Theory and Experiment in the work of Alonzo Church and Emil Post

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Abstract

While most mathematicians would probably agree that 'experimentation' together with an 'empirical' attitude – both understood in their most general sense – can be important methods of mathematical discovery, this is often obscured in the final presentation of the results for the sake of mathematical elegance. In this paper it will be shown how this "method" has played a significant role in the work of two major contributors to the rather abstract discipline called mathematical logic, namely Alonzo Church and Emil Post.

1 Introduction

In this paper it will be investigated in what way 'experiment' and 'empirical evidence' played an important role in the work of two mathematicians/logicians – Alonzo Church and Emil Post. Both made significant contributions to computer science, although at the time they wrote down their results it didn't even exist yet. In Sec. 2 it will be shown in what way Post had to rely on the more 'experimental' work of testing out specific cases in order to get a grip on certain formal systems called tag systems and how this led him to the at that time innovating idea that the Entscheidungsproblem might not be solvable. In investigating Church's work, it will be explained how the notion of 'empirical evidence' played a significant role in the development of his ideas leading to his seminal 1936 paper (Sec. 3). From this perspective it is interesting to confront what is here called Post's second thesis with Church's thesis. This will be discussed in Sec. 4. The general purpose of this paper is to add strength to the idea that neither 'experiment' nor 'empirical evidence' are some special kind of method intervening on the usual methods of mathematics but are simply part of the way mathematicians work when they are confronted with certain problems, each 'method' getting its meaning through the specific problems at hand (Sec. 5).[1] Indeed, through the analyses offered here, it will be demonstrated how notions such as 'experiment' and 'empirical approach' are simply part of a practice, arising from and dissipating in new mathematical results and problems. In this way the author would like to support the idea, stated in Epstein et al. [1992], "*that theory and experiment feed on each other, and that the mathematical community stands to benefit from a more complete exposure to the*

[1]This is why 'experiment' and 'empirical evidence' will be glossed (See Garfinkel and Sacks [1970]): they will not be given an exact definition, but should be understood as being defined by the specific context they are used in – "exhibited in telling". The author is convinced that this "glossing practice" is the only way to talk about 'experiment' and 'empirical' in this context. Predefining these concepts would suggest the idea of a special method intervening on the usual methods of mathematics, contradicting the general ideas proposed in this paper. In this paper, a glossed term x, will be indicated as 'x'.

experimental process."

2 From solvability to unsolvability: Emil Post's frustrating problem of 'Tag'

The 1931 incompleteness result from Gödel [1931] and the negative solution of the decision problem for certain systems of symbolic logic by Church [1936a] and Turing [1936] are among the most important results of 20th century mathematics and logic.[2] Less well-known is that a young mathematician was already working on related problems in 1921.[3]

2.1 Generalizations to forms: solving decision problems.

Emil Leon Post got his B.S. at City College in New York in 1917 and then went to Columbia University where he wrote his Ph.D. He followed Cassius J. Keyser's seminar on the massive three volume work by Bertrand Russel and Alfred North Whitehead, *Principia Mathematica.* As many others, Post believed that *Principia* could be the complete formalization of the whole of mathematics. Moreover, he understood that in order for *Principia* to be such a complete formalization, it had to be decidable and complete. Together with Lewis [1918], *Principia* was thus the main influence on Post's dissertation.

[2]Followed by the negative solution for the Enstcheidungsproblem (See Turing [1936]; Church [1936b], Church [1936c])

[3]The analysis in this section is mainly based on the results presented in De Mol [2006], where a more detailed analysis of Post's early work can be found.

In order to prove that *Principia* is complete and decidable, its propositional part has to fulfill these conditions. This was exactly what Post proved in his Ph.D.[4] He developed the two-valued truth-table method and used it to show that the propositional calculus is complete, consistent and decidable.[5] However, of more significance here is that Post's main goal of this dissertation was to develop a *general* theory of propositions as is clear from the title of the published version of his Ph.D. (Post [1921]): *Introduction to a general theory of elementary propositions.* He wanted to recover the "*generality of outlook which characterized symbolic logic*" which the authors of *Principia* had given up.[6] Indeed, one could say that to capture logic and mathematics in its most general form in order to study its most general properties – and not just one particular system – was and has remained one of the main motivations in Post's career.

In his dissertation Post announces what one could interpret as a kind of research project. He sees two possible directions for further research rooted in the idea of generalization. First of all, understanding *Principia* as but one particular development of the theory of elementary propositions, Post wants

[4]Less well-known is the fact that similar results were already obtained in 1918 by Hilbert and Bernays. For more information see Zach [1999]

[5]He proved that his truth-table method could be used as an algorithmic procedure that determines whether a given formula is yes/no derivable in the system.

[6]"But owing to the particular purpose the authors had in view they decided not to burden their work with more than was absolutely necessary for its achievements, and so gave up the generality of outlook which characterized symbolic logic. [...]we might take cognizance of the fact that the system of 'Principia' is but one particular development of the theory [...] and so [one] might construct a general theory of such developments." (Post [1921], pp. 163–164)

to construct a far more general theory. To this end he constructed two further generalizations. The first is the generalization of two-valued to many-valued logics, the second the construction of what he later called *systems in canonical form A*[7] – two results he regarded as "instruments of generalization" to "study systems of symbolic logic" and ultimately mathematics.[8]

A second possible direction for further research Post proposes is the extension of the decidability and completeness results for the propositional calculus to other portions of *Principia*.

After he had finished his Ph.D. Post became a Procter fellow at Princeton University from 1920–1921, where he further explored the developments sketched in his Ph.D.[9] He set himself the (ambitious) goal to find a *positive* solution for the decision problem for the restricted functional calculus[10] – the

[7]In his dissertation this was called *generalization by postulation.* Without going into the details of how these systems precisely work, it should be mentioned that within such a system strings can be inferred through finite symbol manipulation. Furthermore it is important to notice the explicit use of the word "form". Instead of constructing one specific system with a specific interpretation, Post constructs a form – a general framework – which can then be used as a kind of mold into which several systems fit.

[8]"[...] but we believe that [...] this broadened outlook upon the theory will serve to prepare us for a similar analysis of that complete system [of *Principia Mathematica*], and so ultimately of mathematics."(Post [1921], p. 164)

[9]He didn't directly publish the main results of this research for several reasons. Only in 1941 he submitted a manuscript called *Absolutely unsolvable decision problems and relatively undecidable propositions. Account of an anticipation.* to the American Journal of Mathematics, discussing these results. The manuscript was not accepted in this form – although a abbreviated version of it was published as Post [1943] – and was finally posthumously published by Martin Davis as Post [1965].

[10]As Martin Davis mentions in the introduction to the collected works of Emil Post (Davis [1994]): "[since] *Principia* was intended to formalize all of existing mathematics,

problem which later became to be known, through Ackerman and Hilbert [1928], as the Entscheidungsproblem.[11] However, instead of trying to find a direct proof, he wanted to make use of his above mentioned more general systems in canonical form A, since he believed that finding a solution for the decision problem for this simpler and more general form might be more straightforward. Thus in order to solve the decision problem for the functional calculus *positively*, he first proved that it could be reduced to a system in canonical form A.[12] Finding then a *positive* solution for systems in canonical form A would lead to an equivalent solution for restricted functional calculus.[13] However, to find such a solution was not that straightforward and he thus started to work on another related problem, namely the problem to determine for any two expressions of a given system, what substitutions would make those expressions identical, i.e., the unification problem. A solution was not immediately at hand though, "[this] *general problem proving intractable*". In order to solve it, he used a technique already familiar to him: he abstracted from the original problem to find a more simple form for a solution of the problem. This form Post called the problem of "tag".

Post was proposing no less than to find a single algorithm for all of mathematics."

[11]Post thus did not talk about the Entscheidungsproblem, nor of decision problems. He used the term 'finiteness problem' in connection to decision problems.

