Some Computational Aspects of Inconsistency-Adaptive Logics *

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Abstract

This paper concerns a goal directed proof procedure for the propositional fragment of the adaptive logic **ACLuN1**. The procedure forms an algorithm for final derivability and may easily be generalized for the propositional fragment of all flat adaptive logics. The aim is to articulate a procedure that, if extended to the predicative level, provides criteria for final derivability.

1 Aim of this Paper

Adaptive logics are intended for characterizing inference relations that lack a positive test.[1] The characterization has a specific metalinguistic standard format. This format provides the logic with a semantics and a proof theory and warrants soundness, completeness, and a set of properties of the logic.[2] The first adaptive logics were inconsistency-adaptive. The articulation of other adaptive logics provided increasing insight in the underlying mechanisms and required that adaptive logics were systematized in a new way. This systematization is presented in [8] and will be followed here.[3]

Inference relations for which there is no positive test abound in both everyday and scientific reasoning processes. The importance of adaptive logics derives from there. This applies especially to the dynamic proof theory of adaptive logics. Indeed, this proof theory is intended for *explicating* actual reasoning, a task that cannot be accomplished by definitions, semantic systems, and other more abstract characterizations.

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[1] In other words, there is no systematic procedure that, for every set of premises Γ and for every conclusion A, leads after finitely many steps to a "yes" if A is a consequence of Γ. Remark that the consequence relation defined by classical logic is undecidable, but that there is a positive test for it—see [16] for such matters.

[2] Only part of these results are written up, viz. in [9].

[3] Some flat adaptive logics were described as formula-preferential systems in [17]—see also [1]. It is not clear whether this may be done for all adaptive logics, but the approach was a useful challenge for the Ghent group and indirectly led to the research on the generalized proofs of metatheoretic results.

The dynamics of the proof theory provides from the absence of a positive test. For most consequence relations, the dynamics is double. The *external* dynamics is well known: as new premises become available, consequences derived from the earlier premise set may be withdrawn. In other words, the external dynamics results from the non-monotonic character of the consequence relation—the fact that, for some Γ, Δ and A, $\Gamma \vdash A$ but $\Gamma \cup \Delta \nvdash A$. The *internal* dynamics is very different from the external one. Even if the premise set is constant, certain formulas are considered as derived at some stage of the proof, but are considered as not derived at a later stage. For any consequence relation, insight in the premises is only gained by deriving consequences from them. In the absence of a positive test, this results in the internal dynamics.[4]

The structure of dynamic proofs differs in two main respects from that of usual proofs. The first concerns annotated proofs. Apart from (i) a line number, (ii) a formula, (iii) the line numbers of the formulas from which the formula is derived, and (iv) the rule by which the formula is derived (the latter two are the justification of the line), dynamic proofs also contain (v) a *condition*. Intuitively, these are formulas that are supposed to be false, or, to be more precise, formulas the truth of which is not required by the premises. The second main difference is that, apart from the deduction rules that allow one to add lines to the proof, there is a *marking definition*. The underlying idea is as follows. As the proof proceeds, more formulas are derived from the premises. In view of these formulas, some conditions may turn out not to hold. The lines at which such conditions occur are *marked*. Formulas derived on marked lines are taken not to be derived from the premises. In other words, they are considered as 'out'. One way to understand the procedure is as follows. As the proof proceeds, one's insight in the premises improves. More particularly, some of the conditions that were introduced earlier may turn out not to obtain.

As we have seen, the marking definition determines the dynamics of the proof. For any stage of the proof, the definition settles which lines are marked and which lines are unmarked. This leads to a precise definition of *derivability at a stage*. Notwithstanding the precise character of this notion, we also want a more stable form of derivability, which is called *final derivability*. The latter does not depend on the stage of the proof. Nor does it depend on the way in which a specific proof from a set of premises proceeds. It is an abstract and stable relation between a set of premises and a conclusion. A different way for putting this is that final derivability refers to a stage of the proof at which the marks have become stable. Final derivability should be sound and strongly complete with respect to the semantics. For any adaptive logic **AL**, A should be finally derivable from Γ ($\Gamma \vdash_{\mathbf{AL}} A$) if and only if A is a semantic consequence of Γ ($\Gamma \vDash_{\mathbf{AL}} A$).

Now we come to the problem this paper is about. Suppose that one constructs a dynamic proof from a set of premises. At any point in time, this proof will be finite. It will reveal what is derivable from the premises at that stage of the proof. But obviously we are interested in final derivability. Whence the question: what does a proof at a stage reveal about final derivability?

[4] The Weak consequence relation from [19] and [20]—see [14] and [15] for an extensive study of such consequence relations—is monotonic. Nevertheless, its proof theory necessarily displays an internal dynamics because there is no positive test for it—see [6] and [10]. Some logics for which there is a positive test, may nevertheless be characterized in a nice way in terms of a dynamic proof theory—see [7].

First of all, as there is not even a positive test for the consequence relation, there is no algorithm for final derivability. So, one has at best some *criteria* that decide, for specific A and Γ, whether A is finally derivable from Γ.

