On the interpretation of early Italian bartering problems

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In an interesting article in the *Journal of the British Society for the History of Mathematics*, John Mason expresses his surprise at the solution method adopted for bartering problems which involve cash. He cites a problem from Piero della Francesca in a translation by Judith Field (2004, 17):

> Two men want to barter. One has cloth, the other wool. The piece of cloth is worth 15 ducats. He puts it up for barter at 20 and 1/3 in ready money. A cento of wool is worth 7 ducats. What price for barter so that neither is cheated?

Mason originally expected the solution to be based on the proportion of the barter value to the original value with the barter value being "either 20 + 20/3 ducats or to 20 + 15/3 ducats, depending on which value the 1/3 is intended to act upon" However Piero's solution appears to be different (Mason 2007, 161):

> This computation intrigued me because I was astonished at the sequence of calculations: first reduce by the ready money paid (as a fraction of the barter price), and only then compare barter prices. It seemed to me that in a modern economy it would be more natural to carry out one of the calculations I considered, since the ready money to be paid is a cash value, and the bartering inflation refers to the noncash-traded amounts.

Thus Piero subtracts one third of 20 from 20, which leaves 13 1/3 and the same value from 15 which becomes 8 1/3. The proportion of these two values is hence the fair barter profit to be applied by both parties. Though Mason lists several other examples which follow the same solution method, he does not provide an explanation why this particular method is adopted in abbaco treatises and in later printed books. Given that this way of calculating was in use for over two centuries, not only in Italy but in several European countries, this particular bartering practice needs an explanation. We will demonstrate that his astonishment is based on a wrong interpretation and even more so, a wrong translation of the original problem. We will provide an explanation by placing these early bartering problems within the specific context of Medieval Italian merchant practices.

The original problem by Piero, in Gino Arrighi's transcription from the manuscript, is formulated as follows (f. 8r; Arrighi, 1970, 49)

> Sono doi che voglano baractare, l'uno à panno e l'altro à lana. La peçça del panno vale 15 ducati et mectela a baracto 20 et sì ne vole 1/3 de contanti; et il cento de la

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lana vale 7 ducati a contanti. Che la dèi mectere a baracto a ciò che nisuno non sia ingannato?

A literal translation of the medieval Italian would be as follows[2]:

> There are two [men] that want to barter. One has cloth, the other has wool. The piece of cloth is worth 15 ducats. And he puts this to barter [at] 20 and of this he wants 1/3 in cash. And a hundred of wool is worth 7 ducats in cash. What shall they put for barter so that not one of them is being cheated?

Formulated this way, there is little room for doubt. The one with the cloth wants 20 ducats per piece of cloth, *of which one third in cash*. Obviously then, a third of the value refers to the barter value of 20. The amount of cash per piece is thus 20/3. To know the barter value of the cloth without the cash one has to subtract the cash from it, being 13 1/3. That Mason wants to add one third of the value rather than subtracting it stems from the wrong translation of "et sì ne vole 1/3 de contanti".

Is this interpretation the correct one for all bartering problems of this type in abbaco treatises? Let us look for further clues. Mason provides pointers to several abbaco treatises in which bartering problem appear with a cash value. The earliest he discusses are problems 33, 86, and 87 of Paolo Dagomari's *Trattato d'aritmetica*, written in 1339. He describes problem 86 as a problem which "involves grain to be bartered at 15s but valued at 12, with one-third in ready money, in exchange for *orzo* (?) at 10s.". The word *orzo* should pose no problems as it is the modern Italian word for barley. In our translation:[3]

> There are two that want to barter together. The one has grain and the other has barley. And the one with the grain which is valued at 12 s. puts it in barter at 15 s. per bushel. And he wants from the one with the barley one third of the value in cash. And from what remains he will get barley. And a bushel of barley values 10 s. Asked is what they arrive at in this barter so that none is left cheated.

Here also, the meaning of the problem is different from the one paraphrased by Mason. It is not the person with the grain who puts in the cash, but the other one. Furthermore, the enunciation clearly specifies that the second person should deliver one third of the value in cash and the rest in barley and this conforms with our interpretation.

Was Paolo the first to deal with cash values in bartering problems? We checked all available transcriptions of abbaco treatises before Paolo's *Trattato*. The earliest one is probably the *Columbia algorismus* (Columbia, X 511, A l3) published by Vogel (1977).

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[2] My translation. For a discussion on the translation of abbaco treatises see my recent critical edition Heeffer (2008, 132).

