences of $\Gamma^\Diamond$ are all suspect: there is disagreement about at least one factor of every minimal Dab-consequence of $\Gamma^{\Diamond}$, and it is not determined which one. In line with previous work, all these factors will be called *unreliable* (rather than suspect).

Definition 17: $U(\Gamma)$, *the set of formulas that are unreliable with respect to* Γ, *is the set of factors of the minimal Dab-consequences of* $\Gamma^{\Diamond}$.

Remark that $U(\Gamma) \subseteq \mathcal{F}^p$, where $\mathcal{F}^p$ is the set of primitive formulas. The same idea gives us the abnormal part of a model with respect to primitive formulas:

Definition 18: $Ab_p(M) = \{A \in \mathcal{F}^p \mid M \models \exists(\Diamond A \wedge \Diamond{\sim}A)\}$.

We now select the models in which only unreliable primitive formulas behave abnormally:

Definition 19: *M is a* $\mathsf{RM}^{\star}$*-model of* $\Gamma^{\Diamond}$ *iff it is a* RM*-model of* $\Gamma^{\Diamond}$ *and* $Ab_p(M) \subseteq U(\Gamma)$.

Definition 20: *M is a* $\mathsf{F}^{\star}$*-model of* $\Gamma^{\Diamond}$ *iff it is a* F*-model of* $\Gamma^{\Diamond}$ *and* $Ab_p(M) \subseteq U(\Gamma)$.

These definitions enable us to define the extended Rescher–Manor consequence relations:

Definition 21: $\Gamma \vdash_{Weak^{\star}} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{RM}^{\star}} \Diamond A$.

Definition 22: $\Gamma \vdash_{Strong^{\star}} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{RM}^{\star}} \Box A$.

Definition 23: $\Gamma \vdash_{C\text{-}based^{\star}} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{RM}^{\star}} \Box_{\Gamma} A$.

Definition 24: $\Gamma \vdash_{Free^{\star}} A$ *iff* $\Gamma^{\Diamond} \models_{\mathsf{F}^{\star}} \Box A$.

Example 2. Let $\Gamma = \{p \wedge q, {\sim}p \wedge ({\sim}q \vee r), s\}$. The *MCS* of Γ are $\{p \wedge q, s\}$ and $\{{\sim}p \wedge ({\sim}q \vee r), s\}$ and $U(\Gamma) = \{p\}$. There are five kinds of worlds that verify one of the *MCS*. They may be characterized as follows with respect to the letters p, q, r, and s:

(1) p, q, r, s
(2) $p, q, {\sim}r, s$
(3) ${\sim}p, q, r, s$
(4) ${\sim}p, {\sim}q, r, s$
(5) ${\sim}p, {\sim}q, {\sim}r, s$

Of each kind of worlds, some verify t and others $\sim t$, and similarly for all other (predicative) primitive formulas (that are independent of previously considered ones).

All RM-models of $\Gamma^{\Diamond}$ have worlds of kinds 1 and/or 2 as well as worlds of kinds 3 and/or 4 and/or 5. It is easily seen that RM*-models have only worlds of kinds 1 and 3. Indeed, these are the only models in which $Ab_p(\Gamma) \subseteq U(\Gamma) = \{p\}$. For any other primitive formula A, the RM*-models of Γ verify either $\Box A$ or $\Box{\sim}A$, and hence $A \notin Ab_p(\Gamma)$. As some of them verify $\Box A$ and others $\Box{\sim}A$, neither of these formulas is verified by all RM*-models of Γ.

Let $Cn_{\mathit{Free}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathit{Free}} A\}$, and similarly for the other consequence relations.

Theorem 8: *For all* Γ, $Cn_{\mathit{Free}^\star}(\Gamma)$, $Cn_{\mathit{Strong}^\star}(\Gamma)$, *as well as* $Cn_{C\text{-}\mathit{based}^\star}(\Gamma)$ *are consistent sets.*

Proof. The consequence sets comprise the formulas verified by all worlds of a set of S5-models. Hence they are consistent. □

Theorem 9: *For all* Γ, $Cn_{\mathit{Free}^\star}(\Gamma)$, $Cn_{\mathit{Strong}^\star}(\Gamma)$, *as well as* $Cn_{C\text{-}\mathit{based}^\star}(\Gamma)$ *are closed under* CL.

