A Deontic Logic Framework Allowing for Factual Detachment — Appendix

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Abstract

This Appendix contains in part A the semantic characterization of logics **CDPM.2d** and **CDPM.2e** defined in [3]. Soundness and completeness are proven. In part B the logical framework presented in [3] is generalized such that it is able to deal with nested permissible contexts.

Key words: conditional deontic logic, adaptive logic, detachment, modus ponens, defeasible reasoning

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A. Semantics

Although the semantics that I introduce in this Appendix are very similar to Goble's semantics of **CDPM.2c** in [2, 1], they vary from the former in the following aspects:

- An actual world variant of the semantics is used here in order to model factual premises in an intuitive way.
- The semantics have to deal with a language enriched by modal operators O^{i} and O^{p}, symbols $\bullet_{\mathsf{i}}$ and $\bullet_{\mathsf{p}}$, and the additional principles characterizing them.
- The language is weaker than Goble's in the sense that nested occurrences of modal operators are not allowed.

A.1. Language

The language used for the logics defined in [3] is built up by propositional atoms, denoted by $\mathcal{A}$, the classical connectives, $\top, \bot$, a dyadic modal operator O, monadic modal operators $\mathsf{O}^{\mathsf{i}}, \mathsf{O}^{\mathsf{p}}$ and symbols $\bullet_{\mathsf{i}}, \bullet_{\mathsf{p}}$. We use for (classical) propositional formulas the letters A, B, C, D, E, F and denote by $\mathcal{P}$ the set of all propositional formulas. Let $\mathcal{L}'$ consist of all formulas of the form $\mathsf{O}(A \mid B), \mathsf{O}^{\mathsf{i}}A, \mathsf{O}^{\mathsf{p}}A, \bullet_{\mathsf{i}}\mathsf{O}(A \mid B), \bullet_{\mathsf{p}}\mathsf{O}(A \mid B)$ and the set of all propositional letters. Our set of wffs $\mathcal{L}$ is then defined by the $\langle \neg, \vee, \wedge, \supset \rangle$-closure of $\mathcal{L}'$. We use for formulas in $\mathcal{L}$ lower case greek letters. As usually done, we define $\mathsf{P}(A \mid B) =_{\mathrm{df}} \neg\mathsf{O}(\neg A \mid B)$.

A.2. Syntactic characterization

First, in order to recapitulate the definitions from [3], we state again the syntactic rules used to define logics **CDPM.2d**$^+$ and **CDPM.2e**$^+$:

$$\text{If } \vdash A \equiv B, \text{ then } \vdash \mathsf{O}(C \mid A) \equiv \mathsf{O}(C \mid B) \qquad \text{(RCE)}$$

$$\text{If } \vdash A \equiv B, \text{ then } \vdash \mathsf{O}(A \mid C) \equiv \mathsf{O}(B \mid C) \tag{CRE}$$

$$\text{If } \vdash A \equiv B, \text{ then } \vdash \mathsf{O}^{\mathsf{i}}A \equiv \mathsf{O}^{\mathsf{i}}B \tag{EO$^{\mathsf{i}}$}$$

$$\text{If } \vdash A \equiv B, \text{ then } \vdash \mathsf{O}^{\mathsf{p}}A \equiv \mathsf{O}^{\mathsf{p}}B \tag{EO$^{\mathsf{p}}$}$$

$$\text{If } \vdash A \equiv B, \text{ then } \vdash \bullet_{\mathsf{i}}\mathsf{O}(A \mid C) \equiv \bullet_{\mathsf{i}}\mathsf{O}(B \mid C) \tag{CRE$^{\mathsf{i}}$}$$

$$\text{If } \vdash A \equiv B, \text{ then } \vdash \bullet_{\mathsf{i}}\mathsf{O}(C \mid A) \equiv \bullet_{\mathsf{i}}\mathsf{O}(C \mid B) \tag{RCE$^{\mathsf{i}}$}$$

$$\text{If } \vdash A \equiv B, \text{ then } \vdash \bullet_{\mathsf{p}}\mathsf{O}(A \mid C) \equiv \bullet_{\mathsf{p}}\mathsf{O}(B \mid C) \tag{CRE$^{\mathsf{p}}$}$$

$$\text{If } \vdash A \equiv B, \text{ then } \vdash \bullet_{\mathsf{p}}\mathsf{O}(C \mid A) \equiv \bullet_{\mathsf{p}}\mathsf{O}(C \mid B) \tag{RCE$^{\mathsf{p}}$}$$

$$\text{If } \vdash B \supset C, \text{ then } \vdash \mathsf{P}(B \mid A) \supset \big(\mathsf{O}(B \mid A) \supset \mathsf{O}(C \mid A)\big) \tag{RCPM}$$

$$\begin{split}\text{If } \vdash D \supset \neg A, \text{ then } \vdash \big(\big(\mathsf{P}(D \mid B \wedge C) \vee \mathsf{O}(D \mid B \wedge C)\big) \wedge \\ B \wedge C \wedge \mathsf{P}(B \wedge C \mid B) \wedge \mathsf{O}(A \mid B)) \supset \bullet_{\mathsf{p}}\mathsf{O}(A \mid B)\end{split} \tag{Ep}$$

$$\begin{split}\text{If } \vdash A \supset \neg C \text{ and } \vdash A \supset \neg D, \text{ then} \\ \vdash \big(\mathsf{O}(A \mid B \wedge C) \wedge \mathsf{O}(D \mid B)\big) \supset \bullet_{\mathsf{p}}\mathsf{O}(A \mid B \wedge C)\end{split} \tag{CTDR}$$

$$\begin{split}\text{If } \vdash D \supset \neg A, \text{ then } \vdash \big(\big(\mathsf{P}(D \mid B \wedge C) \vee \mathsf{O}(D \mid B \wedge C)\big) \wedge \\ B \wedge C \wedge \mathsf{O}(A \mid B)) \supset \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)\end{split} \tag{oV-E$^{\mathsf{i}}$}$$

Furthermore, the following axioms are needed:

$$\vdash \mathsf{P}(\top \mid A) \tag{CP}$$

$$\vdash \mathsf{O}(C \mid A \wedge B) \supset \mathsf{O}(B \supset C \mid A) \tag{S}$$

$$\vdash \big(\mathsf{O}(A \mid C) \wedge \mathsf{O}(B \mid C) \wedge \mathsf{P}(A \wedge B \mid C)\big) \supset \mathsf{O}(A \wedge B \mid C) \tag{CPAND}$$

$$\vdash \big(\mathsf{O}(B \mid A) \wedge \mathsf{P}(B \wedge C \mid A)\big) \supset \mathsf{O}(B \mid A \wedge C) \tag{WRM}$$

$$\vdash \big(\mathsf{O}(A \mid B \wedge C) \wedge \mathsf{P}(A \mid \neg B \wedge C)\big) \supset \mathsf{O}(B \supset A \mid C) \tag{PS'}$$

$$\vdash \big(\mathsf{O}(A \mid B) \wedge B \wedge \neg \bullet_{\mathsf{p}} \mathsf{O}(A \mid B)\big) \supset \mathsf{O}^{\mathsf{p}}A \tag{FD$^{\mathsf{p}}$}$$

$$\vdash \big(\mathsf{O}(A \mid B) \wedge B \wedge \neg \bullet_{\mathsf{i}} \mathsf{O}(A \mid B)\big) \supset \mathsf{O}^{\mathsf{i}}A \tag{FD$^{\mathsf{i}}$}$$

$$\vdash \big(\mathsf{O}(A \mid B) \wedge \neg A \wedge B\big) \supset \bullet_{\mathsf{i}}\mathsf{O}(A \mid B) \tag{fV}$$

Definition A.1. Logic $\mathbf{CDPM.2d}^+$ is defined by all the rules stated above (with exception of (PS'))[2], $\mathbf{CDPM.2e}^+$ is defined as $\mathbf{CDPM.2d}^+$ with exception of (S), which is replaced by (PS'). Let $\mathbf{L}^+$ from now on be any of the two logics (if not specified beforehand).

A.3. The neighborhood semantics

One of the basic ideas for the neighborhood semantics is that propositions are interpreted in terms of sets of worlds. For each obligation type (such as $\mathsf{O}^{\mathsf{i}}, \mathsf{O}^{\mathsf{p}}, \ldots$) each world has associated with it propositions, i.e. sets of worlds. The idea is that an obligation $\mathsf{O}^{\mathsf{i}}A$ is true at a world w, in case A is one of its associated propositions with respect to O^{i}. The generalization in terms of conditional obligations is canonical. In this case worlds are associated with

[2]This rule follows directly from (S).

ordered pairs of propositions representing conditional obligations. We are going to make use of an actual world. The reason is that we are going to work with premise sets containing propositional formulas representing given facts, which can be better modelled like that.

Let a dyadic neighborhood frame F be a tuple $\langle W, \mathcal{O}, \mathcal{N}^{\mathrm{i}}, \mathcal{N}^{\mathrm{p}}, \mathcal{O}^{\mathrm{i}}, \mathcal{O}^{\mathrm{p}} \rangle$ where W is a set of worlds and $\mathcal{O} : W \to \wp(\wp(W) \times \wp(W))$, $\mathcal{N}^{\mathrm{i}} : W \to \wp(\wp(W) \times \wp(W))$, $\mathcal{N}^{\mathrm{p}} : W \to \wp(\wp(W) \times \wp(W))$, $\mathcal{O}^{\mathrm{i}} : W \to \wp(\wp(W))$, $\mathcal{O}^{\mathrm{p}} : W \to \wp(\wp(W))$. Thus, $\mathcal{O}$, $\mathcal{N}^{\mathrm{i}}$ and $\mathcal{N}^{\mathrm{p}}$ assign to each world $w \in W$ a set of ordered propositions, i.e., $\mathcal{O}_w, \mathcal{N}^{\mathrm{i}}_w, \mathcal{N}^{\mathrm{p}}_w \subseteq \wp(W) \times \wp(W)$,[3] and $\mathcal{O}^{\mathrm{i}}$ and O^{p} assign to each world a proposition, i.e., $\mathcal{O}^{\mathrm{i}}_w, \mathcal{O}^{\mathrm{p}}_w \subseteq \wp(W)$. A model M on frame F is a triple $\langle F, @, v \rangle$ where $@ \in W$ is the actual world and $v : \mathcal{A} \to \wp(W)$. A propositional atom is mapped into the set of worlds in which it is supposed to hold. We define $M \models \varphi$ iff $M, @ \models \varphi$, $F \models \varphi$ iff for all models M defined on the basis of frame F, $M \models \varphi$, and $\mathcal{F} \models \varphi$ (where $\mathcal{F}$ is a set of frames) iff for all $F \in \mathcal{F}$, $F \models \varphi$. Furthermore, where $w \in W$, we have the following requirements for our models:

$$M, w \models p \text{ iff } w \in v(p), \text{ where } p \in \mathcal{A} \tag{M-p}$$

$$M, w \models \mathsf{O}(A \mid B) \text{ iff } \langle |B|_M, |A|_M \rangle \in \mathcal{O}_w \tag{M-$\mathcal{O}$}$$

$$M, w \models \bullet_{\mathsf{i}}\mathsf{O}(A \mid B) \text{ iff } \langle |B|_M, |A|_M \rangle \in \mathcal{N}^{\mathrm{i}}_w \tag{M-$\mathcal{N}^{\mathrm{i}}$}$$

$$M, w \models \bullet_{\mathsf{p}}\mathsf{O}(A \mid B) \text{ iff } \langle |B|_M, |A|_M \rangle \in \mathcal{N}^{\mathrm{p}}_w \tag{M-$\mathcal{N}^{\mathrm{p}}$}$$

$$M, w \models \mathcal{O}^{\mathrm{i}} A \text{ iff } |A|_M \in \mathcal{O}^{\mathrm{i}}_w \tag{M-$\mathcal{O}^{\mathrm{i}}$}$$

$$M, w \models \mathcal{O}^{\mathrm{p}} A \text{ iff } |A|_M \in \mathcal{O}^{\mathrm{p}}_w \tag{M-$\mathcal{O}^{\mathrm{p}}$}$$

where $|\varphi|_M =_{\mathrm{df}} \{w \in W \mid M, w \models \varphi\}$. For the classical connectives the definitions are as usual:

$$M, w \models \neg\varphi \text{ iff } M, w \not\models \varphi \tag{M-$\neg$}$$

$$M, w \models \varphi \vee \psi \text{ iff } M, w \models \varphi \text{ or } M, w \models \psi \tag{M-$\vee$}$$

$$M, w \models \varphi \wedge \psi \text{ iff } M, w \models \varphi \text{ and } M, w \models \psi \tag{M-$\wedge$}$$

$$M, w \models \varphi \supset \psi \text{ iff } M, w \models \neg\varphi \vee \psi \tag{M-$\supset$}$$

$$M, w \models \top \tag{M-$\top$}$$

$$M, w \not\models \bot \tag{M-$\bot$}$$

We write $\overline{W'} =_{\mathrm{df}} W \setminus W'$ where $W' \subseteq W$ for a given frame $F = \langle W, \mathcal{O}, \mathcal{N}^{\mathrm{i}}, \mathcal{N}^{\mathrm{p}}, \mathcal{O}^{\mathrm{i}}, \mathcal{O}^{\mathrm{p}} \rangle$. In order to define our **CDPM** systems we also need the following conditions on frames. For all $X, Y, Z \subseteq W$ and $w \in W$ we demand:

$$\langle W, W \rangle \in \mathcal{O}_w \tag{F-CN}$$

[3] We follow Goble's writing convention and write the argument of the mappings that constitute frames as subscripts.

$$\text{If } Y \subseteq Z; \langle X, Y\rangle \in \mathcal{O}_w \text{ and } \langle X, \overline{Y}\rangle \notin \mathcal{O}_w, \text{ then } \langle X, Z\rangle \in \mathcal{O}_w \qquad \text{(F-RCPM)}$$

$$\text{If } \langle X \cap Y, Z\rangle \in \mathcal{O}_w, \text{ then } \langle X, \overline{Y} \cup Z\rangle \in \mathcal{O}_w \qquad \text{(F-S)}$$

$$\text{If } \langle X, Y\rangle \in \mathcal{O}_w \text{ and } \langle X, \overline{Y \cap Z}\rangle \notin \mathcal{O}_w, \text{ then } \langle X \cap Z, Y\rangle \in \mathcal{O}_w \qquad \text{(F-WRM)}$$

$$\text{If } \langle X, Y\rangle \in \mathcal{O}_w; \langle X, Z\rangle \in \mathcal{O}_w \text{ and } \langle X, \overline{Y \cap Z}\rangle \notin \mathcal{O}_w, \text{ then } \langle X, Y \cap Z\rangle \in \mathcal{O}_w \qquad \text{(F-CPAND)}$$

$$\langle X, \emptyset\rangle \notin \mathcal{O}_w \qquad \text{(F-CP)}$$

For the **e**-version of our system we need:

$$\text{If } \langle Y \cap Z, X\rangle \in \mathcal{O}_w \text{ and } \langle \overline{Y} \cap Z, \overline{X}\rangle \notin \mathcal{O}_w, \text{ then } \langle Z, \overline{Y} \cup X\rangle \in \mathcal{O}_w \qquad \text{(F-PS')}$$

In order to model detachment we are in need of the following conditions on frames:

$$\text{If } \langle Y, X\rangle \in \mathcal{O}_w; w \in Y; \text{ and } \langle Y, X\rangle \notin \mathcal{N}^{\mathrm{i}}_w, \text{ then } X \in \mathcal{O}^{\mathrm{i}}_w \qquad \text{(F-FD}^{\mathrm{i}}\text{)}$$

$$\text{If } \langle Y, X\rangle \in \mathcal{O}_w; w \in Y; \text{ and } \langle Y, X\rangle \notin \mathcal{N}^{\mathrm{p}}_w, \text{ then } X \in \mathcal{O}^{\mathrm{p}}_w \qquad \text{(F-FD}^{\mathrm{p}}\text{)}$$

$$\text{If } w \in Y \cap Z; \langle Y, \overline{Y \cap Z}\rangle \notin \mathcal{O}_w; \langle Y, X\rangle \in \mathcal{O}_w; Z' \subseteq \overline{X}; \text{ and } \big(\langle Y \cap Z, \overline{Z'}\rangle \notin \mathcal{O}_w \text{ or } \langle Y \cap Z, Z'\rangle \in \mathcal{O}_w\big), \text{ then } \langle Y, X\rangle \in \mathcal{N}^{\mathrm{p}}_w \qquad \text{(F-E}^{\mathrm{p}}\text{)}$$

$$\text{If } \langle Y \cap Z, X\rangle, \langle Y, Z'\rangle \in \mathcal{O}_w; Z' \subseteq \overline{Z}; \text{ and } Z' \subseteq \overline{X}, \text{ then } \langle Y \cap Z, X\rangle \in \mathcal{N}^{\mathrm{p}}_w \qquad \text{(F-CTDR)}$$

$$\text{If } \langle Y, X\rangle \in \mathcal{O}_w; w \in Y; \text{ and } w \notin X; \text{ then } \langle Y, X\rangle \in \mathcal{N}^{\mathrm{i}}_w \qquad \text{(F-fV)}$$

$$\text{If } w \in Y \cap Z; \langle Y, X\rangle \in \mathcal{O}_w; \big(\langle Y \cap Z, \overline{Z'}\rangle \notin \mathcal{O}_w \text{ or } \langle Y \cap Z, Z'\rangle \in \mathcal{O}_w\big); \text{ and } Z' \subseteq \overline{X}, \text{ then } \langle Y, X\rangle \in \mathcal{N}^{\mathrm{i}} \qquad \text{(F-oV-E}^{\mathrm{i}}\text{)}$$

A.4. Soundness

Note that for the proofs in the Appendix I sometimes write LHS $\overset{X}{=}$ RHS if the equation between LHS and RHS holds due to Lemma X. Obviously most of the following results and their proofs resemble results proven by Goble for his **CDPM** systems.

