Adaptive Strategies and Finite-Conditional Premise Sets

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Abstract

The standard format of adaptive logics makes use of two so-called strategies: reliability and minimal abnormality. While these are fairly well-known and frequently applied, the question of whether and when the two strategies are equi-epressive has so far remained unaddressed.

In this paper, we show that for a specific, yet significant class of premise sets, the consequence set of an adaptive logic that uses the minimal abnormality strategy can be expressed by another adaptive logic that uses the reliability strategy. The basic idea is that we close the set of abnormalities under conjunction. We show that the consequence sets obtained by both logics from a premise set Γ is identical if and only if Γ is *finite-conditional*. The latter property is specified in terms of a well-known characterization of minimal abnormality. In addition, we discuss other (stronger) properties of premise sets that have been considered in the literature, showing each of them to imply finite-conditionality.

Keywords: adaptive logics, strategies, computational complexity, preferential semantics

1 Aim of this Paper

The standard format of adaptive logics (henceforth ALs) has been proposed as a unifying framework to model defeasible reasoning forms [6, 7, 29, 8]. ALs have been developed for various purposes: to capture the dynamics inherent to belief revision [35], to model reasoning with inconsistent information [2, 4, 3], to obtain strong yet conflict-tolerant deontic logics [13, 28], as the underlying

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logic of naive set theory [38], etc.[1] The standard format unifies these different systems in terms of one basic underlying structure, thereby allowing us to study their generic properties. In addition, the parametric characterization of ALs in standard format (see the next paragraph) provides modularity, a simple recipe to develop new logics and variants, to fine-tune logics for specific applications, and to compare them.

Every AL in standard format is defined from three parameters: (i) a compact Tarski-logic[2] **L**, (ii) a set of abnormalities Ω, and (iii) a strategy. Throughout this paper, we shall assume that **L** is supraclassical. This assumption is very common in the literature on the standard format[3] and is crucial for the aim and results of this paper.

The AL strengthens **L** in a non-monotonic way, by assuming abnormalities to be false "as much as possible". The latter phrase becomes ambiguous as soon as the premise set **L**-entails a disjunction of abnormalities, but none of its disjuncts. This is where the strategy comes into play. Let us try to give a rough idea of its role here; exact definitions are given in Section 2.

According to the reliability strategy (henceforth simply *reliability*), an abnormality A_i is "out" whenever it is a disjunct of a minimally **L**-derivable disjunction of abnormalities $A_1 \vee \ldots \vee A_n$. That is, *reliability* leaves open the option that all the A_i in this disjunction might be true, whence the assumption of their falsehood becomes rejected altogether. The minimal abnormality strategy (henceforth simply *minimal abnormality*) takes a slightly more fine-grained approach. Roughly speaking, in cases like this it allows us to still rely on the assumption that at least some of the abnormalities A_i are false, though we do not know which ones.

In most discussions and applications of the standard format, the two strategies are considered separately, as two different ways to strengthen **L**, or even as two different "styles of reasoning" [21].[4] *Reliability* is syntactically more straightforward, has a more cautious consequence relation, and is computationally less arduous, whereas *minimal abnormality* is more natural from a semantic viewpoint, yields a stronger consequence relation, and is computationally more complex. Indeed, it is a well-established fact that the upper bound complexity of ALs that use the minimal abnormality strategy is in general higher than that of the corresponding ALs that use reliability (see Section 6.1), and that for a given AL, its minimal abnormality-variant is always at least as strong as its reliability-variant (see Section 2.4).

However, the question when (i.e., for what kind of logics and/or premise sets) the two strategies are equi-expressive has not been investigated. Are the

[1]This list is by far not exhaustive. For a longer list and numerous references, see e.g. [9, 29].

[2]A Tarski-logic is one whose consequence relation is reflexive, transitive and monotonic (see also Section 2).

[3]See e.g. [6, 7, 29, 9]. One exception is [23], where **L** is conceived in terms of a multi-consequence relation, and where no classical connectives are assumed to be available in the object language.

[4]In his [27], Straßer links the difference between both strategies to the debate on whether or not so-called floating conclusions [15] should be validated by non-monotonic logics; he argues that reliability invalidates them, whereas minimal abnormality allows for them.

strategies at all that different when it comes to the respective consequence relations they yield? Or are there (interesting, significant) cases in which one may obtain the same consequence set using another strategy?

The aim of this paper is to spell out exactly under which conditions a consequence set obtained by minimal abnormality can also be obtained by reliability. That is, let **AL1** be defined by the triple (i) **L**, (ii) Ω and (iii) *minimal abnormality.* Where $\Omega^{\wedge}$ denotes the closure of Ω under conjunction, let **AL2** be defined from the triple (i) **L**, (ii) $\Omega^{\wedge}$ and (iii) *reliability.* We show that the set of **AL1**-consequences of Γ equals the set of **AL2**-consequences of Γ if and only if Γ is *finite-conditional.* The latter property is derived from a well-known alternative characterization of the minimal abnormality strategy in a straightforward way. It is moreover a property which, as we show, holds for an interesting range of applications of ALs (see especially Section 5.3).

This result has some interesting corollaries. For instance, it yields a significant reduction of the upper bound complexity of the **AL1**-consequence set for this specific class of premise sets, and it shows that for the premise sets within the aforementioned class, one may readily use the proof theory of **AL2** to illuminate inferences that are validated by **AL1**.

The outline of this paper is as follows. We present the standard format of ALs in Section 2, with special attention to the differences between the two strategies.[5] Next, we explain how it is possible that **AL1** and **AL2** often yield the same consequence set (Section 3). In Section 4, we introduce and discuss the notion of finite-conditionality, and show that for all finite-conditional premise sets Γ, $Cn_{\mathbf{AL1}}(\Gamma) = Cn_{\mathbf{AL2}}(\Gamma)$. Our proofs rely essentially on certain well-known metatheoretic properties of ALs, some basic observations concerning ALs based on sets of abnormalities of the type $\Omega^{\wedge}$ (spelled out in Section 3), and classical logic properties. In Section 5 we list a number of weaker criteria on premise sets that imply finite-conditionality and demonstrate that in many applications these criteria are met. In Section 6 we present some corollaries concerning complexity and the so-called *Distribution* property (see Section 2 for its definition). Section 7 concludes the paper and poses some related questions for future research.

2 The Adaptive Logics Framework

In this paper, we will restrict ourselves to the semantics of ALs in order to provide a concise presentation. This is a typical selection semantics in the vein of [26, 16, 17]: from the set of **L**-models of Γ, a subset is selected. In the case of ALs, the selection depends on the set of abnormalities and the strategy being used. A proof theory in terms of conditional, defeasible derivations and a corresponding syntatic consequence relation can be found e.g. in [6].

[5]Our presentation is in line with that of [6], though we use a slightly different notation and we do not presuppose any restriction on the set of abnormalities – we return to this point in Section 2.1.

2.1 Preliminaries

Where X, Y are sets, we write $X \subseteq_f Y$ ($X \subset_f Y$) to denote that X is a finite (proper) subset of Y. Let $\wp(X)$ be the power set of X, and $\wp_f(X) = \{Y \mid Y \subseteq_f X\}$. Where $\prec$ is a binary relation on the set X, let $\mathsf{min}_{\prec}(X) =_{\mathsf{df}} \{x \in X \mid \text{for no } y \in X, y \prec x\}$.

Let $\mathcal{W}$ be the set of well-formed formulas of **L** a given logic **L**. It is assumed in this paper that $\mathcal{W}$ is closed under the connectives $\neg$ and $\vee$; $\wedge$ is defined from these in the standard way. We use $A, B, \ldots$ as metavariables for members of $\mathcal{W}$; $\Gamma, \Delta, \ldots$ as metavariables for subsets of $\mathcal{W}$; and $\mathbb{A}, \mathbb{B}, \ldots$ as metavariables for subsets of $\wp(\mathcal{W})$.

The set of all **L**-models (henceforth simply *models*) is denoted by $\mathcal{M}$; members of $\mathcal{M}$ are denoted by $M, M', \ldots$. Let $\models\ \subseteq \mathcal{M} \times \mathcal{W}$ denote the satisfiability relation for **L**. We assume that, for all $M \in \mathcal{M}$ and all $A, B \in \mathcal{W}$, (C1) $M \models \neg A$ iff $M \not\models A$ and (C2) $M \models A \vee B$ iff $M \models A$ or $M \models B$. We moreover assume that (C3) if every $\Gamma' \subseteq_f \Gamma$ has models, then also Γ has models.

Let $\mathcal{M}(\Gamma) = \{M \in \mathcal{M} \mid M \models A \text{ for all } A \in \Gamma\}$. The semantic consequence relation $\Vdash\ \subseteq \wp(\mathcal{W}) \times \mathcal{W}$ of **L** is defined from $\mathcal{M}$ and $\models$ as follows: $\Gamma \Vdash A$ iff $M \models A$ for all $M \in \mathcal{M}(\Gamma)$. Finally, $Cn(\Gamma) =_{\mathsf{df}} \{A \mid \Gamma \Vdash A\}$.

In view of the construction of $\Vdash$, **L** is is a Tarski-logic. In other words, the consequence relation Cn of **L** has the following three basic properties: monotonicity ($Cn(\Gamma) \subseteq Cn(\Gamma \cup \Gamma')$), transitivity (where $\Gamma' \subseteq Cn(\Gamma)$, $Cn(\Gamma \cup \Gamma') \subseteq Cn(\Gamma)$), and reflexivity ($\Gamma \subseteq Cn(\Gamma)$). By (C1) and (C2) above, the connectives $\neg$ and $\vee$ behave classically in **L**. Finally, by (C1) and (C3) we can derive that **L** is compact ($\Gamma \Vdash A$ iff there is a $\Gamma' \subseteq_f \Gamma$ such that $\Gamma' \Vdash A$).

The following notation will also be useful: where $M \in \mathcal{M}$ and $\Delta \subseteq \mathcal{W}$, $\Delta(M) = \{A \in \Delta \mid M \models A\}$. We will use this notation i.a. to represent what is usually called the *abnormal part* $Ab(M)$ of a model, given a fixed set of abnormalities Ω. In our notation, we have $Ab(M) = \Omega(M)$.

Let $\Delta^{\wedge}$ denote the closure of Δ under conjunction, i.e., the smallest set $\Theta \supseteq \Delta$ which has the property: if $A, B \in \Delta^{\wedge}$, then $A \wedge B \in \Delta^{\wedge}$. Let $^{\neg}\Delta =_{\mathsf{df}} \{\neg A \mid A \in \Delta\}$. Where Δ is finite and non-empty, let $\bigwedge \Delta$ ($\bigvee \Delta$) denote the conjunction (disjunction) of all the members of Δ. Where $\Delta = \{A\}$, let $\bigwedge \Delta = \bigvee \Delta = A$.

When giving concrete examples, we shall assume that **L** is propositional classical logic with atoms $p, q, \ldots$.

The set of abnormalities Ω is a subset of $\mathcal{W}$. Note that we do not assume that Ω is defined in terms of a logical form as is usually done in the context of the standard format for ALs. This restriction is motivated in terms of a specific conception of what it means for $\Vdash^{\mathsf{r}}_{\Omega}$ and $\Vdash^{\mathsf{m}}_{\Omega}$ to correspond to *formal* logics. In the current paper, we abandon this restriction, as this allows us to present our results in a more generic form, and to use simple examples. It is important to note that all the meta-theory that is used in this paper is not dependent on the former restriction. Also, for all examples given in this paper there are analogous examples that respect the restriction on Ω.

2.2 Minimal Abnormality

The basic idea behind the minimal abnormality strategy is to select from $\mathcal{M}(\Gamma)$ those models that verify a $\subset$-minimal set of abnormalities. A formula is a consequence iff it holds in all the selected models.

Definition 1 $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \{M \in \mathcal{M}(\Gamma) \mid$ *for no* $M' \in \mathcal{M}(\Gamma), \Omega(M') \subset \Omega(M)\}$.

Definition 2 $\Gamma \Vdash^{\mathsf{m}}_{\Omega} A$ *iff* $M \models A$ *for every* $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$.

The semantics of minimal abnormality can be equivalently rephrased as a preferential semantics in the vein of [26]. That is, where $M, M' \in \mathcal{M}$, let $M \prec_{\Omega} M'$ iff $\Omega(M) \subset \Omega(M')$. It can easily be checked that $\mathsf{min}_{\prec_{\Omega}}(\mathcal{M}(\Gamma)) = \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$.

The following is proven in [6] for the case where Ω is characterized in terms of a specific logical form, and generalized to arbitrary sets Ω in [1]:

Theorem 1 ([1], Theorem 4.3) *If* $M \in \mathcal{M}(\Gamma)$, *then there is an* $M' \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ *with* $\Omega(M') \subseteq \Omega(M)$.

Equivalently, every relation $\prec_{\Omega}$ is smooth w.r.t. every set $\mathcal{M}(\Gamma)$.[6] Hence, $\Vdash^{\mathsf{m}}_{\Omega}$ falls within the well-known class **P** of smooth preferential systems, as defined and studied in the classical paper [16] (note though that unlike [16] we allow for infinite premise sets). As a result, this consequence relation satisfies a number of basic meta-theoretic properties such as cumulativity, consistency preservation, left and right absorption, etc. – we refer to [18] for definitions and an elaborate discussion of these.