What if no criterion enables one to conclude from the proof whether certain formulas are or are not finally derivable from the premise set? The answer or rather the answers to this question are presented in [3]. Roughly, the answers go as follows. First, there is a characteristic semantics for derivability at a stage. Next, it can be shown that, as the dynamic proof proceeds, the insight in the premises provided by the proof never decreases and may increase. In other words, derivability at a stage provides an estimate for final derivability, and, as the proof proceeds, this estimate may become better, and never becomes worse. In view of all this, derivability at a stage gives one exactly what one might expect, viz. a fallible but *sensible* estimate of final derivability. At any stage of the proof, one has to decide (obviously on the basis of pragmatic considerations) whether one will continue the proof or rely on present insights. This is fully in line with the contemporary view on rationality.[5]

Nevertheless, one should apply a criterion for final derivability whenever one can. This motivated the search for such criteria—see [3], [11] and [12]. Unfortunately, most of these criteria are complex and only transparent for people that are well acquainted with dynamic proofs. Recently, we started work in terms of goal directed proofs. The idea is not to formulate a criterion, but rather to specify a specific proof procedure that functions as a criterion. The proof procedure is applied to $\Gamma \vdash_{\mathbf{AL}} A$. Whenever the proof procedure stops, it is possible to conclude from the resulting proof whether or not $\Gamma \vdash_{\mathbf{AL}} A$. Preparatory work on the propositional fragment of **CL** (classical logic) is presented in [13] and some first results on the proof procedure for inconsistency-adaptive logics are presented in this paper.

The results concern the propositional level only. So, all references to logical systems concern the *propositional fragments* only. At this level the proof procedure forms an algorithm for final derivability (for finite premise sets). Indeed, if the proof procedure is applied to $A_1, \ldots, A_n \vdash_{\mathbf{AL}} B$, it always stops after finitely many steps. If, at the last stage of the proof, B is derived on an unmarked line, then B is finally derivable from $A_1, \ldots, A_n$; if B is not derived on an unmarked line, it is not finally derivable from $A_1, \ldots, A_n$. The main interest of the proof procedure, however, lies in the fact that it may be extended to the predicative level and there provides a criterion for final derivability if it stops.

In Section 2, I briefly present the inconsistency-adaptive logic **ACLuN1** and its dynamic proof theory. In Section 3, I present the goal-directed proof procedure for **CL**. This will make the matter easily understood by everyone. The proof procedure is upgraded to the adaptive logic **ACLuN1** in Section 4.

2 The Inconsistency-Adaptive Logic ACLuN1

In proof theoretic terms, the central difference between paraconsistent logics and inconsistency-adaptive logics can be described very easily. In a (monotonic)

[5] A different matter is whether the proof is carried out in an efficient way, that is: efficient with respect to obtaining a reliable (but fallible) estimate of final derivability. The goal directed proofs presented in subsequent sections of this paper offer means to obtain efficient proofs, but clearly more research on this problem is desirable.

paraconsistent logic some deduction *rules* of **CL** are invalid; in an inconsistency-adaptive logic, some *applications* of deduction rules of **CL** are invalid.

The original application context that led to inconsistency-adaptive logics is, in my view, still one of the most clarifying ones. Suppose that a theory T was intended to be consistent and hence was formulated with **CL** as its underlying logic. Suppose next that T turns out to be inconsistent. Of course, one will want to replace T by some consistent improvement T'. Typically, one does not just trow away T, restarting from scratch. One *reasons* from T in order to locate the inconsistency or inconsistencies and in order to locate constraints for the replacement T'. Needless to say, logic alone is not sufficient to find the justified replacement T'. If T is an empirical theory, one will need at least new factual data (observations, experiments, and so on). If T is a mathematical theory, one will need more conceptual analysis. However, logic is able to *locate* the inconsistencies in T. What we need is an interpretation of T that is 'as consistently as possible'. Let me phrase this in intuitive terms. At points where T is inconsistent, some deduction rules of **CL** cannot apply—if they all do, one obtains a trivial interpretation of T, an interpretation according to which *every* sentence of the language is a theorem of T. But where T is consistent, all deduction rules of **CL** should apply.

Consider an extremely simple propositional example. Suppose that the axioms of T are the set $\{p, {\sim}p \vee r, q, {\sim}q \vee s, {\sim}p\}$. From these premises, r should not be derived by Disjunctive Syllogism. Indeed, ${\sim}p \vee r$ is just an obvious weakening of ${\sim}p$. If one were to derive r from the premises, then, by the same reasoning, one should derive ${\sim}r$ from p and ${\sim}p \vee {\sim}r$, which also is an obvious weakening of ${\sim}p$. However, if one interprets the premises as consistently as possible, one should derive s from them, viz. by Disjunctive Syllogism from q and ${\sim}q \vee s$. Indeed, while the premises require p to behave inconsistently (require $p \wedge {\sim}p$ to be true), they do not require q to behave inconsistently (they do not require $q \wedge {\sim}q$ to be true).

Let me phrase the matter differently. T turns out to be inconsistent but, as it was intended to be consistent, should be interpreted as consistently as possible. Given that T is inconsistent, one will move 'down' to a paraconsistent logic—a logic that allows for inconsistencies. If a formula turns out to be inconsistent on the paraconsistent reading of T, one cannot apply certain rules of **CL** to it. Thus, even on the paraconsistent interpretation of T, $p \wedge {\sim}p$ is true. But now consider $p \wedge ({\sim}p \vee r)$. Given the meaning of conjunction and disjunction, it is equivalent to $(p \wedge {\sim}p) \vee r$. According to **CL**, $p \wedge {\sim}p$ cannot be true, and hence r is true. However, the premises state that $p \wedge {\sim}p$ is true. So, if one wants to reason sensibly *from* these premises, one cannot rely on the **CL**-presupposition that $p \wedge {\sim}p$ is bound to be false. However, where the paraconsistent reading of T does not require some formula to behave inconsistently, one still should apply all **CL**-rules because the theory was meant as consistent. Thus the premises affirm $q \wedge ({\sim}q \vee s)$, which is equivalent to $(q \wedge {\sim}q) \vee s$. As the premises do not require $q \wedge {\sim}q$ to be true, it should be taken to be false and one should conclude to s.

Inconsistency-adaptive logics provide a precise and coherent formulation of the intuitions behind the two preceding paragraphs.

An adaptive logic is characterized by the following triple:[6]

[6]I restrict the discussion to *flat* adaptive logics. Apart from these, there are prioritized

(i) A *lower limit logic*: a monotonic logic.
(ii) A *set of abnormalities*: a set of formulas characterized by a logical form.
(iii) An *adaptive strategy*: this specifies what it means to interpret the premises "as normally as possible".