[12]In fact this reduction was not direct. He first constructed a second canonical form – form B – showed its equivalence with systems in canonical form A, and then reduced the functional calculus to a canonical form B, since this reductions seemed more straightforward.

[13]If he would have thought at that time that the Entscheidungsproblem might be unsolvable, he would have had to reduce systems in canonical form A to the functional calculus

2.2 'Experimenting' with the problem of "tag"

Starting from the idea of finding a positive solution for the Entscheidungsproblem, using his methods of generalization and abstraction, Post was led to his frustrating problem of "tag". A form of "tag" is defined as follows. Given a positive integer v, and an alphabet $\Sigma = \{0, 1, ..., \mu - 1\}$ consisting of μ symbols. With each of these symbols one associates a word over the alphabet:

$$\begin{array}{rcl} 0 & \rightarrow & a_{0,1}a_{0,2}.....a_{0,v_0} \\ 1 & \rightarrow & a_{1,1}a_{1,2}.....a_{1,v_1} \\ ... & .... & .... \\ \mu - 1 & \rightarrow & a_{\mu-1,1}a_{\mu-1,2}.....a_{\mu-1,v_{\mu-1}} \end{array}$$

Now, given an initial string A over the alphabet, "tag" at the right end of the string the word associated with the leftmost symbol of A, and remove at the left end the first v symbols. Apply this tagging and removing operations on the new resulting string A', which results in a new string A'',... Post gives the following very simple example: $A = \{1, 0\}$; $1 \rightarrow 1101$; $0 \rightarrow 00$; $v = 3$. If the initial string is "10101001110101111001" the first productions lead to the following strings:

10101001110101111001
01001110101111001110 1
01110101111001110100
1010111100111010000
011110011101000000

....

Post formulated two forms of the problem of "Tag". In its first form the problem is to find for a given tag system a finite process (an algorithm) which decides for any initial string, whether the iterative tag process terminates – produces the empty string – yes/no. In its second form, where an initial string is considered being part of the tag system, the problem for a given system is to find a finite process for determining for any arbitrary sequence over the alphabet whether it will yes/no be produced by the system.[14]

As was already stated, Post obtained the form of "tag" in trying to find a solution for the unification problem. Moreover, it also popped up in connection with the decision problem for systems in canonical form A involving functions of more than one argument. The problem thus became a vital stepping stone in the further development of his program:

> The general problem proving intractable, successive simplifications thereof were considered, one of the last being this problem of "tag". Again, after the finiteness problem for systems in canonical form A involving primitive functions of only one argument was solved, an attempt to solve the problem for systems going, it seemed, but a little beyond this one argument case, led once more essentially to the selfsame problem of "tag". The solution of this problem thus appeared as a vital stepping stone in any further progress to be made.Post [1965], p. 361

[14]As Post remarks it is the problem in its second form which arose in connection with the decision problem (Post [1965], p. 362).

In the beginning he was very optimistic about finding a solution for the decision problem through tag systems given their apparent simplicity. As he had already experienced, generalizing or abstracting from a system to solve a problem often leads to more "simple" forms i.e. they lend themselves more easily to find a solution for the problem at hand, because they can be manipulated in a more straightforward way. It was exactly this "simplicity" that forced Post into a more 'experimental' approach of working out specific cases by varying several parameters in order, amongst other things, to deduce more general classes from these cases. Indeed, in order to get a mathematically rigorous grip on these systems, if possible, you first have to find out how these systems behave under certain conditions. For certain of these conditions, you don't need to 'test' anything on paper. For example, it is trivial to see why the class of tag systems with $v = 1$ is solvable, you don't even have to write anything down to understand this. But what about the class of tag systems with $v = 2, \mu = 2$? As Post himself remarks, this case already asked for considerable effort and he considered the proof of the solvability of this class as the major result of his work as a Procter fellow.[15] But how would one start with such a proof?[16] More generally, how can one start with any mathematical proof for these systems without having any information about

[15] [...] the problem of "tag" was made the major project of the writer's tenure of a Procter fellowship in mathematics at Princeton during the academic year 1920-1921.[...] And the major success of that project was the complete solution of the problem for all bases in which μ and v were both 2.[...] this special case $\mu = v = 2$ involved considerable labor." (Post [1965], p. 362.)

[16] The proof by Post was never published. The author has proven the result by herself: it consists of several classes of cases, and one part of the proof was based on observations made using a computer.

what kind of behaviour several initial conditions can lead to, without knowing about the link between the length of the words and v, without having a clue about what the effect is of varying v and μ,...? Of course, as was already shown, some of these questions can be answered by pure reasoning, but most of these questions can only be answered – or even posed – theoretically by first having gone through several "tests". Tag systems simply don't allow for a direct theoretical intuition due to their abstractness.

Post indeed tested several cases, varying the parameters, and found three classes of behaviour: termination, periodicity and divergence.[17] Divergent behaviour was further divided into two subclasses: fluctuating behaviour and strings that keep on growing forever (Post [1965], p. 362):

> Where the process does not terminate, it is readily seen that according as the lengths of the resulting sequences are bounded, or unbounded, the resulting infinite sequence of the sequences will, from some point on, become periodic, or the length of the n-th sequence will increase indefinitely with n. In the first case the second form of the problem is again immediately solvable, while in the second case the solution would follow if a method were also found for determining of any given length of sequence a point in the process beyond which all derived sequences were of length greater than that given length.[18]

[17] In Post's example the string "010001011" will result in a NILL, while the string "10111011101000000" will lead to periodic behaviour – period 6.

[18] After this description Post added the following footnote: "In this analysis we may have gone somewhat further than is justified by the notes." (Post [1965], p. 362), a comment which further points at the time Post investigated in this kind of 'experimental' research.

As was already stated, Post was able to completely solve the class of systems where $v \leq 2$ and $\mu \leq 2$. However, for systems where v and/or μ become greater than 2 Post made the following observations:

> While considerable effort was expanded on the case $\mu = 2, v > 2$, but little progress resulted, such a simple basis as $0 \rightarrow 00, 1 \rightarrow 1101, v = 3$, proving intractable. For a while the case $v = 2, \mu > 2$, seemed to be more promising, since it seemed to offer a greater chance of a finely graded series of problems. But when this possibility was explored in the early summer of 1921, it rather led to an overwhelming confusion of classes of cases, with the solution of the corresponding problem depending more and more on problems of ordinary number theory.

Post observed behaviour he had not expected at all. Simple though as they may seem, tag systems indeed give rise to intractable and complex behaviour. Even the simple example given above is still not known to be decidable or universal (despite the availability of the computer). About this example Post remarks in a footnote (Post [1965], p. 363):

> Numerous initial sequences actually tried led in each case to termination or periodicity, usually the latter. It might be noted that an easily derived "probability" prognostication suggested that in this case periodicity was to be expected.[19]

[19]It should be remarked here that for small initial conditions, the tag system mentioned by Post indeed always terminates or becomes periodic. Of course he was not able to test larger initial conditions, since then one might have to go through millions of iteration steps

From this quote it is not only clear that Post indeed tested several cases, e.g. by trying out "numerous initial condition", but that he even developed a certain probabilistic method to predict the behaviour of the system.
After nine months of work, Post came to a conclusion he had not expected at all for mathematics, let alone for such 'primitive form of mathematics':

> Since it had been our hope that the known difficulties of number theory would, as it were, be dissolved in the particularities of this more primitive form of mathematics, the solution of the general problem of "tag" appeared hopeless, and with it our entire program of the solution of finiteness problems. This *frustration* [my emphasis], however, was largely based on the assumption that "tag" was but a minor, if essential, stepping stone in this wider program.Post [1965], p. 363.