Lemma A.1. *For any model $M = \langle F, @, v\rangle$, where $F = \langle W, \mathcal{O}, \mathcal{N}^{\mathrm{i}}, \mathcal{N}^{\mathrm{p}}, \mathcal{O}^{\mathrm{i}}, \mathcal{O}^{\mathrm{p}}\rangle$, (i) $|\varphi \wedge \psi|_M = |\varphi|_M \cap |\psi|_M$; (ii) $|\varphi \vee \psi|_M = |\varphi|_M \cup |\psi|_M$; (iii) $|\neg\varphi|_M = \overline{|\varphi|_M}$; (iv) $|\top|_M = W$; (v) $|\bot|_M = \emptyset$.*

Proof. Ad (i): $|\varphi \wedge \psi|_M = \{w \in W \mid M, w \models \varphi \wedge \psi\} = \{w \in W \mid M, w \models \varphi, \psi\} = \{w \in W \mid M, w \models \varphi\} \cap \{w \in W \mid M, w \models \psi\} = |\varphi|_M \cap |\psi|_M$. Ad (ii): analogous. Ad (iii): $|\neg\varphi|_M = \{w \in W \mid M, w \models \neg\varphi\} = \{w \in W \mid M, w \not\models \varphi\} = W \setminus \{w \in W \mid M, w \models \varphi\} = \overline{|\varphi|_M}$. Ad (iv): $|\top|_M = \{w \in W \mid M, w \models \top\} = W$. Ad (v): analogous. □

Lemma A.2. *For any model $M = \langle F, @, v\rangle$, where $F = \langle W, \mathcal{O}, \mathcal{N}^{\mathrm{i}}, \mathcal{N}^{\mathrm{p}}, \mathcal{O}^{\mathrm{i}}, \mathcal{O}^{\mathrm{p}}\rangle$, (i), if $F \models \varphi \supset \psi$, then $|\varphi|_M \subseteq |\psi|_M$, and, (ii), if $F \models \varphi \equiv \psi$, then $|\varphi|_M = |\psi|_M$.*

Proof. Ad (i): Suppose there is a $w \in W$ for which $M, w \models \varphi \wedge \neg\psi$. $M' = \langle F, w, v\rangle$ obviously satisfies the model conditions (M-p), (M-$\mathcal{O}$), (M-$\mathcal{N}^{\mathsf{i}}$), (M-$\mathcal{N}^{\mathsf{p}}$), (M-$\mathcal{O}^{\mathsf{i}}$), (M-$\mathcal{O}^{\mathsf{p}}$), (M-$\neg$), (M-$\vee$), (M-$\wedge$) and (M-$\supset$) since M satisfies them. But then $F \not\models \varphi \supset \psi$—a contradiction. Ad (ii): This is an immediate consequence of (i). □

Theorem A.1. $\mathbf{L}^+$ *is sound with respect to the class of frames* $\mathcal{F}$ *that meet the appropriate frame conditions. In case of* $\mathbf{CDPM.2d}^+$ *the frame conditions are* (F-CN)*,* (F-RCPM)*,* (F-S)*,* (F-WRM)*,* (F-CPAND)*,* (F-CP)*,* (F-FD$^{\mathsf{i}}$)*,* (F-FD$^{\mathsf{p}}$)*,* (F-E$^{\mathsf{p}}$)*,* (F-CTDR)*,* (F-fV) *and* (F-oV-E$^{\mathsf{i}}$)*. In case of* $\mathbf{CDPM.2e}^+$ *we replace* (F-S) *by* (F-PS')*.*

Proof. The proof is very simple: we thus show only for a few rules paradigmatically that they are valid in all models of the respective frames. Let $\mathcal{F}$ be our respective class of frames and let $M = \langle F, @, v\rangle$ be an arbitrary model on an arbitrary frame $F \in \mathcal{F}$.

We begin with (RCPM): Let $F \models B \supset C$. Assume that $M, @ \models \mathsf{P}(B \mid A) \wedge \mathsf{O}(B \mid A)$. Then $M, @ \models \mathsf{P}(B \mid A), \mathsf{O}(B \mid A)$ and thus, $M, @ \models \neg\mathsf{O}(\neg B \mid A), \mathsf{O}(B \mid A)$. Hence, $\langle |A|_M, |B|_M\rangle \in \mathcal{O}_@$ and $\langle |A|_M, |\neg B|_M\rangle \notin \mathcal{O}_@$. Thus by Lemma A.1 (iii), $\langle |A|_M, \overline{|B|_M}\rangle \notin \mathcal{O}_@$. Furthermore, by Lemma A.2 (i), $|B|_M \subseteq |C|_M$. Since $\mathcal{F}$ validates (F-RCPM), $\langle |A|_M, |C|_M\rangle \in \mathcal{O}_@$. Hence, $M, @ \models \mathsf{O}(C \mid A)$ and thus, $M, @ \models (\mathsf{P}(B \mid A) \wedge \mathsf{O}(B \mid A)) \supset \mathsf{O}(C \mid A)$. Hence, $M \models (\mathsf{P}(B \mid A) \wedge \mathsf{O}(B \mid A)) \supset \mathsf{O}(C \mid A)$. Since M and F were arbitrary, $\mathcal{F} \models \mathsf{P}(B \mid A) \supset \big(\mathsf{O}(B \mid A) \supset \mathsf{O}(C \mid A)\big)$.

For (WRM): Assume that $M, @ \models \mathsf{O}(B \mid A) \wedge \mathsf{P}(B \wedge C \mid A)$. Then $M, @ \models \mathsf{O}(B \mid A), \neg\mathsf{O}(\neg(B \wedge C) \mid A)$. Thus, $\langle |A|_M, |\neg(B \wedge C)|_M\rangle \overset{A.1iii}{=} \big\langle |A|_M, \overline{|B \wedge C|_M}\big\rangle \overset{A.1i}{=} \big\langle |A|_M, \overline{|B|_M \cap |C|_M}\big\rangle \notin \mathcal{O}_@$. Thus, since F validates (F-WRM), $\langle |A|_M \cap |C|_M, |B|_M\rangle \overset{A.1i}{=} \langle |A \wedge C|_M, |B|_M\rangle \in \mathcal{O}_@$. Hence, $M, @ \models \mathsf{O}(B \mid A \wedge C)$ and thus, $M, @ \models \big(\mathsf{O}(B \mid A) \wedge \mathsf{P}(B \wedge C \mid A)\big) \supset \mathsf{O}(B \mid A \wedge C)$. Hence, $M \models \big(\mathsf{O}(B \mid A) \wedge \mathsf{P}(B \wedge C \mid A)\big) \supset \mathsf{O}(B \mid A \wedge C)$. Since M and F were arbitrary, $\mathcal{F} \models \big(\mathsf{O}(B \mid A) \wedge \mathsf{P}(B \wedge C \mid A)\big) \supset \mathsf{O}(B \mid A \wedge C)$.

For (PS'): Assume that $M, @ \models \mathsf{O}(A \mid B \wedge C) \wedge \mathsf{P}(A \mid \neg B \wedge C)$. Thus, $M, @ \models \mathsf{O}(A \mid B \wedge C), \mathsf{P}(A \mid \neg B \wedge C)$. Thus, $M, @ \models \mathsf{O}(A \mid B \wedge C), \neg\mathsf{O}(\neg A \mid \neg B \wedge C)$. Hence, $\langle |B \wedge C|_M, |A|_M\rangle \in \mathcal{O}_@$ and $\langle |\neg B \wedge C|_M, |\neg A|_M\rangle \notin \mathcal{O}_@$. Hence, by Lemma A.1, $\langle |B|_M \cap |C|_M, |A|_M\rangle \in \mathcal{O}_@$ and $\big\langle \overline{|B|_M} \cap |C|_M, \overline{|A|_M}\big\rangle \notin \mathcal{O}_@$. Since $\mathcal{F}$ satisfies (F-PS'), $\big\langle |C|_M, \overline{|B|_M} \cup |A|_M\big\rangle \in \mathcal{O}_@$. By Lemma A.1, $\langle |C|_M, |\neg B \vee A|_M\rangle \in \mathcal{O}_@$. By Lemma A.2 (ii), $\langle |C|_M, |B \supset A|_M\rangle \in \mathcal{O}_@$ and thus, $M, @ \models \mathsf{O}(B \supset A \mid C)$. Hence, $M, @ \models \big(\mathsf{O}(A \mid B \wedge C) \wedge \mathsf{P}(A \mid \neg B \wedge C)\big) \supset \mathsf{O}(B \supset A \mid C)$. Thus, $M\big(\mathsf{O}(A \mid B \wedge C) \wedge \mathsf{P}(A \mid \neg B \wedge C)\big) \supset \mathsf{O}(B \supset A \mid C)$. Since F and M were arbitrary, $\mathcal{F} \models \big(\mathsf{O}(A \mid B \wedge C) \wedge \mathsf{P}(A \mid \neg B \wedge C)\big) \supset \mathsf{O}(B \supset A \mid C)$.

For (EO$^{\mathsf{i}}$): Let $F \models A \equiv B$. Assume that $M, @ \models \mathsf{O}^{\mathsf{i}}A$. Thus, $|A|_M \in \mathcal{O}^{\mathsf{i}}_@$. By Lemma A.2 (ii), $|A|_M = |B|_M$. Thus, $|B|_M \in \mathcal{O}^{\mathsf{i}}_@$. Hence, $M, @ \models \mathsf{O}^{\mathsf{i}}B$. Thus, $M, @ \models \mathsf{O}^{\mathsf{i}}A \supset \mathsf{O}^{\mathsf{i}}B$. Hence, $M \models \mathsf{O}^{\mathsf{i}}A \supset \mathsf{O}^{\mathsf{i}}B$. Since M and F were arbitrary, $\mathcal{F} \models \mathsf{O}^{\mathsf{i}}A \supset \mathsf{O}^{\mathsf{i}}B$. The other direction is analogous.

For (FD$^{\mathsf{i}}$): Assume that $M, @ \models \mathsf{O}(A \mid B) \wedge B \wedge \neg \bullet_{\mathsf{i}} \mathsf{O}(A \mid B)$ and thus, $M, @ \models \mathsf{O}(A \mid B), B$ and $M, @ \not\models \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)$. Hence, $\langle |B|_M, |A|_M \rangle \in \mathcal{O}_@$, $@ \in |B|_M$ and $\langle |B|_M, |A|_m \rangle \notin \mathcal{N}^{\mathsf{i}}_@$. Since $\mathcal{F}$ satisfies (F-FD$^{\mathsf{i}}$), $|A|_M \in \mathcal{O}^{\mathsf{i}}_@$ and thus, $M, @ \models \mathsf{O}^{\mathsf{i}}A$. Hence, $M, @ \models \big(\mathsf{O}(A \mid B) \wedge B \wedge \neg \bullet_{\mathsf{i}} \mathsf{O}(A \mid B)\big) \supset \mathsf{O}^{\mathsf{i}}A$. Hence, $M \models \big(\mathsf{O}(A \mid B) \wedge B \wedge \neg \bullet_{\mathsf{i}} \mathsf{O}(A \mid B)\big) \supset \mathsf{O}^{\mathsf{i}}A$. Since M and F were arbitrary, we have $\mathcal{F} \models \big(\mathsf{O}(A \mid B) \wedge B \wedge \neg \bullet_{\mathsf{i}} \mathsf{O}(A \mid B)\big) \supset \mathsf{O}^{\mathsf{i}}A$.

The other cases are shown in a similar way. □

A.5. Completeness

Completeness for our logics can be proven in a similar way as Goble proved completeness of his (**C**)**DPM** systems. We proceed in two steps:

1. We prove model-completeness by means of a canonical model $\dot{M}$ and adjusted model conditions. We show that for each non-theorem φ of $\mathbf{L}^+$ there is such a $\dot{M}$ falsifying φ.

2. Using filtration techniques on the canonical model $\dot{M}$ we arrive at an alternative model $\overset{\star}{M}$ on a frame $\overset{\star}{F}$ that satisfies the respective frame conditions. For each non-theorem φ of $\mathbf{L}^+$ we have an $\overset{\star}{M}$ which falsifies φ. This suffices to prove completeness and decidability.

A.5.1. Model completeness

First we define a frame for a canonical model for $\mathbf{L}^+$. Let $\dot{F} = \langle \dot{W}, \dot{\mathcal{O}}, \dot{\mathcal{N}}^{\mathsf{i}}, \dot{\mathcal{N}}^{\mathsf{p}}, \dot{\mathcal{O}}^{\mathsf{i}}, \dot{\mathcal{O}}^{\mathsf{p}} \rangle$. $\dot{W}$ contains all maximal $\mathbf{L}^+$-consistent sets of formulas in $\mathcal{L}$. We have the following assignments for all $w \in \dot{W}$:[4]

$$\dot{\mathcal{O}}_w = \{\langle X, Y \rangle \mid X \subseteq \dot{W}, Y \subseteq \dot{W}, \exists A \exists B (X = [A], Y = [B] \text{ and } \mathsf{O}(B \mid A) \in w\},$$

$$\dot{\mathcal{N}}^{\mathsf{i}}{}_w = \{\langle X, Y \rangle \mid X \subseteq \dot{W}, Y \subseteq \dot{W}, \exists A \exists B (X = [A], Y = [B] \text{ and } \bullet_{\mathsf{i}} \mathsf{O}(B \mid A) \in w\},$$

$$\dot{\mathcal{N}}^{\mathsf{p}}{}_w = \{\langle X, Y \rangle \mid X \subseteq \dot{W}, Y \subseteq \dot{W}, \exists A \exists B (X = [A], Y = [B] \text{ and } \bullet_{\mathsf{p}} \mathsf{O}(B \mid A) \in w\},$$

$$\dot{\mathcal{O}}^{\mathsf{i}}{}_w = \{X \subseteq \dot{W} \mid \exists A (X = [A] \text{ and } \mathsf{O}^{\mathsf{i}}A \in w\},$$

$$\dot{\mathcal{O}}^{\mathsf{p}}{}_w = \{X \subseteq \dot{W} \mid \exists A (X = [A] \text{ and } \mathsf{O}^{\mathsf{p}}A \in w\},$$

where $[\varphi] =_{\mathrm{df}} \{w \in \dot{W} \mid \varphi \in w\}$. Now we can define a canonical model $\dot{M} = \langle \dot{F}, \dot{@}, \dot{v} \rangle$. Let for every atomic formula p

$$\dot{v} : p \mapsto \{w \in \dot{W} \mid p \in w\}$$

Lemma A.3. *For any φ and ψ, (i) $[\varphi \wedge \psi] = [\varphi] \cap [\psi]$; (ii) $[\varphi \vee \psi] = [\varphi] \cup [\psi]$; (iii) $[\neg\varphi] = \overline{[\varphi]}$; (iv) $[\top] = \dot{W}$; (v) $[\bot] = \emptyset$.*

[4] For sake of readability we use from now on "∃" and "∀" in set descriptions in the canonical reading "there is" and "for all".

Proof. Ad (i): $[\varphi \wedge \psi] = \{w \in \dot{W} \mid \varphi \wedge \psi \in w\} \overset{(1)}{=} \{w \in \dot{W} \mid \varphi, \psi \in w\} = \{w \in \dot{W} \mid \varphi \in w\} \cap \{w \in \dot{W} \mid \psi \in w\} = [\varphi] \cap [\psi]$ where (1) is due to the fact that w is a maximal consistent extension. The other cases are shown in a similar way. □

Lemma A.4. *For any φ and ψ, (i) $[\varphi] \subseteq [\psi]$ iff $\vdash \varphi \supset \psi$, and (ii) $[\varphi] = [\psi]$ iff $\vdash \varphi \equiv \psi$.*

Proof. This was proven in an analogous way in Goble [2, 1]. For (i), suppose $[\varphi] \subseteq [\psi]$ but $\nvdash \varphi \supset \psi$. Then $\{\varphi, \neg\psi\}$ is consistent and so has a maximal consistent extension, w. $\varphi \in w$ so $w \in [\varphi]$. Hence $w \in [\psi]$, which is to say $\psi \in w$, contrary to the consistency of w since $\neg\psi \in w$. Therefore, $\vdash \varphi \supset \psi$. Further, if $\vdash \varphi \supset \psi$, then since maximal consistent extensions are closed under provable implications, it is automatic that for any $w' \in [\varphi], w' \in [\psi]$, or $[\varphi] \subseteq [\psi]$. Part (ii) follows immediately from (i). □

Lemma A.5. *For all $\varphi \in \mathcal{L}$ and all $w \in \dot{W}, \dot{M}, w \models \varphi$ iff $\varphi \in w$ (or, $|\varphi|_{\dot{M}} = [\varphi]$).*

Proof. This is shown by induction over the length of φ. The case that φ is a propositional letter is trivial, since $\dot{M}, w \models \varphi$ iff $w \in \dot{v}(\varphi) = \{w' \in \dot{W} \mid \varphi \in w'\}$. Let φ now be a propositional formula. Suppose for the subformulas φ_1 and φ_2 of φ the equivalence holds. Now let $\varphi = \varphi_1 \wedge \varphi_2$. We have $\dot{M}, w \models \varphi$ iff $\dot{M}, w \models \varphi_1$ and $\dot{M}, w \models \varphi_2$ iff $\varphi_1, \varphi_2 \in w$ iff $\varphi_1 \wedge \varphi_2 \in w$ due to the fact that w is a maximal consistent extension. The argument is similar for $\varphi = \varphi_1 \vee \varphi_2$, $\varphi = \varphi_1 \supset \varphi_2$ and $\varphi = \neg\varphi_1$. Thus the equivalence holds for all propositional formulas φ ($\star$).

