# On a Partial Decision Method for Dynamic Proofs[⋆] [⋆⋆]

Diderik Batens

Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
Diderik.Batens@rug.ac.be

**Abstract.** This paper concerns a goal directed proof procedure for the propositional fragment of the adaptive logic **ACLuN1**. At the propositional level, it forms an algorithm for final derivability. If extended to the predicative level, it provides a *criterion* for final derivability. This is essential in view of the absence of a positive test. The procedure may be generalized to all flat adaptive logics.

## 1 The Problem

Inference relations for which there is no positive test abound in both everyday and scientific reasoning processes. Adaptive logics are intended for characterizing such inference relations.[1] The characterization has a specific metalinguistic standard format. This format provides the logic with a semantics and with a proof theory, and warrants soundness, completeness, and a set of properties of the logic.[2] The first adaptive logics were inconsistency-adaptive. The articulation of other adaptive logics provided increasing insight in the underlying mechanisms and required that adaptive logics were systematized in a new way. This systematization is presented in [8] and will be followed here.

An especially important feature of adaptive logics is their dynamic proof theory. Indeed, this proof theory is intended for *explicating* actual reasoning—see [18] for a historical example—a task that cannot be accomplished by definitions, semantic systems, and other more abstract characterizations.

The dynamics of the proof theory provides from the absence of a positive test. For most consequence relations, the dynamics is double. The *external* dynamics

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[1] A positive test is a systematic procedure that, for every set of premises $\Gamma$ and for every conclusion $A$, leads after finitely many steps to a "yes" if $A$ is a consequence of $\Gamma$. Remark that the consequence relation defined by classical logic is undecidable, but that there is a positive test for it—see [16] for such matters.

[2] Only part of these results are written up, viz. in [9].

is well known: as new premises become available, consequences derived from the earlier premise set may be withdrawn. In other words, the external dynamics results from the non-monotonic character of the consequence relation—the fact that, for some $\Gamma$, $\Delta$ and $A$, $\Gamma \vdash A$ but $\Gamma \cup \Delta \nvdash A$. The *internal* dynamics is very different from the external one. Even if the premise set is constant, certain formulas are considered as derived at some stage of the proof, but are considered as not derived at a later stage. For any consequence relation, insight in the premises is only gained by deriving consequences from them. In the absence of a positive test, this results in the internal dynamics.[3]

Dynamic proofs differ in two main respects from usual proofs. The first difference concerns annotated versions. Apart from (i) a line number, (ii) a formula, (iii) the line numbers of the formulas from which the formula is derived, and (iv) the rule by which the formula is derived (the latter two are the justification of the line), dynamic proofs also contain (v) a *condition*. Intuitively, this is a set of formulas that are supposed to be false, or, to be more precise, formulas the truth of which is not required by the premises.

The second main difference is that, apart from the deduction rules that allow one to add lines to the proof, there is a *marking definition*. The underlying idea is as follows. As the proof proceeds, more formulas are derived from the premises. In view of these formulas, some conditions may turn out not to hold. The lines at which such conditions occur are *marked*. Formulas derived on marked lines are taken not to be derived from the premises. In other words, they are considered as 'out'. One way to understand the procedure is as follows. As the proof proceeds, one's insight in the premises improves. More particularly, some of the conditions that were introduced earlier may turn out not to hold.

For any stage of the proof, the marking definition settles which lines are marked and which lines are unmarked. This leads to a precise definition of *derivability at a stage*. Notwithstanding the precise character of this notion, we also want a more stable form of derivability, which is called *final derivability*. The latter does not depend on the stage of the proof; nor does it depend on the way in which a specific proof from a set of premises proceeds. It is an abstract and stable relation between a set of premises and a conclusion. A different way for putting this is that final derivability refers to a stage of the proof at which the marks have become stable. Final derivability should be sound and strongly complete with respect to the semantics. For any adaptive logic **AL**, $A$ should be finally derivable from $\Gamma$ ($\Gamma \vdash_{\mathbf{AL}} A$) if and only if $A$ is a semantic consequence of $\Gamma$ ($\Gamma \vDash_{\mathbf{AL}} A$).