While Post was convinced that *Principia* and with it the whole of mathematics would be complete and decidable before he started working on tag systems, he now began to doubt this idea – it even seemed hopeless. Three months after he had given up working with tag systems he wrote down results of which the significance cannot be underestimated. He developed systems in canonical form C [20] and constructed a special class of systems in canonical form C, called systems in normal form. He proved his normal form theorem (reduction of systems in canonical form C to a normal form) of which Marvin

before the system becomes periodic or terminates (if it ever does) – a task which is hardly possible with pencil and paper.

[20]These are are now known as Post production systems and influenced Chomsky's context-free grammars

Minsky, the person who later proved that the decision problem for tag systems is recursively unsolvable (see Minsky [1961]), wrote that he regards this theorem as one of the most beautiful in mathematics Minsky [1961/1962?].[21] After having formulated a thesis comparable to that by Church and Turing[22], he furthermore showed that the decision problem for systems in normal form is unsolvable. After having closed the circle between systems in canonical form A, B, C and systems in normal form by reducing these last systems to a form A, the unsolvability of the decision problems for these other systems naturally followed. He also sketched an informal proof of the incompleteness of systems in normal form.[23] Based on his thesis, he conjectured the generalization of both the unsolvability and the incompleteness of systems in normal form to any system of symbolic logic. As is argued in Mol [2006], it was his research on tag systems, together with the theoretical attitude of finding more and more abstract forms of symbolic logic, that led him to these important results and finally, through systems in normal form, to the rever-

[21] "We have long felt that this result is one of the most beautiful in mathematics. The fact that any formal system can be reduced to Post canonical systems with a single axiom and productions of [a] restricted form [...] is in itself a remarkable discovery, and even more so when we learn that this was found in 1921, long before the formalization of metamathematics became so popular."(Minsky [1961/1962?], p. 1)

[22] This is called Post's thesis by Martin Davis Davis [1982]. While his research on tag systems was basic to his idea of the Entscheidungsproblem being unsolvable, it was the proof of his normal form theorem which was fundamental to the statement of his thesis (see Davis (1994). It should be furthermore remarked that Post understood that his analysis leading to the thesis was only fragmentary, since for "full generality a complete analysis would have to be given of all the possible ways in which the human mind could set up finite processes for generating sequences" (Post [1965], p. 387)

[23] He indicates how this theorem could be proved, but does not give an explicit proof

sal of his entire program: from trying to prove the solvability of the decision problem for *Principia* he was now convinced that it was unsolvable.[24]

3 Church and the λ-calculus

At about the same time Post found his revolutionary results, though not completely worked out in every detail, Church was just starting his career. He arrived at Princeton as an 18-year old boy after having graduated at a preparatory school in Connecticut. He would stay there – except for some short interruptions – until 1967.

3.1 Towards variant systems of logic.

As Church states in an interview with Aspray [1984a], he was already interested in foundational issues as an undergraduate: his first published paper was on the Lorentz transformation (Church [1924]), which is at the foundations of (special) relativity. The object of this paper was to find a set of

[24]However, he did not prove this. In footnote 79 he states: "As to *10 being merely attached to this circle, [an unpublished note] categorically states that a proof of the reducibility of canonical form A to *10 is "nearly completed," and as a result even suggests that the solution of the finiteness problem for *10 would yield the solution of the finiteness problem for all of *Principia Mathematica*". In footnote 90 he further added: "Less certain, however, is our having paused at the time to realize that the completion of the proof of the reducibility of canonical form A to *10 [...] would yield the unsolvability of the latter's finiteness problem. It remains uncertain, therefore, to what extent the writer aticipated [sic] Church's result on the unsolvability of the deducibility problem for the restricted functional calculus."

logically independent postulates that uniquely determine the Lorentz transformation for one dimension. One year later he published a more general paper (Church [1925]) further exploring the concept of independent sets, in relating it to irredundant sets of postulates.[25]

After graduating, he started his Ph.D. at Princeton in 1924 under Oswald Veblen, who was interested in the foundations of mathematics and thus sharpened Church's general interest in the subject, he even urged him to read some of Hilbert's work. Veblen was also interested in the independence of the axiom of choice, as Church remarks in Aspray [1984a], and this was precisely the subject of his dissertation, published as Church [1927]. As is clear from its title, *Alternatives to Zermelo's assumption*:

> The object of this paper is to consider the possibility of setting up a logic in which the axiom of choice is false.

In his Ph.D., Church indeed started from the 'hypothesis' that the axiom of choice could be considered independent from Zermelo-Fraenkel set theory. In this way he wanted to investigate the possible consequences of several alternatives to Zermelo's 'assumption'. Church was well aware of the fact that replacing the axiom of choice by contradictory assumptions was not evident at that time.[26] He even had to convince his supervisor Veblen, probably due to the fact that he wanted to contradict that which seemed intuitively more natural (Aspray [1984a]):

[25] It is interesting to note that Church gives a method by which any set of independent postulates can be made irredundant: he explicitly identifies it as a "mechanical method" (See Church [1925], p. 321).

[26] In Davis [1995] it is further discussesed that at the time, Church's general approach of exploring variant systems of symbolic logic was indeed far from evident.

> The only thing that might have annoyed some mathematicians was the presumption of assuming that maybe the axiom of choice could fail, and that we should look into contrary assumptions.[...] [Veblen] was really the only man supervising it. I sort of had to convince him about some aspects of the axiom of choice. To deny what seems intuitively natural is rather difficult. You tend to slip back into what informally seems more reasonable. I remember from time to time having to explain things to him, but I convinced him that my arguments were sound.

As is noted by Martin Davis, Church (Davis [1995], p. 275):

> [Church] was acutely aware of set theory together with logic as a foundation of mathematics [...] [while] [n]oteworthy contributions to logic and foundations of mathematics were few and far between during the twenties.

His research was indeed explicitly embedded in the context of the foundations of mathematics. Before actually starting with his investigation into the alternatives to Zermelo's assumption, he discusses the problem of completeness "*to prepare the way for the suggestion that there may be one or more additional independent postulates which can be added to the set of postulates 1-5* [...]".[27]

Church considers three main postulates A, B and C wanting to "*inquire into their character, and to derive as many of their consequences*[28] in order to find reasonable alternatives for the axiom of choice. Significant now is the fact

[27] Church [1927], p.186

[28] Church [1927], p.178

that Church considers the derivation of as many consequences as possible as a valuable way to argue for the independence of the axiom of choice. If one of the postulates would involve a contradiction, this process of deriving as many consequences as possible, should reveal it at a given time. If not however, this fact can be regarded as "presumptive evidence" for the independence of the axiom of choice (Church [1927], p. 187):

> If any one of these involve a contradiction it is reasonable to expect that a systematic examination of its properties will ultimately reveal this contradiction. But if a considerable body of theory can be developed on the basis of one of these postulates without obtaining inconsistent results, then this body of theory, when developed, could be used as presumptive evidence that no contradiction exists. If there be two of these postulates neither of which leads to contradiction, then there are corresponding to them two distinct self-consistent second ordinal classes, just as euclidian and Lobachevskian geometry are distinct self-consistent geometries [...]

Indeed, starting from the idea that if a set of postulates is inconsistent, a systematic examination of its properties should ultimately reveal a contradiction, Church concludes that if one is able to develop a considerable amount of theory starting from the assumption without finding a contradiction, one can presumptively conclude that the theory developed might be consistent. Evidence which in its turn adds strength to the hypothesis of the independence of the axiom of choice. In the same paper, this attitude is explicitly identified as an experimental one. After having deduced many of the consequences of the three postulates, Church proposes two more postulates F and G, inconsistent with each other, but "apparently consistent" with postulates 1-5 and C. After the statement of the postulates, Church announces how he wants to proceed (Church [1927], p. 205):

> We shall examine briefly the consequences of each of the postulates just stated when taken in conjunction with Postulates 1-5 and C, taking the same experimental attitude as that which we took in the case of Postulates A, B and C.