Now consider the other cases in $\mathcal{L}'$. Let $\varphi = \mathsf{O}(A \mid B)$. "⇒": In case $\dot{M}, w \models \mathsf{O}(A \mid B)$ we have $\langle |B|_{\dot{M}}, |A|_{\dot{M}} \rangle \in \dot{\mathcal{O}}_w$ and thus $\langle [B], [A] \rangle \in \dot{\mathcal{O}}_w$ by ($\star$). Hence there are A', B' such that $[A'] = [A]$ and $[B'] = [B]$ and $\mathsf{O}(A' \mid B') \in w$. By Lemma A.4 (ii), $\vdash A' \equiv A$ and $\vdash B' \equiv B$. Since w validates (RCE) and (CRE), $\mathsf{O}(A \mid B) \in w$. "⇐": Let $\mathsf{O}(A \mid B) \in w$, then $\langle [A], [B] \rangle \in \dot{\mathcal{O}}_w$. Thus, by ($\star$), $\langle |A|_{\dot{M}}, |B|_{\dot{M}} \rangle \in \dot{\mathcal{O}}_w$. Hence, $\dot{M}, w \models \mathsf{O}(B \mid A)$.

Let $\varphi = \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)$. "⇒": In case $\dot{M}, w \models \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)$ we have $\langle |B|_{\dot{M}}, |A|_{\dot{M}} \rangle \in \dot{\mathcal{N}}^{\mathsf{i}}{}_w$ and thus $\langle [B], [A] \rangle \in \dot{\mathcal{N}}^{\mathsf{i}}{}_w$ by ($\star$). Hence there are A', B' such that $[A'] = [A]$ and $[B'] = [B]$ and $\bullet_{\mathsf{i}}\mathsf{O}(A' \mid B') \in w$. By Lemma A.4 (ii), $\vdash A' \equiv A$ and $\vdash B' \equiv B$. Since w validates (CRE$^{\mathsf{i}}$) and (RCE$^{\mathsf{i}}$), $\bullet_{\mathsf{i}}\mathsf{O}(A \mid B) \in w$. "⇐": Let $\bullet_{\mathsf{i}}\mathsf{O}(A \mid B) \in w$, then $\langle [A], [B] \rangle \in \dot{\mathcal{N}}^{\mathsf{i}}{}_w$. Thus, by ($\star$), $\langle |A|_{\dot{M}}, |B|_{\dot{M}} \rangle \in \dot{\mathcal{N}}^{\mathsf{i}}{}_w$. Hence, $\dot{M}, w \models \bullet_{\mathsf{i}}\mathsf{O}(B \mid A)$.

The case $\varphi = \bullet_{\mathsf{p}}\mathsf{O}(A \mid B)$ is analogous.

Let $\varphi = \mathsf{O}^{\mathsf{i}}A$. "⇒": In case $\dot{M}, w \models \mathsf{O}^{\mathsf{i}}A$ we have $|A|_{\dot{M}} \in \dot{\mathcal{O}}^{\mathsf{i}}{}_w$ and thus $[A] \in \dot{\mathcal{O}}^{\mathsf{i}}{}_w$. Hence there is a A' such that $[A'] = [A]$ and $\mathsf{O}^{\mathsf{i}}A' \in w$. By Lemma A.4, $\vdash A' \equiv A$. Since w validates (EO$^{\mathsf{i}}$), $\mathsf{O}^{\mathsf{i}}A \in w$. "⇐": Let $\mathsf{O}^{\mathsf{i}}A \in w$, then $[A] \in \dot{\mathcal{O}}^{\mathsf{i}}{}_w$. Thus, by ($\star$), $|A|_{\dot{M}} \in \dot{\mathcal{O}}^{\mathsf{i}}{}_w$. Hence, $\dot{M}, w \models \mathsf{O}^{\mathsf{i}}A$.

The case $\varphi = \mathsf{O}^{\mathsf{p}}A$ is analogous.

Now let $\varphi = \varphi_1 \wedge \varphi_2 \in \mathcal{L} \setminus (\mathcal{L}' \cup \mathcal{P})$. By induction hypothesis we suppose the equivalence to be valid for φ_1 and φ_2. We have $\dot{M}, w \models \varphi$ iff $\dot{M}, w \models \varphi_1$ and $\dot{M}, w \models \varphi_2$ iff $\varphi_1, \varphi_2 \in w$ iff $\varphi_1 \wedge \varphi_2 \in w$ due to the fact that w is a maximal consistent extension. The argument is similar for $\varphi = \varphi_1 \vee \varphi_2$, $\varphi = \varphi_1 \supset \varphi_2$ and $\varphi = \neg\varphi_1$. Thus the equivalence holds for all $\varphi \in \mathcal{L}$. □

In order to prove model-completeness we need to restrict our sets of worlds to sets corresponding to expressible propositions on $\dot{M}$. We define,[5] where $M = \langle F, @, v\rangle$ and $F = \langle W, \mathcal{O}, \mathcal{N}^{\mathrm{i}}, \mathcal{N}^{\mathrm{p}}, \mathcal{O}^{\mathrm{i}}, \mathcal{O}^{\mathrm{p}}\rangle$

$$\varepsilon_M =_{\mathrm{df}} \{X \subseteq W \mid \exists B(X = |B|_M)\}$$

Lemma A.6. *For all $w \in \dot{W}$ and $X, Y \in \varepsilon_{\dot{M}}$ there are A and B for which $[A] = X$ and $[B] = Y$ and we have for all such A and B:*

(i) $\langle X, Y\rangle \in \dot{\mathcal{O}}_w$ *iff* $\mathsf{O}(B \mid A) \in w$

(ii) $\langle X, Y\rangle \in \dot{\mathcal{N}}^{\mathrm{i}}{}_w$ *iff* $\bullet_{\mathsf{i}}\mathsf{O}(B \mid A) \in w$.

(iii) $\langle X, Y\rangle \in \dot{\mathcal{N}}^{\mathrm{p}}{}_w$ *iff* $\bullet_{\mathsf{p}}\mathsf{O}(B \mid A) \in w$.

(iv) $X \in \dot{\mathcal{O}}^{\mathrm{i}}{}_w$ *iff* $\mathsf{O}^{\mathsf{i}}A \in w$.

(v) $X \in \dot{\mathcal{O}}^{\mathrm{p}}{}_w$ *iff* $\mathsf{O}^{\mathsf{p}}A \in w$.

Proof. Let $w \in \dot{W}$ and $X, Y \in \varepsilon_{\dot{M}}$ By definition of $\varepsilon_{\dot{M}}$ there are A and B for which $X = |A|_{\dot{M}}$ and $Y = |B|_{\dot{M}}$. By Lemma A.5 we have $[A] = |A|_{\dot{M}} = X$ and $[B] = |B|_{\dot{M}} = Y$.

Ad (i) "⇒": Let $\langle X, Y\rangle \in \dot{\mathcal{O}}_w$. Thus, $\langle [A], [B]\rangle \in \dot{\mathcal{O}}_w$ and thus, $\langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. Hence, by Lemma A.5, $\mathsf{O}(B \mid A) \in w$. "⇐": Let $\langle X, Y\rangle \notin \dot{\mathcal{O}}_w$. Suppose $\mathsf{O}(B \mid A) \in w$, then by Lemma A.5, $\dot{M}, w \models \mathsf{O}(B \mid A)$, then by (M-$\mathcal{O}$), $\langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. Hence, $\langle [A], [B]\rangle \in \dot{\mathcal{O}}_w$ and thus, $\langle X, Y\rangle \in \dot{\mathcal{O}}_w$—a contradiction.

Ad (ii): "⇒": Let $\langle X, Y\rangle \in \dot{\mathcal{N}}^{\mathrm{i}}{}_w$. Then $\langle [A], [B]\rangle \in \dot{\mathcal{N}}^{\mathrm{i}}{}_w$ and thus, $\langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{N}}^{\mathrm{i}}{}_w$. Thus, by Lemma A.5, $\bullet_{\mathsf{i}}\mathsf{O}(B \mid A) \in w$. "⇐": Now let $\langle X, Y\rangle \notin \dot{\mathcal{N}}^{\mathrm{i}}{}_w$. Suppose $\bullet_{\mathsf{i}}\mathsf{O}(B \mid A) \in w$, then by Lemma A.5, $\dot{M}, w \models \bullet_{\mathsf{i}}\mathsf{O}(B \mid A)$ and thus by (M-$\mathcal{N}^{\mathrm{i}}$), $\langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{N}}^{\mathrm{i}}{}_w$. Hence, $\langle [A], [B]\rangle \in \dot{\mathcal{N}}^{\mathrm{i}}{}_w$ and thus, $\langle X, Y\rangle \in \dot{\mathcal{N}}^{\mathrm{i}}{}_w$—a contradiction.

Ad (iii): analogous.

Ad (iv): "⇒": Let $X \in \dot{\mathcal{O}}^{\mathrm{i}}{}_w$ and thus $[A] \in \dot{\mathcal{O}}^{\mathrm{i}}{}_w$. Hence, $|A|_{\dot{M}} \in \dot{\mathcal{O}}^{\mathrm{i}}{}_w$. Thus, $\dot{M}, w \models \mathsf{O}^{\mathsf{i}}A$ and thus by Lemma A.5, $\mathsf{O}^{\mathsf{i}}A \in w$. "⇐": Now let $X \notin \dot{\mathcal{O}}^{\mathrm{i}}{}_w$.

[5] Note that this definition differs from Goble's proposal to the extent that in our case B is a propositional formula, while in Goble's case it was any wff. The reason is, that we don't allow for nested modal operators in this paper.

Suppose $\mathsf{O}^{\mathrm{i}}A \in w$. Then by Lemma A.5, $\dot{M}, w \models \mathsf{O}^{\mathrm{i}}A$ and thus by (M-$\mathcal{O}^{\mathrm{i}}$), $|A|_{\dot{M}} \in \dot{\mathcal{O}}^{\mathrm{i}}{}_w$. Hence, $[A] \in \dot{\mathcal{O}}^{\mathrm{i}}{}_w$ and thus, $X \in \dot{\mathcal{O}}^{\mathrm{i}}{}_w$—a contradiction.

Ad (v): analogous. □

We now modify the frame conditions to form conditions on models. Where $M = \langle F, @, v\rangle$ and $F = \langle W, \mathcal{O}, \mathcal{N}^{\mathrm{i}}, \mathcal{N}^{\mathrm{p}}, \mathcal{O}^{\mathrm{i}}, \mathcal{O}^{\mathrm{p}}\rangle$ we require:

$\langle W, W\rangle \in \mathcal{O}_w$ (M-CN)

For all $X, Y, Z \in \varepsilon_M$, if $Y \subseteq Z$ and $\langle X, Y\rangle \in \mathcal{O}_w$ and $\langle X, \overline{Y}\rangle \notin \mathcal{O}_w$ then $\langle X, Z\rangle \in \mathcal{O}_w$ (M-RCPM)

For all $X, Y, Z \in \varepsilon_M$, if $\langle X \cap Y, Z\rangle \in \mathcal{O}_w$, then $\langle X, \overline{Y} \cup Z\rangle \in \mathcal{O}_w$ (M-S)

For all $X, Y, Z \in \varepsilon_M$, if $\langle X, Y\rangle \in \mathcal{O}_w$ and $\langle X, \overline{Y \cap Z}\rangle \notin \mathcal{O}_w$, then $\langle X \cap Z, Y\rangle \in \mathcal{O}_w$ (M-WRM)

For all $X, Y, Z \in \varepsilon_M$, if $\langle X, Y\rangle \in \mathcal{O}_w, \langle X, Z\rangle \in \mathcal{O}_w$ and $\langle X, \overline{Y \cap Z}\rangle \notin \mathcal{O}_w$, then $\langle X, Y \cap Z\rangle \in \mathcal{O}_w$ (M-CPAND)

For all $X \in \varepsilon_M, \langle X, \emptyset\rangle \notin \mathcal{O}_w$ (M-CP)

For all $X, Y, Z \in \varepsilon_M$, if $\langle Y \cap Z, X\rangle \in \mathcal{O}_w$ and $\langle \overline{Y} \cap Z, \overline{X}\rangle \notin \mathcal{O}_w$, then $\langle Z, \overline{Y} \cup X\rangle \in \mathcal{O}_w$ (M-PS')

For all $X, Y \in \varepsilon_M$, if $\langle Y, X\rangle \in \mathcal{O}_w; w \in Y$; and $\langle Y, X\rangle \notin \mathcal{N}^{\mathrm{i}}_w$, then $X \in \mathcal{O}^{\mathrm{i}}_w$ (M-FD$^{\mathsf{i}}$)

For all $X, Y \in \varepsilon_M$, if $\langle Y, X\rangle \in \mathcal{O}_w; w \in Y$; and $\langle Y, X\rangle \notin \mathcal{N}^{\mathrm{p}}_w$, then $X \in \mathcal{O}^{\mathrm{p}}_w$ (M-FD$^{\mathsf{p}}$)

For all $X, Y, Z, Z' \in \varepsilon_M$, if $w \in Y \cap Z; \langle Y, \overline{Y \cap Z}\rangle \notin \mathcal{O}_w; \langle Y, X\rangle \in \mathcal{O}_w$; $Z' \subseteq \overline{X}$; and $\big(\langle Y \cap Z, \overline{Z'}\rangle \notin \mathcal{O}_w$ or $\langle Y \cap Z, Z'\rangle \in \mathcal{O}_w\big)$, then $\langle Y, X\rangle \in \mathcal{N}^{\mathrm{p}}_w$ (M-E$^{\mathsf{p}}$)

For all $X, Y, Z, Z' \in \varepsilon_M$, if $\langle Y \cap Z, X\rangle, \langle Y, Z'\rangle \in \mathcal{O}_w; Z' \subseteq \overline{Z}$; and $Z' \subseteq \overline{X}$, then $\langle Y \cap Z, X\rangle \in \mathcal{N}^{\mathrm{p}}_w$ (M-CTDR)

For all $X, Y \in \varepsilon_M$, if $\langle Y, X\rangle \in \mathcal{O}_w; w \in Y$; and $w \notin X$; then $\langle Y, X\rangle \in \mathcal{N}^{\mathrm{i}}_w$ (M-fV)

For all $X, Y, Z, Z' \in \varepsilon_M$, if $w \in Y \cap Z; \langle Y, X\rangle \in \mathcal{O}_w; \big(\langle Y \cap Z, \overline{Z'}\rangle \notin \mathcal{O}_w$ or $\langle Y \cap Z, Z'\rangle \in \mathcal{O}_w\big)$; and $Z' \subseteq \overline{X}$, then $\langle Y, X\rangle \in \mathcal{N}^{\mathrm{i}}$ (M-oV-E$^{\mathsf{i}}$)

Theorem A.2. $\mathbf{L}^+$ *is sound and complete with respect to the class of models that meet conditions, as appropriate. In case of* $\mathbf{CDPM.2d}^+$ *the appropriate conditions are* (M-CN), (M-RCPM), (M-S), (M-WRM), (M-CPAND), (M-CP), (M-FD$^{\mathsf{i}}$), (M-FD$^{\mathsf{p}}$), (M-E$^{\mathsf{p}}$), (M-CTDR), (M-fV), *and* (M-oV-E$^{\mathsf{i}}$). *In case of* $\mathbf{CDPM.2e}^+$ (M-S) *is replaced by* (M-PS').

Proof. Soundness is trivial and is shown in a similar way as it was done in Theorem A.1. Some examples: Let $M = \langle F, @, v\rangle$ be a model that satisfies the required properties. For (WRM), Let $M \models \mathsf{O}(B \mid A) \wedge \mathsf{P}(B \wedge C \mid$

A), then $M, @ \models \mathsf{O}(B \mid A), \mathsf{P}(B \wedge C \mid A)$. Thus, $\langle |A|_M, |B|_M\rangle \in \mathcal{O}_@$ and $\langle |A|_M, \overline{|B \wedge C|_M}\rangle \overset{A.1i}{=} \langle |A|_M, \overline{|B|_M \cap |C|_M}\rangle \notin \mathcal{O}_@$. Since M fulfills (M-WRM), $\langle |B|_M \cap |C|_M, |A|_M\rangle \overset{A.1i}{=} \langle |B \wedge C|_M, |A|_M\rangle \in \mathcal{O}_@$ and hence $M, @ \models \mathsf{O}(A \mid B \wedge C)$. Hence $M \models \mathsf{O}(A \mid B \wedge C)$. For (RCPM) let $M \models \mathsf{P}(B \mid A), \mathsf{O}(B \mid A)$ and $\models B \supset C$. Then $M, @ \models \mathsf{P}(B \mid A), \mathsf{O}(B \mid A)$. Thus, $\langle |A|_M, |B|_M\rangle \in \mathcal{O}_@$ and $\langle |A|_M, \overline{|B|_M}\rangle \notin \mathcal{O}_@$. Hence, since M satisfies (M-RCPM) and since $|B|_M \subseteq |C|_M$, $\langle |A|_M, |C|_M\rangle \in \mathcal{O}_@$. Hence, $M, @ \models \mathsf{O}(A \mid C)$ and thus, $M \models \mathsf{O}(A \mid C)$. The other cases are shown analogously.

In order to show completeness let φ be a formula not provable in $\mathbf{L}^+$. Then $\{\neg\varphi\}$ is $\mathbf{L}^+$-consistent and there is, hence, a maximal consistent extension of all $\mathbf{L}^+$ theorems, $\dot{@} \in \dot{W}$, which verifies $\neg\varphi$. Let $\dot{M} = \langle \dot{W}, \dot{\mathcal{O}}, \dot{\mathcal{N}}^{\mathsf{i}}, \dot{\mathcal{N}}^{\mathsf{p}}, \dot{\mathcal{O}}^{\mathsf{i}}, \dot{\mathcal{O}}^{\mathsf{p}}, \dot{@}, \dot{v}\rangle$ be defined as above. We show now that $\dot{M}$ meets the respective model conditions via some paradigmatical examples.