[3] From Arrighi 1964, 75: "E' xono due che barattano insieme, l'uno àe grano e Il'altro àe orzo; e quello che à grano gli mette in baratto lo staio del grano 15 s., che vale 12 s., e vuole il terzo da quello dell'orzo di ciò che monta il suo grano di chontantj; e dell'avanzo se ne togle orzo. Ello staio dell'orzo vale 10 s., adornando quanto glele chonterà in questo baratto acciò che no' rrimangha inghannato".

Vogel himself dated the manuscript in the second half of the 14th century. However, a recent study of the coin list contained in the manuscript is dated between 1278 and 1284, which makes it the earliest extant treatise within the abbaco tradition (Travaini 2003, 88-92). Høyrup suspects it "likely to be a copy of a still earlier treatise" (Høyrup 2007, 31). It contains two barter problems (19 and 20) but none involves money. The anonymous *Livero del l'abbecho* is dated c.1289-1290 and has also two bartering problems without money (Arrighi 1989, 24, 28). The *Tractatus Algorismi* by Jacopo da Firenze is extant in an earliest version of 1307. It is the subject of a recent comprehensive study of the abbaco tradition by Jens Høyrup (2007). However this extensive treatise does not contain any bartering problems. The next available transcription is the *Liber habaci*, dated by van Egmond (1980) to 1310, and is the first to involve cash in a bartering transaction. The enunciation of the single bartering problem is more elaborate and functions as a prototype for later reformulations by Paolo and Piero:[4]

> There are two merchants who want to barter together. The one has wool and the other has cloth. The one with the wool tells the one with the cloth: "how much do you want for the *channa* of your cloth". And he says: "I want 8 lb. (and he knows well that it values not more than 6 lb.) and I want one quarter in cash and I want three quarters in wool. And the wool is valued at 20 lb. per hundred. Asked is what suits him to sell the wool per hundred so that he is not being cheated.

We find here all the elements of the later bartering problems. The problem clearly specifies that one party will deliver one quarter of the value in cash and three quarters in merchandise. The reference to a fair deal becomes a standard formulation in abbaco bartering problems. The solution recipe is the standard formula adopted in later treatises as discussed by Mason:[5]

> You shall do as such, one quarter is asked in cash, say as such: one quarter of eight is 2. The rest until eight is 6. From 2 until 6 is 4, therefore say as such: for every 4 lb. I get 6 lb., how much do I get for 20 lb.? Multiply 20 lb. against 6 lb. this makes 120 lb. Divide by 4 and 30 lb. results from it. This is how much it suits him to get per hundred for this wool.

We have now found an adequate interpretation for the subtraction of the cash value from the barter price, but why is this cash value also subtracted from the original value? This example from the *Liber habaci* already gives us an insight. Obviously, if one takes into account the barter value minus the cash value (here 6 lb.) something also has to be done with the original value of the merchandise (also 6 lb.). In this example these values are the same

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[4] From *Liber habaci*, Biblioteca Nazionale Centrale Firenze, Magl. Cl. XI, 88, transcription by Arrighi 1987, 147: "Sono due merchatanti che volglono barattare insieme, l'uno si à lana e l'altro si à pannj; dice quellj ch'à lla lana a quellj del pannj: che vuo' tu della channa del panno? E que' dice: io ne volglo lb. *viij* (e sa bene che non vale più di lb. *vj*) e volglo il quarto i' d. chontanti e tre quarti volglo in lana. El centinaio della lana vale lb. *xx*, adomando che lgli chonviene vendere il centinaio di questa lana acciò che non sia inghannato". A *channa* is a unit of length of about 2 m.

[5] Ibid: "De' chosì fare. E' domanda il quarto in danari, diray chosì: il quarto d'otto si è *ij*. Insino inn otto si à *vj*, da *ij* insino *vj* si *iiij* or diray chosì: ongnj *iiij* lb. mi mette lb. *vj,* che mi metterà lb. *xx*? Multipricha lb. *xx* via lb. *vj* farà lb. *Cxx*, dividi per *iiij* ne viene *xxx* lb.: chotanto gli chonviene mettere il centinaio di questa lana".

and there would be no profit ratio. However, adapting Mason's original reasoning to the new interpretation, one could still compare the total barter value (here 8 lb.) with the original value (6 lb.) and use this as a profit ratio. Why is it not done this way?