Extending the lower limit logic with the requirement that no abnormality is logically possible results in a monotonic logic, which is called the *upper limit logic*.

Let us at once look at a specific inconsistency-adaptive logic, viz. **ACLuN1**. In this paper, I shall only consider the propositional part.

The *lower limit logic* of **ACLuN1** is **CLuN**. This monotonic paraconsistent logic is just like **CL**, except in that it allows for gluts with respect to negation—whence the name **CLuN**. Axiomatically, **CLuN** is obtained by extending full positive propositional logic with the axiom schema $A \vee \sim A$—see [4] for a study of the full logics **CLuN** and **ACLuN1**, including the semantics. **CLuN** isolates inconsistencies. Indeed, Double Negation, de Morgan rules, and all similar negation reducing rules are *not* validated by **CLuN**. As a result, complex contradictions do not reduce to truth functions of simpler contradictions.[7] There are several versions of **CLuN**. Here I shall suppose that the language contains $\perp$, and that it is characterized by the axiom schema $\perp \supset A$.

The set of abnormalities, Ω, comprises all formulas of the form $A \wedge \sim A$. Extending **CLuN** with the axiom schema $(A \wedge \sim A) \supset B$ results in the upper limit logic, which is **CL**. Below I shall often need to refer to *disjunctions of abnormalities*, which I shall call *Dab*-formulas. From now on an expression of the form $Dab(\Delta)$ will refer to a disjunction of abnormalities; in other words, Δ is a finite subset of Ω and $Dab(\Delta)$ is the disjunction of the members of Δ.

It can be shown that $\Gamma \vdash_{\mathbf{CL}} \perp$ iff there is a finite $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$. So, each of these expressions may be taken to define that Γ is inconsistent. Suppose now that $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$, but that no member of Δ is **CLuN**-derivable from Γ. This means that the premises require some member of Δ to be true, but do not specify which member is true. Precisely this situation requires one to introduce an adaptive *strategy*. One wants to interpret the premises "as normally as possible" (which for the present Ω means "as consistently as possible"), but this phrase is ambiguous. As indicated in (iii), an adaptive strategy disambiguates the phrase.

The oldest known strategy, and the one that is simplest from a proof theoretic point of view, is the *Reliability strategy* from [2].[8] I shall not consider any other strategies in this paper. Let $Dab(\Delta)$ be a *minimal Dab-consequence of* Γ iff $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta')$. Let $U(\Gamma) = \{A \mid A \in \Delta$ for some minimal *Dab*-consequence $Dab(\Delta)$ of $\Gamma\}$ be the set of formulas that are unreliable with respect to Γ. Below, I shall define $\Gamma \vdash_{\mathbf{ACLuN1}} A$, which will be read as "A is finally **ACLuN1**-derivable from Γ". The following Theorem is provable. In plain words it says that A is **ACLuN1**-derivable from Γ iff there is a Δ such that $A \vee Dab(\Delta)$ is **CLuN**-derivable from Γ an no member of Δ is unreliable with respect to Γ.

adaptive logics, which are defined as specific combinations of flat adaptive logics—see [8].

[7] For example, $(p \wedge q) \wedge \sim(p \wedge q) \nvdash_{\mathbf{CLuN}} (p \wedge \sim p) \vee (q \wedge \sim q)$ and $\sim p \wedge \sim\sim p \nvdash_{\mathbf{CLuN}} p \wedge \sim p$. Of course, one still has $(p \wedge \sim p) \wedge \sim(p \wedge \sim p) \vdash_{\mathbf{CLuN}} p \wedge \sim p$.

[8] This is the oldest paper on the matter, but it appeared in a book that took ten years to come out.

Theorem 1 *$\Gamma \vdash_{\mathbf{ACLuN1}} A$ iff there is a $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{CLuN}} A \vee Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

I now move on to the dynamic proof theory. This is identical for all flat adaptive logics, except of course that the rules RU and RC should refer to the right lower limit logic. Let Γ be the set of premises as before. The deduction rules are as follows (in generic format):[9]

PREM If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) A, (iii) $-$, (iv) PREM, and (v) $\emptyset$.

RU If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$ and each of $A_1, \ldots, A_n$ occur in the proof on lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$, (iv) RU, and (v) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee Dab(\Theta)$ and each of $A_1, \ldots, A_n$ occur in the proof on lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) B, (iii) $i_1, \ldots, i_n$, (iv) RC, and (v) $\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta$.

Where

$$A \quad \Delta$$

abbreviates that A occurs in the proof on the condition Δ, the rules may be phrased more transparently as follows:

PREM If $A \in \Gamma$:

$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \vee Dab(\Theta)$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

Apart from the deduction rules, a dynamic proof theory requires a marking definition that depends on the strategy. First, we need to define the set $U_s(\Gamma)$ of formulas that are unreliable at a stage s of a proof. Let $Dab(\Delta)$ be a *minimal Dab-formula* at stage s of the proof iff, at that stage, $Dab(\Delta)$ has been derived on the condition $\emptyset$ *and* there is no $\Delta' \subset \Delta$ for which $Dab(\Delta')$ has been derived on the condition $\emptyset$. Let $U_s(\Gamma) = \{A \mid A \in \Delta$ for some minimal *Dab*-formula $Dab(\Delta)$ at stage s of the proof$\}$. The marking definition for Reliability reads as follows:

Definition 1 *Line i is marked at stage s iff, where Δ is the condition of line i, $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

[9]The only rule that introduces non-empty conditions is RC. In other words, before RC is applied in a proof, the condition of every line will be $\emptyset$.

Lines that are unmarked at one stage may be marked at the next, and vice versa.