2.3 Reliability

The second strategy used in the standard format is rather different in style from the first. We will first introduce its official semantics, after which we consider its original characterization in terms of reliable conditions.[7]

Some notation: let $\mathbb{S}_{\Omega}(\Gamma) =_{\mathsf{df}} \{\Delta \subseteq_f \Omega \mid \Gamma \Vdash \bigvee \Delta\}$, $\mathbb{S}^{-}_{\Omega}(\Gamma) =_{\mathsf{df}} \mathsf{min}_{\subset}(\mathbb{S}_{\Omega}(\Gamma))$, and $\mathsf{U}_{\Omega}(\Gamma) =_{\mathsf{df}} \bigcup \mathbb{S}^{-}_{\Omega}(\Gamma)$.[8] The set $\mathsf{U}_{\Omega}(\Gamma) \subseteq \Omega$ is often called the set of *unreliable abnormalities* w.r.t. $\langle \Gamma, \Omega \rangle$. Whenever $B \in \Omega - \mathsf{U}_{\Omega}(\Gamma)$, it is a *reliable* abnormality w.r.t. $\langle \Gamma, \Omega \rangle$.

When we use *reliability*, we select exactly those models of Γ that verify no reliable abnormalities w.r.t. $\langle \Gamma, \Omega \rangle$:

Definition 3 $\mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma) = \{M \in \mathcal{M}(\Gamma) \mid \Omega(M) \subseteq \mathsf{U}_{\Omega}(\Gamma)\}$.

Definition 4 $\Gamma \Vdash^{\mathsf{r}}_{\Omega} A$ *iff* $M \models A$ *for every* $M \in \mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$.

[6] We say that $\prec\ \subseteq X \times X$ is *smooth w.r.t.* X iff for all $x \in X$, either x is $\prec$-minimal in X, or there is a $\prec$-minimal y in X such that $y \prec x$.

[7] To avoid additional notation, we spell out the second characterization in terms of the semantic consequence relation $\Vdash$. However, in its original formulation (see e.g. [6]) it is defined syntactically, in terms of a consequence relation $\vdash$ which is sound and complete w.r.t. the **L**-semantics.

[8] In the AL literature, $\mathbb{S}^{-}_{\Omega}(\Gamma)$ is usually denoted by $\Sigma(\Gamma)$.

A different characterization of $\mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$ can also be given, which builds on the semantics of minimal abnormality. Although it follows immediately from certain well-known properties of ALs that were proven in [6], we make it explicit here for the ease of reference.

Theorem 2 *The following holds for all* Γ*:*

1. *If* $\mathcal{M}(\Gamma) \neq \emptyset$*, then* $\mathsf{U}_{\Omega}(\Gamma) = \bigcup_{M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)} \Omega(M)$*.*
2. $\mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma) = \{M \in \mathcal{M}(\Gamma) \mid \Omega(M) \subseteq \bigcup_{M' \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)} \Omega(M')\}$*.*

Proof. Ad 1. Follows immediately from Theorem 11.5 and Lemma 4 in [6].
Ad 2. Immediate in view of item 1 and Definition 3. ■

In the remainder, where $\mathsf{x} \in \{\mathsf{r}, \mathsf{m}\}$, let $Cn^{\mathsf{x}}_{\Omega}(\Gamma) = \{A \mid \Gamma \Vdash^{\mathsf{x}}_{\Omega} A\}$; we call this set the set of *adaptive consequences* of Γ (modulo the strategy x).

As promised, we now explain the alternative characterization of $\Vdash^{\mathsf{r}}_{\Omega}$. The idea behind that characterization proceeds in terms of conditional derivations. That is, when we reason non-monotonically from Γ, we derive certain formulas A from Γ *together with a set of assumptions*, each of which is "safe" in view of Γ. In the AL framework, those assumptions are of the form $\neg A$, where A is an abnormality. Hence, it remains to specify which abnormalities we can safely assume to be (classically) false, given a certain premise set. Now, according to *reliability*, these are exactly the abnormalities that are reliable w.r.t. $\langle \Gamma, \Omega \rangle$. In other words, the (classical) negations of reliable abnormalities correspond to the assumptions that are "safe" in view of Γ.

Thus, A follows from Γ, given the set of abnormalities Ω and the reliability strategy, iff $A \in Cn(\Gamma \cup {}^{\neg}(\Omega - \mathsf{U}_{\Omega}(\Gamma)))$. Equivalently, given compactness and supraclassicality of **L**, iff

(†) $\Gamma \Vdash A \vee \bigvee \Theta$ for a $\Theta \subseteq_f \Omega$ such that $\Theta \cap \mathsf{U}_{\Omega}(\Gamma) = \emptyset$

In case (†) holds, we call every such Θ a *safe condition for* $\langle \Gamma, \Omega, A \rangle$. In [6, Theorem 7] is has been shown that (†) holds iff $\Gamma \Vdash^{\mathsf{r}}_{\Omega} A$.

2.4 Comparing the Strategies

To appreciate the importance of the two adaptive strategies, let us briefly point at three differences between them. First, by Definition 1 and Theorem 2.2, $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) \subseteq \mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$ for all Γ. Hence we have:

Theorem 3 ([6], Theorem 11.1) $\Vdash^{\mathsf{r}}_{\Omega} \subseteq \Vdash^{\mathsf{m}}_{\Omega}$.

It can easily be shown that minimal abnormality is often stronger than reliability. We give one example here:

Example 1 *Let* $\Gamma = \{p \vee q, p \vee r, q \vee r\}$ *and* $\Omega = \{p, q\}$*. Then (i)* $\Gamma \Vdash^{\mathsf{m}}_{\Omega} r$*, whereas (ii)* $\Gamma \nVdash^{\mathsf{r}}_{\Omega} r$*. To see why (i) holds, note that there are two types of models* $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$*: those that verify* p *but falsify* q*, and those that verify* q

but falsify p. In both of them, r is true. Hence r follows when the strategy is minimal abnormality.

On the other hand, both p and q are unreliable abnormalities, in view of the (minimal) disjunction $p \vee q$ that follows from Γ. Hence, also models M that verify both p and q yet falsify r are in $\mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$.

Second, there is an important metatheoretic difference between the two strategies. That is, minimal abnormality validates the following principle [18, Section 2.2]:

$Cn^{\mathsf{m}}_{\Omega}(\Gamma) \cap Cn^{\mathsf{m}}_{\Omega}(\Gamma') \subseteq Cn^{\mathsf{m}}_{\Omega}(Cn(\Gamma) \cap Cn(\Gamma'))$ *(Distribution)*

As shown in [18, Section 3.4, Observation 3.4.6], every consequence relation that is defined in terms of a (supraclassical) preferential semantics satisfies Distribution. This feature is particularly interesting in that it entails a number of more familiar properties, including the following (see [18, Section 2.2, Observation 2.2.3]):

(i) If $\Gamma \cup \{A\} \Vdash^{\mathsf{m}}_{\Omega} B$, then $\Gamma \Vdash^{\mathsf{m}}_{\Omega} A \supset B$ *(Deduction Theorem)*
(ii) If $\Gamma \cup \{A\} \Vdash^{\mathsf{m}}_{\Omega} C$ and $\Gamma \cup \{B\} \Vdash^{\mathsf{m}}_{\Omega} C$, then $\Gamma \cup \{A \vee B\} \Vdash^{\mathsf{m}}_{\Omega} C$ *(Disjunction in the Antecedent)*
(iii) If $\Gamma \cup \{A\} \Vdash^{\mathsf{m}}_{\Omega} B$ and $\Gamma \cup \{\neg A\} \Vdash^{\mathsf{m}}_{\Omega} B$, then $\Gamma \Vdash^{\mathsf{m}}_{\Omega} B$ *(Reasoning By Cases)*

Each of these three properties fail for $\Vdash^{\mathsf{r}}_{\Omega}$. The failure of the Deduction Theorem for $\Vdash^{\mathsf{r}}_{\Omega}$ was already noted in [6, Theorem 13.3]. Let us give a simple example to illustrate this point.

Example 2 *Let $\Gamma = \{p \vee q\}$, $\Omega = \{p, q\}$, and $A = p$. Clearly, since $p \vee q$ is not a* minimal *disjunction of abnormalities that follows from $\Gamma \cup \{p\}$, q is reliable w.r.t. $\langle \Gamma \cup \{p\}, \Omega\rangle$. Hence $\Gamma \cup \{p\} \Vdash^{\mathsf{r}}_{\Omega} \neg q$. However, since both p and q are unreliable w.r.t. $\langle \Gamma, \Omega\rangle$, there are models $M \in \mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$ which verify both p and q. Hence $p \supset \neg q$ does not follow from Γ, if we use the reliability strategy.*

The case for (ii) and (iii) proceeds analogous. For (ii), let $\Omega = \{p, q\}$, $\Gamma = \emptyset$, $A = p$, $B = q$, and $C = \neg p \vee \neg q$. For (iii), let $\Omega = \{\neg p \wedge q, \neg p \wedge \neg q\}$, $\Gamma = \emptyset$, $A = q$ and $B = p$.

Third and last, reliability has a lower worst case complexity than minimal abnormality. We will return to this point in Section 6.1 and recall some results from the literature that are relevant in this context.

3 Conjunctions of Abnormalities

In this section, we show how it is possible to approximate the minimal abnormality strategy by means of the reliability strategy, by closing the set of abnormalities Ω under conjunction. Before we turn to some concrete examples, we first note:

Theorem 4 *Where $\Omega \subseteq \Omega' \subseteq \Omega^{\wedge}$, $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \mathcal{M}^{\mathsf{m}}_{\Omega'}(\Gamma)$ for all Γ.*

Proof. "$\subseteq$" Assume that $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) - \mathcal{M}^{\mathsf{m}}_{\Omega'}(\Gamma)$. So there is an $M' \in \mathcal{M}(\Gamma)$: $\Omega'(M') \subset \Omega'(M)$. Since $\Omega \subseteq \Omega'$, it follows immediately that (†) $\Omega(M') \subseteq \Omega(M)$.

Let now $A \in \Omega'(M) - \Omega'(M')$. Note that $A = B_1 \wedge \ldots \wedge B_n$ for $B_1, \ldots, B_n \in \Omega$. It follows that $M \models B_i$ for each $i \leq n$, whereas $M' \not\models B_j$ for a $j \leq n$. Consequently, this B_j is not in $\Omega(M')$, and hence $\Omega(M') \neq \Omega(M)$. By (†), $\Omega(M') \subset \Omega(M)$ — a contradiction to the fact that $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$.

"$\supseteq$" Assume that $M \in \mathcal{M}^{\mathsf{m}}_{\Omega'}(\Gamma) - \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$. So there is an $M' \in \mathcal{M}(\Gamma)$: $\Omega(M') \subset \Omega(M)$. Since $\Omega \subseteq \Omega'$, it follows immediately that (‡) $\Omega'(M') \neq \Omega'(M)$.

Suppose now that $A \in \Omega'(M')$. Hence $A = B_1 \wedge \ldots \wedge B_n$ for $B_1, \ldots, B_n \in \Omega$. Since each $B_i \in \Omega(M')$, also each $B_i \in \Omega(M)$. So $M \models B_i$ for all $i \leq n$. Hence, also $M \models A$. As a result, $\Omega'(M') \subseteq \Omega'(M)$. By (‡), $\Omega'(M') \subset \Omega'(M)$ — a contradiction to the fact that $M \in \mathcal{M}^{\mathsf{m}}_{\Omega'}(\Gamma)$. ■

This implies that, where each of $A, B, A \wedge B$ is in the set of abnormalities, it is safe to ignore $A \wedge B$, when trying to determine the set of minimally abnormal models of Γ. For the adaptive logician, it shows that adding conjunctions of abnormalities will not make any difference for the consequence relation of the adaptive logic of his choice, as long as he uses the minimal abnormality strategy.

For the reliability strategy, the picture is different. Let us start with the positive result:

Theorem 5 *Let $\Omega \subseteq \Omega' \subseteq \Omega^{\wedge}$. Then $\mathcal{M}^{\mathsf{r}}_{\Omega'}(\Gamma) \subseteq \mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$ for all Γ. Hence, $\Vdash^{\mathsf{r}}_{\Omega} \subseteq \Vdash^{\mathsf{r}}_{\Omega'}$.*

Proof. Suppose $M \in \mathcal{M}(\Gamma) - \mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$. It follows by Theorem 2.2 that $M \models A$ for some $A \in \Omega$ such that, for no $M' \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$, $M' \models A$. Note that $A \in \Omega'$. By Theorem 4, there is no $M' \in \mathcal{M}^{\mathsf{m}}_{\Omega'}(\Gamma)$, such that $M' \models A$. But then, by Theorem 2.2 also $M \notin \mathcal{M}^{\mathsf{r}}_{\Omega'}(\Gamma)$. ■

However, the converse of Theorem 5 fails. We illustrate this by means of two simple examples.