To answer that question we must look at Italian merchant practices at the beginning of the fourteenth century. One important breakthrough took place around that time: the emergence of double-entry bookkeeping. Records of stewards of authorities of Genoa in 1278 show no trace of this kind of bookkeeping while by 1340 a complete system of double-entry bookkeeping was established (Littelton 1927, 147). The earliest extant evidence of double-entry bookkeeping is the Farfoli ledger of 1299-1300 (Lee 1977). So the appearance of cash in bartering problems during the first decades of the fourteenth century coincides with the emergence of double-entry bookkeeping practices. Bartering was the dominant practice for traveling merchants during the Middle Ages. When medieval Europe moved to a monetary economy in the thirteenth century, sedentary merchants depended on bookkeeping to oversee multiple simultaneous transactions financed by bank loans. While standard bartering required no elaborate administration, double-entry bookkeeping supported more complex bartering operations involving cash and time. Calculating practices taught in *bottega d'abbaco*, supported the new economy in the same way as double-entry bookkeeping did. If we want to understand these problems we should therefore look at bookkeeping practices.

As is well known, the first printed text on double-entry bookkeeping is Pacioli's *Particularis de Computis Et Scripturis,* treatise XI of distinction nine of his *Summa de arithmetica et geometria* of 1494. Mason cites from the *Scripturis* but oddly not from the chapter 20 on bartering. Pacioli was well aware about the old bartering practices. His unpublished Perugia manuscript (Vat. Lat. 3129, 1478) contains a chapter on bartering with no less than 56 problems (fols. 61^r-83^v). Many of them involve cash. In the *Scripturis* he writes "Bartering is commonly of three kinds: Simple, Complex, and Time" (*Semplice, Composta, a Tempo*) and he explains how to account for bartering in the Journal and Ledger (Crivelli 1924, 46):

> After you have so described it, you can then reduce it to cash value, and as you wish to see the value in cash of such and such goods you will make out the entry in the Memorandum in whatever kind of money you desire; as it does not matter, providing that the book-keeper afterwards transfers the entry to the Journal and Ledger and reduces the amount to the standard money which you have adopted.

Our bartering problems involving cash are thus of the complex type and Pacioli provides an example of how to note down the value of bartered merchandise for a transaction which involves one third in cash. In Pacioli's terminology *Per* stands for debit and *A* for credit (Crivelli 1924, 47):

> *Per Bellidi* ginger in bulk or packages. *A* sugar of such and such a kind, so many packages, weighing so much. Received ginger from so-and-so in exchange for sugar carried out in this manner: viz., I valued the sugar at 24 ducats per hundred, on condition that I should receive 1/3 in cash, and the ginger to be valued at so

many ducats per hundred, for which ginger I should give so many loaves of sugar, weighing so much, which if paid for in cash are worth 20 ducats per hundred, and for said ginger he received sugar, so many loaves, each

valued at ....L[ire]   S[oldi]   G[rossi]   P[icioli]

Unfortunately Pacioli gives no numerical entries but explains that one should debit the cash (you receive) and credit the sugar (you barter). Furthermore "that which is more in the cash entry will nevertheless be missing per contra in the sugar, and this you are to correct". So let us reconstruct the bookkeeping transactions for the original example by Piero given the balance sheet equation: Assets = Liabilities + Owners Equity, and using Pacioli's [Debit // Credit] notation system:

Assets =   Liabilities   +   Owners Equity

1) Spend 1/3 cash of barter price

[0 // 20/3]   [20/3 // 0]

2) Deliver products for barter from stock

[15 // 0]

3) Receive barter goods at barter value

[0 // 20]

4) Book profit   [0 // 20 – 15]

One third of the barter value is paid in ready money and therefore credited from cash assets and debited from OE. The products we deliver have a booking value of 15 (multiplied by the number of items) while we receive goods valued at 20. The difference has to be booked as profit to maintain the balance and therefore we credit OE with the difference, being 20 – 15. We now can see that the profit or difference between the booked value and the barter value has to be the same as the difference between the two values used to determine the fair profit ratio. Thus the calculation of the barter value of the second party, x depends on:

$$\frac{20-\frac{1}{3}(20)}{15-\frac{1}{3}(20)} = \frac{x}{7}$$

Subtracting the two values $20-\frac{1}{3}(20)$ and $15-\frac{1}{3}(20)$ results in the profit 5.

Conclusion

That a seemingly basic problem from the abbaco tradition, which follows practices that were in use for over two centuries, gives rise to a feeling of astonishment for modern scholars on the history of mathematics is rather interesting. That we have to base ourselves on the socio-economical context of mercantilism to understand the solution of the problem is even more so. The case demonstrates that starting from modern conceptions and looking for corresponding ones in a historical context is often not the best way to study history. Ideas, methods and practices, even mathematical ones, are best understood in their historical socio-economical context.

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