To complete the picture, I list the definitions concerning final derivability. These definitions are the same for all adaptive logics.

Definition 2 *A is* finally derived *from Γ on line i of a proof at stage s iff A is derived on line i, line i is not marked at stage s, and any extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

Definition 3 $\Gamma \vdash_{\mathbf{AL}} A$ *(A is* finally **AL**-derivable *from Γ) iff A is finally derived on a line of a proof from Γ.*

Here is a very simple dynamic proof from $\Gamma = \{(p \wedge q) \wedge t,\ {\sim}p \vee r,\ {\sim}q \vee s,\ {\sim}p \vee {\sim}q,\ t \supset {\sim}p\}$.

1	$(p \wedge q) \wedge t$		PREM	$\emptyset$
2	${\sim}p \vee r$		PREM	$\emptyset$
3	${\sim}q \vee s$		PREM	$\emptyset$
4	${\sim}p \vee {\sim}q$		PREM	$\emptyset$
5	$t \supset {\sim}p$		PREM	$\emptyset$
6	r	1, 2	RC	$\{p \wedge {\sim}p\}$
7	s	1, 3	RC	$\{q \wedge {\sim}q\}$
8	$(p \wedge {\sim}p) \vee (q \wedge {\sim}q)$	1, 4	RU	$\emptyset$
9	$p \wedge {\sim}p$	1, 5	RU	$\emptyset$

Up to stage 7 of the proof, all lines are unmarked. At stage 8, lines 6 and 7 are marked because $U_8(\Gamma) = \{p \wedge {\sim}p, q \wedge {\sim}q\}$. At stage 9, only line 6 is marked because $U_9(\Gamma) = \{p \wedge {\sim}p\}$. It is easily seen that, if 1–5 are the only premises, then the marks will remain unchanged in all extensions of the proof. So, r is not a final consequence of Γ where as for example s is a final consequence of Γ.

Important remark I have supposed before that the language contains $\perp$. This means that classical negation can be defined within the language, viz. by $\neg A =_{df} A \supset \perp$. In other words, **CLuN** is an extension of **CL**. It contains **CL**, $\neg$ functioning as the **CL**-negation, and moreover contains the paraconsistent negation $\sim$. In the original application context mentioned in the second paragraph of this section, the premises belong to the $\perp$-free and $\neg$-free fragment of the language—of course, there are different application contexts as well. Even in the original application context, the presence of $\neg$ is useful in that it greatly simplifies metatheoretic proofs and technical matters in general, and does not in any way hamper the limitations imposed by the application context.[10] The presence of $\neg$ also greatly simplifies the goal directed proof procedure that will serve as a criterion for final derivability and will be presented in Section 4.

[10] Some five years ago, the Ghent logic group became convinced that it is harmless as well as useful, for all adaptive logics, to extend the language and the lower limit logic in such a way that all classical connectives belong to the lower limit logic. This holds even if these connectives do not occur in the premises or in the conclusions a user is interested in—see [5] for an example.

3 Goal Directed Proofs for Classical Logic

The 'defeasible' conditions that occur in dynamic proofs of adaptive logics suggest that one might try to obtain a kind of dynamic proofs with 'prospective' conditions. This led to a specific form of goal directed proofs, which later turned out to be useful for devising criteria for final derivability. However, let us start with goal directed proofs for **CL**.

The general idea is that one constructs a proof for $A_1, \ldots A_n \vdash_{\mathbf{CL}} B$. Let us consider an example: a goal directed proof for $p \supset (q \wedge s), \neg(q \vee r) \vdash_{\mathbf{CL}} \neg p$.[11] As a first step, one introduces the main goal:

1	$\neg p$		GOAL	$\{\neg p\}$

In other words, one writes down the truism that one would have $\neg p$ on the condition $\{\neg p\}$, that is: if one had $\neg p$. This first step is meant to remind one that one is looking for the formula that occurs in the condition, viz $\neg p$. In view of this formula, one introduces a premise from which it may be obtained, and next analyses the premise:

2	$p \supset (q \wedge s)$		PREM	$\emptyset$
3	$\neg p$	2	$\supset$E	$\{\neg(q \wedge s)\}$

Line 3 says that, in view of 2, one has $\neg p$ on the condition that one has $\neg(q \wedge s)$. As $\neg(q \wedge s)$ cannot be obtained by analysing a premise, one analyses $\neg(q \wedge s)$ and proceeds thus:

4	$\neg p$	3	C$\neg\wedge$E	$\{\neg q\}$
5	$\neg(q \vee r)$		PREM	$\emptyset$
6	$\neg q$	5	$\neg\vee$E	$\emptyset$
7	$\neg p$	4, 6	Trans	$\emptyset$

At line 7, the main goal was obtained on the empty condition, which means that the proof is completed.

Two remarks are useful before I list the rules. First, there are algorithms to transform goal directed proofs to other kinds of proofs, for example axiomatic proofs or Fitch-style proofs. Next, it is easily seen that a formula A is derivable on the condition Δ just in case $\bigwedge(\Delta) \supset A$ is **CL**-derivable from the premises, where "$\bigwedge(\Delta) \supset$" is the empty string if $\Delta = \emptyset$.

In order to keep the rules as readable as possible, I shall write A_Δ to indicate that A occurs in the proof (or may be added to the proof) on the condition Δ. The rules for constructing a goal directed proof for $\Gamma \vdash_{\mathbf{CL}} G$ are as follows. First, there are rules for introducing the main goal and the premises:

Goal Introduce $G_{\{G\}}$.

Prem If $A \in \Gamma$, then $A_\emptyset$ may be introduced.

Formula analysing rules (two formulas below the horizontal line indicate variants of the rule):

$\supset$E $$\frac{(A \supset B)_\Delta}{B_{\Delta \cup \{A\}} \quad \neg A_{\Delta \cup \{\neg B\}}}$$

[11] In order to simplify Section 4, I write the classical negation as $\neg$ even in the context of **CL**.