In other words, while Church was working on highly abstract problems, closely connected with the foundations of mathematics, his method for studying these problems was, from a certain point of view, clearly less abstract.
One year later another paper by Church was published *On the law of the excluded middle* (Church [1928]) of which the purpose is clearly in line with the ideas sketched in his Ph.D.:

> [The purpose of this paper is] to discuss the possibility of a system of logic in which the law of the excluded middle is not assumed [...] (Church [1928])

Again it is clear that Church was not interested in the study of one ultimate system of logic. On the contrary, he wanted to consider variant systems of symbolic logic rooted in the idea that there cannot be one absolute system of symbolic logic.
Four years later he published the first of two major papers in which the ideas and methods already present in his earlier work become even more apparent. It were these papers which led to λ-calculus and ultimately Church's thesis.

3.2 An Inconsistent Set of Postulates

In Church [1932] Church developed a system of postulates to serve as a foundation for logic and mathematics – a system of logic adequate for the development of mathematics, and thus comparable to the ambitions of the authors of *Principia*. This set of postulates however had to be "*free of some of the complications entailed by Bertrand Russel's theory of types, and* [at the same time had to avoid] *the well known paradoxes* [...]"[29]. Unlike the authors of *Principia*, Church however did not claim any absoluteness for his proposed set of postulates, an attitude clearly inspired by his former work:

> We do not attach any character of uniqueness or absolute truth to any particular system of logic.

While he did not give explicit reasons for this kind of attitude towards logic in his former work, Church now adds strength to his approach by making statements about the connections between an abstract theory and the reasons why it is developed – its 'application'. In this respect he connects, by analogy, the existence of alternative geometries with the existence of alternative systems of symbolic logic:

> The entities of formal logic are abstractions, invented because of their use in describing and systematizing facts of experience or observation, and their properties, determined in rough outline by this intended use, depend for their exact character on the arbitrary choice of the inventor. We may draw the analogy of a three dimensional geometry used in describing physical space [...]

[29] Church [1933], p. 839

> In building the geometry, the proposed application to physical space serves as a rough guide in determining what properties the abstract entities shall have, but does not assign these properties completely. Consequently there may be, and actually are, more than one geometry whose use is feasible in describing physical space. Similarly, there exist, undoubtedly, more than one formal system whose use as a logic is feasible, and of these systems one may be more pleasing or more convenient than another, but it cannot be said that one is right and the other wrong.

Indeed, the fact that any system of formal logic is always developed with a certain goal in mind, rooted in certain experiences and observations, implies that there cannot be one ultimate system of logic. This does not mean that the logic is completely determined by its application. On the contrary, "*in developing this formal structure reference to the proposed application must be held irrelevant*". [30] Given this rather pragmatic attitude towards variant systems of logic, there remains only one criterion to reject or accept (be it on a presumptive basis) a given system of logic: its consistency. Given the non-existence of a general method to prove consistency the only reasonable attitude left to apply this criterion to a given system of logic, is an 'empirical' one (Church [1932], p. 348):

> Whether the system of logic which results from our postulates is adequate for the development of mathematics, and whether it is wholly free from contradiction, are questions which we cannot answer except by conjecture. Our proposal is to seek at least an

[30]Church [1932], p. 349

> empirical answer to these questions by carrying out in some detail a derivation of the consequences of our postulates, and it is hoped either that the system will turn out to satisfy the conditions of adequacy and freedom from contradiction or that it can be made to do so by modifications or additions.

This attitude is repeated by Church in a reply to a letter to Gödel, dated july 27, 1932.[31] In answering the question posed by Gödel of whether there is any other way to prove the consistency of Church's set of postulates besides proving it consistent relative to type or set theory, Church answers:[32]

> In fact, the only evidence for the freedom from contradiction of *Principia Mathematica* is the empirical evidence arising from the fact that the system has been in use for some time, many of its consequences have been drawn, and no one has found a contradiction. If my system be really free from contradiction, then an equal amount of work in deriving its consequences should pro-

[31]It should be noted here that Church later admitted that he was among those at that time who believed that "*Gödel's incompleteness theorem might be found to depend on peculiarities of type theory [...] in a way that would show this results to have less universal significance than he was claiming for them.*" (Church in a letter to John Dawson, dated July 25, 1983, reprinted in Sieg [1997]). This is already clear from this reply to Gödel, in which he states amongst other things, that he "*has been unable to see, however, that your conclusions in 4 [Gödel's second incompleteness theorem] of this paper apply to my system.*", Gödel [2003a], p. 369

[32]The exact question posed by Gödel is: "*In case the system is consistent, won't it then be possible to interpret the fundamental concepts in a system with type theory, or in the axiom system of set theory, and can one make the consistency plausible at all in any other way than through such an interpretation?*" , 17 June, 1932, Gödel [2003a], p. 367

> vide an equal weight of empirical evidence for its freedom from contradiction.(Gödel [2003a], p. 368)

This 'empirical' attitude was further pursued in Church [1933]. Having learned in the meantime that some of his postulates lead to a contradiction, the list was revised. Furthermore 42 new theorems were proven to follow from this new set of postulates and a basis to develop a theory of positive integers in the set of postulates was added. In this paper Church beautifully summarizes his 'empirical' approach to logic and mathematics as follows (Church [1933], p. 842):

> Our present project is to develop the consequences of the foregoing set of postulates, until a contradiction is obtained from them, or until the development has been carried so far consistently as to make it empirically probably that no contradiction can be obtained from them. And in this connection it is to be remembered that just such empirical evidence, although admittedly inconclusive, is the only existing evidence of the freedom from contradiction of any system of mathematical logic which has a claim to adequacy.

However, soon after the publication of this paper it would be shown by Kleene and Rosser – Church's Ph.D. students – that he had not deduced enough consequences out of the system: they showed that Church's set of postulates leads to a contradiction in their Kleene and Rosser [1935][33] – a result which clearly illustrates the problematic character of Church's, or any other, 'empirical' attitude.

[33]The proof itself is from early 1934

3.3 λ - The Ultimate Operator

In the meantime Kleene's attention had shifted to a subpart of Church's set of postulates, now known as the λ-calculus.[34] He was working on his Ph.D. replying to the program Church proposed at the end of Church [1933] (Church [1932], p. 864):

> Our program is to develop the theory of positive integers on the basis which we have just been describing, and then, by known methods or appropriate modifications of them, to proceed to a theory of rational numbers and a theory of real numbers.

Kleene's original Ph.D. topic was indeed to develop a theory of positive integers in Church's set of postulates.[35] It was published in two parts in 1935 (Kleene [1935a] Kleene [1935b]) and contained the development of such a theory in the λ-calculus.[36] Already from the first paragraph it is clear what Kleene had learned, or at least inherited, from Church (Kleene [1935a], p. 153):

> Our object is to demonstrate empirically that the system is adequate for the theory of positive integers, by exhibiting a con-

[34]The λ-operator was introduced in order to clarify the notation for functions.

[35]In Aspray [1984b], Kleene says: "*Church, in the last paragraph or the last page of his second paper on the foundation of logic, proposed the problem of developing the theory of positive integers on the basis of his system. There was a ready-made Ph.D. thesis problem. With my very limited knowledge of the area at that time, I don't think I could have dreamed up a problem for myself. It proved to be a challenging problem, and I did it.*"

[36]After Kleene and Rosser had shown that Church's set of postulates was inconsistent, Kleene rewrote his dissertation taking into account this result, although his Ph.D. had already been accepted in September 1933.

> struction of a significant portion of the theory within the system. By carrying out the construction on the basis of a certain subset of Church's formal axioms, we show that this portion at least of the theory of positive integers can be deduced from logic without the use of the notions of *negation*, *class*, and *description*.