Proof. As for the previous theorem. □

Theorem 10: $Cn_{\mathit{Free}}(\Gamma) \subseteq Cn_{\mathit{Free}^\star}(\Gamma)$ *for all* Γ, *and* $Cn_{\mathit{Free}}(\Gamma) \subset Cn_{\mathit{Free}^\star}(\Gamma)$ *for some* Γ. *Similarly for the other extended consequence relations.*

Proof. That $Cn_{\mathit{Free}}(\Gamma) \subseteq Cn_{\mathit{Free}^\star}(\Gamma)$ for all Γ is obvious in view of the fact that the F*-models of $\Gamma^{\Diamond}$ are a subset of the F-models of $\Gamma^{\Diamond}$. By the same reasoning, $Cn_{\mathit{Strong}}(\Gamma) \subseteq Cn_{\mathit{Strong}^\star}(\Gamma)$, $Cn_{\mathit{Weak}}(\Gamma) \subseteq Cn_{\mathit{Weak}^\star}(\Gamma)$, and $Cn_{C\text{-}\mathit{based}}(\Gamma) \subseteq Cn_{C\text{-}\mathit{based}^\star}(\Gamma)$.

Example 2 shows that the inclusions are strict for some Γ: q and r are elements of $Cn_{\mathit{Free}^\star}(\Gamma) - Cn_{\mathit{Free}}(\Gamma)$, $Cn_{\mathit{Strong}^\star}(\Gamma) - Cn_{\mathit{Strong}}(\Gamma)$, $Cn_{\mathit{Weak}^\star}(\Gamma) - Cn_{\mathit{Weak}}(\Gamma)$, and $Cn_{C\text{-}\mathit{based}^\star}(\Gamma) - Cn_{C\text{-}\mathit{based}}(\Gamma)$. □

In Example 2, the RM*-models of Γ are those for which $Ab_p(\Gamma) = \{p\}$. This derives from the fact that the only minimal Dab-consequence of Γ is $\Diamond p \wedge \Diamond{\sim}p$. It is instructive to consider an example in which there is a dependence between the abnormalities.

Example 3. Let $\Gamma = \{\sim p \wedge s, \sim q, p \vee q, r\}$. The *MCS* are $\{\sim p \wedge s, \sim q, r\}$, $\{\sim p \wedge s, p \vee q, r\}$ and $\{\sim q, p \vee q, r\}$. The only minimal *Dab*-consequence of Γ is $Dab\{p, q\}$ and $U(\Gamma) = \{p, q\}$. There are three kinds of worlds that verify one of the *MCS*:

(1) $p, \sim q, r, s$
(2) $p, \sim q, r, \sim s$
(3) $\sim p, q, r, s$
(4) $\sim p, \sim q, r, s$

The RM-models of $\Gamma^{\Diamond}$ have worlds of kinds 1 and/or 2, worlds of kind 3, as well as worlds of kind 4. RM*-models have only worlds of kinds 1, 3, and 4. Indeed, these are the only models in which $Ab_p(\Gamma) \subseteq U(\Gamma) = \{p, q\}$. Remark that s is a Weak consequence, but not a Free, Strong, or C-based consequence of Γ. However, s is a Free⋆,[7] Strong⋆, and C-based⋆ consequence of Γ.

As appears from Examples 2 and 3, the restriction to RM*-models warrants that all primitive formulas that do not belong to $U(\Gamma)$ are either all true in all worlds of the model or all false in all worlds of the model. If, for some primitive formula $A \notin U(\Gamma)$, $\Gamma^{\Diamond}$ does not have RM-models that verify $\Box A$ (respectively $\Box{\sim}A$), then all RM*-models verify $\Box{\sim}A$ (respectively $\Box A$), and hence ${\sim}A$ (respectively A) is a Free⋆, Strong⋆, and C-based⋆ consequence of Γ. It is interesting to see that this leads to a stepwise restriction on the models. In Example 2, $U(\Gamma) = \{p\}$. Some RM-models of the premises verify $\Box q$, and none verify $\Box{\sim}q$. The effect is that all RM*-models of the premises verify $\Box q$. Among RM-models of the premises that verify $\Box q$, some verify $\Box r$ and none verify $\Box{\sim}r$. As a result, all RM*-models of the premises verify $\Box r$. So, notwithstanding the fact that some RM-models of the premises verify $\Box{\sim}r$, all RM*-models of the premises verify $\Box r$.