∨E $$\frac{(A \vee B)_{\Delta}}{A_{\Delta \cup \{\neg B\}} \quad B_{\Delta \cup \{\neg A\}}}$$

∧E $$\frac{(A \wedge B)_{\Delta}}{A_{\Delta} \quad B_{\Delta}}$$

≡E $$\frac{(A \equiv B)_{\Delta}}{(A \supset B)_{\Delta} \quad (B \supset A)_{\Delta}}$$

¬¬E $$\frac{\neg\neg A_{\Delta}}{A_{\Delta}}$$

¬⊃E $$\frac{\neg(A \supset B)_{\Delta}}{A_{\Delta} \quad \neg B_{\Delta}}$$

¬∨E $$\frac{\neg(A \vee B)_{\Delta}}{\neg A_{\Delta} \quad \neg B_{\Delta}}$$

¬∧E $$\frac{\neg(A \wedge B)_{\Delta}}{(\neg A \vee \neg B)_{\Delta}}$$

¬≡E $$\frac{\neg(A \equiv B)_{\Delta}}{(A \vee B)_{\Delta} \quad (\neg A \vee \neg B)_{\Delta}}$$

Condition analysing rules:

C⊃E $$\frac{A_{\Delta \cup \{B \supset C\}}}{A_{\Delta \cup \{\neg B\}} \quad A_{\Delta \cup \{C\}}}$$

C∨E $$\frac{A_{\Delta \cup \{B \vee C\}}}{A_{\Delta \cup \{B\}} \quad A_{\Delta \cup \{C\}}}$$

C∧E $$\frac{A_{\Delta \cup \{B \wedge C\}}}{A_{\Delta \cup \{B, C\}}}$$

C≡E $$\frac{A_{\Delta \cup \{B \equiv C\}}}{A_{\Delta \cup \{B, C\}} \quad A_{\Delta \cup \{\neg B, \neg C\}}}$$

C¬¬E $$\frac{A_{\Delta \cup \{\neg\neg B\}}}{A_{\Delta \cup \{B\}}}$$

C¬⊃E $$\frac{A_{\Delta \cup \{\neg(B \supset C)\}}}{A_{\Delta \cup \{B, \neg C\}}}$$

C¬∨E $$\frac{A_{\Delta \cup \{\neg(B \vee C)\}}}{A_{\Delta \cup \{\neg B, \neg C\}}}$$

C¬∧E $$\frac{A_{\Delta \cup \{\neg(B \wedge C)\}}}{A_{\Delta \cup \{\neg B\}} \quad A_{\Delta \cup \{\neg C\}}}$$

C¬≡E $$\frac{A_{\Delta \cup \{\neg(B \equiv C)\}}}{A_{\Delta \cup \{\neg B, C\}} \quad A_{\Delta \cup \{B, \neg C\}}}$$

We need three further rules to warrant that the system is complete:[12]

Trans $$\frac{\begin{array}{c} A_{\Delta \cup \{B\}} \\ B_{\Delta'} \end{array}}{A_{\Delta \cup \Delta'}}$$

[12] See [13] for the proof that G is derivable from $A_1, \ldots, A_n$ by the present rules iff $A_1, \ldots, A_n \vdash_{\mathbf{CL}} G$. That paper also deals with the case of an infinite premise set.

EM $\dfrac{\begin{array}{c}A_{\Delta\cup\{B\}}\\ A_{\Delta'\cup\{\neg B\}}\end{array}}{A_{\Delta\cup\Delta'}}$

EFQ If $A \in \Gamma$, G may be introduced on the condition $\{\neg A\}$.

The last rule is a somewhat unusual version of Ex Falso Quodlibet: one obtains the main goal if one is able to obtain the negation of a premise. A useful derivable rule is

EM0 $\dfrac{A_{\Delta\cup\{\neg A\}}}{A_{\Delta}}$

There is no room to discuss the proof heuristics in the present paper. However, I need to say at least a few things about it. First, if A is derived on the condition Δ at some line and $\Delta \subseteq \Theta$, then A should not be derived on the condition Θ at any subsequent line. This is essential for warranting that the proofs stop. Next, any line added to the proof should be added in order to derive a member of some condition in the proof. This is essential for the goal directed character of the proofs. Where one proceeds in order to derive a member A of some condition, one may introduce a premise B iff A is a positive part of B, and one may derive B from some C by a formula analysing rule iff A is a positive part of B. That A *is a positive part of* B is defined as follows:

(i) A is a positive part of each of the following: A, $A \wedge B$, $B \wedge A$, $A \vee B$, $B \vee A$, $B \supset A$, $A \equiv B$, and $B \equiv A$;
(ii) A is a negative part of $\neg A$, $A \supset B$, $A \equiv B$, and $B \equiv A$;
(iii) if A is a negative part of B, then $\neg A$ is a positive part of B;
(iv) if A is a positive part of B and B is a positive part of C, then A is a positive part of C;
(v) if A is a positive part of B and B is a negative part of C, then A is a negative part of C;
(vi) if A is a negative part of B and B is a positive part of C, then A is a negative part of C;
(vii) if A is a negative part of B and B is a negative part of C, then A is a positive part of C.

Finally, some lines in the proofs will be marked, indicating that one should not try to derive the formulas that occur in the condition of the line. A first reason to mark a line is that its condition is redundant.

M1 Where A is derived on the condition Δ at line i, line i is D-marked if $A \in \Delta$ or there is a line at which A is derived on the condition Δ' and $\Delta' \subset \Delta$.