While Church's empirical approach *might* have been disappointing when his set of postulates turned out to be inconsistent, it would show very fruitful during further research on the λ-calculus.
As is stated by Rosser in his Rosser [1984], Church first mentioned the idea of every effectively calculable function from positive integers being λ-definable in a conversation in late 1933 (after Rosser had told him about his latest function in λ-calculus).[37] Again according to Rosser, Church was convinced about the equivalence between λ-definability and effective calculability in early 1934. This idea however, was not evident at all:

> Before research was done, no one guessed the richness of this subsystem. Who would have guessed that this formulation, generated as I have described to clarify the notation for functions, has implicit in it the notion (not known in mathematics in 1931 in a precise version) of all functions on the positive integers (or on the natural numbers) for which there are algorithms? (Kleene [1981a], p. 54)

Indeed, Kleene himself had not expected that λ-calculus would have been so powerful, and it was not he but Church he first came up with this idea

[37] A more detailed account of the events preceding the first official statement of Church's thesis can be found in Sieg [1997]

of identifying λ-definability with calculability. One important trigger for Church's idea was probably given by Kleene's definition of the predecessor function in λ-calculus:[38]

> When I brought this result to Church, he told me that he had just about convinced himself that there is no λ-definition of the predecessor function. The discovery that the predecessor function is after all λ-definable excited our interest in what functions are not just definable in the full system but actually λ-definable. The exploration of this became a major subproject for my Ph.D. thesis. Of course, I did develop a great deal of theory of positive integers in Churchs formalism, using many λ-definitions in the process. (Kleene [1981a] p. 57)

From that moment on, the search for effectively calculable functions which are λ-definable became a more explicit research goal. Kleene gradually unravelled the amazing computational power of the λ-calculus, in being able to show that each example of an effective calculable function he and Church could think of, was indeed λ-definable:

> We [Church and Kleene] kept thinking of specific such functions, and of specific operations for proceeding from such functions to others. I kept establishing the functions to be λ-definable and the operations to preserve λ-definability. (Kleene [1981a] p. 57)

[38]In Kleene [1981a] he explains that he got the idea of how to λ-define the predecessor function at the dentist in late January or early in February in 1932.

However, as was stated before, it was not Kleene but Church who first thought about an explicit identification between λ-calculus and effective calculability. In fact, when Church first proposed his 'thesis' to Kleene, he:

> [...] sat down to disprove it by diagonalizing out of the class of the λ- definable functions. But, quickly realizing that the diagonalization cannot be done effectively, I became overnight a supporter of the thesis.(Kleene [1981a], p. 59)

Of more significance here is the fact that it was not "*the concept [of the thesis] itself but rather [the] results established about it*" (Kleene [1981b], p. 49) that led Church to his 'conjecture'. As is pointed out by Sieg [1997], the reason for proposing the identification was, what Sieg calls, the 'quasi-empirical' fact expressed by Church in a letter to Bernays, dated January 23, 1935:

> The most important results of Kleene's thesis concern the problem of finding a formula to represent a given intuitively defined function of positive integers (it is required that the formula shall contain no other symbol than λ, variables, and parentheses). The results of Kleene are so general and the possibilities of extending them apparently so unlimited that one is led to the conjecture that a formula can be found to represent any particular constructively defined function of positive integers whatever.(Quoted in Sieg [1997], p. 155)

In March 1935, the equivalence between λ-definability and general recursiveness was established, and on April 19, 1935, Church publicly announced his thesis for the first time to the American Mathematical Society,[39] however not in terms of λ-definability, but in terms of general recursiveness.[40] On the basis of this result, he proved that the Entscheidungsproblem is unsolvable (Church [1936b] and Church [1936c]).

4 Church's thesis vs. Post's thesis: definition or hypothesis?

In 1936, Church, Post and Turing [41] tried to formally capture the respective intuitive notions of effective calculability, solvability and computability in their respective formalisms: λ-calculus and general recursiveness, formulation 1 and Turing machines.[42] While Turing's proposal is of course at least as interesting as Post's and Church's, it will not be discussed here.[43] Of more significance here is the fact that Post and Church clearly opposed each other with regard to the interpretation of what is now generally known as a thesis, having as many forms as there are equivalent formalisms. But back in 1936 Church nor Post were talking of their result as a thesis, it was Kleene who first talked about Church's result in terms of a thesis in his Kleene [1952].

[39]An abstract of this talk is published as Church [1935]

[40]A discussion on why Church did first publicly announce his thesis in terms of general recursiveness can be found in'Sieg [1997]

[41]While Turing's and Church's paper were written independently of each other, Post's paper, although written independently of Turing's, was not independent of Church's, since he refers to Church's paper.

[42]These were all shown to be equivalent

[43]In Gandy [1988] and Sieg [1997] a more detailed analysis of Turing's thesis can be found.

Church himself clearly regarded his thesis as a definition. He identified effective calculability with general recursiveness (and λ-calculus) by definition:[44]

> We now *define* [m.i.] the notion, already discussed, of an *effectively calculable* function of positive integers by identifying it with the notion of a recursive function of positive integers (or of a λ-definable function of positive integers). This *definition* [m.i.] is thought to be justified by the considerations which follow, so far as a positive justification can ever be obtained for the selection of a formal definition to correspond to an intuitive notion.(Church [1936a], p.356)

As Church points out, to give a formal definition for an intuitive notion is problematical. One can try to give a justification but only insofar as such a justification is possible for this kind of definition. One can merely try to give some good reasons, but nothing more should be expected from it. Emil Post, on the contrary, rejected the idea of calling such identifications definitions. Post had already formulated a 'thesis' similar to that of Church in 1921, in identifying the notion of a generated set with his systems in normal form.[45] In 1936 he came up with another such 'identification', in developing his *Formulation 1*,[46] which will be called Post's second thesis here.

[44]In this respect, Sieg [1997], p. 155 uses the term 'definitional identification'

[45]This identification of a generated set with a normal set, was first called Post's thesis by Martin Davis Davis [1982]. For more information on Post's first thesis the reader is referred to Gandy [1988], Davis [1994], Stillwell [2004] and Mol [2006]

[46]The details of formulation 1 will not be discussed here. It should be noted though that Post's formulation 1 and Turing's machines are quasi-identical. Post however did not prove the unsolvability of any decision problem, at least not in this paper, nor did he

As is pointed out in Davis [1994], Post was not satisfied with the analysis of an algorithmic process in terms of general recursiveness or λ-definability:

> Believing that the Herbrand-Gödel notion of general recursiveness and the Church-Kleene notion of λ- definability were both lacking in that neither constituted a "fundamental" analysis of the notion of algorithmic process, Post proposed as suitably "fundamental" the operations of marking an empty "box" or erasing the mark in a marked box.

In his Post [1936] however, the notion of an algorithmic process is not described in terms of calculability or computability, but in terms of solvability. Indeed the goal of formulation 1 was to describe mathematically what is meant with a general method to solve any decision problem which is intuitively considered solvable (Post [1936],p. 103):[47]

> We have in mind a *general problem* consisting of a class of *specific problems*. A solution of the general problem will then be one which furnishes an answer to each specific problem. In the following formulation of such a solution [...] [i.e. formulation 1].

He furthermore defined a whole set of notions in terms of solvability such as *applicability* to a general problem, a *1-solution*, a general problem being

provide an explicit description of the equivalent of a Universal Turing machine. This is probably *one* of the reasons why this paper is now less well-known.

[47]He does not mention notions such as "algorithmic procedure", "calculability" or "computability". Only once at the end of the paper "effective calculability" is used, with respect to Church's identification. It is however clear that although Post works in terms of solvability, he identified this notion with the above mentioned notions of "calculability", "computability" and "algorithmic" process.

1-given,... in identifying the notion of what is intuitively solvable with solvability in formulation 1.