Consider a dynamic proof from a set of premises. At any point in time, the proof will be finite. It will reveal what is derivable from the premises at that stage of the proof. But obviously we are interested in final derivability. Whence

[3] The Weak consequence relation from [20] and [21]—see [14] and [15] for an extensive study of such consequence relations—is monotonic. Nevertheless, its proof theory necessarily displays an internal dynamics because there is no positive test for it—see [6] and [10]. Some logics for which there is a positive test, may nevertheless be characterized in a nice way in terms of a dynamic proof theory—see [7].

the question: what does a proof at a stage reveal about final derivability? As there is no positive test for the consequence relation, there is no algorithm for final derivability. So, one has at best some *criteria* that decide, for specific $A$ and $\Gamma$, whether $A$ is finally derivable from $\Gamma$.

What if no criterion enables one to conclude from the proof whether certain formulas are or are not finally derivable from the premise set? The answer or rather the answers to this question are presented in [3]. Roughly, they go as follows. First, there is a characteristic semantics for derivability at a stage. Next, it can be shown that, as the dynamic proof proceeds, the insight in the premises provided by the proof never decreases and may increase.[4] In other words, derivability at a stage provides an estimate for final derivability, and, as the proof proceeds, this estimate may become better, and never becomes worse. In view of all this, derivability at a stage gives one exactly what one might expect, viz. a fallible but *sensible* estimate of final derivability.[5] At any stage of the proof, one has to decide (obviously on the basis of pragmatic considerations) whether one will continue the proof or rely on present insights. This is fully in line with the contemporary view on rationality.[6]

Needless to say, one should apply a criterion for final derivability whenever one can. This motivated the search for such criteria—see [3], [11] and [12]. Unfortunately, most of these criteria are complex and only transparent for people that are well acquainted with dynamic proofs. Recently, it turned out that a specific kind of goal directed proofs offer a way out in this respect. The idea is not to formulate a criterion, but rather to specify a specific proof procedure that functions as a criterion. The proof procedure is applied to $\Gamma \vdash_{\mathbf{AL}} A$. Whenever the proof procedure stops, it is possible to conclude from the resulting proof whether or not $\Gamma \vdash_{\mathbf{AL}} A$. Preparatory work on the propositional fragment of **CL** (classical logic) is presented in [13] and some first results on the proof procedure for inconsistency-adaptive logics are presented in this paper.

The present paper is restricted to the propositional level. So, all references to logical systems concern the propositional fragments only. At this level the proof procedure forms an *algorithm* for final derivability: if the proof procedure is applied to $A_1, \ldots, A_n \vdash B$, it always stops after finitely many steps. If, at the last stage of the proof, $B$ is derived on an unmarked line, then $B$ is finally derivable from $A_1, \ldots, A_n$; if $B$ is not derived on an unmarked line, it is not finally derivable from $A_1, \ldots, A_n$. However, the proof procedure may be extended to the predicative level and there provides a criterion for final derivability if it stops. The main interest of the procedure lies there.

[4] More particularly, this insight increases if informative steps are added to the proof, where "informative step" is clearly definable—see [3].

[5] This estimate is defined in terms of the proof theory, and the latter explicates actual reasoning. So, the estimate should not be confused with approximations that may be obtained by certain computational procedures.

[6] Needless to say, some proofs provide more efficient estimates of final derivability than others. The goal directed proofs presented in this paper offer means to obtain efficient proofs, but more research on this problem is desirable.

In Section 2, I briefly present the inconsistency-adaptive logic **ACLuN1** and its dynamic proof theory. In Section 3, the goal-directed proof is applied to **CL**. This will make the matter easily understood by everyone. The proof procedure for the adaptive logic **ACLuN1** is spelled out in Section 4.

## 2 The Inconsistency-Adaptive Logic ACLuN1

The central difference between paraconsistent logics and inconsistency-adaptive logics can be easily described in proof theoretic terms. In a (monotonic) paraconsistent logic some deduction *rules* of **CL** are invalid; in an inconsistency-adaptive logic, some *applications* of deduction rules of **CL** are invalid.