For (M-WRM): Let $X, Y, Z \in \varepsilon_{\dot{M}}$, $\langle X, Y\rangle \in \dot{\mathcal{O}}_w$, and $\langle X, \overline{Y \cap Z}\rangle \notin \dot{\mathcal{O}}_w$. There are A, B such that $[A] = X$, $[B] = Y$ and $\mathsf{O}(B \mid A) \in w$. By Lemma A.5, $M, w \models \mathsf{O}(B \mid A)$ and thus, $\langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. Furthermore there is a C such that $|C|_{\dot{M}} = Z$. Suppose, $\mathsf{O}(\neg(B \wedge C) \mid A) \in w$. Then, $\langle [A], [\neg(B \wedge C)]\rangle \in \dot{\mathcal{O}}_w$ and thus by Lemma A.5, $\langle |A|_{\dot{M}}, |\neg(B \wedge C)|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. But then, since $\langle |A|_{\dot{M}}, |\neg(B \wedge C)|_{\dot{M}}\rangle \overset{A.1iii}{=} \langle |A|_{\dot{M}}, \overline{|B \wedge C|_{\dot{M}}}\rangle \overset{A.1i}{=} \langle |A|_{\dot{M}}, \overline{|B|_{\dot{M}} \cap |C|_{\dot{M}}}\rangle = \langle X, \overline{Y \cap Z}\rangle$, $\langle X, \overline{Y \cap Z}\rangle \in \dot{\mathcal{O}}_w$—a contradiction. Thus, $\mathsf{O}(\neg(B \wedge C) \mid A) \notin w$ and thus $\mathsf{P}(B \wedge C \mid A) \in w$. Since w validates (WRM), $\mathsf{O}(B \mid A \wedge C) \in w$. By Lemma A.5, $\dot{M}, w \models \mathsf{O}(B \mid A \wedge C)$. Thus, $\langle |A \wedge C|_{\dot{M}}, |B|_{\dot{M}}\rangle \overset{A.1i}{=} \langle |A|_{\dot{M}} \cap |C|_{\dot{M}}, |B|_{\dot{M}}\rangle = \langle X \cap Z, Y\rangle \in \dot{\mathcal{O}}_{\dot{M}}$.

For (M-RCPM) let $X, Y, Z \in \varepsilon_{\dot{M}}$, $Y \subseteq Z$, $\langle X, Y\rangle \in \dot{\mathcal{O}}_w$, and $\langle X, \overline{Y}\rangle \notin \dot{\mathcal{O}}_w$. There are A, B such that $[A] = X$, $[B] = Y$ and $\mathsf{O}(B \mid A) \in w$. Furthermore, there is a C such that $|C|_M = Z$. By Lemma A.5, $M, w \models \mathsf{O}(B \mid A)$ and hence $\langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. Since, by Lemma A.5, $\overline{|B|_{\dot{M}}} = \overline{Y}$, by Lemma A.1 (iii), $|\neg B|_{\dot{M}} = \overline{Y}$. By Lemma A.6 (i), $\mathsf{O}(\neg B \mid A) \notin w$ since $\langle X, \overline{Y}\rangle \notin \dot{\mathcal{O}}_w$, and thus $\mathsf{P}(B \mid A) \in w$. By Lemma A.4 (i), $\vdash B \supset C$ since $[B] \subseteq |C|_M \overset{A.5}{=} [C]$. Since w validates all $\mathbf{L}^+$-theorems and (RCPM), $\mathsf{O}(C \mid A) \in w$. Thus, by Lemma A.5, $\dot{M}, w \models \mathsf{O}(C \mid A)$ and thus, $\langle |A|_{\dot{M}}, |C|_{\dot{M}}\rangle \overset{A.5}{=} \langle X, Z\rangle \in \dot{\mathcal{O}}_w$.

For (M-FD$^{\mathsf{i}}$): Let $X, Y \in \varepsilon_{\dot{M}}$. By Lemma A.6 there are A and B such that $[A] = X$ and $[B] = Y$. Now let $\langle Y, X\rangle \in \dot{\mathcal{O}}_w$, $w \in Y$, and $\langle Y, X\rangle \notin \dot{\mathcal{N}}^{\mathsf{i}}{}_w$. Since $w \in Y$ we have $w \in [B]$ and thus $B \in w$. By Lemma A.6 (i), $\mathsf{O}(A \mid B) \in w$. By Lemma A.6 (ii), $\bullet_{\mathsf{i}}\mathsf{O}(A \mid B) \notin w$ and thus $\neg \bullet_{\mathsf{i}} \mathsf{O}(A \mid B) \in w$ since w is maximal consistent. Thus, since w validates (FD$^{\mathsf{i}}$), $\mathsf{O}^{\mathsf{i}}A \in w$. By Lemma A.6 (iv), $X \in \dot{\mathcal{O}}^{\mathsf{i}}{}_w$.

For (M-FD$^{\mathsf{p}}$): the proof is analogous.

For (M-E$^{\mathsf{p}}$): Let $X, Y, Z, Z' \in \varepsilon_{\dot{M}}$. By Lemma A.6 there are A, B, C and D for which $[A] = X$, $[B] = Y$, $[C] = Z$ and $[D] = Z'$. Suppose the antecedent of (M-E$^{\mathsf{p}}$) is true. By Lemma A.3 (i) we have $[B] \cap [C] = [B \wedge C]$. Thus, since $w \in [B \wedge C]$, $B \wedge C \in w$. By Lemma A.3 (iii), $[\neg(B \wedge C)] = \overline{[B \wedge C]} = \overline{[B] \cap [C]} = \overline{Y \cap Z}$. Thus, by Lemma A.6 (i), $\mathsf{O}(\neg(B \wedge C) \mid B) \notin w$ since by hypothesis

$\langle Y, \overline{Y \cap Z}\rangle \notin \dot{\mathcal{O}}_w$. By Lemma A.6 (i), $\mathsf{O}(A \mid B) \in w$ since $\langle Y, X\rangle \in \dot{\mathcal{O}}_w$. Since $Z' \subseteq \overline{X}$, $[D] \subseteq \overline{[A]}$ and thus by Lemma A.3 (iii), $[D] \subseteq [\neg A]$. Thus, by Lemma A.4 (i), $\vdash D \supset \neg A$. Now we have by hypotheses, (a) $\langle Y \cap Z, \overline{Z'}\rangle \notin \dot{\mathcal{O}}_w$, or, (b), $\langle Y \cap Z, Z'\rangle \in \dot{\mathcal{O}}_w$. Note that by Lemma A.3 (iii), $\overline{[D]} = [\neg D]$. Thus in case (a) we have by Lemma A.6 (i), $\mathsf{O}(\neg D \mid B \wedge C) \notin w$, and thus $\neg\mathsf{O}(\neg D \mid B \wedge C) \in w$ which is equivalent to $\mathsf{P}(D \mid B \wedge C) \in w$. In case (b) we have by Lemma A.6 (i), $\mathsf{O}(D \mid B \wedge C) \in w$. Since w validates (Ep), $\bullet_{\mathsf{p}}\mathsf{O}(A \mid B) \in w$. By Lemma A.6 (iii), $\langle Y, X\rangle \in \dot{\mathcal{N}}^{\mathrm{p}}{}_w$.

For the remaining conditions the proofs are analogous.

Thus, our model $\dot{M}$ satisfies all the model-conditions. By Lemma A.5, $\dot{M}, \dot{@} \models \neg\varphi$ and thus, $\dot{M}, \dot{@} \not\models \varphi$. Hence, $\dot{M} \not\models \varphi$. By contraposition we have that if φ is valid in all models which meet the appropriate conditions, then φ is provable in $\mathbf{L}^{+}$. □

A.5.2. Frame completeness and decidability

As shown in [2, 1], the canonical models $\dot{M} = \langle \dot{F}, \dot{@}, \dot{v}\rangle$ do not suffice to prove frame completeness. The problem is that $\dot{F}$ does not in general satisfy the appropriate frame conditions (as demonstrated by Goble for the monadic case with the permitted inheritance principle). Let me demonstrate the problem by means of (F-RCPM): Let $X, Y, Z \subseteq \dot{W}$ such that $Y \subseteq Z; \langle X, Y\rangle \in \dot{\mathcal{O}}_w$ and $\langle X, \overline{Y}\rangle \notin \dot{\mathcal{O}}_w$. There are, by the definition of $\dot{\mathcal{O}}_w$, A and B for which $X = [A], Y = [B]$ and

$$\mathsf{O}(B \mid A) \in w \tag{1}$$

Now suppose $\mathsf{O}(\neg B \mid A) \in w$. Then $\langle [A], [\neg B]\rangle \in \dot{\mathcal{O}}_w$. However, by Lemma A.3 (iii), $\langle [A], [\neg B]\rangle = \langle [A], \overline{[B]}\rangle$. Thus, $\langle X, \overline{Y}\rangle \in \dot{\mathcal{O}}_w$—a contradiction. Thus,

$$\mathsf{O}(\neg B \mid A) \notin w \tag{2}$$

And hence due to the maximal consistency of w,

$$\mathsf{P}(B \mid A) \in w \tag{3}$$

Now, in case there would be a C such that $[C] = Z$ it would be easy to prove frame completeness. Since we have in that case $[B] \subseteq [C]$, we get, by Lemma A.4 (i),

$$\vdash B \supset C \tag{4}$$

Since w satisfies (RCPM) we have $\mathsf{P}(B \mid A) \supset (\mathsf{O}(B \mid A) \supset \mathsf{O}(C \mid A))$ due to (4). By (1) and (3) we get via modus ponens, $\mathsf{O}(C \mid A) \in w$. Hence, $\langle [A], [C]\rangle = \langle X, Z\rangle \in \dot{\mathcal{O}}_w$.

However, the problem is that we have no guarantee that there is such a C.

On the basis of a given model $\dot{M} = \langle \dot{W}, \dot{\mathcal{O}}, \dot{\mathcal{N}}^{\mathrm{i}}, \dot{\mathcal{N}}^{\mathrm{p}}, \dot{\mathcal{O}}^{\mathrm{i}}, \dot{\mathcal{O}}^{\mathrm{p}}, \dot{@}, \dot{v}\rangle$ we construct a model $\overset{\star}{M} = \langle \overset{\star}{F}, \overset{\star}{@}, \overset{\star}{v}\rangle$ on a frame $\overset{\star}{F} = \langle \overset{\star}{W}, \overset{\star}{\mathcal{O}}, \overset{\star}{\mathcal{N}}^{\mathrm{i}}, \overset{\star}{\mathcal{N}}^{\mathrm{p}}, \overset{\star}{\mathcal{O}}^{\mathrm{i}}, \overset{\star}{\mathcal{O}}^{\mathrm{p}}\rangle$ by filtration in the following way.

Let Φ be a finite set of formulas closed under subformulas, i.e., if $\varphi \in \Phi$ and

ψ is a subformula of φ, then $\psi \in \Phi$, and let $\top, \bot \in \Phi$. Furthermore, let $\hat{\Phi}$ be the closure of Φ under truth-functions, i.e., $\hat{\Phi}$ is the smallest set of formulas such that $\Phi \subseteq \hat{\Phi}$ and if $\varphi, \psi \in \hat{\Phi}$, then $\varphi \wedge \psi \in \hat{\Phi}$, $\varphi \vee \psi \in \hat{\Phi}$ and $\neg\varphi \in \hat{\Phi}$. Note that $\top, \bot \in \hat{\Phi}$, and that $\hat{\Phi}$ itself is closed under subformulas.

Now let $\Psi = \Phi \cap \mathcal{P}$ and $\hat{\Psi}$ be again the closure of Ψ under truth-functions.

We define an equivalence relation $\sim_\Psi^{\dot{M}}$ on $\dot{W}$ such that, for all $w, w' \in \dot{W}$:

$$w \sim_\Psi^{\dot{M}} w' \text{ iff } \forall\varphi\big(\text{if } \varphi \in \Psi \text{ then } (\dot{M}, w \models \varphi \text{ iff } \dot{M}, w' \models \varphi)\big).$$

Lemma A.7. *For all $w, w' \in \dot{W}$, if $w \sim_\Psi^{\dot{M}} w'$, then for all $A \in \hat{\Psi}$, ($\dot{M}, w \models A$ iff $\dot{M}, w' \models A$).*

Proof. Suppose $w \sim_\Psi^{\dot{M}} w'$. The proof is by induction on the length of A. For all $A \in \Psi$ the statement holds by definition. If $A = A_1 \wedge A_2$ or $A = A_1 \vee A_2$ or $A = \neg A_1$, for some A_1, A_2, then the result follows directly from the inductive hypothesis. □

It is important to note that $\sim_\Psi^{\dot{M}}$ partitions $\dot{W}$ into finitely many equivalence classes $[w]$ for $w \in \dot{W}$, where $[w] = \{w' \in \dot{W} \mid w' \sim_\Psi^{\dot{M}} w\}$.

We may now, for each equivalence class, select a member $\overset{\star}{w} \in [w]$ (not necessarily w itself) and define $\overset{\star}{W}$ as the set of all these selected representants. Let $\big[\dot{@}\big]$ be represented by $\overset{\star}{@} =_{\text{df}} \dot{@}$. The following fact follows directly from the definitions.

Lemma A.8. *(i) $\overset{\star}{W} \subseteq \dot{W}$; (ii) $\overset{\star}{W}$ is finite; (iii) for all $w' \in \dot{W}$ there is a $\overset{\star}{w} \in \overset{\star}{W}$ such that $w' \sim_\Psi^{\dot{M}} \overset{\star}{w}$; (iv) for all $w, w' \in \overset{\star}{W}$, if $w \neq w'$ then it is not the case that $w \sim_\Psi^{\dot{M}} w'$.*

Some more writing conventions: for $X \subseteq \dot{W}$, let $X{\downarrow} =_{\text{df}} X \cap \overset{\star}{W}$.

The assignments $\overset{\star}{\mathcal{O}}, \overset{\star}{\mathcal{N}}{}^{\text{i}}, \overset{\star}{\mathcal{N}}{}^{\text{p}}, \overset{\star}{\mathcal{O}}{}^{\text{i}}, \overset{\star}{\mathcal{O}}{}^{\text{p}}$ fullfil the following conditions for each $w \in \overset{\star}{W}$:

$$\begin{aligned}&\langle X, Y\rangle \in \overset{\star}{\mathcal{O}}_w \text{ iff } \exists A \exists B\big(A, B \in \hat{\Psi} \text{ and } X = |A|_{\dot{M}}{\downarrow} \text{ and}\\ &\quad Y = |B|_{\dot{M}}{\downarrow} \text{ and } \langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w\big)\end{aligned} \qquad (\text{D}\mathcal{O}\star)$$

$$\begin{aligned}&\langle X, Y\rangle \in \overset{\star}{\mathcal{N}}{}^{\text{i}}{}_w \text{ iff } \exists A \exists B\big(A, B \in \hat{\Psi} \text{ and } X = |A|_{\dot{M}}{\downarrow}\\ &\quad \text{and } Y = |B|_{\dot{M}}{\downarrow} \text{ and } \langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{N}}{}^{\text{i}}{}_w\big)\end{aligned} \qquad (\text{D}\mathcal{N}^{\text{i}}\star)$$

$$\begin{aligned}&\langle X, Y\rangle \in \overset{\star}{\mathcal{N}}{}^{\text{p}}{}_w \text{ iff } \exists A \exists B\big(A, B \in \hat{\Psi} \text{ and } X = |A|_{\dot{M}}{\downarrow}\\ &\quad \text{and } Y = |B|_{\dot{M}}{\downarrow} \text{ and } \langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{N}}{}^{\text{p}}{}_w\big)\end{aligned} \qquad (\text{D}\mathcal{N}^{\text{p}}\star)$$

$$X \in \overset{\star}{\mathcal{O}}{}^{\text{i}}{}_w \text{ iff } \exists A\big(A \in \hat{\Psi} \text{ and } X = |A|_{\dot{M}}{\downarrow} \text{ and } |A|_{\dot{M}} \in \dot{\mathcal{O}}{}^{\text{i}}{}_w\big) \qquad (\text{D}\mathcal{O}^{\text{i}}\star)$$

$$X \in \overset{\star}{\mathcal{O}}{}^{\text{p}}{}_w \text{ iff } \exists A\big(A \in \hat{\Psi} \text{ and } X = |A|_{\dot{M}}{\downarrow} \text{ and } |A|_{\dot{M}} \in \dot{\mathcal{O}}{}^{\text{p}}{}_w\big) \qquad (\text{D}\mathcal{O}^{\text{p}}\star)$$

For all atomic formulas p we demand that $\overset{\star}{v} : p \mapsto \dot{v}(p){\downarrow}$.