Post expected that formulation 1 would turn out to be equivalent to general recursiveness, but immediately adds that its purpose is not simply to:

> [...] present a system of a certain logical potency but also, in its restricted field, of psychological fidelity. In the latter sense wider and wider formulations are contemplated. On the other hand, our aim will be to show that all such are logically reducible to formulation 1. We offer this conclusion at the present moment as a *working hypothesis*. And to our mind such is Church's identification of effective calculability with recursiveness. [...] The success of the above program would, for us, change this hypothesis not so much to a definition or to an axiom but to a *natural law*. Only so, it seems to the writer, can Gödel's theorem concerning the incompleteness of symbolic logics of a certain general type and Church's results on the recursive unsolvability of certain problems be transformed into conclusions concerning all symbolic logics and methods of solvability.(Post [1936], p. 105)

Post considered his formulation 1 as a system of psychological fidelity: generalizing Gödel's incompleteness theorem or Church's proof of the unsolvability of certain decision problems to all symbolic logics and methods of solvability depends on the "faith" one can have in identifications such as that proposed by Post. In that sense, he suggests to contemplate as wide a variety of formulations as possible, each of which should be shown to be reducible to formulation 1. Indeed, at the time Post wrote this paper he considered his conclusions concerning the 'power' of formulation 1, merely as a working hypothesis. It is only if the program of finding more and more formulations reducible to formulation 1 has proven its worth that the hypothesis can be considered as a *natural* law. It is exactly at this point that Post criticizes Church's "definitional identification". After having noticed in footnote 8 that Church's, Kleene's and Rosser's work already carries the identification beyond the working hypothesis stage, in having shown that λ-definability and general recursiveness are equivalent, Post continues:

> But to mask this identification under a definition hides the fact that a fundamental discovery in the limitations of the mathematicizing power of *Homo Sapiens* has been made and blinds us to the need of its continual verification.(Post [1936], p. 105)

Indeed Post does not accept Church's definitional identification as the correct interpretation of such an identification. Rather it should be regarded as a hypothesis or law of nature, to be continually verified, and is thus subject to inductive reasoning. In a letter to Gödel, dated October 30, 1938 Post again emphasizes the significance of the hypothetical character of e.g. Church's thesis (Gödel [2003b], p. 171):

> the absolute unsolvability of [a] problem has but a basis in the nature of physical induction at least in my work and I still think in any work.

Considering identifications such as those offered by Post as a hypothesis indeed implies that an unsolvable decision problem can only presumptively be considered absolute: it is true only in supposing the validity of the identification. In the quote Post also underlines the fact that this is not only true for his but also for the work of others. But why did he add this small comment? It might be the case that this is a further reaction on Church. Indeed, after Post's paper had been published criticizing the definitional interpretation, Church replied by one, among many, of his sharp reviews:[48]

> [Post] does not, however, regard his formulation as certainly to be identified with effectiveness in the ordinary sense, but takes this identification as a "working hypothesis" in need of continual verification. To this the reviewer would object that effectiveness in the ordinary sense has not been given an exact definition, and hence the working hypothesis in question has not an exact meaning. To define effectiveness as computability by an arbitrary machine, subject to restrictions of finiteness, would seem to be an adequate representation of the ordinary notion, and if this is done the need for a working hypothesis disappears.(Church [1937a])

Church explicitly opposes the possibility of regarding his 'definition' as a 'hypothesis' and gives two objections. Firstly he finds that Post has not given an exact definition of the ordinary notion of effectiveness so that the 'working hypothesis' is ambiguous. Secondly, if one does provide an adequate representation for the notion of effectiveness it should simply not be regarded as a working hypothesis but as a definition. The first critique is partly true, since

[48] Gandy gives a short discussion of this review in Gandy [1988]. A general paper on Church and his work as a reviewer for the Bulletin of Symbolic Logic can be found in Enderton [1998].

Post indeed never makes an explicit identification between his formulation 1 and effectiveness. However, as was argued before, Post understood formulation 1 as the formal equivalent of the intuitive notion of solvability by a general method, and not directly as an identification with effectiveness. Only after having developed this formalism he compares this 'hypothetical identification' with Church's definitional one. The second objection, that there is no need for a 'working hypothesis' when one has given an exact definition of effectiveness only seems to mean that Church preferred definition to further argument as is stated in Gandy [1988].[49]

This small "quarrel" between Church and Post becomes the more remarkable in the light of Church's reluctance to publicly announce his "definition". It was only after the establishment of the equivalence between general recursiveness and λ-calculus that Church made his thesis available to the public in 1935. As is argued by Sieg:

> I claim [...] that Church was reluctant to put forward the thesis in writing – until the equivalence of λ-definability and general recursiveness had been established. The fact that the thesis was formulated in terms of recursiveness indicates also that λ-definability

[49]It should be furthermore noted that Church's opinion here is contradicted by what happened after 1936. Until now, hundreds of people are still doing research on the continual verification of what is generally called the Church-Turing thesis: not everyone wants to accept it as is clear from the literature, often due to certain misinterpretations. Many counter-examples have been suggested, especially in relation to the physical version of the thesis, although it seems to be the case that none of them has lead to a falsification. A good overview of the proposed counter-examples, together with a well-argued rejection of these "falsifications" is given by Cotogno [2003].

> was at first, even by Church, not viewed as one among equally natural definitions of effective calculability [...] (Sieg [1997], p. 157)[50]

Gandy adds further strength to the idea that this equivalence proof played an important role in Church's formulation of the thesis. In discussing the 'argument by confluence', the significance of equivalence proofs to strengthen the theses, he points out a footnote in Church [1936a]. Here Church states that this kind of argument adds strength to his further justifications for his definitional identification, "*so far as a positive justification can ever be obtained for the selection of a formal definition to correspond to an intuitive notion.*" It was exactly this kind of argument Post had in mind in order to get the identification beyond the working hypothesis stage, this argument being understood as 'presumptive evidence', to use Church's words, for the validity of the identification. Did Church indeed interpret the evidence offered by the equivalence proof in the sense Post understood it (and in the sense he used such evidence in his earlier work), or was it merely a theoretically convincing fact, used to state a more naturally convincing definition, having no link whatsoever with his needing more evidence for λ-calculus' computational powers? Whatever the right answer might be, Church's rather strong

[50]The author would like to add one small comment here. It is indeed true that λ-definability is not a natural definition of effective calculability. However, it still remains the case that it was through λ-calculus and not general recursiveness that Church first had this idea. Furthermore, as is pointed out in Kleene [1981b], general recursiveness is as unnatural a definition for effective calculability as is λ-calculus. The author is thus not completely convinced by the reason given by Sieg here for Church's reluctance. As to what other possible reasons Church might have had, one can only guess.

reaction against Post's critique remains rather strange in the light of the work preceding the public announcement. As was argued here, more than once Church seems to have relied on what was called here an 'empirical' attitude. The significance the equivalence between general recursiveness and λ-definability has played in the first public announcement of Church's thesis, only seems to be in line with this kind of attitude, although it is partly contradicted by his critique on Post.

5 Discussion

As was shown, already in his Ph.D. thesis Post set himself the goal of finding the most general *form* of systems of symbolic logic in order to be able to study their most fundamental and general properties. One of his basic methods was thus to develop more abstract forms, because he did not want to be preoccupied with specific logical concepts but rather with the outward form of symbolic logic.[51] Because of their more abstract and simple character, he hoped that finding a positive solution for the decision problem for *Principia* might be more straightforward. In the course of this research he ended up with his form of "tag". The only way to get a grip on these systems is

[51] As he states at the beginning of his Post [1965], p. 343, one of the chief differences in method between his work and that done by Gödel, Church and Turing: "is its preoccupation with the outward forms of symbolic expressions, and possible operations thereon, rather than with logical concepts as clothed in, or reflected by, correspondingly particularized symbolic expressions, and operations thereon. [...] it also allows a greater freedom of method and technique."

the above discussed 'experimental' approach. With such systems one can no longer rely on an analysis based on the logical concepts involved, since one is left with pure string manipulating systems. This together with the inherent intractability of tag systems made such an approach necessary.[52] Of course, this does not exclude other methods. On the contrary, based on the observations one can further develop the theory. One can e.g. define classes of tag systems and classes of problems.[53] In Post's case this approach even led to fundamental theoretical results, as was shown in Sec. 2.2. In constructing more and more abstract and simple forms, in order to get a grip on a hard theoretical problem, 'experimentation' became necessary, which then led to a reversal of the ideas put forward by the theoretical assumptions.