Example 3 *Let $\Gamma = \{p \vee q, p \vee r, q \vee r\}$ and $\Omega = \{p, q\}$. As we saw in Example 1, $\Gamma \Vdash^{\mathsf{m}}_{\Omega} r$, whereas $\Gamma \nVdash^{\mathsf{r}}_{\Omega} r$.*

Let now $\Omega' = \{p, q, p \wedge q\}$. In that case, $\Gamma \Vdash^{\mathsf{r}}_{\Omega'} r$. To see why, note that $p \wedge q$ does not follow classically from Γ. Nor does any minimal disjunction of abnormalities which contains $p \wedge q$. Hence, although p and q are both unreliable abnormalities, $p \wedge q$ is not, and hence it is false in every model $M \in \mathcal{M}^{\mathsf{r}}_{\Omega'}(\Gamma)$. As a consequence, $\Gamma \Vdash^{\mathsf{r}}_{\Omega'} r$.

Example 4 *Let $\Gamma = \{p \vee q, p \vee r, q \vee r\}$ and $\Omega = \{p, q, r\}$. Note that $\Gamma \nVdash^{\mathsf{r}}_{\Omega} \neg(p \wedge q \wedge r)$ — since each of the abnormalities p, q, r is unreliable, there is an $M \in \mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$ that verifies all of them.*

Let now $\Omega' = \Omega \cup \{p \wedge q, p \wedge r, q \wedge r\}$ and $\Omega'' = \Omega' \cup \{p \wedge q \wedge r\}$.

Note first that, by Theorem 4, $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \mathcal{M}^{\mathsf{m}}_{\Omega'}(\Gamma) = \mathcal{M}^{\mathsf{m}}_{\Omega''}(\Gamma)$. Also, every $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ verifies all members of one of the following three sets:

$$\Delta_1 = \{p, q, p \wedge q\}, \quad \Delta_2 = \{p, r, p \wedge r\}, \quad \Delta_3 = \{q, r, q \wedge r\}$$

and falsifies all other members of Ω''. By Theorem 2.1, each of the following holds:

(i) p, q, r *are unreliable abnormalities w.r.t.* $\langle \Gamma, \Omega \rangle$*;*
(ii) $p, q, r, p \wedge q, p \wedge r, q \wedge r$ *are unreliable abnormalities w.r.t.* $\langle \Gamma, \Omega' \rangle$*;*
(iii) $p, q, r, p \wedge q, p \wedge r, q \wedge r$ *are unreliable abnormalities w.r.t.* $\langle \Gamma, \Omega'' \rangle$*.*

In view of (i) and (ii), we can infer that there are models $M \in \mathcal{M}^{\mathsf{r}}_{\Omega'}(\Gamma) \subseteq \mathcal{M}^{\mathsf{r}}_{\Omega}(\Gamma)$ *such that* $M \models p \wedge q \wedge r$*. Hence,* $\Gamma \not\Vdash^{\mathsf{r}}_{\Omega} \neg(p \wedge q \wedge r)$ *and* $\Gamma \not\Vdash^{\mathsf{r}}_{\Omega'} \neg(p \wedge q \wedge r)$*.*

However, $p \wedge q \wedge r$ *is* not *unreliable w.r.t.* $\langle \Gamma, \Omega'' \rangle$*, since it is false in every model* $M \in \mathcal{M}^{\mathsf{m}}_{\Omega''}(\Gamma)$*. Hence,* $\Gamma \Vdash^{\mathsf{r}}_{\Omega''} \neg(p \wedge q \wedge r)$*.*

As the examples illustrate, when we add conjunctions of abnormalities, the resulting logic is stronger than the one we started with. In particular, the more conjunctions of abnormalities we add, the closer – so it seems – we get to $\Vdash^{\mathsf{m}}_{\Omega}$. Still, it remains to be specified when exactly the two sets of adaptive consequences of Γ coincide.

4 The Equivalence Result

In this section, we show that $Cn^{\mathsf{m}}_{\Omega}(\Gamma) = Cn^{\mathsf{r}}_{\Omega^{\wedge}}(\Gamma)$ whenever Γ is *finite-conditional*. This property relates to an alternative characterization of $\Vdash^{\mathsf{m}}_{\Omega}$, which we first spell out. The characterization is already implicit in the soundness and completeness proofs of ALs from [6]; here we make it explicit and elaborate on it.

4.1 Alternative Characterization of Minimal Abnormality

Safe Condition-Sets According to *reliability*, we may infer A from Γ iff A follows from Γ together with a (finite) number of assumptions (negations of abnormalities), which are "safe" in view of Γ (see page 6 and (†)).

We can do something similar for the minimal abnormality strategy. However, instead of relying on *one* set $\Theta \subseteq \neg\Omega$ of safe assumptions, we have to rely on a *set* of such Θ. If we can guarantee that at least one of those sets of assumptions is "safe", we can infer A.

We now rephrase this in terms of abnormalities and make it exact.

Definition 5 *Let* $\mathbb{D} \subseteq \wp_f(\Omega)$ *be a* condition-set *for* $\langle \Gamma, \Omega, A \rangle$ *iff* $\Gamma \Vdash A \vee \bigvee \Delta$ *for all* $\Delta \in \mathbb{D}$*.* $\mathbb{D}$ *is a* safe condition-set *for* $\langle \Gamma, \Omega, A \rangle$ *iff (i) it is a condition-set for* $\langle \Gamma, \Omega, A \rangle$ *and (ii) for every* $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$*, there is a* $\Delta \in \mathbb{D}$ *such that* $\Delta \cap \Omega(M) = \emptyset$*.*

Clause (i) of Definition 5 means that, for each set of assumptions on which we rely, A should follow from Γ together with this set of assumptions. Clause (ii) implies that in every minimally abnormal model, there should be at least one such set of assumptions such that each member of it holds in this model.[9]

The following theorem is an immediate consequence of Theorem 9 and Lemma 4 from [6].

Theorem 6 $\Gamma \Vdash^{\mathsf{m}}_{\Omega} A$ *iff there is a safe condition-set for* $\langle \Gamma, \Omega, A \rangle$.

Being Finite-Conditional In some cases, $\Gamma \Vdash^{\mathsf{m}}_{\Omega} A$, but all safe condition-sets for $\langle \Gamma, \Omega, A \rangle$ are infinite.

Example 5 *Let* $\Gamma = \{p_i \vee p_j \mid i, j \in \mathbb{N}, i \neq j\} \cup \{q \vee p_i \mid i \in \mathbb{N}\}$; $\Omega = \{p_i \mid i \in \mathbb{N}\}$. *Then every safe condition-set for* $\langle \Gamma, \Omega, q \rangle$ *is a superset of* $\mathbb{D} = \{\{p_i\} \mid i \in \mathbb{N}\}$. *The reason is that for each* $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ *there is an* $i \in \mathbb{N}$ *for which* $\Omega(M) = \Omega \setminus \{p_i\}$.

For cases like the one in Example 5, one may ask whether the interpretation in terms of safe assumptions is still sensible. That is, what does it mean exactly, that we can rely on at least one of infinitely many sets of assumptions? We will not answer this question positively or negatively, as doing so would take us far beyond the scope of the present paper. Suffice to say that in some interesting cases this problem does not arise.

Definition 6 *We call* Γ finite-conditional w.r.t. Ω *iff for all* $A \in Cn^{\mathsf{m}}_{\Omega}(\Gamma)$, *there is a finite, safe condition-set for* $\langle \Gamma, \Omega, A \rangle$.

It can be easily observed that, whenever $\{\Omega(M) \mid M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)\}$ is finite, then Γ is finite-conditional w.r.t. Ω. As a consequence, all the premise sets presented in examples 1–4 were finite-conditional w.r.t. to the resp. sets of abnormalities. This raises the question: given some **L** and Ω, for what sub-classes of premise sets do we get finite-conditionality? We return to this in Section 5, where we provide a number of increasingly stronger restrictions on Γ and Ω, each of which implies the finite-conditionality of Γ w.r.t. Ω. We will see in Section 5.3 that for many premise sets that are used in practical applications, such as finite premise sets, finite conditionality indeed holds.

Nevertheless, in view of Example 5, it is clear that premise sets that are not finite-conditional can be readily constructed whenever some basic conditions are fulfilled. We briefly discuss these here, before we focus again on the class of finite-conditional premise sets.

First, if Ω consists of infinitely many logically independent formulas, and there is a formula B such that B is itself contingent w.r.t. each abnormality, then we can construct a premise set which is not finite-conditional. Let us make this formally precise.

Definition 7 Ω *is* independent *iff for all* $\Delta, \Theta \subseteq \Omega$, $\Delta \Vdash \bigvee \Theta$ *implies* $\Theta \cap \Delta \neq \emptyset$.

[9] The term "(safe) condition-set" is new, but it is clearly foreshadowed in the soundness and completeness proofs for ALs. We discuss this more in the context of Remark 1 in Section 5.2.

Definition 8 *B is* contingent w.r.t. Ω *iff for each* $\Delta \subseteq \Omega$, $\mathcal{M}(\Delta \cup {}^{\neg}(\Omega - \Delta)) \neq \emptyset$ *implies* $\mathcal{M}(\Delta \cup {}^{\neg}(\Omega - \Delta) \cup \{B\}) \neq \emptyset \neq \mathcal{M}(\Delta \cup {}^{\neg}(\Omega - \Delta) \cup \{\neg B\})$.

Note that according to Definition 7, Ω is independent iff for each $\Delta \subset \Omega$ there is a model M such that $\Omega(M) = \Delta$. That B is contingent w.r.t. Ω can be equally expressed by: whenever there is an **L**-model M with the abnormal part Δ (i.e., $\Omega(M) = \Delta$) then there are M_B and $M_{\neg B}$ with the same abnormal part (i.e., $\Omega(M_B) = \Omega(M_{\neg B}) = \Delta$) such that $M_B \models B$ and $M_{\neg B} \models \neg B$ (see also Lemma 2 below).

Theorem 7 *If Ω is infinite and independent and there is an Ω-contingent B, then there is a premise set that is not finite-conditional.*

Instead of proving this theorem we show it on a more general level: Ω only needs to have an independent subset, and B should only be contingent w.r.t all subsets of this infinite subset.[10]

Definition 9 $\Omega' \subseteq \Omega$ *is* Ω-independent *iff for all* $\Delta \subseteq \Omega'$ *and for all* $\Theta \subseteq \Omega$, $\Delta \Vdash \bigvee \Theta$ *implies* $\Theta \cap \Delta \neq \emptyset$.

The following lemma illustrates this property:

Lemma 1 *Where Ω is infinite and $\Omega' \subseteq \Omega$: Ω' is Ω-independent iff for each $\Delta \subseteq \Omega'$ there is an* **L**-*model M for which $\Omega(M) = \Delta$.*

Proof. ($\Rightarrow$) Suppose there is a $\Delta \subseteq \Omega'$ such that there is no **L**-model M with $\Omega(M) = \Delta$. Hence, $\Delta \cup {}^{\neg}(\Omega - \Delta)$ is **L**-trivial. By the compactness of **L** there are $\Delta_f \subseteq_f \Delta$ and $\Theta_f \subseteq \Omega - \Delta$ such that $\Delta_f \cup {}^{\neg}\Theta_f$ is **L**-trivial. Case 1: $\Delta \subset \Omega$. Hence, by classical properties of $\vee$ and $\neg$, $\Delta_f \Vdash \bigvee \Theta_f$. Since $\Delta_f \cap \Theta_f = \emptyset$ this shows that Ω' is not Ω-independent. Case 2: $\Delta = \Omega$. Since Ω is infinite there is a $A \in \Omega - \Delta_f$. Since Δ_f is trivial, also $\Delta_f \Vdash A$. This again shows that Ω' is not Ω-independent.

($\Leftarrow$) Suppose Ω' is not Ω-independent. Hence, there are $\Delta \subseteq \Omega'$ and $\Theta \subseteq \Omega$ for which $\Delta \cap \Theta = \emptyset$ and $\Delta \Vdash \bigvee \Theta$. Thus, there is no **L**-model M for which $\Omega(M) = \Delta$. ■

Definition 10 *Where $\Omega' \subseteq \Omega$: B is* contingent w.r.t. $\langle \Omega', \Omega \rangle$ *iff for each* $\Delta \subseteq \Omega'$, $\mathcal{M}(\Delta \cup {}^{\neg}(\Omega - \Delta)) \neq \emptyset$ *implies* $\mathcal{M}(\Delta \cup {}^{\neg}(\Omega - \Delta) \cup \{B\}) \neq \emptyset \neq \mathcal{M}(\Delta \cup {}^{\neg}(\Omega - \Delta) \cup \{\neg B\})$.