A set of formulas Δ will be called *flatly inconsistent* iff $A, \neg A \in \Delta$ for some A. The second reason to mark lines is that they lead us to unwanted applications of ex falso quodlibet.[13]

[13] The underlying idea is that one first tries to obtain the goal in a strictly goal directed way, applying formula analysing rules and condition analysing rules and applying Trans provided it results in marking one of the lines to which it is applied. If this does not succeed, one attempts to obtain the goal by applying EM and Trans. If this does not succeed, one attempts to obtain the goal by EFQ. Further marking rules are extremely useful to speed up the strictly goal directed search.

M2 Where A is derived on the condition Δ at line i, and no application of EFQ occurs in the proof, line i is D-marked if Δ is flatly inconsistent or, for some $B \in \Delta$, $\neg B$ is derived on the empty condition.

4 Goal Directed Proofs for ACLuN1

In order to tackle our problem in terms of goal directed proofs, we need proofs whose lines contain two conditions:

$i \qquad A \qquad\qquad\qquad \ldots \quad \ldots \qquad \Delta \qquad \Theta$

The first condition, Δ, will be called the D-condition. It contains the formulas that one needs to derive in order to unconditionally obtain A. The second condition, Θ, will be called the I-condition. It contains abnormalities that should not belong to $U(\Gamma)$ in order for A to be derivable from the premises. Put differently, the occurrence of the above line i in a proof from Γ warrants that

$$\Gamma \cup \Delta \vdash_{\mathbf{CLuN}} A \vee Dab(\Theta)\,.$$

In order to show that

$$\Gamma \vdash_{\mathbf{ACLuN1}} G$$

one needs to show that

$$\Gamma \vdash_{\mathbf{CLuN}} G \vee Dab(\Theta)\,.$$

In other words, one needs a line like the displayed line i at which $A = G$, $\Delta = \emptyset$, and $\Theta \cap U(\Gamma) = \emptyset$.

I now extend the goal directed system described in Section 3 to a system for **ACLuN1**.

The rules First we need to upgrade the rules from Section 3 for the new proof format. This is simply done by adding a second condition: for GOAL and PREM, the second condition is $\emptyset$. For all other rules, if they have one (local) premise line, one adds Θ as the second condition on the premise line as well as on the conclusion line; if they have two premise lines, one adds as the second condition: Θ on the first premise line, Θ' on the second premise line, and $\Theta \cup \Theta'$ on the conclusion line. Here are two examples:

$$\supset\!\mathrm{E} \qquad \frac{(A \supset B)_{\Delta,\Theta}}{B_{\Delta\cup\{A\},\Theta} \quad \neg A_{\Delta\cup\{\neg B\},\Theta}}$$

$$\mathrm{Trans} \qquad \frac{\begin{array}{c}A_{\Delta\cup\{B\},\Theta}\\ B_{\Delta',\Theta'}\end{array}}{A_{\Delta\cup\Delta',\Theta\cup\Theta'}}$$

Next we need rules for the paraconsistent negation. Two of them have no effect on the I-condition; the other two have.

$$\sim\!\mathrm{E} \qquad \frac{\sim\! A_{\Delta,\Theta}}{\neg A_{\Delta,\Theta\cup\{A\wedge\sim A\}}}$$

$$\neg\!\sim\!\mathrm{E} \qquad \frac{\neg\!\sim\! A_{\Delta,\Theta}}{A_{\Delta,\Theta}}$$

C$\sim$E $\dfrac{A_{\Delta \cup \{\sim B\}, \Theta}}{A_{\Delta \cup \{\neg B\}, \Theta}}$

C$\neg\sim$E $\dfrac{A_{\Delta \cup \{\neg\sim B\}, \Theta}}{A_{\Delta \cup \{B\}, \Theta \cup \{B \wedge \sim B\}}}$

It is instructive to check what these rules precisely mean, and why they are correct. We shall soon see the use of two further rules (they have different names because they serve a different purpose—see below):

I-GOAL Where $\Delta \subseteq \Omega$, $Dab(\Delta)_{Dab(\Delta), \emptyset}$ may be introduced

X-GOAL Where $\Delta \subseteq \Omega$, $Dab(\Delta)_{Dab(\Delta), \emptyset}$ may be introduced

The following rule is permissible. Making it obligatory greatly simplifies the required proof procedure

IC $\dfrac{Dab(\Lambda \cup \Lambda')_{\Delta, \Theta \cup \Lambda'}}{Dab(\Lambda \cup \Lambda')_{\Delta, \Theta}}$

The procedure In order to decide whether $A_1, \ldots, A_n \vdash_{\mathbf{ACLuN1}} G$, one has to find out whether there is a goal directed proof in which G is derived on an empty D-condition and on an I-condition Δ for which $\Delta \cap U(\Gamma) = \emptyset$. The proof procedure for $\Gamma \vdash_{\mathbf{ACLuN1}} G$ will consist of three phases—I shall restrict attention to finite Γ. I shall first describe the three phases, and next present an informal proof for the correctness of the procedure. The procedure starts in phase 1, may move to phases 2 and 3, and returns to phase 1. During phases 2 and 3, some line may be I-marked (marked in view of its I-condition). A phase stops if no lines can be added in view of conditions introduced during that phase.

Phase 1. One attempts to derive $G_{\emptyset, \Theta}$ for some Θ. Three cases have to be considered.

(1.1) $G_{\emptyset, \emptyset}$ is derived. Then $\Gamma \vdash_{\mathbf{ACLuN1}} G$.

(1.2) $G_{\emptyset, \Theta}$ is derived, say at line i. The procedure moves to phase 2 and later returns to phase 1. Then
  - (1.2.1) if line i is not I-marked, $\Gamma \vdash_{\mathbf{ACLuN1}} G$.
  - (1.2.2) if line i is I-marked, one attempts to derive $G_{\emptyset, \Theta'}$ for some $\Theta' \not\supseteq \Theta$.