I now return to the considered application context, and spell out the precise interpretation of the extension. What is clear by now is this: if A is derivable from a participant's claims, and all primitive subformulas in A are reliable (not suspect), then A is implicitly agreed upon by all participants. The reliability of the primitive subformulas functions here as an explication of the vague phrase "is contradicted." Example 3 nicely illustrates why we need to proceed in terms of *Dab*-formulas — *disjunctions* of abnormalities — rather than in terms of abnormalities. In that example, there is disagreement about either p or q, but we do not have sufficient information to decide about which one, and hence both p and q are unreliable.

[7] That s is a Free⋆-consequence of Γ is mentioned here for the sake of completeness, but cannot be seen from the listed kinds of worlds. The F-models of Γ need not verify a *MCS* of Γ, but all verify $\Box r$ — this leaves us with eight kinds of worlds of the sort mentioned in the list. The F*-models of Γ moreover all verify $\Box s$ — this leaves us with four kinds of worlds.

The claims implicitly agreed upon by all participants should obviously be closed under CL and they are, as appears from Theorem 9. This is illustrated by Example 2: the participants implicitly agree on r because it is a CL-consequence of two formulas, q and $\sim q \vee r$, that are asserted by different participants, but that all participants implicitly agree upon. However, implicit agreement cannot be expressed in terms of the participants' claims only. Consider the following example.

Example 4. Let $\Gamma = \{p \wedge q, \sim p, s \wedge (p \vee r)\}$. The *MCS* are $\{p \wedge q, s \wedge (p \vee r)\}$ and $\{\sim p, s \wedge (p \vee r)\}$, and $U(\Gamma) = \{p\}$. The kinds of worlds:

(1) p, q, r, s
(2) $p, q, \sim r, s$
(3) $\sim p, q, r, s$
(4) $\sim p, \sim q, r, s$

The RM-models of $\Gamma^{\Diamond}$ have worlds of kinds 1 and/or 2 as well as worlds of types 3 and/or 4. RM*-models have only worlds of types 1 and 3. Remark that q and r are not Free, Strong, or C-based consequences of Γ, but they are Free*, Strong*, and C-based* consequences of Γ.

The interesting formula is r. No participant states r. The only participant that even mentions r is the one stating $s \wedge (p \vee r)$.[8] But that formula and its conjunct $p \vee r$ contain the unreliable primitive subformula p. It follows that there is no implicit agreement on $p \vee r$. So, where does the implicit agreement on r derive from? The answer is simple: the consistent subgroup represented by the *MCS* $\{\sim p, s \wedge (p \vee r)\}$ states r.

So, in line with the Rescher–Manor approach, the extensions proceed in terms of *MCS*. The set of formulas implicitly agreed upon by all participants is the CL-deductive closure of the unsuspect formulas (those that contain no unreliable primitive subformulas) that are (implicitly or explicitly) stated by a consistent subgroup.

Incidentally, this is a major difference with the adaptive logic AJ that Joke Meheus defines from D2 in [12].[9] There, the 'consistent core' is the CL-deductive closure of the unsuspect formulas stated by some *participant* — this does not include r in the last example. In the present paper, the consistent core is the CL-deductive closure of the unsuspect formulas stated by a consistent *subgroup*.

[8] There might be several participants stating precisely the same things. Whether this is so would only show in a multiset.

[9] There are some minor differences, for example that AJ, in line with D2, delivers triviality if there is a self-inconsistent premise.

6. *The Argued Consequence Relation*

We have already seen that the Argued consequence relation warrants neither consistency nor closure under CL. Of course, it is simple enough to define an 'extension' of it:

Definition 25: $\Gamma \vdash_{Argued^\star} A$ *iff* $\Gamma^\Diamond \models_{\mathsf{RM}^\star} \Diamond A$ *and* $\Gamma^\Diamond \not\models_{\mathsf{RM}^\star} \Diamond{\sim}A$.