Lemma 2 *Where Ω' is Ω-independent and Ω is infinite: B is contingent w.r.t. $\langle \Omega', \Omega \rangle$ iff for all $\Delta \subseteq \Omega'$ and all $\Theta \subseteq \Omega$, ($\Delta \Vdash \bigvee \Theta$ iff $\Delta \cup \{B\} \Vdash \bigvee \Theta$ iff $\Delta \cup \{\neg B\} \Vdash \bigvee \Theta$).*

[10]This generalization is motivated by various specific ALs to which Definition 7 does not apply. For instance, consider the case of inconsistency-ALs which use as their abnormalities all formulas of the form $A \wedge {\sim}A$ (where $\sim$ is a paraconsistent negation). Here we may have that $A \wedge {\sim}A \vdash B \wedge {\sim}B$ for a $B \neq A$, e.g. when $A = B \wedge {\sim}B$.

Proof. ($\Rightarrow$) Suppose B is B is contingent w.r.t. $\langle \Omega', \Omega \rangle$. Case 1: $\Delta \cup \{B\} \Vdash \bigvee \Theta$ where $\Delta \subseteq \Omega'$ and $\Theta \subseteq \Omega$. Hence, $\Delta \cup \neg\Theta \cup \{B\}$ is **L**-trivial. By the Ω-independence of Ω' and since Ω is infinite we know by Lemma 1 that $\Delta \cup \neg(\Omega - \Delta)$ is **L**-non-trivial. Hence, by the supposition also $\Delta \cup \neg(\Omega - \Delta) \cup \{B\}$ and $\Delta \cup \neg(\Omega - \Delta) \cup \{\neg B\}$ are **L**-non-trivial. By the triviality of $\Delta \cup \neg\Theta \cup \{B\}$ this means that $\Delta \cap \Theta \neq \emptyset$. Hence $\Delta \Vdash \bigvee \Theta$ and $\Delta \cup \{\neg B\} \Vdash \bigvee \Theta$.

Case 2: $\Delta \cup \{\neg B\} \Vdash \bigvee \Theta$. In an analogous way we can show that $\Delta \Vdash \bigvee \Theta$ and $\Delta \cup \{B\} \Vdash \bigvee \Theta$.

Case 3: $\Delta \Vdash \bigvee \Theta$. Then by the monotonicity of **L** also $\Delta \cup \{B\} \Vdash \bigvee \Theta$ and $\Delta \cup \{\neg B\} \Vdash \bigvee \Theta$.

($\Leftarrow$) Suppose B is not contingent w.r.t. $\langle \Omega', \Omega \rangle$. Hence, there is a $\Delta \subseteq \Omega'$ for which $\Delta \cup \neg(\Omega - \Delta)$ is **L**-non-trivial, but either $\Delta \cup \neg(\Omega - \Delta) \cup \{B\}$ or $\Delta \cup \neg(\Omega - \Delta) \cup \{\neg B\}$ is **L**-trivial. Hence, either $\Delta \cup \{\neg B\} \Vdash \bigvee \Theta$ or $\Delta \cup \{B\} \Vdash \bigvee \Theta$ while $\Delta \nVdash \bigvee \Theta$. ∎

Theorem 8 *If (a) Ω has an infinite Ω-independent subset $\Omega' = \{A_i \mid i \in \mathbb{N}\}$ and (b) B is contingent w.r.t. $\langle \Omega', \Omega \rangle$, then $\Gamma = \{A_i \vee A_j \mid 1 \leq i < j\} \cup \{B \vee A_i \mid 1 \leq i\}$ is not finite-conditional.*

Proof. Note first that for each $i \in \mathbb{N}$ there is an $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ such that $\Omega'(M) = \Omega' - \{A_i\}$. Assume this is not so. Then $(\Omega' - \{A_i\}) \cup \{B\} \Vdash A_i$. By (a), (b) and Lemma 2 also $\Omega' - \{A_i\} \Vdash A_i$, in contradiction to (a).

Assume there is a finite safe condition-set $\mathbb{D}$ for $\langle \Gamma, \Omega, B \rangle$. Let $\Delta \in \mathbb{D}$. Hence, $\Gamma \Vdash B \vee \bigvee \Delta$. By classical properties of $\vee$ and $\neg$, $\Gamma \cup \{\neg B\} \Vdash \bigvee \Delta$. Obviously, $\Gamma \cup \{\neg B\}$ and $\Omega' \cup \{\neg B\}$ are **L**-equivalent. Hence, $\Omega' \cup \{\neg B\} \Vdash \bigvee \Delta$. By (a), (b) and Lemma 2, $\Omega' \Vdash \bigvee \Delta$. By (a), there is an $i \in \mathbb{N}$ such that $A_i \in \Delta$. Hence, each $\Delta \in \mathbb{D}$ contains some A_i. Let n be the maximal i with this property. Since there is a model M with $\Omega'(M) = \Omega' - \{A_{n+1}\}$ and thus for all $\Delta \in \mathbb{D}$ we have $\Omega'(M) \cap \Delta \neq \emptyset$, this is a contradiction. ∎

4.2 The Equivalence Result

Being finite-conditional turns out to be a necessary and sufficient condition for whether $Cn^{\mathsf{m}}_{\Omega}(\Gamma) = Cn^{\mathsf{r}}_{\Omega^{\wedge}}(\Gamma)$. This will be shown in the current section. In view of Theorem 6 it suffices to show that,

Theorem 9 $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge}} A$ *iff there is a finite safe condition-set for $\langle \Gamma, \Omega, A \rangle$.*

We first show the left-to-right direction (Corollary 1, see below) and then the right-to-left direction (Theorem 11, see below).

Let $\Delta^{\wedge k} = \{A_1 \wedge \ldots \wedge A_j \mid A_1, \ldots, A_j \in \Delta \text{ and } j \leq k\}$. We have:

Theorem 10 *If $\{\Delta_1, \ldots, \Delta_k\}$ is a safe condition-set for $\langle \Gamma, \Omega, A \rangle$, then $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge k}} A$.*

Proof. Let $\mathbb{D} = \{\Delta_1, \ldots, \Delta_k\}$ be our safe condition-set for $\langle \Gamma, \Omega, A \rangle$ and let $\mathbb{E}$ be such that it contains all sets $\Theta = \{A_1, \ldots, A_k\}$ where $A_i \in \Delta_i$ for all $i \leq k$.

Our proof relies on two insights. First,

Fact 1 *$\Gamma \Vdash A \vee \bigvee_{\Theta \in \mathbb{E}} \bigwedge \Theta$.*

Note first that $\mathbb{E}$ is finite since each Δ_i is finite. Second, since $\Gamma \Vdash A \vee \bigvee \Delta_i$ for each $i \leq k$, also $\Gamma \Vdash A \vee \bigwedge_{i \leq k} \bigvee \Delta_i$. The fact then follows immediately by classical properties of $\vee$ and $\wedge$. Our second fact is,

Fact 2 *For each $\Theta \in \mathbb{E}$ and each $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$, $M \not\models \bigwedge \Theta$.*

This holds since by Definition 5, there is a $i \leq k$ such that $M \not\models B$ for all $B \in \Delta_i$ and each $\Theta \in \mathbb{E}$ contains at least one member of Δ_i.

The rest of the proof is now straight-forward: Since by Theorem 4, $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \mathcal{M}^{\mathsf{m}}_{\Omega^{\wedge k}}$, Fact 2 also applies to all $M \in \mathcal{M}^{\mathsf{m}}_{\Omega^{\wedge k}}(\Gamma)$. By Theorem 2.2, each $M \in \mathcal{M}^{\mathsf{r}}_{\Omega^{\wedge k}}(\Gamma)$ falsifies $\bigwedge \Theta$ for all $\Theta \in \mathbb{E}$. Hence, by Fact 1, $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge k}} A$. ■

For a concrete example of how the reduction works, consider again Example 4 from Section 3. We saw there that $\Gamma \Vdash^{\mathsf{m}}_{\Omega} \neg(p \wedge q \wedge r)$. Note that $\mathbb{D} = \{\{p\}, \{q\}, \{r\}\}$ is a safe condition-set for $\langle \Gamma, \Omega, \neg(p \wedge q \wedge r)\rangle$. From $\mathbb{D}$, we obtain a single safe condition w.r.t. $\langle \Gamma, \Omega^{\wedge 3}, \neg(p \wedge q \wedge r)\rangle$.

The converse of Theorem 10 fails. That is, one may need at least m conditions $\Delta \subseteq_f \Omega$ in order to obtain a safe condition-set for $\langle \Gamma, \Omega, A\rangle$, even though $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge k}} A$ for a $k < m$. This is illustrated by the following example.

Example 6 *$\Gamma = \{p \vee q, r \vee s\}$, $\Omega = \{p, q, r, s\}$. Then $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge 2}} (\neg p \vee \neg q) \wedge (\neg r \vee \neg s)$; the corresponding safe condition is $\{p \wedge q, r \wedge s\}$. However, every safe condition-set for $\langle \Gamma, \Omega, (\neg p \vee \neg q) \wedge (\neg r \vee \neg s)\rangle$ is a superset of $\mathbb{D} = \{\{p, r\}, \{q, r\}, \{p, s\}, \{q, s\}\}$.*

Let us now turn to the set of all consequences of a given Γ. First, recall that $\Vdash^{\mathsf{r}}_{\Omega^{\wedge k}} \subseteq \Vdash^{\mathsf{r}}_{\Omega^{\wedge}}$ (see Theorem 5). Hence by Theorem 10,

Corollary 1 *If there is a finite, safe condition-set for $\langle \Gamma, \Omega, A\rangle$, then $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge}} A$.*

We can also prove the converse of Corollary 1.

Theorem 11 *If $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge}} A$, then there is a finite, safe condition-set for $\langle \Gamma, \Omega, A\rangle$.*

Proof. While in the previous proof we started with a safe-condition set $\mathbb{D}$ and constructed from it the set $\mathbb{E}$ that gave us a reliable condition, we now proceed inversely. Since $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge}} A$, we know that there is a finite set $\mathbb{E}$ of finite subsets of Ω such that $\Gamma \Vdash A \vee \bigvee_{\Theta \in \mathbb{E}} \bigwedge \Theta$ such that for each $\Theta \in \mathbb{E}$, $\bigwedge \Theta \notin \mathsf{U}_{\Omega^{\wedge}}(\Gamma)$. Let $\mathbb{D}$ be the finite set of all sets $\Delta = \bigcup_{\Theta \in \mathbb{E}} \{A_\Theta\}$ where $A_\Theta \in \Theta$,

Fact 3 *For all $\Delta \in \mathbb{D}$, $\Gamma \Vdash A \vee \bigvee \Delta$.*

This follows by simple classical properties of $\vee$ and $\wedge$. Note also that,

Fact 4 *For each $M \in \mathcal{M}^{\mathsf{m}}_{\Omega^{\wedge}}(\Gamma)$, there is a $\Delta \in \mathbb{D}$ such that $M \not\models \bigvee \Delta$.*

The reason is that by Theorem 2.2, $M \not\models \bigwedge \Theta$ for all $\Theta \in \mathbb{E}$. Hence, for each $\Theta \in \mathbb{E}$, there is a $A_\Theta \in \Theta$ such that $M \not\models A_\Theta$. Since $\bigcup_{\Theta \in \mathbb{E}} \{A_\Theta\} \in \mathbb{D}$, the fact is proven.

In order to complete the main proof we only have to recall that by Theorem 4, $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \mathcal{M}^{\mathsf{m}}_{\Omega^\wedge}(\Gamma)$. Hence, by Fact 3 and Fact 4, $\mathbb{D}$ is a finite safe condition-set for $\langle \Gamma, \Omega, A \rangle$. ■

Putting Theorems 10 and 11 together, we obtain:

Corollary 2 *If $\Gamma \Vdash^{\mathsf{r}}_{\Omega^\wedge} A$, then there is a $k \in \mathbb{N}$ such that $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge k}} A$.*

Note that the k can get arbitrarily high, as the following example shows:

Example 7 *Let $k \in \mathbb{N}$, $\Omega = \{p_1, \ldots, p_k\}$ and $\Gamma = \{q \vee p_i \mid 1 \leq i \leq k\} \cup \left\{\bigvee_{1 \leq i \leq k} \bigwedge_{1 \leq j \leq k, j \neq i} p_j\right\}$. Note that $\Gamma \Vdash^{\mathsf{r}}_{\Omega^{\wedge k}} q$ while $\Gamma \nVdash^{\mathsf{r}}_{\Omega^{\wedge k-1}} q$. The reason is that for each $M \in \mathcal{M}^{\mathsf{r}}_{\Omega^{\wedge k}}(\Gamma)$ we have $\Omega(M) = \Omega - \{p_i\}$ for some $i \leq k$. For each such model $M \models q$ since $M \models q \vee p_i$ and $M \not\models p_i$. The situation is different for $\mathcal{M}^{\mathsf{r}}_{\Omega^{\wedge k-1}}(\Gamma)$ since there is a model M for which $\Omega(M) = \Omega$ and $M \not\models q$ as the reader can easily verify.*

Finally, Theorem 6 and Theorem 9 allow us to characterize the class of all premise sets Γ for which $Cn^{\mathsf{m}}_{\Omega}(\Gamma) = Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma)$:

Corollary 3 *Γ is finite-conditional w.r.t. Ω iff $Cn^{\mathsf{m}}_{\Omega}(\Gamma) = Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma)$.*

In order to see this suppose Γ is finite-conditional w.r.t. Ω and $A \in Cn^{\mathsf{m}}_{\Omega}(\Gamma)$. Hence, there is a finite safe condition-set for $\langle \Gamma, \Omega, A \rangle$. Hence, by Theorem 9, $A \in Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma)$. Similarly, if $A \in Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma)$ then there is a finite safe condition-set for $\langle \Gamma, \Omega, A \rangle$ and hence by Theorem 6, $A \in Cn^{\mathsf{m}}_{\Omega}(\Gamma)$. Now suppose Γ is not finite-conditional w.r.t. Ω. Then there is an $A \in Cn^{\mathsf{m}}_{\Omega}(\Gamma)$ such that there is a safe-condition for $\langle \Gamma, \Omega, A \rangle$ but there is no finite one. By Theorem 9, $A \notin Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma)$.