Lemma A.9. *For all $\overset{\star}{w} \in \overset{\star}{W}$, there is a formula $B \in \hat{\Psi}$ such that $|B|_{\dot{M}}{\downarrow} = \{\overset{\star}{w}\}$.*

Proof. The following proof only differs minimally from Goble's. First we have for all $\overset{\star}{w}, \overset{\star}{w}' \in \overset{\star}{W}$, if $\overset{\star}{w} \neq \overset{\star}{w}'$ then there is a formula A such that $A \in \hat{\Psi}$ and $\overset{\star}{w} \in |A|_{\dot{M}}$ and $\overset{\star}{w}' \notin |A|_{\dot{M}}$. For suppose otherwise. Suppose $\overset{\star}{w} \neq \overset{\star}{w}'$ but for every $A \in \hat{\Psi}$ if $\overset{\star}{w} \in |A|_{\dot{M}}$ then $\overset{\star}{w}' \in |A|_{\dot{M}}$. Then $\overset{\star}{w} \sim^{\dot{M}}_{\Psi} \overset{\star}{w}'$, for consider any $B \in \Psi$, hence $B \in \hat{\Psi}$. If $\dot{M}, \overset{\star}{w} \models B$, then $\overset{\star}{w} \in |B|_{\dot{M}}$, so by supposition $\overset{\star}{w}' \in |B|_{\dot{M}}$, and thus $\dot{M}, \overset{\star}{w}' \models B$. Suppose then that $\dot{M}, \overset{\star}{w}' \models B$, i.e., $\overset{\star}{w}' \in |B|_{\dot{M}}$, but that it is not the case that $\dot{M}, \overset{\star}{w} \models B$. Then $\dot{M}, \overset{\star}{w} \models \neg B$ and $w \in |\neg B|_{\dot{M}}$. Since $\neg B \in \hat{\Psi}$, by supposition, $\overset{\star}{w}' \in |\neg B|_{\dot{M}}$, or $\dot{M}, \overset{\star}{w}' \models \neg B$. That means $\dot{M}, \overset{\star}{w}' \not\models B$—a contradiction. Hence, if $\dot{M}, \overset{\star}{w}' \models B$, then $\dot{M}, \overset{\star}{w} \models B$, and so $\dot{M}, \overset{\star}{w} \models B$ iff $\dot{M}, \overset{\star}{w}' \models B$, which suffices for $\overset{\star}{w} \sim^{\dot{M}}_{\Psi} \overset{\star}{w}'$. But if $\overset{\star}{w} \neq \overset{\star}{w}'$ then it is not the case that $\overset{\star}{w} \sim^{\dot{M}}_{\Psi} \overset{\star}{w}'$, by Lemma A.8 (iv), a contradiction. Therefore, it must be the case that if $\overset{\star}{w} \neq \overset{\star}{w}'$, there is a $A \in \hat{\Psi}$ such that $\overset{\star}{w} \in |A|_{\dot{M}}$ and $\overset{\star}{w}' \notin |A|_{\dot{M}}$. For each $\overset{\star}{w}'$ such that $\overset{\star}{w}' \neq \overset{\star}{w}$, select one such formula, and call it $A_{\overset{\star}{w}'}$. Let $\Lambda = \{\gamma_i \mid i \in I\}$ be the set of all such formulas $A_{\overset{\star}{w}'}$ for all $\overset{\star}{w}' \neq \overset{\star}{w}$. Λ is finite since $\overset{\star}{W}$ is finite. Let $B_{\overset{\star}{w}} = \bigwedge \Lambda = \bigwedge_I \gamma_i$ be the conjunction of all the members of Λ. $B_{\overset{\star}{w}} \in \hat{\Psi}$ since each conjunct $\gamma_i \in \hat{\Psi}$ and $\hat{\Psi}$ is closed under truth-functions. We now show that $|B_{\overset{\star}{w}}|_{\dot{M}}\!\downarrow = \{\overset{\star}{w}\}$.

(i) Suppose $x \in |B_{\overset{\star}{w}}|_{\dot{M}}\!\downarrow$. So $x \in |B_{\overset{\star}{w}}|_{\dot{M}}$ and $x \in \overset{\star}{W}$. Suppose $x \neq \overset{\star}{w}$. Then there is a formula $A_x \in \Psi$ such that $\overset{\star}{w} \in |A_x|_{\dot{M}}$ and $x \notin |A_x|_{\dot{M}}$. We have $|B_{\overset{\star}{w}}|_{\dot{M}} = |\bigwedge_I \gamma_i|_{\dot{M}} \overset{A.1i}{=} \bigcap_I |\gamma_i|_{\dot{M}}$. Hence, $|B_{\overset{\star}{w}}|_{\dot{M}} \subseteq |\gamma_i|_{\dot{M}}$ for all $i \in I$. Note that $A_x = \gamma_j$ for a $j \in I$. Since $x \in |B_{\overset{\star}{w}}|_{\dot{M}}$, $x \in |A_x|_{\dot{M}}$—a contradiction. Therefore, if $x \in |B_{\overset{\star}{w}}|_{\dot{M}}$, $x = \overset{\star}{w}$ and so $x \in \{\overset{\star}{w}\}$. Thus $|B_{\overset{\star}{w}}|_{\dot{M}}\!\downarrow \subseteq \{\overset{\star}{w}\}$.

(ii) Suppose $x \in \{\overset{\star}{w}\}$, i.e., $x = \overset{\star}{w}$. Thus $x \in \overset{\star}{W}$. For all $\gamma_i \in \Lambda$, $x \in |\gamma_i|_{\dot{M}}$. Hence, $\dot{M}, x \models \gamma_i$ for all $i \in I$. Consequently, $\dot{M}, x \models \bigwedge_I \gamma_i$. But $\bigwedge_I \gamma_i = B_{\overset{\star}{w}}$, hence $\dot{M}, x \models B_{\overset{\star}{w}}$. That is to say, $x \in |B_{\overset{\star}{w}}|_{\dot{M}}$, and therefore $x \in |B_{\overset{\star}{w}}|_{\dot{M}}\!\downarrow$. Thus, $\{\overset{\star}{w}\} \subseteq |B_{\overset{\star}{w}}|\!\downarrow$. Therefore, by (i) and (ii) together, $|B_{\overset{\star}{w}}|_{\dot{M}}\!\downarrow = \{\overset{\star}{w}\}$, as required for the Lemma. □

Lemma A.10. *For all $w \in \overset{\star}{W}$ we have: $\overset{\star}{M}, w \models A$ iff $\dot{M}, w \models A$.*

Proof. This is shown by induction. Let $A \in \mathcal{A}$, then $\overset{\star}{M}, w \models A$ iff $w \in \overset{\star}{v}(A)$ iff $w \in \dot{v}(A)\!\downarrow$ iff $w \in \dot{v}(A) \cap \overset{\star}{W}$ iff (since $w \in \overset{\star}{W}$) $w \in \dot{v}(A)$ iff $\dot{M}, w \models A$. Now by induction hypothesis let the lemma hold for B and C. Let $A = B \wedge C$. Then $\overset{\star}{M}, w \models A$ iff $\overset{\star}{M}, w \models B, C$ iff $\dot{M}, w \models B, C$ iff $\dot{M}, w \models B \wedge C$. The argument is similar for $A = B \vee C$, $A = B \supset C$ and $A = \neg B$. □

Lemma A.11. $|A|_{\overset{\star}{M}} = |A|_{\dot{M}}\!\downarrow$.

Proof. $|A|_{\overset{\star}{M}} = \{w \in \overset{\star}{W} \mid \overset{\star}{M}, w \models A\} \overset{(1)}{=} \{w \in \overset{\star}{W} \mid \dot{M}, w \models A\} = \{w \in \dot{W} \mid \dot{M}, w \models A\} \cap \overset{\star}{W} = |A|_{\dot{M}}\!\downarrow$, where (1) is due to Lemma A.10. □

Lemma A.12. *For all $X \subseteq \overset{\star}{W}$, there is a formula B such that $B \in \hat{\Psi}$ and $X = |B|_{\dot{M}}\!\downarrow$.*

Proof. Let $X \subseteq \overset{\star}{W}$. Then $X = \{x_1, \ldots, x_n\}$ is finite, since $\overset{\star}{W}$ is finite. By Lemma A.9, there is an $A_i \in \hat{\Psi}$ for each $x_i \in X$ such that $|A_i|_{\dot{M}}{\downarrow} = \{x_i\}$. Let $A_X = A_1 \vee \cdots \vee A_n$. Since $\hat{\Psi}$ is closed under classical connectives, $A_X \in \hat{\Psi}$. $|A_X|_{\dot{M}}{\downarrow} = |\bigvee_{i=1}^{n} A_i|_{\dot{M}}{\downarrow} \overset{A.11}{=} |\bigvee_{i=1}^{n} A_i|_{\overset{\star}{M}} = \{w \in \overset{\star}{W} \mid \overset{\star}{M}, w \models \bigvee_{i=1}^{n} A_i\} = \{w \in \overset{\star}{W} \mid \overset{\star}{M}, w \models A_1 \text{ or } \ldots \text{ or } \overset{\star}{M}, w \models A_n\} = \bigcup_{i=1}^{n}\{w \in \overset{\star}{W} \mid \overset{\star}{M}, w \models A_i\} = \bigcup_{i=1}^{n} |A_i|_{\overset{\star}{M}} \overset{A.11}{=} \bigcup_{i=1}^{n} |A_i|_{\dot{M}}{\downarrow} = \bigcup_{i=1}^{n}\{x_i\} = X$. □

Lemma A.13. *For all $A, B \in \hat{\Psi}$, (i) if $|A|_{\dot{M}}{\downarrow} \subseteq |B|_{\dot{M}}{\downarrow}$, then $|A|_{\dot{M}} \subseteq |B|_{\dot{M}}$; (ii) if $|A|_{\dot{M}}{\downarrow} = |B|_{\dot{M}}{\downarrow}$, then $|A|_{\dot{M}} = |B|_{\dot{M}}$.*

Proof. Let $A, B \in \hat{\Psi}$ such that $|A|_{\dot{M}}{\downarrow} \subseteq |B|_{\dot{M}}{\downarrow}$. Take any $w \in |A|_{\dot{M}}$. By Lemma A.8 (iii), there is a $\overset{\star}{w} \in \overset{\star}{W}$ such that $w \sim_{\Psi}^{\dot{M}} \overset{\star}{w}$. Since $\dot{M}, w \models A$, by Lemma A.7, $\dot{M}, \overset{\star}{w} \models A$. Hence, $\overset{\star}{w} \in |A|_{\dot{M}}$, and, since $\overset{\star}{w} \in \overset{\star}{W}$, $\overset{\star}{w} \in |A|_{\dot{M}}{\downarrow}$. Thus, $\overset{\star}{w} \in |B|_{\dot{M}}{\downarrow}$ and hence, $\overset{\star}{w} \in |B|_{\dot{M}}$ or $\dot{M}, \overset{\star}{w} \models B$. Thus, since $B \in \hat{\Psi}$ and $w \sim_{\Psi}^{\dot{M}} \overset{\star}{w}$, $\dot{M}, w \models B$ by Lemma A.7. Thus, $w \in |B|_{\dot{M}}$. (ii) follows immediately. □

Lemma A.14. *(i) $|\varphi|_{\dot{M}}{\downarrow} \cap |\psi|_{\dot{M}}{\downarrow} = |\varphi \wedge \psi|_{\dot{M}}{\downarrow}$; (ii) $|\varphi|_{\dot{M}}{\downarrow} \cup |\psi|_{\dot{M}}{\downarrow} = |\varphi \vee \psi|_{\dot{M}}{\downarrow}$; (iii) $\overline{|\varphi|_{\dot{M}}{\downarrow}} = |\neg\varphi|_{\dot{M}}{\downarrow}$ (where the complement is interpreted w.r.t. frame $\overset{\star}{F}$).*

Proof. Ad (i): $|\varphi \wedge \psi|_{\dot{M}}{\downarrow} = |\varphi \wedge \psi|_{\dot{M}} \cap \overset{\star}{W} \overset{(1)}{=} (|\varphi|_{\dot{M}} \cap |\psi|_{\dot{M}}) \cap \overset{\star}{W} = (|\varphi|_{\dot{M}} \cap \overset{\star}{W}) \cap (|\psi|_{\dot{M}} \cap \overset{\star}{W}) = |\varphi|_{\dot{M}}{\downarrow} \cap |\psi|_{\dot{M}}{\downarrow}$ where (1) is due to Lemma A.1 (i).

Ad (ii): $|\varphi \vee \psi|_{\dot{M}}{\downarrow} \overset{(2)}{=} (|\varphi|_{\dot{M}} \cup |\psi|_{\dot{M}}){\downarrow} = (|\varphi|_{\dot{M}} \cup |\psi|_{\dot{M}}) \cap \overset{\star}{W} = (|\varphi|_{\dot{M}} \cap \overset{\star}{W}) \cup (|\psi|_{\dot{M}} \cap \overset{\star}{W}) = |\varphi|_{\dot{M}}{\downarrow} \cup |\psi|_{\dot{M}}{\downarrow}$ where (2) is due to Lemma A.1 (ii).

Ad (iii): $\overline{|\varphi|_{\dot{M}}{\downarrow}} = (\overset{\star}{W} \setminus |\varphi|_{\dot{M}}{\downarrow}) \cap \overset{\star}{W} = (\overset{\star}{W} \setminus (|\varphi|_{\dot{M}} \cap \overset{\star}{W})) \cap \overset{\star}{W} = (\overset{\star}{W} \setminus |\varphi|_{\dot{M}}) \cap \overset{\star}{W} = (\dot{W} \setminus |\varphi|_{\dot{M}}) \cap \overset{\star}{W} \overset{(3)}{=} |\neg\varphi|_{\dot{M}} \cap \overset{\star}{W} = |\neg\varphi|_{\dot{M}}{\downarrow}$ where (3) is due to Lemma A.1 (iii). □

Lemma A.15. *If $\dot{M} = \langle \dot{W}, \dot{\mathcal{O}}, \dot{\mathcal{N}}^{\mathsf{i}}, \dot{\mathcal{N}}^{\mathsf{p}}, \dot{\mathcal{O}}^{\mathsf{i}}, \dot{\mathcal{O}}^{\mathsf{p}}, \dot{@}, \dot{v} \rangle$, defined as above, satisfies conditions* {M-X | X ∈ $\boldsymbol{X}$} *where* $\boldsymbol{X} \subseteq$ {CN, RCPM, S, WRM, CPAND, CP, PS', FD$^{\mathsf{i}}$, FD$^{\mathsf{p}}$, E$^{\mathsf{p}}$, CTDR, fV, oV-E$^{\mathsf{i}}$}, *then $\overset{\star}{F}$ satisfies conditions* {F-X | X ∈ $\boldsymbol{X}$}.

Proof. We demonstrate the proof via some paradigmatical rules.

For (F-PS'): Let $X, Y, Z \subseteq \overset{\star}{W}$, $\langle Y \cap Z, X \rangle \in \overset{\star}{\mathcal{O}}_w$ and $\langle \overline{Y} \cap Z, \overline{X} \rangle \notin \overset{\star}{\mathcal{O}}_w$. To show: $\langle Z, \overline{Y} \cup X \rangle \in \overset{\star}{\mathcal{O}}_w$. By (D$\mathcal{O}\star$), there are $E, F \in \hat{\Psi}$ for which $|E|_{\dot{M}}{\downarrow} = Y \cap Z$, $|F|_{\dot{M}}{\downarrow} = X$ and $\langle |E|_{\dot{M}}, |F|_{\dot{M}} \rangle \in \dot{\mathcal{O}}_w$. Furthermore, by Lemma A.12, there are $A, B, C \in \hat{\Psi}$ such that $|A|_{\dot{M}}{\downarrow} = X$, $|B|_{\dot{M}}{\downarrow} = Y$ and $|C|_{\dot{M}}{\downarrow} = Z$. By Lemma A.14 (i), $|B|_{\dot{M}}{\downarrow} \cap |C|_{\dot{M}}{\downarrow} = |B \wedge C|_{\dot{M}}{\downarrow}$. Thus $|E|_{\dot{M}}{\downarrow} = |B \wedge C|_{\dot{M}}{\downarrow}$ and by Lemma A.13 and since $B \wedge C \in \hat{\Psi}$, $|E|_{\dot{M}} = |B \wedge C|_{\dot{M}}$. Also by Lemma A.13, $|F|_{\dot{M}} = |A|_{\dot{M}}$. Thus, $\langle |B \wedge C|_{\dot{M}}, |A|_{\dot{M}} \rangle \overset{A.1i}{=} \langle |B|_{\dot{M}} \cap |C|_{\dot{M}}, |A|_{\dot{M}} \rangle \in \dot{\mathcal{O}}_w$. Suppose $\langle \overline{|B|_{\dot{M}}} \cap |C|_{\dot{M}}, \overline{|A|_{\dot{M}}} \rangle \in \dot{\mathcal{O}}_w$. Note that $\langle \overline{|B|_{\dot{M}}} \cap |C|_{\dot{M}}, \overline{|A|_{\dot{M}}} \rangle \overset{A.1iii}{=} \langle |\neg B|_{\dot{M}} \cap |C|_{\dot{M}}, |\neg A|_{\dot{M}} \rangle \overset{A.1i}{=} \langle |\neg B \wedge C|_{\dot{M}}, |\neg A|_{\dot{M}} \rangle \in \dot{\mathcal{O}}_w$. Now by (D$\mathcal{O}\star$), $\langle |\neg B \wedge C|_{\dot{M}}{\downarrow}, |\neg A|_{\dot{M}}{\downarrow} \rangle \in$

$\overset{\star}{\mathcal{O}}_w$. Note that $\langle|\neg B\wedge C|_{\dot M}\downarrow, |\neg A|_{\dot M}\downarrow\rangle \overset{A.14i}{=} \langle|\neg B|_{\dot M}\downarrow\cap|C|_{\dot M}\downarrow, |\neg A|_{\dot M}\downarrow\rangle \overset{A.14iii}{=} \langle\overline{|B|_{\dot M}\downarrow}\cap |C|_{\dot M}\downarrow, \overline{|A|_{\dot M}\downarrow}\rangle \in \overset{\star}{\mathcal{O}}_w$. However, now we have $\langle \overline{Y}\cap Z, \overline{X}\rangle \in \overset{\star}{\mathcal{O}}_w$—a contradiction. Hence, $\langle \overline{|B|_{\dot M}} \cap |C|_{\dot M}, \overline{|A|_{\dot M}}\rangle \notin \dot{\mathcal{O}}_w$. Since $\dot M$ satisfies (M-PS'), $\langle |C|_{\dot M}, \overline{|B|_{\dot M}} \cup |A|_{\dot M}\rangle \in \dot{\mathcal{O}}_w$. By Lemma A.1 (ii) and (iii), $\langle |C|_{\dot M}, |\neg B\vee A|_{\dot M}\rangle \in \dot{\mathcal{O}}_w$. Thus, by (D$\mathcal{O}\star$) and since $C,\neg B\vee A\in\hat\Psi$, $\langle |C|_{\dot M}\downarrow, |\neg B\vee A|_{\dot M}\downarrow\rangle \in \overset{\star}{\mathcal{O}}_w$. By Lemma A.14 (ii) and (iii), $\langle |C|_{\dot M}\downarrow, \overline{|B|_{\dot M}\downarrow}\cup|A|_{\dot M}\downarrow\rangle \in \overset{\star}{\mathcal{O}}_w$ and thus $\langle Z, \overline{Y}\cup X\rangle \in \overset{\star}{\mathcal{O}}_w$.