However, the Argued* consequence relation does not always extend the Argued consequence relation as appears from:

Example 5. Let $\Gamma = \{p \wedge q, {\sim}p \wedge (r \vee {\sim}q)\}$. The *MCS* are $\{p \wedge q\}$ and $\{{\sim}p \wedge (r \vee {\sim}q)\}$, and $U(\Gamma) = \{p\}$. The types of worlds if we consider only the letters p, q, and r:

(1) p, q, r
(2) $p, q, {\sim}r$
(3) ${\sim}p, q, r$
(4) ${\sim}p, {\sim}q, r$
(5) ${\sim}p, {\sim}q, {\sim}r$

The RM-models of $\Gamma^\Diamond$ have worlds of types 1 and/or 2 as well as worlds of types 3 and/or 4 and/or 5. RM*-models have only worlds of types 1 and 3.

The selection leads to a gain: r is not an Argued consequence, but is an Argued*-consequence of Γ. The selection also leads to a loss: ${\sim}p \vee {\sim}r$ is an Argued consequence, but is not an Argued*-consequence of Γ.

In other words, we do *not* have $Cn_{Argued}(\Gamma) \subseteq Cn_{Argued^\star}(\Gamma)$. Something very simple is going on here. A is an Argued consequence of Γ iff A is and ${\sim}A$ is not a Weak consequence of Γ; A is an Argued* consequence of Γ iff A is and ${\sim}A$ is not a Weak* consequence of Γ. As the Weak* consequence in general extends the Weak consequence set, the transition from Argued-consequences to Argued*-consequences involves a gain as well as a loss.

All this is not very important for the present application context. I did not find any sensible interpretation for the Argued consequence relation anyway, and the same holds for the Argued* consequence relation. Moreover, the gain obtained by the transition from the Strong consequence relation to the Strong* consequence relation, seems to offer everything one might hope to obtain from the Argued consequence relation. Another aspect that makes Strong* consequence sets interesting is that, unlike Argued and Argued* consequence sets, they are, for any set of premises, consistent and closed under CL.

In the following section, where I list the dynamic proof theories for the original and extended Rescher–Manor consequence relations, I include the proof theory for the Argued and Argued* consequence relations.

7. *Dynamic Proof Theory*

For those not familiar with dynamic proofs, I briefly mention some basics. A line in a proof consists of five elements: (i) a line number, (ii) the wff derived, (iii) the numbers of the lines from which the wff is derived, (iv) the rule applied, and (v) *the condition* of the line (usually a set of formulas). A line is called *unconditional* iff its condition is $\emptyset$; otherwise it is called conditional.

Apart from the rules for adding lines to a proof, there is a definition of the lines that are *marked* at a stage of the proof. Whether a line is marked depends on its condition and on the formulas derived in the proof at the stage.

A formula A is *finally derived* at a stage iff it occurs in an unmarked line i at the stage and, whenever i is marked in an extension of the proof, it is non-marked in a further extension. $\Gamma \vdash_{\mathsf{L}} A$ denotes that A is finally derivable from Γ by the adaptive logic L. That $\Gamma \vdash_{\mathsf{L}} A$ iff $\Gamma \models_{\mathsf{L}} A$ indicates that the proof theory is sound and complete with respect to the semantics. To save some space, I skip the soundness and completeness proofs. They are straightforward in view of previous results on adaptive logics.

Let $\bigwedge(\Gamma)$ denote the conjunction of the members of the finite set Γ.

Let us first consider the original Rescher–Manor consequence relations. The condition of a line is always a set of premises. The common rules are the premise rule and the unconditional rule (which does not introduce a new condition):

PREM Any $\Diamond A \in \Gamma^{\Diamond}$ may be added to the proof with the justification PREM and $\{A\}$ as its condition.

RU If $B_1, \ldots, B_n \vdash_{\mathsf{S5}} A$ ($n \geq 0$), and $B_1, \ldots, B_n$ occur in the proof with, respectively, the conditions $\Delta_1, \ldots, \Delta_n$, then one may add A to the proof with the condition $\Delta_1 \cup \ldots \cup \Delta_n$.

That even premise lines are conditional is the result of the fact that $A \in \Gamma$ is not even a Weak consequence of Γ unless A is self-consistent. If $n = 0$ in RU, A is a $\mathsf{S5}$-theorem and hence is derivable on an *unconditional* line.

For the Weak consequence relation, we need the following conditional rule:

RCW From $\Diamond A$ on the condition Δ and $\Diamond B$ on the condition Θ, to derive $\Diamond(A \wedge B)$ on the condition $\Delta \cup \Theta$.