Note that Corollary 3 expresses an equivalence. Thus, we gain a necessary and sufficient condition that a premise set Γ has to fulfill —namely finite-conditionality w.r.t. Ω— so that the consequence sets $Cn^{\mathsf{m}}_{\Omega}(\Gamma)$ and $Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma)$ coincide.

In view of the fact that for many $\langle \mathbf{L}, \Omega \rangle$, finite conditionality holds only if we restrict the consequence relations to a sub-class of premise sets (see Theorem 8), it is important to notice that Corollary 3 does not establish a 'global' alternative characterization of $\Vdash^{\mathsf{m}}_{\Omega}$ in the sense that $\Vdash^{\mathsf{m}}_{\Omega} = \Vdash^{\mathsf{r}}_{\Omega^\wedge}$. This equality holds only restricted to finite-conditional premise sets.

5 Sufficient Conditions for Finite-Conditionality

In this section, we discuss some alternative criteria known from the literature that warrant the finite-conditionality of premise sets. An overview of the results in this section is provided in Figure 1. Unlike finite-conditionality, which is both sufficient and necessary for $Cn^{\mathsf{m}}_{\Omega}(\Gamma) = Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma)$ (see Corollary 3), the criteria

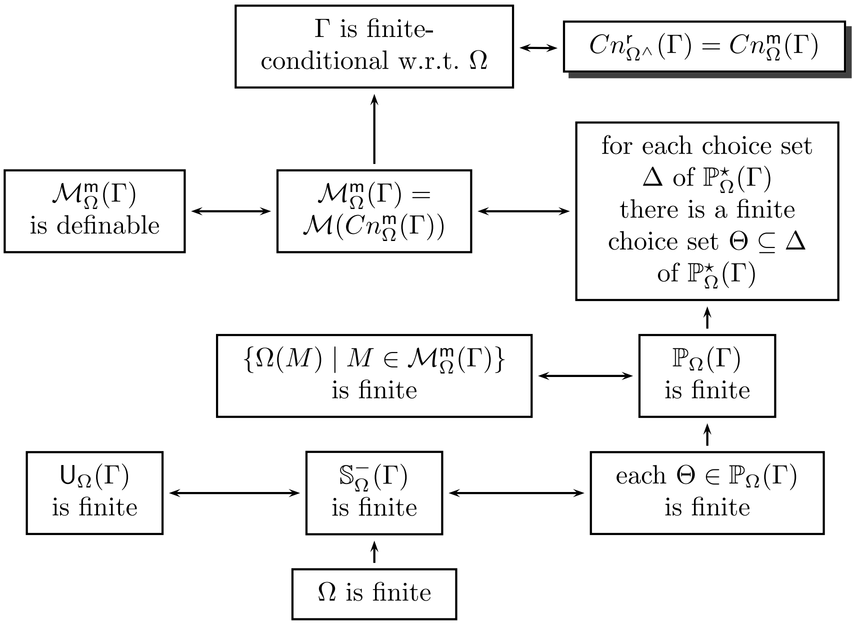

Figure 1: Overview of the restrictions

discussed in this section are (in general) merely sufficient. What makes them independently interesting is the fact that often they may be easier to verify (such as the fact that Ω is finite). Additionally, it is interesting to notice that the domain of finite-conditional premise sets is a proper super-class of the domain of premise sets that satisfy these properties.

5.1 Definability

There is a sufficient condition for our reduction, which relates to a well-known topic in the study of non-monotonic consequence relations. We call $\mathcal{M}' \subseteq \mathcal{M}$ *definable* iff there is a Θ such that $\mathcal{M}' = \mathcal{M}(\Theta)$. Definability can be alternatively expressed as follows:

Theorem 12 *$\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ is definable iff $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$.*

Proof.($\Rightarrow$) Suppose Θ is such that (1) $\mathcal{M}(\Theta) = \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$. This implies that $\Theta \subseteq Cn^{\mathsf{m}}_{\Omega}(\Gamma)$. Hence, (2) $\mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma)) \subseteq \mathcal{M}(\Theta)$. By Definition 1, (3) $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) \subseteq \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$. By (1), (2) and (3), $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$. ($\Leftarrow$) Trivial. ∎

Definability is a sufficient criterion for the adequacy of our reduction, as the following theorem shows:

Theorem 13 *If $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$, then Γ is finite-conditional w.r.t. Ω.*

Proof. Suppose the antecedent is true and assume $\Gamma \Vdash^{\mathsf{m}}_{\Omega} A$ such that all safe condition-sets for $\langle \Gamma, \Omega, A\rangle$ are infinite. Let $\mathbb{D}$ be such an infinite safe condition-set (which exists by Theorem 6). Hence, for each model $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ there is a $\Delta_M \in \mathbb{D}$ such that $\Omega(M) \cap \Delta_M = \emptyset$ and $\mathbb{D}' = \{\Delta_M \mid M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)\}$ is an infinite safe condition-set. By our antecedent this immediately means that $Cn^{\mathsf{m}}_{\Omega}(\Gamma) \cup \{\bigvee \Delta \mid \Delta \in \mathbb{D}'\}$ has no **L**-models. By the compactness of **L**, there is a finite subset $\mathbb{E}$ of $\mathbb{D}'$ such that $Cn^{\mathsf{m}}_{\Omega}(\Gamma) \Vdash \neg \bigwedge_{\Delta\in\mathbb{E}} \bigvee \Delta$ and hence by our assumption, $\Gamma \Vdash^{\mathsf{m}}_{\Omega} \neg \bigwedge_{\Delta\in\mathbb{E}} \bigvee \Delta$. Hence, each model $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ falsifies some $\bigvee \Delta$ for some $\Delta \in \mathbb{E}$. But that shows that $\mathbb{E}$ is a finite safe condition-set for $\langle \Gamma, \Omega, A\rangle$,—a contradiction. ∎

Corollary 4 *If $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ is definable, then Γ is finite-conditional w.r.t. Ω.*

The converse of Theorem 13 fails, in view of the following example:

Example 8 *Let $\Omega = \{p_i \mid i \in \mathbb{N}\}$ and $\Gamma = \{p_i \vee p_j \mid i, j \in \mathbb{N}, i \neq j\}$. It is easily seen that $Cn(\Gamma) = Cn^{\mathsf{m}}_{\Omega}(\Gamma)$ which by Theorems 3 and 4 immediately implies that $Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma) = Cn^{\mathsf{m}}_{\Omega}(\Gamma)$. Thus, by Corollary 3, Γ is finite-conditional. Note also that there is a model $M \in \mathcal{M}(\Gamma) = \mathcal{M}(Cn(\Gamma))$ such that $M \models p_i$ for each $i \in \mathbb{N}$. In contrast, every minimally abnormal model of Γ falsifies exactly one abnormality p_i ($i \in \mathbb{N}$). Hence, $M \notin \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$.*

In [25, Chapter 5], Schlechta studies so-called *definability-preserving structures.* When translated to the current context, the triple $\langle \mathbf{L}, \Omega, \mathsf{x}\rangle$ is a definability-preserving structure iff for all Γ, $\mathcal{M}^{\mathsf{x}}_{\Omega}(\Gamma)$ is definable. In other words, a definability-preserving structure based on **L** is one that ensures that the set of selected models can always be characterized by means of a (possibly infinite) set of formulas, i.e., there is a set of formulas Δ such that $\mathcal{M}^{\mathsf{x}}_{\Omega}(\Gamma) = \mathcal{M}(\Gamma \cup \Delta)$.

Since by Definition 3, $\mathcal{M}^{\mathsf{r}}_{\Omega}\Gamma = \mathcal{M}(\Gamma \cup \neg(\Omega - \mathsf{U}_{\Omega}(\Gamma)))$, every structure $\langle \mathbf{L}, \Omega, \mathsf{r}\rangle$ is definability-preserving. For the minimal abnormality strategy, this does not hold — see again 8.

Relying on Theorem 12, Theorem 13, and Corollary 3, we have:

Corollary 5 *If $\langle \mathbf{L}, \Omega, \mathsf{m}\rangle$ is definability-preserving, then $\Vdash^{\mathsf{m}}_{\Omega} \; = \; \Vdash^{\mathsf{r}}_{\Omega^\wedge}$.*

In view of Example 8 it is clear that whenever Ω has an infinite subset with logically independent formulas (see Theorem 8) then $\langle \mathbf{L}, \Omega, \mathsf{m}\rangle$ is not definability-preserving. On the other hand, we have definability-preserving structures whenever, for instance, we deal with finite sets of abnormalities Ω.

5.2 Syntactic Criteria in View of Γ

Recall the second clause from the definition of a safe condition-set $\mathbb{D}$ for $\langle \Gamma, \Omega, A\rangle$: (ii) for each $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$, there is a $\Delta \in \mathbb{D}$ such that $\Delta \cap \Omega(M) = \emptyset$.

Now interestingly, it is possible to avoid talk of minimally abnormal models, yet still fully characterize (ii), by looking at disjunctions of abnormalities that follow from Γ (see e.g., [5]). Let us briefly explain how this works.

Recall that, according to the reliability strategy, the sets in $\mathbb{S}^{-}_{\Omega}(\Gamma)$ contain the abnormalities that we cannot safely assume to be false in view of Γ (see Definition 3.) However, we may also interpret $\mathbb{S}^{-}_{\Omega}(\Gamma)$ in a more fine-grained way and, as we now show, this is what happens according to the minimal abnormality strategy. That is, as long as we ensure that at least one disjunct of each disjunction of abnormalities is taken to be true, we can assume all others to be false. In other words, instead of looking at the set $\bigcup \mathbb{S}^{-}_{\Omega}(\Gamma)$, we may look at the set of all *choice sets*[11] of $\mathbb{S}^{-}_{\Omega}(\Gamma)$. Moreover, where Θ, Θ' are two such choice sets and $\Theta \subset \Theta'$, we may ignore Θ', as it represents an interpretation of $\mathbb{S}^{-}_{\Omega}(\Gamma)$ that is more abnormal than the one given by Θ.

Let $\mathbb{P}_{\Omega}(\Gamma)$ be the set of all $\subset$-minimal choice sets of $\mathbb{S}^{-}_{\Omega}(\Gamma)$.[12] We have:

Theorem 14 (Lemma 4, [6]) *If Γ has models, then $\mathbb{P}_{\Omega}(\Gamma) = \{\Omega(M) \mid M \in \mathcal{M}^{\mathrm{m}}_{\Omega}(\Gamma)\}$.*

Remark 1 *Consequently, if Γ has models, then $\mathbb{D}$ is a safe condition-set for $\langle \Gamma, \Omega, A\rangle$ iff (i) for all $\Delta \in \mathbb{D}$, $\Gamma \Vdash A \vee \bigvee \Delta$ and (ii') for every $\Theta \in \mathbb{P}_{\Omega}(\Gamma)$ there is a $\Delta \in \mathbb{D}$ such that $\Theta \cap \Delta = \emptyset$.*

This observation shows that the notion safe condition-set was clearly foreshadowed in the literature on adaptive logics since it is well-known that A is an adaptive consequence of Γ according to the minimal abnormality strategy iff (i) and (ii) above are fulfilled (see e.g., [5, Theorem 8]). Hence, the characterization of adaptive consequences in terms of safe condition-sets is well-known, just there was no name for these sets.[13]

This insight helps us in the remainder to specify conditions that are merely a function of $\mathbb{P}_{\Omega}(\Gamma)$. These conditions were already studied in the context of combined and prioritized adaptive logics [34]. It turns out that the weakest of these conditions is still stronger than the property of being finite-conditional.

We first recall some basic observations from [34]:

Theorem 15 *Each of the following holds:*

0. Ω is finite implies $\mathbb{S}^{-}_{\Omega}(\Gamma)$ is finite;
1. $\mathbb{S}^{-}_{\Omega}(\Gamma)$ is finite iff each $\Theta \in \mathbb{P}_{\Omega}(\Gamma)$ is finite iff $\mathsf{U}_{\Omega}(\Gamma)$ is finite;
2. If each $\Theta \in \mathbb{P}_{\Omega}(\Gamma)$ is finite, then $\mathbb{P}_{\Omega}(\Gamma)$ is finite.

Note that the converse of item 2 fails. Take for instance $\Gamma = \{p_i \mid i \in \mathbb{N}\}$ where $\Omega = \Gamma$. Obviously $\mathbb{P}_{\Omega}(\Gamma) = \{\Omega\}$ and Ω is infinite.