For (F-RCPM): Let $X,Y,Z\subseteq \overset{\star}{W}$, $Y\subseteq Z$, $\langle X,Y\rangle\in\overset{\star}{\mathcal{O}}_w$, and $\langle X,\overline{Y}\rangle\notin\overset{\star}{\mathcal{O}}_w$. By (D$\mathcal{O}\star$), there are $A,B\in\hat\Psi$ such that $|A|_{\dot M}\downarrow = X, |B|_{\dot M}\downarrow = Y$ and $\langle |A|_{\dot M}, |B|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$. Furthermore, by Lemma A.12, there is a $C\in\hat\Psi$ for which $Z=|C|_{\dot M}\downarrow$. Suppose, $\langle |A|_{\dot M}, |\neg B|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$. Then by (D$\mathcal{O}\star$), $\langle |A|_{\dot M}\downarrow, |\neg B|_{\dot M}\downarrow\rangle \overset{A.14iii}{=} \langle |A|_{\dot M}\downarrow, \overline{|B|_{\dot M}\downarrow}\rangle = \langle X,\overline{Y}\rangle\in\overset{\star}{\mathcal{O}}_w$—a contradiction. Thus, $\langle |A|_{\dot M}, |\neg B|_{\dot M}\rangle \overset{A.1iii}{=} \langle |A|_{\dot M}, \overline{|B|_{\dot M}}\rangle\notin\dot{\mathcal{O}}_w$. By Lemma A.13, $|B|_{\dot M}\subseteq|C|_{\dot M}$, since $|B|_{\dot M}\downarrow\subseteq|C|_{\dot M}\downarrow$. Since $\dot M$ satisfies (M-RCPM), $\langle |A|_{\dot M}, |C|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$. Thus, by (D$\mathcal{O}\star$), $\langle |A|_{\dot M}\downarrow, |C|_{\dot M}\downarrow\rangle = \langle X,Z\rangle\in\overset{\star}{\mathcal{O}}_w$.

For (F-E$^{\mathsf{p}}$): Let $X,Y,Z,Z'\in\overset{\star}{W}$ such that, (a), $w\in Y\cap Z$, (b), $\langle Y,\overline{Y\cap Z}\rangle\notin\overset{\star}{\mathcal{O}}_w$, (c), $\langle Y,X\rangle\in\overset{\star}{\mathcal{O}}_w$, (d), $Z'\subseteq\overline{X}$, and either, (e), $\langle Y\cap Z,\overline{Z'}\rangle\notin\overset{\star}{\mathcal{O}}_w$, or, (f), $\langle Y\cap Z, Z'\rangle\in\overset{\star}{\mathcal{O}}_w$. To show: $\langle Y,X\rangle\in\overset{\star}{\mathcal{N}^{\mathsf{p}}}_w$. By (D$\mathcal{O}\star$) and (c) there are $A,B\in\hat\Psi$ for which $|A|_{\dot M}\downarrow = X$, $|B|_{\dot M}\downarrow = Y$ and $\langle |B|_{\dot M}, |A|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$. By Lemma A.12 there is a $C\in\hat\Psi$ such that $|C|_{\dot M}\downarrow = Z$. Suppose $\langle |B|_{\dot M}, \overline{|B\wedge C|_{\dot M}}\rangle\in\dot{\mathcal{O}}_w$, then by Lemma A.1 (iii), $\langle |B|_{\dot M}, |\neg(B\wedge C)|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$. Now by (D$\mathcal{O}\star$) and since $B,\neg(B\wedge C)\in\hat\Psi$, $\langle |B|_{\dot M}\downarrow, |\neg(B\wedge C)|_{\dot M}\downarrow\rangle\in\overset{\star}{\mathcal{O}}_w$. Then by Lemma A.14, $\langle |B|_{\dot M}\downarrow, |\neg(B\wedge C)|_{\dot M}\downarrow\rangle = \langle |B|_{\dot M}\downarrow, \overline{|B\wedge C|_{\dot M}\downarrow}\rangle = \langle |B|_{\dot M}\downarrow, \overline{|B|_{\dot M}\downarrow\cap|C|_{\dot M}\downarrow}\rangle = \langle Y,\overline{Y\cap Z}\rangle\in\overset{\star}{\mathcal{O}}_w$—a contradiction with (b). Thus, $\langle |B|_{\dot M}, \overline{|B\wedge C|_{\dot M}}\rangle \overset{A.1i}{=} \langle |B|_{\dot M}, \overline{|B|_{\dot M}\cap|C|_{\dot M}}\rangle\notin\dot{\mathcal{O}}_w$. By Lemma A.12 there is a $D\in\hat\Psi$ for which $|D|_{\dot M}\downarrow = Z'$. Thus $|D|_{\dot M}\downarrow\subseteq\overline{|A|_{\dot M}\downarrow}$ and thus by Lemma A.14 (iii), $|D|_{\dot M}\downarrow\subseteq|\neg A|_{\dot M}\downarrow$. By Lemma A.13 and since $D,\neg A\in\hat\Psi$, $|D|_{\dot M}\subseteq |\neg A|_{\dot M} \overset{A.1iii}{=} \overline{|A|_{\dot M}}$. Case (e): Suppose $\langle |B|_{\dot M}\cap|C|_{\dot M}, \overline{|D|_{\dot M}}\rangle\in\dot{\mathcal{O}}_w$, then by Lemma A.1 (i) and (iii), $\langle |B\wedge C|_{\dot M}, |\neg D|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$. By (D$\mathcal{O}\star$) and since $B\wedge C,\neg D\in\hat\Psi$, $\langle |B\wedge C|_{\dot M}\downarrow, |\neg D|_{\dot M}\downarrow\rangle\in\overset{\star}{\mathcal{O}}_w$. By Lemma A.14 (i) and (iii), $\langle |B|_{\dot M}\downarrow\cap|C|_{\dot M}\downarrow, \overline{|D|_{\dot M}\downarrow}\rangle = \langle Y\cap Z,\overline{Z'}\rangle\in\overset{\star}{\mathcal{O}}_w$—a contradiction with (e). Thus, $\langle |B|_{\dot M}\cap|C|_{\dot M}, \overline{|D|_{\dot M}}\rangle\notin\dot{\mathcal{O}}_w$. Case (f): By (D$\mathcal{O}\star$) there are $E,F\in\hat\Psi$ such that $|E|_{\dot M}\downarrow = Y\cap Z$, $|F|_{\dot M}\downarrow = Z'$ and $\langle |E|_{\dot M}, |F|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$. We have $|E|_{\dot M}\downarrow = |B|_{\dot M}\downarrow\cap|C|_{\dot M}\downarrow \overset{A.14i}{=} |B\wedge C|_{\dot M}\downarrow$. Thus, by Lemma A.13 and since $E, B\wedge C\in\hat\Psi$, $|E|_{\dot M} = |B\wedge C|_{\dot M} \overset{A.1i}{=} |B|_{\dot M}\cap|C|_{\dot M}$. Thus, $\langle |B|_{\dot M}\cap|C|_{\dot M}, |F|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$. Also by Lemma A.13, since $|F|_{\dot M}\downarrow = Z' = |D|_{\dot M}\downarrow$, $|F|_{\dot M} = |D|_{\dot M}$. Thus, $\langle |B\wedge C|_{\dot M}, |D|_{\dot M}\rangle\in\dot{\mathcal{O}}_w$.

Since $\dot M$ satisfies (M-E$^{\mathsf{p}}$), $\langle |B|_{\dot M}, |A|_{\dot M}\rangle\in\dot{\mathcal{N}^{\mathsf{p}}}_w$. Hence, by (D$\mathcal{N}^{\mathsf{p}}\star$), $\langle Y,X\rangle\in\overset{\star}{\mathcal{N}^{\mathsf{p}}}_w$.

For (F-WRM): Consider $X,Y,Z\in\overset{\star}{W}$. Let $\langle X,Y\rangle\in\overset{\star}{\mathcal{O}}_w$ and $\langle X,\overline{Y\cap Z}\rangle\notin$

$\overset{\star}{\mathcal{O}}_w$. By (D$\mathcal{O}\star$), there are $A, B \in \hat{\Psi}$ such that $|A|_{\dot{M}}\downarrow = X$, $|B|_{\dot{M}}\downarrow = Y$ and $\langle |A|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. By Lemma A.12, there is a $C \in \hat{\Psi}$ such that $|C|_{\dot{M}}\downarrow = Z$. Suppose that $\left\langle |A|_{\dot{M}}, \overline{|B|_{\dot{M}} \cap |C|_{\dot{M}}}\right\rangle \in \dot{\mathcal{O}}_w$. Due to the fact that $\left\langle |A|_{\dot{M}}, \overline{|B|_{\dot{M}} \cap |C|_{\dot{M}}}\right\rangle \overset{A.1i}{=} \left\langle |A|_{\dot{M}}, \overline{|B \wedge C|_{\dot{M}}}\right\rangle \overset{A.1iii}{=} \langle |A|_{\dot{M}}, |\neg(B\wedge C)|_{\dot{M}}\rangle$, we have, $\langle |A|_{\dot{M}}, |\neg(B\wedge C)|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. By (D$\mathcal{O}\star$), $\langle |A|_{\dot{M}}\downarrow, |\neg(B\wedge C)|_{\dot{M}}\downarrow\rangle \in \overset{\star}{\mathcal{O}}_w$. However, $\langle |A|_{\dot{M}}\downarrow, |\neg(B\wedge C)|_{\dot{M}}\downarrow\rangle \overset{A.14iii}{=} \left\langle |A|_{\dot{M}}\downarrow, \overline{|B \wedge C|_{\dot{M}}\downarrow}\right\rangle \overset{A.14i}{=} \left\langle |A|_{\dot{M}}\downarrow, \overline{|B|_{\dot{M}}\downarrow \cap |C|_{\dot{M}}\downarrow}\right\rangle = \langle X, \overline{Y \cap Z}\rangle$. Thus, $\langle X, \overline{Y\cap Z}\rangle \in \overset{\star}{\mathcal{O}}_w$—a contradiction. Thus, $\left\langle |A|_{\dot{M}}, \overline{|B|_{\dot{M}} \cap |C|_{\dot{M}}}\right\rangle \notin \dot{\mathcal{O}}_w$. Since $\dot{M}$ satisfies (M-WRM), $\langle |A|_{\dot{M}} \cap |C|_{\dot{M}}, |B|_{\dot{M}}\rangle \overset{A.1i}{=} \langle |A \wedge C|_{\dot{M}}, |B|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. Since $A \wedge C \in \hat{\Psi}$ (since $\hat{\Psi}$ is closed under the classical connectives and $A, C \in \hat{\Psi}$) and $B \in \hat{\Psi}$, by (D$\mathcal{O}\star$), $\langle |A \wedge C|_{\dot{M}}\downarrow, |B|_{\dot{M}}\downarrow\rangle \overset{A.14i}{=} \langle |A|_{\dot{M}}\downarrow \cap |C|_{\dot{M}}\downarrow, |B|_{\dot{M}}\downarrow\rangle = \langle X \cap Z, Y\rangle \in \overset{\star}{\mathcal{O}}_w$.

The other cases are shown in a similar way and are left to the reader. □

Now we show that $\dot{M}$ and $\overset{\star}{M}$ are equivalent modulo $\hat{\Phi}$.

Lemma A.16. *For all $\psi \in \hat{\Phi}$ and all $w \in \overset{\star}{W}$, $\dot{M}, w \models \psi$ iff $\overset{\star}{M}, w \models \psi$.*

Proof. We show the equivalence by induction on the length of ψ. The equivalence holds for all propositional formulas ψ by Lemma A.10.

Let now $\psi = \mathsf{O}(A \mid B)$. Note that $A, B \in \hat{\Psi}$. $\overset{\star}{M}, w \models \mathsf{O}(A \mid B)$ iff $\langle |B|_{\overset{\star}{M}}, |A|_{\overset{\star}{M}}\rangle \in \overset{\star}{\mathcal{O}}_w$ iff (by Lemma A.11) $\langle |B|_{\dot{M}}\downarrow, |A|_{\dot{M}}\downarrow\rangle \in \overset{\star}{\mathcal{O}}_w$. By (D$\mathcal{O}\star$), there are $A', B' \in \hat{\Psi}$ such that $|A'|_{\dot{M}}\downarrow = |A|_{\dot{M}}\downarrow$, $|B'|_{\dot{M}}\downarrow = |B|_{\dot{M}}\downarrow$ and $\langle |B'|_{\dot{M}}, |A'|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$. Since $A, A', B, B' \in \hat{\Psi}$, by Lemma A.13, $|A|_{\dot{M}} = |A'|_{\dot{M}}$ and $|B|_{\dot{M}} = |B'|_{\dot{M}}$. Thus, $\langle |B|_{\dot{M}}, |A|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$ and thus, $\dot{M}, w \models \mathsf{O}(A \mid B)$. Let now $\dot{M}, w \models \mathsf{O}(A \mid B)$. Then $\langle |B|_{\dot{M}}, |A|_{\dot{M}}\rangle \in \dot{\mathcal{O}}_w$ and thus by (D$\mathcal{O}\star$), $\langle |B|_{\dot{M}}\downarrow, |A|_{\dot{M}}\downarrow\rangle \in \overset{\star}{\mathcal{O}}_w$. By Lemma A.11, $\langle |B|_{\overset{\star}{M}}, |A|_{\overset{\star}{M}}\rangle \in \overset{\star}{\mathcal{O}}_w$ and thus, $\overset{\star}{M}, w \models \mathsf{O}(A \mid B)$.

Let $\psi = \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)$. Note that $A, B \in \hat{\Psi}$. $\overset{\star}{M}, w \models \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)$ iff $\langle |B|_{\overset{\star}{M}}, |A|_{\overset{\star}{M}}\rangle \in \overset{\star}{\mathcal{N}}{}^{\mathsf{i}}{}_w$ iff (by Lemma A.11) $\langle |B|_{\dot{M}}\downarrow, |A|_{\dot{M}}\downarrow\rangle \in \overset{\star}{\mathcal{N}}{}^{\mathsf{i}}{}_w$. By (D$\mathcal{N}^{\mathsf{i}}\star$), there are $A', B' \in \hat{\Psi}$ such that $|A'|_{\dot{M}}\downarrow = |A|_{\dot{M}}\downarrow$, $|B'|_{\dot{M}}\downarrow = |B|_{\dot{M}}\downarrow$ and $\langle |B'|_{\dot{M}}, |A'|_{\dot{M}}\rangle \in \dot{\mathcal{N}}{}^{\mathsf{i}}{}_w$. Since $A, A', B, B' \in \hat{\Psi}$, by Lemma A.13, $|A|_{\dot{M}} = |A'|_{\dot{M}}$ and $|B|_{\dot{M}} = |B'|_{\dot{M}}$. Thus, $\langle |B|_{\dot{M}}, |A|_{\dot{M}}\rangle \in \dot{\mathcal{N}}{}^{\mathsf{i}}{}_w$ and thus, $\dot{M}, w \models \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)$. Let now $\dot{M}, w \models \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)$. Then $\langle |B|_{\dot{M}}, |A|_{\dot{M}}\rangle \in \dot{\mathcal{N}}{}^{\mathsf{i}}{}_w$ and thus by (D$\mathcal{O}\star$), $\langle |B|_{\dot{M}}\downarrow, |A|_{\dot{M}}\downarrow\rangle \in \overset{\star}{\mathcal{N}}{}^{\mathsf{i}}{}_w$. By Lemma A.11, $\langle |B|_{\overset{\star}{M}}, |A|_{\overset{\star}{M}}\rangle \in \overset{\star}{\mathcal{N}}{}^{\mathsf{i}}{}_w$ and thus, $\overset{\star}{M}, w \models \bullet_{\mathsf{i}}\mathsf{O}(A \mid B)$.

The case $\psi = \bullet_{\mathsf{p}}\mathsf{O}(A \mid B)$ is shown analogously.