A line with Δ as fifth element is *marked* iff $\sim\Diamond \bigwedge(\Delta)$ has been derived on an unconditional line — in other words, iff $\sim\Diamond \bigwedge(\Delta)$ is a $\mathsf{S5}$-theorem.

In view of Theorem 2, we are only interested in derived formulas of the form $\Diamond A$. It is easily seen (i) that such a formula is derivable iff it is derivable on a condition $\{B_1, \ldots, B_n\} \subseteq \Gamma$, and (ii) that this is the case iff $B_1, \ldots, B_n \vdash_{\mathsf{CL}} A$. The line will *not* be marked in any extension of the proof iff $\{B_1, \ldots, B_n\}$ is consistent, in other words iff A is a CL-consequence of a *MCS* of Γ. I leave it to the reader to apply the same form of analysis to the subsequent proof theories.[10]

For the Strong consequence relation, the proofs are governed by RCW together with[11]

RCS From $\Diamond A$ on the condition Δ, to derive $\Box A$ on the condition Δ.

It is useful to apply the following derivable thickening rule:

Th From $\Diamond A$ on the condition Δ, to derive $\Diamond A$ on the condition $\Delta \cup \Theta$.

Marking is most easily introduced in two steps. A line is W-marked iff it is marked for the Weak consequence relation. A W-unmarked line at which $\Box A$ has been derived by RCS on a condition Δ is S-marked iff, for some condition Θ of a W-unmarked line, there is no W-unmarked line at which $\Box A$ has been derived on condition $\Theta \cup \Theta'$ (for some $\Theta' \supseteq \emptyset$).[12] Any line that is derived from a S-marked line is itself S-marked. A line is marked iff it is either W-marked or S-marked.

For the Free consequence relation, the proofs are governed by RCW together with

RCF From $\Diamond A$ on the condition Δ, to derive $\Box A$ on the condition Δ.

A line is W-marked iff it is marked for the Weak consequence relation. A line at which $\Box A$ has been derived on the condition Δ is F-marked iff there is a $\Theta \subseteq \Gamma$ such that $\sim\Diamond \bigwedge(\Theta \cup \Delta)$ occurs in the proof in an unconditional line, and $\sim\Diamond \bigwedge(\Theta)$ does not occur in the proof in an unconditional line. Any line that is derived from a F-marked line is itself F-marked. A line is marked iff it is either W-marked or F-marked.

For the C-based consequence relation, the proofs are governed by RCW and RCC, which actually is just like RCS unless that the modality is different:

RCC From $\Diamond A$ on the condition Δ, to derive $\Box_\Gamma A$ on the condition Δ.

[10] A central difference is that, although there is no positive test for the Weak consequence relation, it is monotonic. The other consequence relations are non-monotonic.

[11] Although the rule RCS does not explicitly introduce a new condition, it may be considered as a conditional rule. Indeed, lines at which formulas of the form $\Box A$ are derived are subject to stricter marking conditions — see below in the text — than lines at which formulas of the form $\Diamond A$ are derived. Similarly for other rules below.

[12] Hence, the line will not be S-marked in some extension of the proof iff A is a CL-consequence of each *MCS* of Γ. Most lines with $\Box A$ as second element are S-marked when they are introduced; they are unmarked if the condition is fulfilled at a later stage. If a further premise is introduced in the proof by PREM, all lines in which $\Box A$ was derived will be S-marked. The marks may be removed after an application of rule Th, but some of the lines may be W-marked at a later stage of the proof.

Here too, the derivable thickening rule Th is useful. Where B is a wff, let $\#_\Gamma(\Delta) = 0$ iff $\Delta \not\subseteq \Gamma$; otherwise let $\#_\Gamma(\Delta)$ be the cardinality of Δ .

A line is W-marked iff it is marked for the Weak consequence relation. A W-unmarked line at which $\Box_\Gamma A$ has been derived by RCC on some condition Δ, is C-marked iff, for some condition Θ of a non-W-marked line in the proof, (i) $\#_\Gamma(\Delta) < \#_\Gamma(\Theta)$ or (ii) $\#_\Gamma(\Delta) = \#_\Gamma(\Theta)$ and $\Box A$ has not been derived on the condition Θ. Any line that is derived from a C-marked line is itself C-marked. A line is marked iff it is either W-marked or C-marked.