[11] Δ is a *choice set* of $\mathbb{A}$ iff $\Delta \cap \Theta \neq \emptyset$ for all $\Theta \in \mathbb{A}$.

[12] In the AL literature, $\mathbb{P}_{\Omega}(\Gamma)$ is usually denoted by $\Phi(\Gamma)$.

[13] When adaptive logicians speak about conditions, what they usually have in mind is the dynamic proof theory that comes with adaptive logics. Formulas in dynamic proofs are derived on conditions, i.e., finite sets of abnormalities that are considered false. Each strategy is associated with a specific retraction mechanism in view of which lines are marked whose condition is to be considered unsafe. A formula A is derivable on a condition Δ iff $A \vee \bigvee \Delta$ is **L**-derivable from Γ. In view of Remark 1 this motivates our terminological choice "safe *condition*-set".

There is a stronger criterion in terms of choice sets than the ones in Theorem 15 that is equivalent to the definability of $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$. This provides us a bridge to the results of Section 5.1. Where $\mathbb{P}^{\star}_{\Omega}(\Gamma) = \{\Omega - \Delta \mid \Delta \in \mathbb{P}_{\Omega}(\Gamma)\}$ we define:

($\star$) for each choice set Δ of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$, there is a finite choice set Θ of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$ such that $\Theta \subseteq \Delta$

Remark that if Θ is a finite choice set of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$, then by Theorem 14, for every $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$, there exists an $A \in \Theta$ such that $A \notin \Omega(M)$, and hence $\Gamma \Vdash^{\mathsf{m}}_{\Omega} \bigvee \neg\Theta$. Also, by Theorem 14, if $\Gamma \Vdash^{\mathsf{m}}_{\Omega} \bigvee \neg\Theta$ then Θ is a finite choice set of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$.

The following simple observation relates ($\star$) to the conditions in Theorem 15:

Theorem 16 *If $\mathbb{P}_{\Omega}(\Gamma)$ is finite, then* ($\star$).

Proof. Note first:

Fact 5 *If $\mathbb{A}$ is a finite set of sets and Δ is a choice set of $\mathbb{A}$ then there is a finite $\Delta' \subseteq \Delta$ that is a choice set of $\mathbb{A}$.*

Let Δ be such a choice set. Then for every $\Theta \in A$ there is at least one $A_\Theta \in \Delta$. Let $\Delta' = \bigcup_{\Theta \in \mathbb{A}} \{A_\Theta\}$. Since $\mathbb{A}$ is finite, so is Δ'.

Our theorem follows immediately by this fact and since the finiteness of $\mathbb{P}_{\Omega}(\Gamma)$ implies the finiteness of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$. ■

The converse of Theorem 16 fails as demonstrated in the following example.

Example 9 *Take $\Omega = \{p_i \mid i \in \mathbb{N}\}$ and $\Gamma = \{p_0 \vee p_1, p_2 \vee p_3, \ldots\}$. Then $\mathbb{P}_{\Omega}(\Gamma)$ consists of all sets $\{p_{2n+\epsilon_n} \mid n \in \mathbb{N}\}$ where each ϵ_n is either 0 or 1. Hence, $\mathbb{P}_{\Omega}(\Gamma)$ is infinite. It is easy to see that for each choice set Δ of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$ there is a choice set $\Theta \subseteq \Delta$ of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$ that has the form $\{p_{2n}, p_{2n+1}\}$ where $n \geq 0$.*

Finally, we relate ($\star$) to definability as defined in Section 5.1:

Theorem 17 $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) = \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$ *iff* ($\star$).

Proof. ($\Rightarrow$) Suppose there is an infinite choice set Δ of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$ such that there is no finite choice set Θ of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$ for which $\Theta \subset \Delta$. Hence there is no finite $\Theta \subset \Delta$ for which $\Gamma \Vdash^{\mathsf{m}}_{\Omega} \bigvee \neg\Theta$. Note that by Theorem 14, we have (‡) for each $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$, there is a $A \in \Delta$ such that $M \models \neg A$. Assume there is no $M \in \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$ such that $M \models B$ for all $B \in \Delta$. Hence, $Cn^{\mathsf{m}}_{\Omega}(\Gamma) \cup \Delta$ is **L**-trivial which means by the compactness of **L** that there is a finite $\Theta \subset \Delta$ such that $Cn^{\mathsf{m}}_{\Omega}(\Gamma) \Vdash \bigvee \neg\Theta$. By Theorem 14 and since we always have $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) \subseteq \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$, Θ is a choice set of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$,—a contradiction. Hence, our assumption is wrong which together with (‡) shows that the left hand side of our theorem fails.

($\Leftarrow$) Suppose the left hand side of our theorem is false and hence there is a $M \in \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma)) - \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ (since $\mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma) \subseteq \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$). This means that for each $M' \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)$ there is a $A' \in \Omega(M) - \Omega(M')$. Assume now that the right hand side of our theorem is true. Note that $\Delta = \{A' \mid M' \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma)\}$ is a choice set of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$. Hence, there is a finite $\Theta \subseteq \Delta$ such that Θ is a choice set of $\mathbb{P}^{\star}_{\Omega}(\Gamma)$. Hence, $\Gamma \Vdash^{\mathsf{m}}_{\Omega} \bigvee \neg\Theta$ and thus $M \notin \mathcal{M}(Cn^{\mathsf{m}}_{\Omega}(\Gamma))$,—a contradiction. ■

With Theorem 13 and Corollary 3 this immediately shows that $(\star)$ is a sufficient condition for the adequacy of our reduction. By Theorems 15 and 16 this also holds for all the other conditions presented in this section. Hence, this completes our meta-theoretic substantiation of Figure 1.

5.3 The Conditions in Practice

We now mention some adaptive logics from the literature and show in which cases their premise sets satisfy the conditions mentioned above. This underwrites our claim that these conditions apply to an interesting class of applications of the AL framework. For the sake of space, our characterizations of the logics will be rather loose; exact definitions and illustrations can be found in the literature cited.

Finite Set of Abnormalities As noted above, there are few ALs in the literature which are defined in terms of a finite set of abnormalities. The reason is that ALs are usually based on (i) an **L** which ranges over an infinite language (be it propositional or predicative), and (ii) an Ω that is defined in terms of a logical form F (see also Section 2.1). Given that one may plug in infinitely many variables into F, this results in an infinite number of abnormalities. Moreover, since the variables are logically independent in **L**, we may construct examples of infinite premise sets for which $\Vdash^{\mathsf{m}}_{\Omega}$ is stronger than $\Vdash^{\mathsf{r}}_{\Omega^{\wedge}}$ — see also Theorem 8 in Section 4.1.

An example of an AL for which Ω is not characterized by a logical form can be found in [31]. There the idea is that the members of Ω represent negative assumptions that correspond to (defeasible) conceptual knowledge. Incidentally, the specific set of abnormalities construed by Urbaniak is finite.

The AL characterizations of Dung's framework for abstract argumentation presented in [30] are based on a language which uses only finitely many atoms. As a result, their set of abnormalities is also finite.

In [36], ALs are defined which use a finite number of predicates, constants and variables, and therefore only have finitely many abnormalities. The aim of these logics is to model reasoning with vague predicates, and more particularly, to handle cases such as the Sorites paradox. Note that such reasoning usually involves only a finite number of predicates, whence the restriction of the language is fairly harmless in this context.

Finite $\mathbb{S}^{-}_{\Omega}(\Gamma)$ and Finite Γ Let us now move to a slightly more general level, and consider the condition that $\mathbb{S}^{-}_{\Omega}(\Gamma)$ is finite for a specific Γ. It turns out that for many concrete logics, this condition is satisfied at the propositional level as soon as Γ itself is finite.

For the propositional fragments of the inconsistency-adaptive logic $\mathbf{CLuN^{m}}$ this fact is well-known – see [22, Proposition 3.1] for a direct proof. In view of the proofs, it can be easily generalized to the logic $\mathbf{CLuNs^m}$ (see e.g. [7, Chapter 7]) and its modal extension in [11]. The main point is that in these logics, the members Δ of $\mathbb{S}^-_\Omega(\Gamma)$ are such that they contain only propositional variables that occur in Γ. Hence, if Γ is finite, so is $\mathbb{S}^-_\Omega(\Gamma)$.

The same fact holds for the modal adaptive logics $\mathbf{AR^m_1}$ and $\mathbf{AR^m_2}$ from [35] – see Appendix F, and in particular Proposition 5.1 in that paper.These logics model the defeasible inference from $\Box A$ (it is believed that A) to A.

In [21], a number of ALs are defined which characterize the Rescher-Manor consequence relations from [24] and generalize them to the predicate level. This is done by extending first order predicate logic with a dummy operator $\bullet$, and translating premises by prefixing them with this operator. For instance, where the premise set is $\Gamma = \{Pa, Qa, \forall x\neg(Px \wedge Qx)\}$, we obtain $\Gamma^\bullet = \{\bullet Pa, \bullet Qa, \bullet\forall x(\neg Px \wedge Qx)\}$. The abnormalities are all formulas of the form $\bullet A \wedge \neg A$; thus the AL allows for the (defeasible) inference of e.g. Pa from $\bullet Pa$, $\forall x\neg(Px \wedge Qx)$ from $\bullet\forall x\neg(Px \wedge Qx)$, etc.

It can easily be verified that in these systems (even at the predicative level), the set $\mathbb{S}^-_{\Omega^\bullet}(\Gamma^\bullet)$ is finite whenever Γ is finite. This follows immediately from Corollary A.11(ii) in [21]. In our notation, this corollary reads as follows:

($\clubsuit$) $\mathbb{P}_\Omega(\Gamma^\bullet) = \{\{\bullet A \wedge \neg A \mid A \in \Gamma - \Lambda\} \mid \Lambda \text{ is a maximal consistent subset of } \Gamma\}$

If $\Gamma^\bullet$ (and hence Γ) is finite, then by ($\clubsuit$) also each $\Theta \in \mathbb{P}_\Omega(\Gamma^\bullet)$ is finite. So we can apply Theorem 14 to derive that also $\mathbb{S}^-_\Omega(\Gamma)$ is finite. In other words, if we restrict ourselves to the intended application of these logics (i.e. premise sets of the type $\Gamma^\bullet$), then all finite premise sets are finite-conditional.

As a final example, consider the propositional version of the logics for reasoning with "plausible information" [32, 33].[14] These are defined on the basis of a normal modal logic (most often $\mathbf{K}$ or $\mathbf{T}$), and have as their set of abnormalities $\Omega_\Diamond = \{\Diamond A \wedge \neg A \mid A \text{ is an atom or its negation}\}$.

Here again, we can easily show that only abnormalities which use atoms that occur in Γ can be unreliable w.r.t. $\langle\Gamma, \Omega_\Diamond\rangle$. We will give a rough sketch of the argument here, assuming that the underlying modal logic $\mathbf{L}$ is characterized by a Kripke-semantics.[15]

Let M be a model of Γ and suppose that $M \models \Diamond A \wedge \neg A$, where the atom that occurs in A does not occur in Γ. Consider a $\mathbf{K}$-model M' which is exactly as M, except that in all possible worlds (including the actual world) it verifies A and falsifies $\neg A$. Note that, since M' agrees with M on all atoms that occur in Γ (in all possible worlds), M' is also a model of Γ. It can easily be verified that $\Omega_\Diamond(M') = \Omega_\Diamond(M) \setminus \{\Diamond A \wedge \neg A\}$. Hence M is not a minimally abnormal model of Γ. It follows that no minimally abnormal model of Γ verifies any abnormality

[14]In the papers cited, these logics are used as a stepping stone towards prioritized ALs. Here various degrees of plausibility are distinguished, and the translation to the modal language is generalized accordingly.

[15]To be more precise: every model M is defined in terms of a set of possible worlds W, a valuation function $v : W \times \mathcal{W} \to \{0,1\}$, an accessibility relation $R \subseteq W \times W$ and an actual world @. We define $M \models A$ iff A is true in the actual world of M, according to v.

$\Diamond A \wedge \neg A$ which contains an atom that does not occur in Γ.[16]

Finite $\mathbb{P}_\Omega(\Gamma)$ Some readers may think that the above observation can be generalized to *all* (propositional) adaptive logics. However, this is not true. In fact, sometimes every $A \in \Omega$ is such that, for infinitely many $B \in \Omega$, $\{A\} \Vdash B$. Hence whenever a premise set Γ **L**-entails some disjunction of abnormalities, infinitely many other disjunctions of abnormalities follow from Γ.

Examples of this type are the logics $\mathbf{AR^3_m}$ and $\mathbf{AR^4_m}$ for belief revision from [35], and the deontic logics $\mathbf{P2.1^r}$ from [20] and $\mathbf{P2.2^m}$ from [10]. As their respective sets of abnormalities (and, in the case of the deontic logics, also their monotonic core) are of a rather specific type, we shall not define them here.