Let $\psi = \mathsf{O}^{\mathsf{i}}A$. Note that $A \in \hat{\Psi}$. $\overset{\star}{M}, w \models \mathsf{O}^{\mathsf{i}}A$ iff $|A|_{\overset{\star}{M}} \in \overset{\star}{\mathcal{O}}{}^{\mathsf{i}}{}_w$ iff (by Lemma A.11) $|A|_{\dot{M}}\downarrow \in \overset{\star}{\mathcal{O}}{}^{\mathsf{i}}{}_w$. By (D$\mathcal{O}^{\mathsf{i}}\star$), there is a $A' \in \hat{\Psi}$ such that $|A'|_{\dot{M}}\downarrow = |A|_{\dot{M}}\downarrow$, and $|A'|_{\dot{M}} \in \dot{\mathcal{O}}{}^{\mathsf{i}}{}_w$. Since $A, A' \in \hat{\Psi}$, by Lemma A.13, $|A|_{\dot{M}} = |A'|_{\dot{M}}$. Thus, $|A|_{\dot{M}} \in \dot{\mathcal{O}}{}^{\mathsf{i}}{}_w$ and thus, $\dot{M}, w \models \mathsf{O}^{\mathsf{i}}A$. Let now $\dot{M}, w \models \mathsf{O}^{\mathsf{i}}A$. Then $|A|_{\dot{M}} \in \dot{\mathcal{O}}{}^{\mathsf{i}}{}_w$ and thus by (D$\mathcal{O}^{\mathsf{i}}\star$), $|A|_{\dot{M}}\downarrow \in \overset{\star}{\mathcal{O}}{}^{\mathsf{i}}{}_w$. By Lemma A.11, $|A|_{\overset{\star}{M}} \in \overset{\star}{\mathcal{O}}{}^{\mathsf{i}}{}_w$ and thus, $\overset{\star}{M}, w \models \mathsf{O}^{\mathsf{i}}A$.

The case $\psi = \mathsf{O}^{\mathsf{p}}A$ is shown analogously.

We still have to show that our statement holds for $\varphi \in (\mathcal{L} \cap \hat{\Psi}) \setminus (\mathcal{P} \cup \mathcal{L}')$. As induction hypothesis, suppose that the equivalence holds for $\psi_1, \psi_2 \in \hat{\Phi}$. Let $\psi = \psi_1 \wedge \psi_2$. Then $\overset{\star}{M}, w \models \psi_1 \wedge \psi_2$ iff $\overset{\star}{M}, w \models \psi_1, \psi_2$ tiff (by induction hypothesis) $\dot{M}, w \models \psi_1, \psi_2$ iff $\dot{M}, w \models \psi_1 \wedge \psi_2$. The cases $\psi = \psi_1 \vee \psi_2, \psi = \psi_1 \supset \psi_2$ and $\psi = \neg\psi_1$ are shown similarly. Thus, the equivalence holds for all formulas $\psi \in \hat{\Phi}$. □

Corollary A.1. *For all $\psi \in \hat{\Phi}$, $\dot{M} \models \psi$ iff $\overset{\star}{M} \models \psi$.*

Proof. $\dot{M} \models \psi$ iff $\dot{M}, \dot{@} \models \psi$ iff (by Lemma A.16 and since $\overset{\star}{@} = \dot{@}$) $\overset{\star}{M}, \overset{\star}{@} \models \psi$ iff $\overset{\star}{M} \models \psi$. □

Theorem A.3. $\mathbf{L}^+$ *is complete with respect to the class of frames that satisfy the appropriate conditions.*

Proof. The proof is similar to the proof of Theorem A.2. Take again a formula ψ such that $\nvdash_{\mathbf{L}^+} \psi$. The model $\dot{M} = \langle \dot{F}, \dot{@}, \dot{v} \rangle$ constructed for Theorem A.2 meeting the respective model conditions was such that $\dot{M}, \dot{@} \nvDash \psi$ and thus $\dot{M} \nvDash \psi$. We choose now Φ to be the set of subformulas of ψ. We construct $\overset{\star}{M}$ on basis of $\dot{M}$ as above. By Corollary A.1, $\overset{\star}{M} \nvDash \psi$. By Lemma A.15, $\overset{\star}{F}$ satisfies the respective frame conditions. Therefore, there is a model in the respective class of frames that meets the respective frame conditions. By contraposition and generalization, if a formula ψ is valid with respect to that class, it must be provable in $\mathbf{L}^+$. □

The following two corollary follow immediately.

Corollary A.2. $\mathbf{L}^+$ *is sound and complete with respect to the class of all finite frames that meet the appropriate frame conditions.*

Corollary A.3. $\mathbf{L}^+$ *has the finite model property.*

Corollary A.4. $\mathbf{L}^+$ *is decidable.*

A.6. Dealing with (finite) premise sets

In order to work with (finite) premise sets $\Gamma \subset \mathcal{L}$ we define:

$$\Gamma \models_{\mathcal{F}} \varphi \text{ iff for all } F \in \mathcal{F} : \Gamma \models_F \varphi$$
$$\Gamma \models_F \varphi \text{ iff for all } M = \langle \mathcal{F}, @, v \rangle : \text{if } M \models \Gamma, \text{ then } M \models \varphi$$
$$\Gamma \vdash_{\mathbf{L}^+} \varphi \text{ iff } \vdash_{\mathbf{L}^+(\Gamma)} \varphi,$$

where $\mathbf{L}^+(\Gamma)$ is $\mathbf{L}^+$ enriched by axioms γ for all $\gamma \in \Gamma$.

Let $\Gamma \subset \mathcal{L}$ and $\varphi \in \mathcal{L}$.

Lemma A.17. $\vdash_{\mathbf{L}^+(\Gamma)} \varphi$ *iff* $\vdash_{\mathbf{L}^+} \bigwedge \Gamma \supset \varphi$.

Proof. "$\Rightarrow$": Since $\vdash_{\mathbf{L}^+(\Gamma)} \varphi$, for all $\mathbf{L}^+(\mathbf{\Gamma})$-extensions w', $\varphi \in w'$. Suppose $\nvdash_{\mathbf{L}^+} \bigwedge \Gamma \supset \varphi$. Thus $\{\bigwedge \Gamma, \neg\varphi\}$ is $\mathbf{L}^+$-consistent and hence there is a maximal consistent $\mathbf{L}^+$-extension w for which $\bigwedge \Gamma, \neg\varphi \in w$. Obviously, due to the maximal consistency, $\gamma \in w$ for all $\gamma \in \Gamma$. But then w is also a maximal consistent $\mathbf{L}^+(\mathbf{\Gamma})$-extension—a contradiction.

"$\Leftarrow$": Suppose $\nvdash_{\mathbf{L}^+(\Gamma)} \varphi$. Thus, there is a maximal consistent $\mathbf{L}^+(\Gamma)$-extension w such that $\neg\varphi \in w$. Furthermore, $\gamma \in w$ for all Γ and hence, due to the maximal consistency, $\bigwedge \Gamma \in w$. Obviously, w is also an $\mathbf{L}^+$-extension. But, due to the consistency of w, $\bigwedge \Gamma \supset \varphi \notin w$—a contradiction. □

Lemma A.18. *$M \models \bigwedge \Gamma \supset \varphi$ iff (if $M \models \Gamma$, then $M \models \varphi$).*

Proof. "$\Rightarrow$": $M \models \bigwedge \Gamma \supset \varphi$ iff $M \models \neg(\bigwedge \Gamma) \vee \varphi$ iff ($M \models \neg(\bigwedge \Gamma)$ or $M \models \varphi$). Also, $M \models \Gamma$ iff $M \models \bigwedge \Gamma$ (due to (M-$\wedge$)). Thus, if $M \models \Gamma$, then $M \models \varphi$.

"$\Leftarrow$": Suppose $M \not\models \bigwedge \Gamma \supset \varphi$, then $M \not\models \neg(\bigwedge \Gamma) \vee \varphi$. Then, by (M-$\vee$), it is not the case that ($M \models \neg(\bigwedge \Gamma)$ or $M \models \varphi$). Thus, $M \not\models \neg(\bigwedge \Gamma)$ and $M \not\models \varphi$, and hence, $M \models \bigwedge \Gamma$ and $M \not\models \varphi$—a contradiction. □

Theorem A.4. *Where $\mathbf{L}^+ \in \{\mathbf{CDPM.2d}^+, \mathbf{CDPM.2e}^+\}$ and $\mathcal{F}$ is the appropriate class of frames, $\Gamma \models_{\mathcal{F}} \varphi$ iff $\Gamma \vdash_{\mathbf{L}^+} \varphi$.*

Proof. $\Gamma \models_{\mathcal{F}} \varphi$ iff for all $F \in \mathcal{F} : \Gamma \models_F \varphi$ iff for all $F \in \mathcal{F}$ and for all $M = \langle F, @, v\rangle$: if $M \models \Gamma$, then $M \models \varphi$ iff (Lemma A.18) for all $F \in \mathcal{F}$ and for all $M = \langle F, @, v\rangle : M \models \bigwedge \Gamma \supset \varphi$ iff (Theorem A.3) $\vdash_{\mathbf{L}^+} \bigwedge \Gamma \supset \varphi$ iff (Lemma A.17) $\Gamma \vdash_{\mathbf{L}^+} \varphi$. □

A.7. Deontic detachment

$$\vdash \big(\mathsf{O}(A \mid C) \wedge \mathsf{P}(A \wedge B \mid C) \wedge \mathsf{O}(B \mid A \wedge C)\big) \supset \mathsf{O}(B \mid C) \qquad \text{(DDP1)}$$

$$\vdash \big(\mathsf{O}(A \mid \top) \wedge \mathsf{P}(A \wedge B \mid \top) \wedge \mathsf{O}(B \mid A)\big) \supset \mathsf{O}(B \mid \top) \qquad \text{(DDP}\top\text{1)}$$

$$\begin{aligned}\vdash \big(&\mathsf{O}(A \mid C) \wedge \mathsf{P}(A \wedge B \mid C) \wedge \mathsf{P}(B \mid \neg A \wedge C) \wedge \mathsf{O}(B \mid A \wedge C)\big) \\ &\supset \mathsf{O}(B \mid C)\end{aligned} \qquad \text{(DDP2)}$$

$$\vdash \big(\mathsf{O}(A \mid \top) \wedge \mathsf{P}(A \wedge B \mid \top) \wedge \mathsf{P}(B \mid \neg A) \wedge \mathsf{O}(B \mid A)\big) \supset \mathsf{O}(B \mid \top) \qquad \text{(DDP}\top\text{2)}$$

Theorem A.5. *In* $\mathbf{CDPM.2d}^+$ (DDP1) *and* (DDP$\top$1) *are valid.*

Proof. By (S) and $\mathsf{O}(B \mid A \wedge C)$ we get $\mathsf{O}(A \supset B \mid C)$. $\mathsf{P}(A \wedge (A \supset B) \mid C)$ is a consequence of (CRE) and $\mathsf{P}(A \wedge B \mid C)$. By (CPAND), $\mathsf{O}(A \mid C)$, $\mathsf{P}(A \wedge (A \supset B) \mid C)$ and $\mathsf{O}(A \supset B \mid C)$ we have $\mathsf{O}(A \wedge (A \supset B) \mid C)$. Thus, by (CRE), $\mathsf{O}(A \wedge B \mid C)$. By this, (RCPM) and $\mathsf{P}(A \wedge B \mid C)$ we get $\mathsf{O}(B \mid C)$. (DDP$\top$1) follows immediately. □

Theorem A.6. *In* $\mathbf{CDPM.2e}^+$ (DDP2) *and* (DDP$\top$2) *are valid.*

Proof. The proof is similar to the one above. Since we don't have (S), but instead the weaker (PS'), we need the additional hypothesis $\mathsf{P}(B \mid \neg A \wedge C)$ in order to derive $\mathsf{O}(A \supset B \mid C)$ from $\mathsf{O}(B \mid A \wedge C)$. The rest of the proof is identical to the proof of Theorem A.5. □

B. Modeling nested permissible contexts

As pointed out in the main paper, the generic enhancement $\mathbf{L}^+$ for deontic logics presented in Section 3 is not able to model nested permissible contexts. These are cases in which we have a permissible context C to B but not $\mathsf{P}(C \mid B)$. The idea was there to focus on the explication of the adaptive handling of detachment and hence not to introduce additional complications. However, as will be demonstrated in this section, the logical framework can be enhanced with this ability by introducing some additional techniques.

Recall that $\langle C_1, \ldots, C_n \rangle$ is a *permissive sequence* from C_1 to C_n iff, for all $i < n$ (a) $\vdash C_{i+1} \supset C_i$ and (b) $\mathsf{P}(C_{i+1} \mid C_i)$. Moreover, C is a *permissible context* to B iff there is a permissive sequence from B to C.

We have already noticed that the permissive sequences characterizing permissible contexts have indeed sometimes a length of more than 1. An instance was given by the asparagus example where we have $\mathsf{O}(\neg f \mid \top)$, $\mathsf{P}(a \mid \top)$ and $\mathsf{P}(f \wedge a \mid a)$, but not $\mathsf{P}(f \wedge a \mid \top)$. Evidently $f \wedge a$ describes a permissible context to $\top$.

B.1. Generalizing $\mathbf{L}^+$ *for nested permissible contexts*

How can permissible sequences be formally modeled? The idea is to make use of an additional permission operator $\mathbb{P}(A \mid B)$ that expresses that A is a permissible context to B. It is axiomatized as follows:

$$\text{If } \vdash A \supset B, \text{ then } \mathsf{P}(A \mid B) \supset \mathbb{P}(A \mid B) \tag{P-Ps}$$

$$\vdash \big(\mathbb{P}(B \mid A) \wedge \mathbb{P}(C \mid B)\big) \supset \mathbb{P}(C \mid A) \tag{Ps-T}$$

By these axioms we can derive $\mathbb{P}(f \wedge a \mid \top)$ from $\mathsf{P}(a \mid \top)$ and $\mathsf{P}(f \wedge a \mid a)$, as desired. More generally, we are able to derive $\mathbb{P}(C_n \mid C_1)$ from $\mathsf{P}(C_2 \mid C_1), \ldots, \mathsf{P}(C_n \mid C_{n-1})$ (where for all $i < n$, $\vdash C_{i+1} \supset C_i$) by multiple applications of (P-Ps) and (Ps-T).

Now we can adjust the axiomatization of our generic enhancement $\mathbf{L}^+$ of the base logic $\mathbf{L}$ from Section 3 so that it can model precisely the more general notions from Section 2.

$$\begin{aligned}\text{If } \vdash D \supset \neg A \text{ and } \vdash C \supset B, \text{ then } &\big(\big(\mathsf{P}(D \mid C) \vee \mathsf{O}(D \mid C)\big) \wedge C \wedge \\ &\mathbb{P}(C \mid B) \wedge \mathsf{O}(A \mid B)\big) \supset \bullet_{\mathsf{p}}\mathsf{O}(A \mid B)\end{aligned} \tag{Ep-g}$$

$$\begin{aligned}\text{If } \vdash A \supset \neg D, \vdash A \supset \neg C, \text{ and } \vdash C \supset B, \text{ then } &\\ \big(\mathsf{O}(D \mid C) \wedge \mathsf{O}(A \mid B) \wedge \neg\mathbb{P}(C \mid B)\big) \supset \bullet_{\mathsf{p}}\mathsf{O}(D \mid C)&\end{aligned} \tag{CTDR-g}$$

The idea behind (Ep-g) is that if $\mathsf{O}(A \mid B)$ is excepted in C, then the proper obligation to bring about A should not be detached from $\mathsf{O}(A \mid B)$. Hence, in this case $\bullet_{\mathsf{p}}\mathsf{O}(A \mid B)$ is derived. Rule (CTDR-g) concerns strong CTD obligations. Given that $\mathsf{O}(D \mid C)$ is a strong CTD obligation to $\mathsf{O}(A \mid B)$, the proper obligation to bring about D should not be detachable from $\mathsf{O}(D \mid C)$. Hence, $\bullet_{\mathsf{p}}\mathsf{O}(D \mid C)$ is derived.

The rules (fV) resp. (oV-ei) that manage the blocking of instrumental detachment in case an obligation is factually violated resp. in case there is a more specific obligation incompatible with it can remain as they were defined in Section 3, since permissible contexts do not play a role for them.

Definition B.1. Given a base logic $\mathbf{L}$ we define $\mathbf{L}_{\mathbb{P}}^{+}$ to be $\mathbf{L}$ enriched by the axioms (P-Ps), (Ps-T), (Ep-g), (CTDR-g), (fV), (oV-ei), (CRE$^{\mathsf{i}}$), (RCE$^{\mathsf{i}}$), (CRE$^{\mathsf{p}}$), (RCE$^{\mathsf{p}}$), (EO^{i}), (EO^{p}), (FD$^{\mathsf{p}}$), and (FD$^{\mathsf{i}}$).

The underlying logic for the following examples is again an enriched $\mathbf{CDPM.2}\alpha$ where $\alpha \in \{\mathbf{d}, \mathbf{e}\}$, i.e., $\mathbf{CDPM.2}\alpha_{\mathbb{P}}^{+}$.

Example B.1. Let us again have a look at the asparagus example (PA). One of the counter-intuitive consequences of $\mathbf{CDPM.2}\alpha^{+}$ is $\bullet_{\mathsf{p}}\mathsf{O}(f \mid f \wedge a)$ which is derivable by (CTDR). It is easy to see that this is not anymore derivable by $\mathbf{CDPM.2}\alpha_{\mathbb{P}}^{+}$. The reason is that $\mathbb{P}(f \wedge a \mid \top)$ is derivable (given $\mathsf{P}(a \mid \top)$ and $\mathsf{P}(f \wedge a \mid a)$) and hence (CTDR-g) is not applicable in such a way that $\bullet_{\mathsf{p}}\mathsf{O}(f \mid f \wedge a)$ is derivable.

There is still a drawback to the idea as it was presented so far. Take for instance the premises of the Forrester paradox: $\mathsf{O}(\neg k \mid \top)$ and $\mathsf{O}(g \mid k)$. Note that there are models[6] in which k is a permissible context to $\top$, that is to say, models in which $\mathbb{P}(k \mid \top)$ is verified. Take for instance the model that validates $\mathsf{P}(k \vee x \mid \top)$ and $\mathsf{P}(k \mid k \vee x)$. Moreover, there is a model that validates $\mathbb{P}(k \mid \top)$ even if there is no permissive sequence from $\top$ to k. As a consequence, $\neg\mathbb{P}(k \mid \top)$ is not derivable and hence (CTDR-g) is not applicable in order to derive $\bullet_{\mathsf{p}}\mathsf{O}(g \mid k)$.