For the Argued consequence relation, the proofs are governed by RCW.

A line is W-marked iff it is marked for the Weak consequence relation. If two lines that are not W-marked, contain $\Diamond A$ and $\Diamond{\sim}A$ as their second elements respectively, then both lines are A-marked. A line is marked iff it is either W-marked or A-marked.

We now come to the extended consequence relations. In these proofs, the condition is a couple of sets: the first set contains premises, the second primitive formulas.

The common rules are

PREM* Any $\Diamond A \in \Gamma^\Diamond$ may be added to the proof with the justification PREM and $\{\langle A, \emptyset\rangle\}$ as its fifth element.

RU* If $B_1, \ldots, B_n \vdash_{S5} A$ $(n \geq 0)$, and $B_1, \ldots, B_n$ occur in the proof with, respectively, the conditions $\langle \Delta_1, \Theta_1\rangle, \ldots, \langle \Delta_n, \Theta_n\rangle$, then one may add A to the proof with the condition $\langle \Delta_1 \cup \ldots \cup \Delta_n, \Theta_1 \cup \ldots \cup \Theta_n\rangle$.

Let $p(A)$ be the set of all primitive formulas that occur in A. For the Weak* consequence relation, we need the following conditional rules:

RCW* From $\Diamond A$ on the condition $\langle \Delta, \Theta\rangle$ and $\Diamond B$ on the condition $\langle \Delta', \Theta'\rangle$, to derive $\Diamond(A \wedge B)$ on the condition $\langle \Delta \cup \Delta', \Theta \cup \Theta'\rangle$.

RCA From $\Diamond A$ on the condition $\langle \Delta, \Theta\rangle$ to derive $\Box A$ on the condition $\langle \Delta, \Theta \cup \{p(A)\}\rangle$.

At any stage of the proof, zero or more *Dab*-formulas are derived in an unmarked line on a condition that has an empty second element. Some of these are minimal at the stage. Let $U_s(\Gamma^\Diamond)$ be the set of the factors of the minimal *Dab*-formulas at stage s.

A line with condition $\langle \Delta, \Theta\rangle$ is marked iff (i) ${\sim}\Diamond \bigwedge(\Delta)$ has been derived on an unconditional line, or (ii) $\Theta \cap U_s(\Gamma^\Diamond) \neq \emptyset$.

Obviously (i) corresponds to W-marking. The effect of (ii) is that lines at which RCA is applied and lines derived from such lines are marked because they presuppose that the members of Θ are reliable with respect to $\Gamma^\Diamond$ whereas they actually are not. If $\Box A$ is derived in an unmarked line by RCA, then, for any derived $\Diamond B$, $\Diamond(A \wedge B)$ is derivable. RCA and the connected marking rule warrant that, if someone affirmed A and no one denied it, then

A belongs to the consistent core (is implicitly accepted by all parties in the discussion).

For the Strong* consequence relation, the specific rules are RCW* and RCA together with

RCS* From $\Diamond A$ on the condition $\langle \Delta, \Theta \rangle$, to derive $\Box A$ on the condition $\langle \Delta, \Theta \rangle$.

The rule Th* is derivable and useful:

Th* From $\Diamond A$ on the condition $\langle \Delta, \Theta \rangle$, to derive $\Diamond A$ on the condition $\langle \Delta \cup \Delta', \Theta \rangle$.

A line is W*-marked iff it is marked for the Weak* consequence relation. A W*-unmarked line at which $\Box A$ has been derived on a condition $\langle \Delta, \Theta \rangle$, is S*-marked iff, for some condition $\langle \Delta', \Theta' \rangle$ of a W*-unmarked line, there is no W*-unmarked line at which $\Box A$ has been derived on a condition $\langle \Delta' \cup \Delta'', \Theta'' \rangle$ (for some $\Delta'' \supseteq \emptyset$ and $\Theta'' \supseteq \emptyset$). Any line that is derived from a S*-marked line is itself S*-marked. A line is marked if it is either W*-marked or S*-marked.

For the Free* consequence relation, the proofs are governed by RCW* and RCA together with

RCF* From $\Diamond A$ on the condition $\langle \Delta, \Theta \rangle$, to derive $\Box A$ on the condition $\langle \Delta, \Theta \rangle$.