The original application context that led to inconsistency-adaptive logics—see [2]—is still one of the most clarifying ones. Suppose that a theory $T$ was intended as consistent and was formulated with **CL** as its underlying logic. Suppose next that $T$ turns out to be inconsistent. Of course, one will want to replace $T$ by some consistent improvement $T'$. Typically, one does not just trow away $T$, restarting from scratch. One *reasons* from $T$ in order to locate the inconsistency or inconsistencies and in order to locate constraints for the replacement $T'$. Needless to say, logic alone is not sufficient to find the justified replacement $T'$.[7] However, logic is able to *locate* the inconsistencies in $T$. It can provide one with an interpretation of $T$ that is 'as consistently as possible'. Let me phrase this in intuitive terms. At points where $T$ is inconsistent, some deduction rules of **CL** cannot apply—if they did, the resulting interpretation of $T$ would be trivial in that it would make *every* sentence of the language a theorem of $T$. But where $T$ is consistent, all deduction rules of **CL** should apply.

An extremely simple propositional example will clarify the matter. Consider the theory $T$ that is characterized by the premise set $\{p, {\sim}p \vee r, q, {\sim}q \vee s, {\sim}p\}$. From these premises, $r$ should not be derived by Disjunctive Syllogism. Indeed, ${\sim}p \vee r$ is just an obvious weakening of ${\sim}p$. If one were to derive $r$ from the premises, then, by the same reasoning, one should derive ${\sim}r$ from $p$ and ${\sim}p \vee {\sim}r$, which also is an obvious weakening of ${\sim}p$. However, if one interprets the premises as consistently as possible, one should derive $s$ from them, viz. by Disjunctive Syllogism from $q$ and ${\sim}q \vee s$. Indeed, while the premises require $p$ to behave inconsistently (require $p \wedge {\sim}p$ to be true), they do not require $q$ to behave inconsistently (they do not require $q \wedge {\sim}q$ to be true).

As the matter is central, let me phrase it differently. The theory $T$ from the previous paragraph turns out to be inconsistent. As it was intended to be consistent, it should be interpreted as consistently as possible. Given that $T$ is inconsistent, one will move 'down' to a paraconsistent logic—a logic that allows for inconsistencies. If a formula turns out to be inconsistent on the paraconsistent reading of $T$, one cannot apply certain rules of **CL** to it. Thus, even on the paraconsistent interpretation of $T$, $p \wedge {\sim}p$ is true. But consider $p \wedge ({\sim}p \vee r)$.

[7] If $T$ is an empirical theory, at least new factual data (observations and outcomes of experiments) will be required. If $T$ is a mathematical theory, more conceptual analysis will be required.

Given the meaning of conjunction and disjunction, this formula is equivalent to $(p \wedge \sim p) \vee r$. According to **CL**, $p \wedge \sim p$ cannot be true, and hence $r$ is true. However, the premises state that $p \wedge \sim p$ is true. So, if one wants to reason sensibly *from* these premises, one cannot rely on the **CL**-presupposition that $p \wedge \sim p$ is bound to be false. However, where the paraconsistent reading of $T$ does not require that a specific formula $A$ behaves inconsistently, one may retain the **CL**-presupposition that $A$ is consistent, and hence apply **CL**-rules where they are validated by *this* presupposition. Thus $T$ affirms $q \wedge (\sim q \vee s)$, which is equivalent to $(q \wedge \sim q) \vee s$. As $T$ does not require $q \wedge \sim q$ to be true, it should be taken to be false and one should conclude to $s$.

The intuitive statements from the two preceding paragraphs are given a precise and coherent formulation by inconsistency-adaptive logics.

An adaptive logic is characterized by the following triple:[8]

(i) a monotonic *lower limit logic*,
(ii) a *set of abnormalities* (characterized by a logical form), and
(iii) an *adaptive strategy* (specifying the meaning of "interpreting the premises as normally as possible").

Extending the lower limit logic with the requirement that no abnormality is logically possible results in a monotonic logic, which is called the *upper limit logic*.

Let me illustrate this by the specific inconsistency-adaptive logic **ACLuN1**. In this paper, I shall only consider the propositional level of the logic and I shall consider no other strategy than Reliability.