The reason for this is that all that is guaranteed by (P-Ps) and (Ps-T) is that if there is a permissive sequence from some A to some B then $\mathbb{P}(B \mid A)$. However, the other direction is not ensured. Moreover, there seem to be no simple axiomatic way of doing so. What would have to be expressed is that whenever we have $\mathbb{P}(B \mid A)$ then there is a natural number n such that there is a permissive sequence $\langle C_1, \ldots, C_n \rangle$ where $A = C_1$ and $B = C_n$. However, without means to quantify over propositions and numbers this seems a hopeless enterprise.

Here is where ALs help us out another time. The idea is to interpret a premise set in such a way that B is a permissible context to A, i.e., $\mathbb{P}(B \mid A)$, iff there is an explicit permissive sequence from A to B. Our axioms (P-Ps) and (Ps-T) ensure the right-left direction. Hence, it is the task of the AL to ensure the left-right direction. In order to achieve this, we define the abnormalities $\Omega^{\mathbb{P}} = \{\mathbb{P}(B \mid A) \mid A, B \in \mathcal{P}\}$ and the adaptive logic $\mathbf{PL}_{\mathbb{P}}^{+}$ by the triple $\langle \mathbf{L}_{\mathbb{P}}^{+}, \Omega_{\mathbb{P}}$, reliability$\rangle$.

The reason why this realizes both directions is easy to see. If there is a permission sequence from A to B, then by (P-Ps) and (Ps-T), $\mathbb{P}(B \mid A)$. If

[6] The semantics of $\mathbf{CDPM.2}\alpha_{\mathbb{P}}^{+}$ is defined by means of neighborhood frames similar as the semantics of $\mathbf{CDPM.2}\alpha^{+}$ in Part A. I will give a more precise account of this in Section B.3.

there is no permissive sequence, then $\mathbb{P}(B \mid A)$ is not derivable by (P-Ps) and (Ps-T) and the AL will take care of deriving $\neg\mathbb{P}(B \mid A)$, since $\mathbb{P}(B \mid A)$ is an abnormality. Obviously $\vdash_{\mathbf{L}_{\mathbb{P}}^{+}} \mathbb{P}(B \mid A) \vee \neg\mathbb{P}(B \mid A)$ (presupposing $\neg$ is a classical negation) and hence $\neg\mathbb{P}(B \mid A)$ is adaptively derivable on the condition $\{\mathbb{P}(B \mid A)\}$. In the remainder we indicate such conditional derivations by "RC$\mathbb{P}$" in the adaptive proofs. The following examples are formulated for $\mathbf{PCDPM.2}\alpha_{\mathbb{P}}^{+}$ where $\alpha \in \{\mathbf{d}, \mathbf{e}\}$.

Example B.2. Let us take another look at the Gentle Murderer.

1	$\mathsf{O}(\neg k \mid \top)$	PREM	$\emptyset$
2	$\mathsf{O}(g \mid k)$	PREM	$\emptyset$
3	$\neg\mathbb{P}(k \mid \top)$	RC$\mathbb{P}$	$\{\mathbb{P}(k \mid \top)\}$
4	$\bullet_{\mathsf{p}}\mathsf{O}(g \mid k)$	1,2,3; CTDR-g	$\{\mathbb{P}(k \mid \top)\}$

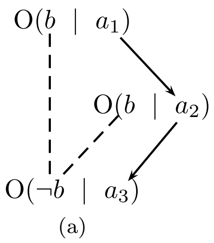
It is easy to see that there is no way of extending the proof in such a way that lines 3 and 4 get marked. Hence, as desired, $\bullet_{\mathsf{p}}\mathsf{O}(g \mid k)$ is a finally derivable in $\mathbf{PCDPM.2}\alpha_{\mathbb{P}}^{+}$.

The following example features nested permissible contexts.

Example B.3. Let $a_{i+1} \vdash a_i$ where $1 \leq i < 3$.

1	$\mathsf{O}(b \mid a_1)$	PREM	$\emptyset$
2	$\mathsf{P}(a_2 \mid a_1)$	PREM	$\emptyset$
3	$\neg\mathsf{P}(a_3 \mid a_1)$	PREM	$\emptyset$
4	$\mathsf{O}(b \mid a_2)$	PREM	$\emptyset$
5	$\mathsf{P}(a_3 \mid a_2)$	PREM	$\emptyset$
6	$\mathsf{O}(\neg b \mid a_3)$	PREM	$\emptyset$
7	$\neg\mathsf{P}(a_3 \mid a_1)$	PREM	$\emptyset$
8	a_3	PREM	$\emptyset$
9	$\mathsf{O}(\neg a_3 \mid a_1)$	7; Def	$\emptyset$
10	$\mathbb{P}(a_2 \mid a_1)$	2; P-Ps	$\emptyset$
11	$\mathbb{P}(a_3 \mid a_2)$	5; P-Ps	$\emptyset$
12	$\mathbb{P}(a_3 \mid a_1)$	10,11; Ps-T	$\emptyset$
13	a_2	8; CL	$\emptyset$
14	$\bullet_{\mathsf{p}}\mathsf{O}(\neg a_3 \mid a_1)$	5,9,10,13; Ep-g	$\emptyset$
15	$\bullet_{\mathsf{i}}\mathsf{O}(\neg a_3 \mid a_1)$	8,9; fV	$\emptyset$
16	$\bullet_{\mathsf{p}}\mathsf{O}(b \mid a_1)$	1,6,12,8; Ep-g	$\emptyset$
17	$\bullet_{\mathsf{p}}\mathsf{O}(b \mid a_2)$	4,6,11,13; Ep-g	$\emptyset$
18	$\bullet_{\mathsf{i}}\mathsf{O}(b \mid a_1)$	1,6,8; oV-Ei	$\emptyset$
19	$\bullet_{\mathsf{i}}\mathsf{O}(b \mid a_2)$	4,6,8; oV-Ei	$\emptyset$

Note that $\mathbb{P}(a_3 \mid a_1)$ although $\neg\mathsf{P}(a_3 \mid a_1)$. The two permissions $\mathsf{P}(a_2 \mid a_1)$ and $\mathsf{P}(a_3 \mid a_2)$ give rise to the nested permissible context a_3 to a_2 where a_2 is a permissible context to a_1. See for an illustration Figure 1a. Note that $\bullet_{\mathsf{p}}\mathsf{O}(b \mid a_1)$ is not derivable by $\mathbf{CDPM.2}\alpha^{+}$. Evidently it is desired, since $\mathsf{O}(b \mid a_1)$ and $\mathsf{O}(b \mid a_2)$ are excepted in a_3 due to $\mathsf{O}(\neg b \mid a_3)$ and $\mathbb{P}(a_3 \mid a_1)$ (resp. $\mathbb{P}(a_3 \mid a_2)$).



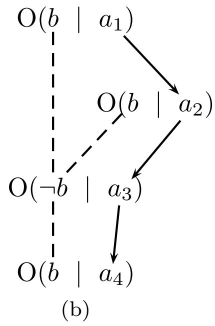


Figure 1: The dashed line indicates an inconsistency between b and $\neg b$. The solid arrow indicates that e.g. a_2 is a permissible context to a_1.

Example B.4. Let us extend the example from above. The reader may wonder what happens if the primary obligation to bring about b in the context a_1 gets reinstated at an even more specific level (see Figure 1b). Suppose for the following that $a_4 \vdash a_3$.

20	$\mathsf{O}(b \mid a_4)$	PREM	$\emptyset$
21	a_4	PREM	$\emptyset$
22	$\mathsf{P}(a_4 \mid a_3)$	PREM	$\emptyset$
23	$\mathbb{P}(a_4 \mid a_3)$	22; P-Ps	$\emptyset$
24	$\bullet_{\mathsf{p}}\mathsf{O}(\neg b \mid a_3)$	6,20,21,23; Ep-g	$\emptyset$
25	$\bullet_{\mathsf{i}}\mathsf{O}(\neg b \mid a_3)$	6,20,21; oV-Ei	$\emptyset$

As desired, due to the second element of lines 24 and 25, proper and instrumental detachment is blocked from $\mathsf{O}(\neg b \mid a_3)$ since it is excepted in a_4 since we have $\mathsf{O}(b \mid a_4)$ and $\mathbb{P}(a_4 \mid a_3)$.

B.2. Adaptively Applying Detachment

In order to apply deontic detachment adaptively we can now proceed analogously to Section 4. Given a premise set Γ we first apply $\mathbf{PL}^+_{\mathbb{P}}$ and then $\mathbf{DL}^+_{\mathbb{P}}$ (as defined in Section 4.2). This is realized by a sequential adaptive logic. **DPL** is characterized by the consequence relation

$$Cn_{\mathbf{DPL}}(\Gamma) = Cn_{\mathbf{DL}^+_{\mathbb{P}}}(Cn_{\mathbf{PL}^+_{\mathbb{P}}}(\Gamma))$$

The marking for abnormalities in $\Omega_{\mathbb{P}}$ is analogous to Definition 2 in the main paper (see Definition B.2 below). We only need to slightly alter the marking for abnormalities in Ω^d. Since in the sequential case $\mathbf{DL}^+_{\mathbb{P}}$ operates on the consequence set of $\mathbf{PL}^+_{\mathbb{P}}$, Dab-formulas over abnormalities in Ω^d that are derived at unmarked lines on conditions that are subsets of $\Omega_{\mathbb{P}}$ have to be taken into account for the marking. Let me give an example.

Example (continues Example B.2). We extend the proof above by the following lines:

	5	k	PREM	$\emptyset$
$\ddagger^{(4)}$	6	$\mathsf{O}^{\mathsf{p}}g$	2,5; cFDp	$\{\bullet_{\mathsf{p}}\mathsf{O}(g \mid k)\}$
	7	$\mathsf{O}^{\mathsf{p}}\neg k$	1; cFDp	$\{\bullet_{\mathsf{p}}\mathsf{O}(\neg k \mid \top)\}$
	8	$\bullet_{\mathsf{i}}\mathsf{O}(\neg k \mid \top)$	1,5; fV	$\emptyset$
	9	$\mathsf{O}^{\mathsf{i}}g$	2,5; cFDi	$\{\bullet_{\mathsf{i}}\mathsf{O}(g \mid k)\}$

Note that the Dab-formula that is responsible for the marking of line 6 has been derived on the condition $\{\mathbb{P}(k \mid \top)\}$ at line 4. It is derivable in $\mathbf{PCDPM.2}\alpha^{+}_{\mathbb{P}}$ that k is not a permissible context to $\top$ (line 3). Given this, it follows further that $\mathsf{O}(g \mid k)$ is a strong CTD obligation to $\mathsf{O}(\neg k \mid \top)$ and hence $\bullet_{\mathsf{p}}\mathsf{O}(g \mid k)$ is derived at line 4. This, however, blocks the detachment at line 6.

Where $\mathsf{Dab}(\Delta_1), \ldots, \mathsf{Dab}(\Delta_m)$ are all minimal disjunctions of abnormalities in $\Omega^{\mathbb{P}}$ derived on the empty condition at stage s, we define $U^{\mathbb{P}}_s(\Gamma) = \{\Delta_1, \ldots, \Delta_m\}$. The marking for $\mathbf{PL}^{+}_{\mathbb{P}}$ is defined as usual for the reliability strategy.

Definition B.2. Line i is †-marked at stage s iff, where $\Delta \subseteq \Omega^{\mathbb{P}}$ is its condition, $\Delta \cap U^{\mathbb{P}}_s(\Gamma) \neq \emptyset$.

Where $\mathsf{Dab}(\Delta'_1), \ldots, \mathsf{Dab}(\Delta'_n)$ are the minimal disjunctions of abnormalities in Ω^d derived at unmarked lines on conditions $\Theta \subseteq \Omega^{\mathbb{P}}$ at stage s, we define $U^d_s(\Gamma) = \{\Delta'_1, \ldots, \Delta'_n\}$.

Definition B.3. Line i is ‡-marked at stage s iff, where Δ is its condition, $\Delta \cap U^d_s(\Gamma) \neq \emptyset$.

Example (continues Example B.3). Prolonging the proof above nicely demonstrates the conditional applications of detachment for the case with nested exceptional contexts.

	20	a_1	8; CL	$\emptyset$
$\ddagger^{(16)}$	21	$\mathsf{O}^{\mathsf{p}}b$	1,20; cFDp	$\{\bullet_{\mathsf{p}}\mathsf{O}(b \mid a_1)\}$
$\ddagger^{(17)}$	22	$\mathsf{O}^{\mathsf{p}}b$	4,13; cFDp	$\{\bullet_{\mathsf{p}}\mathsf{O}(b \mid a_2)\}$
	23	$\mathsf{O}^{\mathsf{p}}\neg b$	6,8; cFDp	$\{\bullet_{\mathsf{p}}\mathsf{O}(\neg b \mid a_3)\}$
	24	$\mathsf{O}^{\mathsf{i}}\neg b$	6,8; cFDi	$\{\bullet_{\mathsf{i}}\mathsf{O}(\neg b \mid a_3)\}$

As expected, factual detachment is neither applicable to $\mathsf{O}(b \mid a_1)$ nor to $\mathsf{O}(b \mid a_2)$. Both are excepted in a_3. Hence, $\mathsf{O}^{\mathsf{p}}\neg b$ is derived at line 23 and $\mathsf{O}^{\mathsf{i}}\neg b$ at line 24.

Example (continues Example B.4). The situation is different if we proceed with the enhanced premise set from Example B.4.

$\ddagger^{(24)}$	26	$\mathsf{O}^{\mathsf{p}}\neg b$	6,8; cFDp	$\{\bullet_{\mathsf{p}}\mathsf{O}(\neg b \mid a_3)\}$
	27	$\mathsf{O}^{\mathsf{p}}b$	20,21; cFDp	$\{\bullet_{\mathsf{p}}\mathsf{O}(b \mid a_4)\}$

$^{\ddagger(25)}$28	$\mathsf{O}^{\mathsf{i}}\neg b$	6,8; cFDi	$\{\bullet_{\mathsf{i}}\mathsf{O}(\neg b \mid a_3)\}$
29	$\mathsf{O}^{\mathsf{i}} b$	20,21; cFDi	$\{\bullet_{\mathsf{i}}\mathsf{O}(b \mid a_4)\}$

In this case we are able to derive the proper and instrumental obligation to bring about b. This is intuitive since $\mathsf{O}(\neg b \mid a_3)$ is excepted in a_4.

B.3. The Semantics

The semantics for our enhanced new lower limit logic $\mathbf{CDPM.2}\alpha^{+}_{\mathbb{P}}$ is defined in a similar way as the semantics of $\mathbf{CDPM.2}\alpha^{+}$. Neighborhood frames are now tuples $\langle W, \mathcal{O}, \mathcal{N}^{\mathrm{i}}, \mathcal{N}^{\mathrm{p}}, \mathcal{O}^{\mathrm{i}}, \mathcal{O}^{\mathrm{p}}, \mathcal{P}^{\star}\rangle$ where $W, \mathcal{O}, \mathcal{N}^{\mathrm{i}}, \mathcal{N}^{\mathrm{p}}, \mathcal{O}^{\mathrm{i}}$, and $\mathcal{O}^{\mathrm{p}}$ are defined as before and $\mathcal{P}^{\star} : W \to (\wp(W) \times \wp(W))$ is used to characterize our new operator $\mathbb{P}$. We add the following requirement for all $w \in W$:

$$M, w \models \mathbb{P}(A \mid B) \text{ iff } \langle |B|_M, |A|_M\rangle \in \mathcal{P}^{\star}_w \qquad (\text{M-}\mathcal{P}^{\star})$$

We have to add two more frame conditions corresponding to the new rules (P-Ps) and (Ps-T), namely

$$\text{For all } X, Y \subseteq W, \text{ if } X \subseteq Y \text{ and } \langle Y, \overline{X}\rangle \notin \mathcal{O}_w, \text{ then } \langle Y, X\rangle \in \mathcal{P}^{\star}_w \qquad (\text{F-P-Ps})$$

$$\text{For all } X, Y, Z \subseteq W, \text{ if } \langle X, Y\rangle, \langle Y, Z\rangle \in \mathcal{P}^{\star}_w, \text{ then } \langle X, Z\rangle \in \mathcal{P}^{\star}_w \qquad (\text{F-Ps-T})$$

Moreover, the frame-conditions for the altered rules (Ep-g) and (CTDR-g) have to be adjusted.

$$\text{For all } X, Y, Z, Z' \subseteq W, \text{ if } X \subseteq \overline{Z'}, Z \subseteq Y, \big(\langle Z, \overline{Z'}\rangle \notin \mathcal{O}_w \text{ or } \langle Z, Z'\rangle \in \mathcal{O}_w\big), w \in Z, \langle Y, Z\rangle \in \mathcal{P}^{\star}_w, \text{ and } \langle Y, X\rangle \in \mathcal{O}_w, \text{ then } \langle Y, X\rangle \in \mathcal{N}^{\mathrm{p}}_w \qquad (\text{F-Ep-g})$$

$$\text{For all } X, Y, Z, Z' \subseteq W, \text{ if } X \subseteq \overline{Z'}, X \subseteq \overline{Z}, Z \subseteq Y, \langle Z, Z'\rangle, \langle Y, X\rangle \in \mathcal{O}_w \text{ and } \langle Y, Z\rangle \notin \mathcal{P}^{\star}_w, \text{ then } \langle Z, Z'\rangle \in \mathcal{N}^{\mathrm{p}}_w \qquad (\text{F-CTDR-g})$$

The soundness and completeness proofs offered for $\mathbf{CDPM.2}\alpha^{+}$ in Part A can be easily adjusted for the altered and additional frame conditions for $\mathbf{CDPM.2}\alpha^{+}_{\mathbb{P}}$.

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