A line is W*-marked iff it is marked for the Weak* consequence relation. A W*-unmarked line at which $\Box A$ has been derived on the condition $\langle \Delta, \Theta \rangle$ is F*-marked iff there is a $\Delta' \subseteq \Gamma$ such that $\sim\Diamond \bigwedge(\Delta' \cup \Delta)$ occurs in the proof on the condition $\langle \emptyset, \emptyset \rangle$, and $\sim\Diamond \bigwedge(\Delta')$ does not occur in the proof on the condition $\langle \emptyset, \emptyset \rangle$. Any line that is derived from a F*-marked line is itself F*-marked. A line is marked if it is either W*-marked or F*-marked.

For the C-based* consequence relation, the proofs are governed by RCW*, RCA, and RCC*, which again is just like RCS* except for the modality:

RCC* From $\Diamond A$ on the condition $\langle \Delta, \Theta \rangle$, to derive $\Box_\Gamma A$ on the condition $\langle \Delta, \Theta \rangle$.

A line is W*-marked iff it is marked for the Weak* consequence relation. A W-unmarked line in which $\Box_\Gamma A$ has been derived by RCC* on a condition $\langle \Delta, \Theta \rangle$ is C*-marked iff, for some condition $\langle \Delta', \Theta' \rangle$ of a W*-unmarked line in the proof, (i) $\#_\Gamma(\Delta) < \#_\Gamma(\Delta')$ or (ii) $\#_\Gamma(\Delta) = \#_\Gamma(\Delta')$ and $\Box A$ has not been derived on some condition $\langle \Delta', \Theta'' \rangle$ (that is: for that Δ' and for an arbitrary Θ''). Any line that is derived from a C*-marked line is itself C*-marked. A line is marked if it is either W*-marked or C*-marked.

For the Argued* consequence relation, the proofs are governed by RCW* and RCA.

A line is W*-marked iff it is marked for the Weak* consequence relation. If two W*-unmarked lines contain, respectively, $\Diamond A$ and $\Diamond\sim A$ as their second elements, then both lines are A*-marked. A line is marked if it is either W*-marked or A*-marked.

The marking of lines in proofs for the extended consequence relations depends in part on the second element of the conditions of lines. This is listed in (ii) of the marking definition for the Weak* consequence relation. Apart from this, the marking definitions for the extended consequence relations are identical to those for the original ones, except that the formulation is more complex because the condition now is a couple of sets. In other words, the extensions depend fully on RCA and on W*-marking.

There is no room for mentioning derivable rules (and shortcuts for marking) in this paper. As usual, these have the advantage to make the proof heuristics more interesting and perspicuous.

8. *Some Comments and Open Problems*

We have seen that the starred consequence relations extend the original ones (except for the Argued consequence relation). From a semantic point of view, they reduce to a further selection of the models. From a proof theoretic point of view, they introduce a second condition and a related marking rule. Moreover, there is a clear and intuitive justification for the extensions (in terms of a consistent core). So, both from a technical and from a conceptual point of view, the extensions seem to strengthen the original Rescher–Manor consequence relations in a natural way (except for the Argued consequence relation).

Readers familiar with adaptive logics will have noticed that the specific selection that leads to the extended consequence relations is based on the Reliability strategy and not on the Minimal Abnormality strategy. This leads to easier definitions of marked lines in the proof theories, but there is also a philosophical point to the matter. As is well known, the Minimal Abnormality strategy leads to a richer consequence set for very specific cases. Here is an example in which the difference matters.

Example 6. Let $\Gamma = \{{\sim}p \wedge {\sim}q, (p \vee q) \wedge (p \vee r) \wedge (q \vee r)\}$. The *MCS* are $\{{\sim}p \wedge {\sim}q\}$ and $\{(p \vee q) \wedge (p \vee r) \wedge (q \vee r)\}$, and $U(\Gamma) = \{p, q\}$. There are six kinds of worlds that verify one of the *MCS*:

(1) ${\sim}p, {\sim}q, r$
(2) ${\sim}p, {\sim}q, {\sim}r$
(3) p, q, r
(4) $p, q, {\sim}r$
(5) $p, {\sim}q, r$
(6) ${\sim}p, q, r$

The RM-models of $\Gamma^{\Diamond}$ have worlds of types 1 and/or 2 as well as worlds of types 3 and/or 4 and/or 5 and/or 6. RM*-models have only worlds of types 1 together with worlds of type 3 and/or 5 and/or 6 — all of these verify r — or worlds of types 2 and 4 — these verify $\sim r$. I leave it to the reader to check that r does not belong to any of the extended consequence sets.