While the notion of 'experimentation' does not seem to have played any significant role, the notion of 'empirical evidence' cannot be ignored in Church's earlier work. It is clear that Church's 'empirical' attitude was closely connected with a rather pragmatic and relativistic position towards logic, as was shown in sec. 3.2. In exploring several systems of symbolic logic, he rejected the idea of one absolute logic. In the end, there is no absolute reason whatsoever to prefer one system above the other.[54] This attitude was partly

[52]Indeed, because of the later proven unsolvability of the decision problem for tag systems, there are (infinitely) many tag systems for which the final behaviour cannot be predicted, since there simply is no general method to predict their behaviour.

[53]For example, Post as well Maslov [1964] and Wang [1963] proved that there are certain conditions which, if satisfied by a certain tag system, imply decidability.

[54]This is e.g. explicitly stated in the following quote from Church [1928], p. 76: "In connection with geometry and other branches of mathematics it is commonly recognized that it is meaningless to ask about the absolute truth of a postulate and that the choice between one of two contrary postulates must be made on the basis of simplicity and

motivated by the possibility of the existence of undecidable propositions: the possible independence of the axiom of choice is a theoretical reason for studying its possible alternatives. In rejecting the idea of an absolute logic, the only reason left for Church to accept or reject a system of symbolic logic is its consistency. However, as long as one does not have a consistency proof, the only reasonable argument to 'trust' a given system is an empirical one: the longer one has worked on it, and the more theorems one has deduced from it, without finding any contradiction, the more probable it becomes that the system is indeed consistent (although one can never be sure). This 'empirical' attitude is necessitated by the fact that there is no general method to prove the consistency of any system. As long as one does not find a proof, one has to rely on what has already been found. In other words, this 'empirical' approach is (again) rooted in a certain mathematical fact about logical and mathematical systems.

When Church's set of postulates was shown to be inconsistent, he had every reason to be more careful with regard to this empirical method, being faced with its possible failure. As was shown in Sec. 3.3., this did not stop him to again draw conclusions on he basis of 'empirical' evidence: after Kleene had λ-defined several functions of positive integers, he became convinced that λ-calculus could be used to define any calculable function. This evidence was fundamental to Church's formulation of the thesis. Research on λ-calculus was not motivated by the concept of the thesis itself, it was not developed as

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serviceability. It seems reasonable to recognize the same thing with regard to the postulates of logic, in particular the law of excluded middle, and to say on this basis that it is meaningless to ask about the truth of the law of excluded middle."

an instrument of an analysis of the notion of a computation, as was clearly the case with Turing's and Post's second thesis. Rather it was, again, the system itself, and the results established about it that led to the thesis.

But in how far is it really grounded to talk about concepts such as 'experimental' and 'empirical evidence' in the work of Church and Post? More generally how far can one go with such notions in relation to logic and mathematics? With the rise of the computer it has become rather convenient to talk about experimental mathematics. Many research groups make intensive use of the computer in the establishment of their results. Typical examples are fractal geometry and chaos theory, branches of mathematics that could not have progressed the way they have without the computer. Many of the results are based on observations and analysis of visual or other output. But what is it exactly that makes it 'normal' to talk about experimental mathematics when the computer is involved, while it is less evident to do this for mathematics in the pre-computer era?

As was already stated at the beginning of this paper, the author is not convinced at all that 'experimentation' or the use of 'empirical evidence' should be considered as some kind of special method, intervening on the usual methods of mathematics. The abstractness and intractability of tag systems made it for Post impossible to proceed in a purely theoretical manner. Instead he had to try out specific cases and particular methods in order to build up an intuition of the systems he was working in. Through this 'experimental' approach it then became possible for Post to draw certain theoretical conclusions and even prove some results concerning tag systems. In calling this approach 'experimental' one should be very careful though. It can only be valuable in this context if one understands that it cannot be isolated from the practice it is used in, and the specific problems Post was confronted with. If one really wants to talk about the notion of an 'experiment' in mathematics one should always start from a specific practice, never forgetting that it is only through a combination and interaction of 'theory' and 'experiment' that results are reached

A similar reasoning can be applied to Church's empirical approach in his earlier work. As was shown Church himself explicitly used terms such as 'presumptive evidence' or 'empirical evidence'. Does this mean that Church's general approach to mathematics should be identified as an empirical one? Not at all. The simple fact of his reaction against Post's calling his definition a working hypothesis shows that this kind of conclusion would not do right to Church's position. Rather one should conclude that given Church's theoretical ideas, his interest in foundational issues and his awareness of the non-absoluteness of any system of symbolic logic, an empirical approach seemed to be the most reasonable. While he was clearly well-aware about the necessity of this approach in this context, motivated as it was by theoretical considerations, this was clearly not the case in first conjecturing his thesis. The concept of λ-definability was not preceded nor developed in function of the theoretical idea that λ-definability is a good formal definition for effective calculability through proving the λ-definability of as many integer functions one can think of. It was only after the 'empirical evidence' became – against all odds – more and more convincing that Church formulated his thesis.

Also in Church's case one cannot but conclude that an 'empirical' approach can only be meaningful here in connection to the problems and practice it becomes apparent with. Indeed talking about the notion of ‘experiment’ or ‘empirical evidence’ without back-linking it to the specific context it appears in, might lead to purely theoretical discussions, losing its link with the reality of the mathematical practice it appears in and thus losing the connection with its own reality. Looking at the practice they are used in one sees that one cannot simply separate ‘experiment’ from ‘theory’, they feed on each other.

6 Conclusion

As was already stated, with the rise of the computer the significance of ‘experiments’ and ‘empirical evidence’ for mathematics has become more and more clear. Important new results in and even new branches of mathematics originated in the possibility of performing ‘experiments’ on computers. In 1992 even a new journal was founded, devoted solely to the subject of experimental mathematics. In the statement of the general philosophy behind the journal however, it is explicitly stated that the notion of an ‘experiment’ should not be confined to computer experiments – “[since some experiments] *are still the result of pencil-and-paper work*”.

While it seems rather obvious to talk about ‘experiments’ and ‘empirical evidence’ in connection to computers, this is often not the case for other parts of mathematics. This is partly due to the fact that the more ‘experimental’ parts of research in mathematics are concealed in the final presentation of the results for the sake of elegance and mathematical rigor. While these are of course important features of mathematics which should not be abandoned by the mathematicians themselves, they can obscure the interaction between ‘experiment’ and ‘theory’ in mathematics. It is thus significant to look at the more “ugly” work preceding the elegance of the publication, especially in those more abstract branches of mathematics. This is not only important from a philosophical point of view, but maybe even more from a pedagogical point of view since it might help to relativize the idea, believed to be true by a majority of people, of mathematics being a kind of abstract totality of eternal truths. To end with a quote from *Experimental Mathematics*:

> Experiment has always been, and increasingly is, an important method of mathematical discovery. [...] Yet this tends to be concealed by the tradition of presenting only elegant, well-rounded, and rigorous results. [...] we consider it anomalous that an important component of the process of mathematical creation is hidden from public discussion. It is to our loss that most of us in the mathematical community are almost always unaware of how new results have been discovered. It is especially deplorable that this knowledge is not made part of the training of graduate students, who are left to find their own way through the wilderness. [...] There is value not only in the discovery itself, but also in the road that leads to it.

References

Ackerman, W. and Hilbert, D. [1928]. *Grundzüge der theoretische Logik*.