Nevertheless, it can easily be shown that each of these logics satisfy the weaker condition that, whenever Γ is finite, then $\mathbb{P}_\Omega(\Gamma)$ is finite. For $\mathbf{AR^m_3}$ and $\mathbf{AR^m_4}$, this follows immediately from Proposition 5.1 in Appendix F of [35]. For the deontic systems, one can apply basically the same reasoning as in [35] — again, we leave the exact proof for another occasion as this would take us far beyond the scope of the present paper.

6 Some Corollaries

In this section we give some immediate corollaries of the basic reduction results from the preceding section. First, we point out how it allows us to reduce the computational complexity of $Cn^{\mathsf{m}}_\Omega(\Gamma)$ whenever Γ is finite-conditional. Second, we show how it gives us a weak variant of the Distribution Property for $\Vdash^{\mathsf{r}}_{\Omega^\wedge}$.

6.1 Computational Complexity of Minimal Abnormality

As was shown in [23, Section 3], whenever Γ is a Σ_{m+1}-set (where $m \geq 0$), Ω is computable, and **L** is computably enumerable, $Cn^{\mathsf{r}}_\Omega(\Gamma)$ has the complexity upper bound Σ^0_{m+3} in the arithmetic hierarchy.[17] For instance, if Γ is computably enumerable, we get the complexity upper bound Σ^0_3 for $Cn^{\mathsf{r}}_\Omega(\Gamma)$. All these estimations turn out to be exact for specific logics such as the inconsistency-adaptive logics $\mathbf{CLuN^r}$ and $\mathbf{CLuN^m}$, as was proved in [14, 23].

In view of the results from the preceding section, we have:

Corollary 6 *Where $m \geq 0$, Γ is a Σ^0_{m+1}-set,* **L** *is computably enumerable, and Ω is computable: if Γ is finite-conditional w.r.t. Ω, then then the upper bound complexity of $Cn^{\mathsf{m}}_\Omega(\Gamma)$ is Σ^0_{m+3}.*

This estimation can be shown to be exact for specific lower limit logics and specific premise sets. We will now give an example.

[16] A similar argument applies to the logic $\mathbf{AD^r}$ from [19], where $\mathbf{L} = \mathbf{S5}$ and $\Omega = \{\Diamond A \wedge \Diamond \neg A \mid A \text{ is an atom}\}$

[17] It would go way beyond the scope of this paper to introduce the reader into the rather involving world of arithmetical complexity. The interested reader is referred to [12] for details.

As before we work with **L** being classical propositional logic, a classical propositional language $\mathcal{L}$ based on an infinite set of atoms: $\mathcal{S} = \{s^n_{i,k,l} \mid i,k,l,n \in \mathbb{N}\} \cup \{q^n_{i,k} \mid i,k,n \in \mathbb{N}\} \cup \{r^n_i \mid i,n \in \mathbb{N}\} \cup \{p_n \mid n \in \mathbb{N}\}$, and the set of abnormalities $\Omega_* = \{q^n_{i,k} \mid i,k,n \in \mathbb{N}\} \cup \{r^n_i \mid i,n \in \mathbb{N}\}$. Let $\Omega^\wedge_*$ be the closure of Ω_* under conjunction.

Theorem 18 ([14, 22])[18] *For each $m \geq 0$, there exists a $\Pi^0_m(\Sigma^0_{m+1})$-set $\Gamma \subseteq \mathcal{W}$ such that $Cn^{\mathsf{r}}_{\Omega^\wedge_*}(\Gamma)$ is Σ^0_{m+3}-complete.*

Sketch of the proof. We denote the standard model of arithmetics by $\mathfrak{N}$. Let $m \geq 0$. We start by fixing an arithmetical Σ_{m+3}-formula $A(v)$ such that the set of numbers n satisfying $A(n)$, i.e. $\{n \in \mathbb{N} \mid \mathfrak{N} \models A(n)\}$, is Σ^0_{m+3}-complete. We can represent $A(v)$ by $\exists x \forall y \exists z B(x,y,z,v)$ where $B(x,y,z,v)$ is a Π_m-formula.

We fix the set $\Gamma_* = \Gamma_s \cup \Gamma_q \cup \Gamma_p \cup \Gamma_{s,q}$ of formulas in $\mathcal{L}$ where

$$\Gamma_s = \{s^n_{i,k,l} \mid i,k,l,n \in \mathbb{N}\}, \quad \Gamma_q = \{q^n_{i,k} \vee r^n_i \mid i,k,n \in \mathbb{N}\},$$

$$\Gamma_p = \{p_n \vee r^n_i \mid i,n \in \mathbb{N}\}, \quad \Gamma_{s,q} = \{\neg s^n_{i,k,l} \vee q^n_{i,k} \mid \mathfrak{N} \models B(i,k,l,n)\}.$$

Given some coding function $\#$ of formulas in $\mathcal{L}$, the set $\{\#A \mid A \in \Gamma_*\}$ is Π^0_m and hence also Σ^0_{m+1} in view of the definition of $\Gamma_{s,q}$. To complete our proof we now show that

$$\Gamma_* \Vdash^{\mathsf{r}}_{\Omega^\wedge_*} p_n \text{ iff } \mathfrak{N} \models A(n)$$

Suppose $\mathfrak{N} \models A(n)$. Hence, in view of $\Gamma_{s,q}$, there is an i such that for all k, $\Gamma_* \Vdash q^n_{i,k}$. By the definition of Γ_*, $r^n_i \notin U_{\Omega^\wedge_*}(\Gamma_*)$. Hence, $\{r^n_i\}$ is a safe condition for p_n. By Corollary 1, $\Gamma_* \Vdash^{\mathsf{r}}_{\Omega^\wedge_*} p_n$.

Suppose now that $\mathfrak{N} \not\models A(n)$. Thus, for all $i,k \in \mathbb{N}$, $\Gamma_* \Vdash q^n_{i,k} \vee r^n_i$ but $\Gamma_* \nVdash q^n_{i,k}$ and $\Gamma_* \nVdash r^n_i$. From this we immediately get (‡) $\bigwedge_{i \in I} r^n_i \in U^{\mathsf{r}}_{\Omega^\wedge_*}(\Gamma_*)$ for all $I \subset_f \mathbb{N}$. In view of Γ_p, the set of all conditions for $\langle \Gamma_*, \Omega_*, p_n \rangle$ is $\wp\big(\{\bigwedge_{i \in I} r^n_i \mid I \subset_f \mathbb{N}\}\big) \setminus \emptyset$. By (‡), neither of these condition is safe for $\langle \Gamma_*, \Omega_*, p_n \rangle$. Hence, by Theorem 11, $\Gamma_* \nVdash^{\mathsf{r}}_{\Omega^\wedge_*} p_n$.

We have shown that $Cn^{\mathsf{r}}_{\Omega^\wedge_*}(\Gamma)$ is Σ^0_{m+3}-hard. In view of the fact that the upper bound complexity Σ^0_{m+3} of $Cn^{\mathsf{r}}_{\Omega}(\Gamma)$ trivially applies also to $Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma)$ we also get that $Cn^{\mathsf{r}}_{\Omega^\wedge_*}(\Gamma)$ is Σ^0_{m+3}-complete. ■

Theorem 19 *For each $m \geq 0$, there is a $\Pi^0_m(\Sigma^0_{m+1})$-set $\Gamma \subseteq \mathcal{W}$ for which $\mathcal{M}^{\mathsf{m}}_{\Omega_*}(\Gamma)$ is definable such that $Cn^{\mathsf{m}}_{\Omega_*}(\Gamma)$ is Σ^0_{m+3}-complete.*

Proof. Let Γ_* be defined as in the proof of Theorem 18. We only need to show that $\mathcal{M}^{\mathsf{m}}_{\Omega_*}(\Gamma_*)$ is definable. By Corollary 4 this implies that Γ_* is finite-conditional w.r.t. Ω_* and thus, by Corollary 3, $Cn^{\mathsf{r}}_{\Omega^\wedge_*}(\Gamma_*) = Cn^{\mathsf{m}}_{\Omega_*}(\Gamma_*)$. Hence, the Σ^0_{m+3}-completeness of $Cn^{\mathsf{m}}_{\Omega_*}(\Gamma)$ follows immediately by Theorem 18.

Note first that the minimal disjunctions of abnormalities derivable via **CL** from Γ_* are,

[18] This proof is a simple variant of [22, Prop. 3.13]. The main difference concerns the last two paragraphs of the proof. Also, instead of using **CLuN** as the lower limit logic, we stick to **CL**.

(1) $\bigcup_{i,k,n\in\mathbb{N}}\{q^n_{i,k} \mid \mathfrak{N} \models \exists z B(i,k,z,n)\} \cup \{q^n_{i,k} \vee r^n_i \mid \mathfrak{N} \not\models \exists z B(i,k,z,n)\}$

In view of this it is easy to see that (2) $M \in \mathcal{M}^{\mathsf{m}}_{\Omega}(\Gamma_*)$ iff $\Omega(M) = \bigcup_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}}\Delta^n_i$, where $\Delta^n_i = \{q^n_{i,k} \mid k \in \mathbb{N}\}$ if $\mathfrak{N} \models \forall y \exists z B(i,y,z,n)$ and else $\Delta^n_i \in \big\{\{q^n_{i,k} \mid k \in \mathbb{N}\}, \{r^n_i\}\big\}$. Let

$$\Theta = \bigcup_{i,k,n\in\mathbb{N}} \{\neg q^n_{i,k} \vee \neg r^n_i \mid \mathfrak{N} \not\models \forall y \exists z B(i,y,z,n)\} \cup \{\neg r^n_i \mid \mathfrak{N} \models \forall y \exists z B(i,y,z,n)\}$$

We now show that $\mathcal{M}(\Gamma_* \cup \Theta) = \mathcal{M}^{\mathsf{m}}_{\Omega_*}(\Gamma_*)$. Let $M \in \mathcal{M}(\Gamma_* \cup \Theta)$. Let $i,k \in \mathbb{N}$. By the definition of Θ and Γ_p, (3) $M \models q^n_{i,k}$ if $\mathfrak{N} \models \forall y \exists z B(i,y,z,n)$. Moreover, by the definition of Θ and Γ_p, (4) whenever $\mathfrak{N} \not\models \forall y \exists z B(i,y,z,n)$ then either $M \models q^n_{i,k} \wedge \neg r^n_i$ or $M \models \neg q^n_{i,k} \wedge r^n_i$. By (3) and (4), $\Omega_*(M)$ is of the form $\bigcup_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}}\Delta^n_i$ where Δ^n_i is characterized as in (2). Hence, by (2) and since $M \in \mathcal{M}(\Gamma_*)$, $M \in \mathcal{M}^{\mathsf{m}}_{\Omega_*}(\Gamma_*)$.

Let now $M \in \mathcal{M}^{\mathsf{m}}_{\Omega_*}(\Gamma_*)$. Hence, $M \in \mathcal{M}(\Gamma_*)$ and by (2) $\Omega_*(M)$ is of the form $\bigcup_{n\in\mathbb{N}}\bigcup_{i\in\mathbb{N}}\Delta^n_i$. From this follows immediately that $M \models \Theta$. Hence, $M \in \mathcal{M}(\Gamma_* \cup \Theta)$. ■

By Corollary 4, this also shows that for each $m \geq 0$, there is a $\Pi^0_m(\Sigma^0_{m+1})$-set $\Gamma \subseteq \mathcal{W}$ that is finite-conditional w.r.t. Ω_* such that $Cn^{\mathsf{m}}_{\Omega_*}(\Gamma)$ is Σ^0_{m+3}-complete.

Remark 2 *In view of Theorem 18 and Theorem 19 it has been demonstrated that the complexity upper bound of Σ^0_3 is reached for a specific $\langle \mathbf{L}, \Omega^\wedge, \mathsf{r}\rangle$ and $\langle \mathbf{L}, \Omega, \mathsf{m}\rangle$. The premise set Γ_* can easily be adjusted for other systems such as the well-known inconsistency adaptive logics where $\mathbf{L} = \mathbf{CLuN}$ and $\Omega = \{{\sim}A \mid A \in \mathcal{W}\}$ (where $\sim$ is the paraconsistent negation of* **CLuN***) or the adaptive systems that employ dummy operators (see Section 5.3).*

The complexity of minimal abnormality is significantly higher than that of reliability: it is not even situated in the arithmetic hierarchy. That is, if Ω is computable and $\mathbf{L}$ is computably enumerable, then for any arithmetical (e.g., computable) set Γ of premises, $Cn^{\mathsf{m}}_{\Omega}(\Gamma)$ has the complexity upper bound Π^1_1 in the analytical hierarchy — as was proved in [23, Section 3]. This estimation is exact for concrete adaptive logics such as the inconsistency-adaptive logics based on **CLuN** and appropriate computable sets of premises (see [23, 37]). Our result shows that for the sub-class of finite-conditional premise sets we are dealing with the same arithmetical complexity class as for the reliability strategy, which is significantly lower in the hierarchy.

6.2 Restricted Distribution for $\Vdash^{\mathsf{r}}_{\Omega^\wedge}$

Given its similarity to $\Vdash^{\mathsf{m}}_{\Omega}$, one may ask whether $\Vdash^{\mathsf{r}}_{\Omega^\wedge}$ satisfies Distribution in general. The answer is negative, as the following example shows.