If the RM*-models are selected by the Minimal Abnormality strategy, only models with $Ab_p(M) = \{p\}$ or $Ab_p(M) = \{q\}$ will be selected. Hence some RM*-models contain only worlds of types 1 and 5, and the others only worlds of types 1 and 6. But then all selected models verify $\Box r$ as well as $\Box(\sim p \vee \sim q)$.

It can easily be shown that, if the extended consequence sets are defined in terms of the Minimal Abnormality strategy, then they are supersets, and for some premises proper supersets, of the extended consequence sets as defined above in terms of the Reliability strategy. While there is no doubt about this, the problem is whether the further extension is justified. Consider again Example 6. The Minimal Abnormality strategy supposes that the two participants disagree about either p or q, *but not about both.* Is it justified to suppose so? The second participant states that at least two of $\{p, q, r\}$ are true. This is the strongest way, within the present framework, in which the second participant can express *not* to exclude $p \wedge q$.

I shall leave the matter here. I do not suggest that further reflection will settle the question about the suitability of the Minimal Abnormality strategy. It rather seems that the relevant arguments refer to features that cannot be expressed within the present framework — see below.

Several open problems deserve to be mentioned. There may be simpler criteria for selecting models. More specifically, there may be criteria that select the RM*-models of Γ directly from the MA-models of Γ, or even from the C-models of Γ, or — the nicest alternative — directly from the S5-models. Similarly for the F*-models of Γ. Such selection criteria for models will have their counterparts in one-shot marking definitions. A different open problem is whether the present results constitute a maximum. Can the extended consequence relations be further extended in such a way that the consequence sets are consistent and closed under CL, and that the extension is intuitively justified?

Rephrasing the logics presented in this paper in the standard format of [4] would require that the paper be thoroughly rewritten. Except for the C-Based consequence relation, the original ones are characterized by adaptive logics in standard format in [3], where the premise set Γ is 'translated' to $\{\sim\neg A \mid A \in \Gamma\}$. Except for the C*-Based consequence relation, the extended consequence relations can be characterized by adaptive logics in standard format under the present translation of Γ to $\{\Diamond A \mid A \in \Gamma\}$. However, this requires a rather different approach from the one in the present paper. I found no way to present these adaptive logics as natural extensions of those for the original Rescher–Manor consequence relations in terms of that approach — and this was the main point I tried to make in the present paper.

An open problem of a very different sort concerns a more realistic approach to explicit and implicit agreements in discussions. This requires at least the three following changes to the present framework. First, it should be possible for the participants to express lack of knowledge or even lack of agreement. In the present framework, a participant can only express (direct or connected) disagreement with respect to a statement by another participant. In real life discussions, participants often express their lack of support for some statement, even if they explicitly refuse to commit themselves to any (direct or connected) disagreement about the statement. If a participant states q and another participant states $\sim q \vee r$, but refuses commitment to q, then r does not belong to the consistent core. Next, it is required that the statements made by a participant are separated (instead of being given as a conjunction) and that the temporal order of all the statements made during the discussion is taken into account. For example, if one participant states p, and another immediately thereafter states $p \vee q$, then the latter statement apparently expresses a refusal to commit to p. If the temporal order is reversed, this needs not be the case: the participant who now first states $p \vee q$ may consider the participant stating p as an authority on the matter, and hence agree with p. Finally, one should take into account that participants may change their minds. Here too the temporal order is essential. As a result, self-inconsistent participants will not be disregarded. Self-inconsistency will rather lead to a revision of a participant's claims. As the present framework (justly and importantly) involves consistent subgroups, this revision will be more sophisticated than the usual mechanisms for belief revision.

As a final remark, let me stress that it was not my intention to defend the Rescher–Manor consequence relations as general inconsistency handling mechanisms. As I have stressed time and again, their application contexts are very restricted. They are justly applied to discussions of the type considered, because in this application context the premises may be constructed as representing clearly distinct sources.

Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
E-mail: `Diderik.Batens@UGent.be`

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