Berlin: Springer.

Aspray, W. [1984a]. Transcript of an interview with Alonzo Church by William Aspray on 17 May 1984 at the University of California, Los Angeles. (The Princeton Mathematics Community, transcript number 5. Available at: http://libweb.princeton.edu/libraries/firestone/rbsc/finding aids/mathoral/ pmc05.htm).

— [1984b]. Transcript of an interview with Stephen C. Kleene and J. Barkley Rosser by William Aspray on 26 April 1984 in Madison, Wisconsin. (The Princeton Mathematics Community, transcript number 23. Available at: http://libweb.princeton.edu/libraries/firestone/rbsc/finding aids/mathoral/ pmc23.htm).

Church, A. [1924]. 'Uniqueness of the Lorentz transformation'. *American Mathematical Monthly*, **31**: 376–382.

— [1925]. 'On irredundant sets of postulates'. *Transactions of the American Mathematical Society*, **27**: 318–328.

— [1927]. 'Alternatives to Zermelo's assumption'. *Transactions of the American Mathematical Society*, **29**: 178–208.

— [1928]. 'On the law of the excluded middle'. *Bulletin of the American Mathematical Society*, **34**: 178–208.

— [1932]. 'A set of postulates for the foundation of logic'. *Annals of mathematics*, **33** (2): 346–366. 2nd series.

— [1933]. ‘A set of postulates for the foundation of logic (second paper)’. *Annals of mathematics*, **34** (4): 839–864. 2nd series.

— [1935]. ‘An unsolvable problem of elementary number theory’. *Bulletin of the American Mathematical Society*, **41**: 332–333. Abstract of a talk given on 19 April 1935, to the American Mathematical Society.

— [1936a]. ‘An unsolvable problem of elementary number theory’. *American Journal of Mathematics*, (58): 345–363. Also published in Davis (1965), 108–109.

— [1936b]. ‘A note on the Entscheidungsproblem’. *The journal of symbolic logic*, **1** (1): 40–41.

— [1936c]. ‘Correction to A note on the Entscheidungsproblem’. *The journal of symbolic logic*, **1** (3): 101–102.

— [1937a]. ‘Review of Post (1936)’. *Journal of Symbolic Logic*, **2** (1): 43.

— [1937b]. ‘Review of Turing (1936)’. *Journal of Symbolic Logic*, **2** (1): 42–43.

Cotogno, P. [2003]. ‘Hypercomputation and the Physical Church-Turing Thesis’. *British Journal for the Philosophy of Science*, (54): 181–223.

Davis, M. [1965]. *The Undecidable. Basic papers on undecidable propositions, unsolvable problems and computable functions*. New York: Raven Press. Corrected republication (2004), Dover publications, New York.

— [1982]. ‘Why Gödel didn’t have Church’s thesis’. *Information and Control*, **54**: 3–24.

— [1994]. 'Emil L. Post. His life and work.' In 'Solvability, Provability, Definability: The collected works of Emil L. Post Post (1994)', xi–xviii.

— [1995]. 'Logic in the twenties'. *The Bulletin of Symbolic Logic*, **1** (3): 273–278.

ENDERTON, H. [1998]. 'Alonzo Church and the reviews'. *Bulletin of Symbolic Logic*, **4** (2): 172–180.

EPSTEIN, D., LEVY, S., and DE LA LAVE, R. [1992]. 'Letter from the editor'. *Experimental Mathematics*, **1** (1): 1–3. Statement of the Philosophy of the journal "Experimental Mathematics", available at: `http://www.math.ethz.ch/EMIS/journals/EM/expmath/philosophy.html`.

GANDY, R. [1988]. 'The confluence of ideas in 1936'. 55–111. Published in Herken (1988).

GARFINKEL, H. and SACKS, H. [1970]. 'On formal structures of practical action'. In 'McKinney and (eds.)', 338–366.

GÖDEL, K. [1931]. 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandtrer Systeme I'. *Monatshefte für Mathematik und Physik*, (38): 173–198. Also published in Davis (1965).

GÖDEL, K. [2003a]. *Collected Works IV: Correspondence A-G*. Oxford: Oxford University Press.

— [2003b]. *Collected Works V: Correspondence H-Z*. Oxford: Oxford University Press.

HERKEN, R. (ed.) [1988]. *The Universal Turing machine*. Oxford: Oxford University Press. Republication (1994), Springer Verlag, New York.

KLEENE, S. C. [1935a]. 'A theory of positive integers in formal logic. Part I'. *American Journal of Mathematics*, **57** (1): 153–173.

— [1935b]. 'A theory of positive integers in formal logic. Part II'. *American Journal of Mathematics*, **57** (2): 219–244.

— [1952]. *Introduction to Metamathematics*, vol. 1 of *Bibliotheca Mathematica*. Amsterdam: North-Holland. 11th Reprint, 1996),.

— [1981a]. 'Origins of recursive function theory'. *IEEE Annals of the history of computing*, **3**: 52–67.

— [1981b]. 'The theory of recursive functions, approaching its centennial'. *Bulletin of the american mathematical society*, **5**: 43–60.

KLEENE, S. C. and ROSSER, J. [1935]. 'The inconsistency of certain formal logics'. *The annals of Mathematics*, **36** (3): 630–636.

LEWIS, C. I. [1918]. *A survey of Symbolic Logic*. Berkeley: University of California Press.

MASLOV, S. J. [1964]. 'On E. L. Post's 'Tag' Problem.' *Trudy Matematicheskogo Instituta imeni V.A. Steklova*, (72): 57–68.

MCKINNEY, J. and (EDS.), E. T. [1970]. *Theoretical Sociology*. New York: Appleton-Century-Crofts.

Minsky, M. [1961]. 'Recursive unsolvability of Post's problem of tag and other topics in the theory of Turing machines'. *Annals of Mathematics*, **74**: 437–455.

— [1961/1962?]. 'A simple direct proof of Post's Normal Form Theorem'. Artificial Intelligence Project – RLE and MIT Computation Center, memo 44.

Mol, L. D. [2006]. 'Closing the circle: An analysis of Emil Post's early work.' *The Bulletin of Symbolic Logic*, **12** (2): 267–289.

Post, E. L. [1921]. 'Introduction to a general theory of elementary propositions.' *American Journal of Mathematics*, (43): 163–185.

— [1936]. 'Finite Combinatory Processes - Formulation 1'. *The Journal of Symbolic Logic*, **1** (3): 103–105. Also published in Davis (1965), 289–291.

— [1943]. 'Formal Reductions of the General Combinatorial Decision Problem'. *American Journal of Mathematics*, (65): 197–215.

— [1965]. 'Absolutely unsolvable problems and relatively undecidable propositions - account of an anticipation'. In 'Davis (1965)', 340–433.

— [1994]. *Solvability, Provability, Definability: The collected works of Emil L. Post*. Boston: Birkhauser. Edited by Martin Davis.

Rosser, J. [1984]. 'Highlights of the history of the lambda-calculus'. *Annals of the history of computing*, **6** (4): 337–349.

Sieg, W. [1997]. 'Step by Recursive Step: Church's analysis of Effective Calculability'. *Bulletin of Symbolic Logic*, **3** (2): 154–180.

STILLWELL, J. [2004]. 'Emil Post and His Anticipation of Gödel and Turing'. *Mathematics Magazine*, **77** (1): 3–14.

TURING, A. [1936]. 'On computable numbers with an application to the Entscheidungsproblem'. *Proceedings of the London Mathematical Society*, (42): 230–265. Also published in Davis (1965), 116–151.

WANG, H. [1963]. 'Tag systems and lag systems'. *Matheamtische Annalen*, **152**: 65–74.

ZACH, R. [1999]. 'Completeness before Post: Bernays, Hilbert, and the development of propositional logic'. *The Bulletin of Symbolic Logic*, **5** (3): 331–366.