Example 10 *Let $\Gamma = \{p \vee q_i \mid i \in \mathbb{N}\} \cup \{q_i \vee q_j \mid i,j \in \mathbb{N}, i \neq j\} \cup \{\neg r \vee q_i \mid i \in \mathbb{N}\} \cup \{s \vee p\}$. Let $\Omega = \{p\} \cup \{q_i \mid i \in \mathbb{N}\}$. Note first that $\Gamma \cup \{r\} \Vdash^{\mathsf{r}}_{\Omega^\wedge} s$: all* *models of* $\Gamma \cup \{r\}$ *verify each* q_i*, and hence all minimally abnormal models of* $\Gamma \cup \{r\}$ *falsify* p*; by disjunctive syllogism, all those models verify* s*.*

However, $\Gamma \nVdash^{\mathsf{r}}_{\Omega^\wedge} r \supset s$*. That is, every safe condition-set for* $\langle \Gamma, \Omega, r \supset s \rangle$ *is a superset of* $\mathbb{D} = \{\{p\}\} \cup \{\{q_i\} \mid i \in \mathbb{N}\}$*. The rest is immediate in view of Theorem 11.*

Nevertheless, on the basis of our current results, we do have a restricted version of the property:

Theorem 20 *If* $Cn(\Gamma) \cap Cn(\Gamma')$ *is finite-conditional w.r.t.* Ω*, then* $Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma) \cap Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma') \subseteq Cn^{\mathsf{r}}_{\Omega^\wedge}(Cn(\Gamma) \cap Cn(\Gamma'))$*.*

Proof. Suppose $Cn(\Gamma) \cap Cn(\Gamma')$ is finite-conditional w.r.t. Ω. Note first that, by Theorems 3 and 4, $Cn^{\mathsf{r}}_{\Omega^\wedge}(\Delta) \subseteq Cn^{\mathsf{m}}_{\Omega^\wedge}(\Delta) = Cn^{\mathsf{m}}_{\Omega}(\Delta)$ for all Δ. It follows that

$$Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma) \cap Cn^{\mathsf{r}}_{\Omega^\wedge}(\Gamma') \subseteq Cn^{\mathsf{m}}_{\Omega}(\Gamma) \cap Cn^{\mathsf{m}}_{\Omega}(\Gamma')$$

By the distribution property for $\Vdash^{\mathsf{m}}_{\Omega}$,

$$Cn^{\mathsf{m}}_{\Omega}(\Gamma) \cap Cn^{\mathsf{m}}_{\Omega}(\Gamma') \subseteq Cn^{\mathsf{m}}_{\Omega}(Cn(\Gamma) \cap Cn(\Gamma'))$$

Finally, by the supposition and Corollary 3,

$$Cn^{\mathsf{m}}_{\Omega}(Cn(\Gamma) \cap Cn(\Gamma')) = Cn^{\mathsf{r}}_{\Omega^\wedge}(Cn(\Gamma) \cap Cn(\Gamma')). \quad \blacksquare$$

Theorem 21 *Each of the following holds:*

(i) *If* Γ *is finite-conditional w.r.t.* Ω *and* $\Gamma \cup \{A\} \Vdash^{\mathsf{r}}_{\Omega^\wedge} B$*, then* $\Gamma \Vdash^{\mathsf{r}}_{\Omega^\wedge} A \supset B$*.*

(ii) *If* $\Gamma \cup \{A \vee B\}$ *is finite-conditional w.r.t.* Ω*,* $\Gamma \cup \{A\} \Vdash^{\mathsf{r}}_{\Omega^\wedge} C$*, and* $\Gamma \cup \{B\} \Vdash^{\mathsf{r}}_{\Omega^\wedge} C$*, then* $\Gamma \cup \{A \vee B\} \Vdash^{\mathsf{r}}_{\Omega^\wedge} B$*.*

(iii) *If* Γ *is finite-conditional w.r.t.* Ω*,* $\Gamma \cup \{A\} \Vdash^{\mathsf{r}}_{\Omega^\wedge} B$*, and* $\Gamma \cup \{\neg A\} \Vdash^{\mathsf{r}}_{\Omega^\wedge} B$*, then* $\Gamma \Vdash^{\mathsf{r}}_{\Omega^\wedge} B$*.*

Proof. Ad (i): Suppose Γ is finite-conditional w.r.t. Ω and $\Gamma \cup \{A\} \Vdash^{\mathsf{r}}_{\Omega^\wedge} B$. By Theorem 3, $\Gamma \cup \{A\} \Vdash^{\mathsf{m}}_{\Omega^\wedge} B$. By Theorem 4, $\Gamma \cup \{A\} \Vdash^{\mathsf{m}}_{\Omega} B$. Since the distribution property holds for $\Vdash^{\mathsf{m}}_{\Omega}$ also the deduction theorem holds. Thus, $\Gamma \Vdash^{\mathsf{m}}_{\Omega} A \supset B$. By Corollary 3, $\Gamma \Vdash^{\mathsf{r}}_{\Omega^\wedge} A \supset B$.

(ii) and (iii) are proven analogously. This is left to the reader. $\blacksquare$

7 Conclusion and Outlook

The results from this paper are significant for various reasons. First, the two adaptive strategies reliability and minimal abnormality are usually presented as two alternative paths of strengthening **L**. However, the question when they are equi-expressive has not been investigated. It was so far not clear (a) in which cases (i.e., for what kind of premise sets) the two strategies are equi-expressive, and (b) what kind of transformation is necessary to achieve this. Concerning (b) we have shown that this can be done in a rather straight-forward way by closing

Ω under (classical) conjunction and concerning (a) we have given a sufficient and necessary criterion: finite-conditionality.

Second, the two strategies come with different meta-theoretic properties. By identifying a class of premise sets for which they are equi-expressive we at the same time identify domains for which their meta-theoretic properties are transferable. We exemplified this insight by means of complexity results and the distribution property.

The research in this paper motivates various questions:

- What effects on the expressive power of the strategies do other manipulations of the set of abnormalities have? For instance, are there other manipulations beside the closure under conjunction that have the same effect? Are there manipulations that allow to express Reliability with Minimal Abnormality?
- Are there variants of the dynamic proof theory for $\langle \mathbf{L}, \Omega, \mathsf{m} \rangle$ that characterize the consequence relation obtained by $\langle \mathbf{L}, \Omega^{\wedge}, \mathsf{r} \rangle$ (without reference to $\Omega^{\wedge}$)? Is there a variant of the selection semantics for Cn^{m}_{Ω} based on minimal abnormal models that offers a straightforward characterization of $Cn^{\mathsf{r}}_{\Omega^{\wedge}}$ (without direct reference to $\Omega^{\wedge}$)?

We have preliminary results for many of these questions which will be presented in future work.

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References

[1] Arnon Avron and Iddo Lev. A formula-preferential base for paraconsistent and plausible non-monotonic reasoning. In *Proceedings of the Workshop on Inconsistency in Data and Knowledge (KRR-4) Int. Joint Conf. on AI (Ijcai 2001)*, pages 60–70, 2001.

[2] Diderik Batens. Inconsistency-adaptive logics. In *Logic at Work. Essays dedicated to the memory of Helena Rasiowa*, pages 445–472. Springer, 1999.

[3] Diderik Batens. A survey of inconsistency-adaptive logics. In Diderik Batens, Chris Mortensen, Graham Priest, and Jean Paul Van Bendegem, editors, *Frontiers of Paraconsistent Logic*, pages 49–73. Research Studies Press, King's College Publications, 2000.

[4] Diderik Batens. Towards the unification of inconsistency handling mechanisms. *Logic and Logical Philosophy*, 8:5–31, 2000. Appeared 2002.

[5] Diderik Batens. A universal logic approach to adaptative logics. *Logica Universalis*, 1:221–242, 2007.

[6] Diderik Batens. A universal logic approach to adaptive logics. *Logica Universalis*, 1:221–242, 2007.

[7] Diderik Batens. *Adaptive Logics and Dynamic Proofs. Mastering the Dynamics of Reasoning, with Special Attention to Handling Inconsistency.* In Progress.

[8] Diderik Batens, Kristof De Clercq, Peter Verdée, and Joke Meheus. Yes fellows, most human reasoning is complex. *Synthese*, 166:113–131, 2009.

[9] Diderik Batens, Christian Straßer, and Peter Verdée. On the transparency of defeasible logics: Equivalent premise sets, equivalence of their extensions, and maximality of the lower limit. *Logique et Analyse*, 207:281–304, 2009.

[10] Mathieu Beirlaen. *Tolerating Normative Conflicts in Deontic Logic*. PhD thesis, Ghent University, 2012.

[11] Mathieu Beirlaen, Christian Straßer, and Joke Meheus. An inconsistency-adaptive deontic logic fornormative conflicts. *Journal of Philosophical Logic*, pages 1–31, To appear.

[12] S. B. Cooper. *Computability theory.* CRC Press, 2004.

[13] Lou Goble. Deontic logic (adapted) for normative conflicts. *Logic Journal of IGPL*, to appear, 2013.

[14] Leon Horsten and Philip Welch. The undecidability of propositional adaptive logic. *Synthese*, 158:41–60, 2007.

[15] John Horty. Skepticism and floating conclusions. *Artifical Intelligence*, 135:55–72, 2002.

[16] Sarit Kraus, Daniel Lehmann, and Menachem Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.

[17] Sten Lindström. A semantic approach to nonmonotonic reasoning: inference operations and choice. *Uppsala Prints and Reprints in Philosophy*, 10, 1994.

[18] David Makinson. General patterns in nonmonotonic reasoning. In *Handbook of Logic in Artificial Intelligence and Logic Programming, vol. III.* Clarendon Press, 1994.

[19] Joke Meheus. An adaptive logic based on Jaśkowski's approach to paraconsistency. *Journal of Philosophical Logic*, 35:539–567, 2006.

[20] Joke Meheus, Mathieu Beirlaen, and Frederik Van De Putte. Avoiding deontic explosion by contextually restricting aggregation. In Guido Governatori and Giovanni Sartor, editors, *Deontic Logic in Computer Science*, pages 148 – 165. Springer, 2010.

[21] Joke Meheus, Christian Straer, and Peter Verdée. Which style of reasoning to choose in the face of conflicting information? *Journal of Logic and Computation*, 2013.

[22] Sergei Odintsov and Stanislav Speranski. On algorithmic properties of propositional inconsistency-adaptive logics. *Logic and Logical Philosopy*, 21(3):209–228, 2012.

[23] Sergei Odintsov and Stanislav Speranski. Computability issues for adaptive logics in expanded standard format. *Studia Logica*, 101(6):1237–1262, 2013.

[24] Nicholas Rescher and Ruth Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179–217, 1970.

[25] Karl Schlechta. *Coherent Systems*. Elsevier, 2004.

[26] Yoav Shoham. A semantical approach to nonmonotonic logics. In M. L. Ginsberg, editor, *Readings in Non-Monotonic Reasoning*, pages 227–249. Morgan Kaufmann, Los Altos, CA, 1987.

[27] Christian Straßer. *Adaptive Logic Characterizations of Defeasible Reasoning. With Applications in Argumentation, Normative Reasoning and Default Reasoning*. PhD thesis, Ghent University, 2010.

[28] Christian Straßer. An adaptive logic framework for conditional obligations and deontic dilemmas. *Logic and Logical Philosophy*, 19:95 – 128, 2010.

[29] Christian Straßer. *Adaptive Logic and Defeasible Reasoning. Applications in Argumentation, Normative Reasoning and Default Reasoning.* Springer, 2014.

[30] Christian Straßer and Dunja Šešelja. Towards the Proof-theoretic Unification of Dung's Argumentation Framework: an Adaptive Logic Approach. *Journal of Logic and Computation*, 21:133–156, 2010.

[31] Rafal Urbaniak. Capturing dynamic conceptual frames. *Logic Journal of IGPL*, 18(3):430–455, 2010.

[32] Frederik Van De Putte. Hierarchic adaptive logics. *Logic Journal of IGPL*, 20(1):45–72, February 2011.

[33] Frederik Van De Putte and Christian Straßer. Extending the standard format of adaptive logics to the prioritized case. *Logique et Analyse*, 220:601–641, 2012.

[34] Frederik Van De Putte and Christian Straßer. Three formats of prioritized adaptive logics: A comparative study. *Logic Journal of IGPL*, 21:127–159, February 2013. Logic Journal of the IGPL; doi: 10.1093/jigpal/jzs004.

[35] Frederik Van De Putte and Peter Verdée. The dynamics of relevance: Adaptive belief revision. *Synthese*, 187(1):1–42, 2012.

[36] Stephan van der Waart van Gulik and Peter Verdée. Modeling sorites reasoning with adaptive fuzzy logic. *Fuzzy Sets and Systems*, 159:1869–1884, 2008.

[37] Peter Verdée. Adaptive logics using the minimal abnormality strategy are Π^1_1-complex. *Synthese*, 167:93–104, 2009.

[38] Peter Verdée. Non-monotonic set theory as a pragmatic foundation of mathematics. *Foundations of Science*, pages 1–26, 2012.