(1.3) The procedure stops and $G_{\emptyset, \Theta}$ is not derived on an unmarked line for any Θ. Then $\Gamma \nvdash_{\mathbf{ACLuN1}} G$.

Phase 2. $G_{\emptyset, \Theta}$ was derived in phase 1 for some Θ, say at line i. Phase 2 starts by applying the I-Goal rule in order to add $Dab(\Theta)_{\{Dab(\Theta)\}, \emptyset}$ to the proof. One attempts to obtain $Dab(\Theta)_{\emptyset, \Lambda}$ for some Λ ($\subseteq \Omega$). Three cases have to be considered.

(2.1) $Dab(\Theta)_{\emptyset, \emptyset}$ is derived: line i is I-marked and the procedure returns to phase 1.

(2.2) $Dab(\Theta)_{\emptyset, \Lambda}$ is derived for some Λ, say at line j. The procedure moves to phase 3 and later returns to phase 2. Then
  - (2.2.1) if line j is I-marked, one attempts to derive $Dab(\Theta)_{\emptyset, \Lambda'}$ for some $\Lambda' \not\supseteq \Lambda$.

(2.2.2) if line j is not I-marked, line i is I-marked and the procedure returns to phase 1.

(2.3) $Dab(\Theta)_{\emptyset,\Lambda}$ is not derived for any Λ when phase 2 stops. Line i is not I-marked and the procedure returns to phase 1.

Phase 3. $G_{\emptyset,\Theta}$ was derived in phase 1 for some Θ, say at line i, and $Dab(\Theta)_{\emptyset,\Lambda}$ was derived in phase 2 for some Λ, say at line j. Phase 3 starts by an application of the rule X-GOAL in order to add $Dab(\Lambda)_{\{Dab(\Lambda)\},\emptyset}$ to the proof. The aim is to derive $Dab(\Lambda)_{\emptyset,\emptyset}$. All lines added in phase 3 should have an empty I-condition. There are two cases to consider.

(3.1) $Dab(\Lambda)_{\emptyset,\emptyset}$ is derived: line j is I-marked and the procedure returns to phase 2.

(3.2) Phase 3 stops without $Dab(\Lambda)_{\emptyset,\emptyset}$ being derived: line j is not I-marked and the procedure returns to phase 2.

In [13] it is proved that the rules from Section 3 are sound and complete with respect to **CL**. That proof can easily be transformed to show that the rules from the present section are sound and complete with respect to **CLuN** in the following sense (for finite Γ):

(1) If $\Gamma \vdash_{\mathbf{CLuN}} G$, then $G_{\emptyset,\emptyset}$ is derived in the dynamic proof for $\Gamma \vdash_{\mathbf{CLuN}} G$. If $\Gamma \nvdash_{\mathbf{CLuN}} G$, then the dynamic proof for $\Gamma \vdash_{\mathbf{CLuN}} G$ stops.

(2) $A_{\emptyset,\Theta}$ is derivable in the dynamic proof for $\Gamma \vdash_{\mathbf{CLuN}} G$ iff $\Gamma \vdash_{\mathbf{CLuN}} A \vee Dab(\Theta)$.

Relying on this, I extend the result to **ACLuN1**. An essential point concerns phase 2. Suppose that $G_{\emptyset,\Theta}$ is derived at line i for some Θ, and that $Dab(\Theta)_{\emptyset,\Lambda}$ is derived for some Λ at line j. It follows that $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Theta \cup \Lambda)$. If $Dab(\Lambda)_{\emptyset,\emptyset}$ is derived in phase 3, then $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Lambda)$, and hence $Dab(\Theta \cup \Lambda)$ is not a minimal Dab-consequence of Γ. So, $\Theta \cap U(\Gamma) = \emptyset$ iff the following holds for all Λ: if $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Theta \cup \Lambda)$, then $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Lambda)$. This condition comes to: if $Dab(\Theta)_{\emptyset,\Lambda}$ is derivable, then $Dab(\Lambda)_{\emptyset,\emptyset}$ is derivable. Precisely this is checked in phase 2: the procedure returns to phase 1 with line i not I-marked iff it holds for all Λ that $Dab(\Lambda)_{\emptyset,\emptyset}$ is derivable whenever $Dab(\Theta)_{\emptyset,\Lambda}$ is derivable. So, if the procedure returns to phase 1 with line i not I-marked, then $\Theta \cap U(\Gamma) = \emptyset$ and hence G is finally derived at line i.

It is easily seen that line i is marked iff $\Theta \cap U(\Gamma) \neq \emptyset$. If, for some Λ, $Dab(\Theta)_{\emptyset,\Lambda}$ is derivable whereas $Dab(\Lambda)_{\emptyset,\emptyset}$ is not derivable, then $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Theta \cup \Lambda)$ whereas $\Gamma \nvdash_{\mathbf{ACLuN1}} Dab(\Lambda)$. But then $\Theta \cap U(\Gamma) \neq \emptyset$.[14]

Positive parts To complete the description of the procedure, I still need to adjust the definition of "positive part of". Two situations have to be distinguished. For steps in phase 3, the definition of "positive part of" from Section 3 is modified by adding that A is a positive part of $\neg{\sim}A$.[15] For phases 1 and 2, the definition of "positive part of" from Section 3 is modified by replacing clauses (ii) and (iii) as follows:

(ii) A is a negative part of $\neg A$, ${\sim}A$, $A \supset B$, $A \equiv B$, and $B \equiv A$;

(iii) if A is a negative part of B, then $\neg A$ and ${\sim}A$ are positive parts of B.

[14] Indeed, if $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Theta \cup \Lambda)$ and $\Gamma \nvdash_{\mathbf{ACLuN1}} Dab(\Lambda)$, there is a non-empty $\Theta' \subseteq \Theta$ and a (possibly empty) $\Lambda' \subseteq \Lambda$ such that $Dab(\Theta' \cup \Lambda')$ is a minimal Dab-consequence of Γ.

[15] In view of C$\sim$E, there is no need to stipulate that ${\sim}A$ is a positive part of $\neg A$.

Examples Goal directed proof for $p, {\sim}p \vee s, r \supset t, {\sim}p \vee q, {\sim}q \vdash_{\mathbf{ACLuN1}} s$.

1	s		GOAL	$\{s\}$	$\emptyset$
2	${\sim}p \vee s$		PREM	$\emptyset$	$\emptyset$
3	s	2	$\vee$E	$\{\neg{\sim}p\}$	$\emptyset$
4	s	3	C$\neg{\sim}$E	$\{p\}$	$\{p \wedge {\sim}p\}$
5	p		PREM	$\emptyset$	$\emptyset$
6	s	4, 5	TRANS	$\emptyset$	$\{p \wedge {\sim}p\}$
7	$p \wedge {\sim}p$		I-GOAL	$\{p \wedge {\sim}p\}$	$\emptyset$
8	$p \wedge {\sim}p$	7	C$\wedge$E	$\{p, {\sim}p\}$	$\emptyset$
9	$p \wedge {\sim}p$	8, 5	TRANS	$\{{\sim}p\}$	$\emptyset$
10	${\sim}p$	2	$\vee$E	$\{\neg s\}$	$\emptyset$
11	${\sim}p \vee q$		PREM	$\emptyset$	$\emptyset$
12	${\sim}p$	11	$\vee$E	$\{\neg q\}$	$\emptyset$
13	${\sim}q$		PREM	$\emptyset$	$\emptyset$
14	$\neg q$	13	${\sim}$E	$\emptyset$	$\{q \wedge {\sim}q\}$
15	${\sim}p$	12, 14	TRANS	$\emptyset$	$\{q \wedge {\sim}q\}$
16	$p \wedge {\sim}p$	9, 15	TRANS	$\emptyset$	$\{q \wedge {\sim}q\}$
17	$q \wedge {\sim}q$		X-GOAL	$\{q \wedge {\sim}q\}$	$\emptyset$
18	$q \wedge {\sim}q$	17	C$\wedge$E	$\{q, {\sim}q\}$	$\emptyset$
19	$q \wedge {\sim}q$	13, 18	TRANS	$\{q\}$	$\emptyset$
20	q	11	$\vee$E	$\{\neg{\sim}p\}$	$\emptyset$

At this point, phase 3 stops—$\neg{\sim}p$ is not **CLuN**-derivable from the premises. So, line 16 is not I-marked and the procedure returns to phase 2. As line 16 is not I-marked, line 6 is I-marked and the procedure returns to phase 1. The procedure then attempts to derive $s_{\emptyset,\Theta}$ in phase 1 for some $\Theta \not\supseteq \{p \wedge {\sim}p\}$, which fails—this includes applications of EFQ which fail because the premises are not $\neg$-inconsistent.

A computer programme that implements the procedure is available. It will be used for presenting examples during the lecture and will soon be on the internet.[16]

References

[1] Arnon Avron and Iddo Lev. Formula-preferential systems for paraconsistent non-monotonic reasoning (an extended abstract). To appear.

[2] Diderik Batens. Dynamic dialectical logics. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic. Essays on the Inconsistent*, pages 187–217. Philosophia Verlag, München, 1989.

[3] Diderik Batens. Blocks. The clue to dynamic aspects of logic. *Logique et Analyse*, 150–152:285–328, 1995. Appeared 1997.

[4] Diderik Batens. Inconsistency-adaptive logics. In Orłowska [18], pages 445–472.

[16] Unpublished papers by members of our research group are available from the internet address `http://logica.rug.ac.be/centrum/writings/`.

[5] Diderik Batens. Zero logic adding up to classical logic. *Logical Studies*, 2:15, 1999. (Electronic Journal: `http://www.logic.ru/LogStud/02/LS2.html`).

[6] Diderik Batens. Towards the unification of inconsistency handling mechanisms. *Logic and Logical Philosophy*, 8:5–31, 2000. Appeared 2002.

[7] Diderik Batens. A dynamic characterization of the pure logic of relevant implication. *Journal of Philosophical Logic*, 30:267–280, 2001.

[8] Diderik Batens. A general characterization of adaptive logics. *Logique et Analyse*, to appear.

[9] Diderik Batens. The need for adaptive logics in epistemology. *Logique et Analyse*, to appear.

[10] Diderik Batens. A strengthening of the Rescher–Manor consequence relations. To appear.

[11] Diderik Batens and Joke Meheus. A tableau method for inconsistency-adaptive logics. In Roy Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes in Artificial Intelligence Vol. 1847, pages 127–142. Springer, 2000.

[12] Diderik Batens and Joke Meheus. Shortcuts and dynamic marking in the tableau method for adaptive logics. *Studia Logica*, 69:221–248, 2001.

[13] Diderik Batens and Dagmar Provijn. Pushing the search paths in the proofs. A study in proof heuristics. *Logique et Analyse*, to appear.

[14] Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 1: The flat case. *Studia Logica*, 58:17–45, 1997.

[15] Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 2: The prioritized case. In Orłowska [18], pages 473–511.

[16] George S. Boolos and Richard J. Jeffrey. *Computability and Logic*. Cambridge University Press, 1989. (Third edition).

[17] Iddo Lev. Preferential systems for plausible non-classical reasoning. Master's thesis, Department of Computer Science, Tel-Aviv University, 2000.

[18] Ewa Orłowska, editor. *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*. Physica Verlag (Springer), Heidelberg, New York, 1999.

[19] Nicholas Rescher. *Hypothetical Reasoning*. North-Holland, Amsterdam, 1964.

[20] Nicholas Rescher and Ruth Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179–217, 1970.