The *lower limit logic* of **ACLuN1** is **CLuN**. This monotonic paraconsistent logic is just like **CL**, except in that it allows for gluts with respect to negation—whence the name **CLuN**. Axiomatically, **CLuN** is obtained by extending full positive propositional logic with the axiom schema $A \vee \sim A$—see [4] for a study of the full logics **CLuN** and **ACLuN1**, including the semantics. **CLuN** isolates inconsistencies. Indeed, Double Negation, de Morgan rules, and all similar negation reducing rules are *not* validated by **CLuN**. As a result, complex contradictions do not reduce to truth functions of simpler contradictions.[9] There are several versions of **CLuN**. I shall suppose that the language contains $\perp$, characterized by the axiom schema $\perp \supset A$, and I shall discuss this convention below.

[8] In this paper I consider only *flat* adaptive logics. Other adaptive logics are the prioritized ones, which are defined as specific combinations of flat adaptive logics—see [8].

[9] For example, $(p \wedge q) \wedge \sim(p \wedge q) \nvdash_{\mathbf{CLuN}} (p \wedge \sim p) \vee (q \wedge \sim q)$ and $\sim p \wedge \sim\sim p \nvdash_{\mathbf{CLuN}} p \wedge \sim p$. Of course, one still has $(p \wedge \sim p) \wedge \sim(p \wedge \sim p) \vdash_{\mathbf{CLuN}} p \wedge \sim p$.

The *set of abnormalities*, $\Omega$, comprises all formulas[10] of the form $A \wedge {\sim}A$.[11] Extending **CLuN** with the axiom schema $(A \wedge {\sim}A) \supset B$ results in the *upper limit logic*, which is **CL**.

Finally, we come to the adaptive *strategy*. Below I shall often need to refer to *disjunctions of abnormalities*, which I shall call *Dab*-formulas. From now on an expression of the form $Dab(\Delta)$ will refer to a disjunction of abnormalities; in other words, $\Delta$ is a finite subset of $\Omega$ and $Dab(\Delta)$ is the disjunction of the members of $\Delta$.[12] Suppose now that $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$, but that no member of $\Delta$ is **CLuN**-derivable from $\Gamma$. This means that the premises require some member of $\Delta$ to be true, but do not specify which member is true. In view of this possibility, one needs to introduce an adaptive strategy. One wants to interpret the premises "as normally as possible" (which for the present $\Omega$ means "as consistently as possible"), but this phrase is ambiguous. As indicated in (iii), an adaptive strategy disambiguates the phrase.

The *Reliability strategy* from [2][13] is the oldest known strategy, and the one that is simplest from a proof theoretic point of view. I shall not consider any other strategies in this paper. Let $Dab(\Delta)$ be a *minimal Dab-consequence of* $\Gamma$ iff $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$ and there is no $\Delta' \subset \Delta$ for which $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta')$. Let $U(\Gamma) = \{A \mid A \in \Delta$ for some minimal *Dab*-consequence $Dab(\Delta)$ of $\Gamma\}$ be the set of formulas that are *unreliable* with respect to $\Gamma$. Below, I shall define $\Gamma \vdash_{\mathbf{ACLuN1}} A$, which will be read as "$A$ is finally **ACLuN1**-derivable from $\Gamma$". The following Theorem is provable. In plain words it says that $A$ is **ACLuN1**-derivable from $\Gamma$ iff there is a $\Delta$ such that $A \vee Dab(\Delta)$ is **CLuN**-derivable from $\Gamma$ and no member of $\Delta$ is unreliable with respect to $\Gamma$.

**Theorem 1.** *$\Gamma \vdash_{\mathbf{ACLuN1}} A$ iff there is a $\Delta \subseteq \Omega$ such that $\Gamma \vdash_{\mathbf{CLuN}} A \vee Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

The dynamic proof theory of any (flat) adaptive logic is characterized by three (generic) rules, except of course that the rules RU and RC should refer to the right lower limit logic. Let $\Gamma$ be the set of premises as before. I now list the official deduction rules.[14] Immediately thereafter I shall mention a shorthand notation that most people will find more transparent.

PREM  If $A \in \Gamma$, one may add a line comprising the following elements: (i) an appropriate line number, (ii) $A$, (iii) $-$, (iv) PREM, and (v) $\emptyset$.

[10] For some logics, the abnormalities are couples consisting of an open formula with $n$ free variables and of an $n$-tuple of elements of the domain.

[11] Some flat adaptive logics are described and studied as formula-preferential systems in [17]—see also—[1]. $\Omega$ is then any set of formulas. It is not clear whether this may be generalized to all adaptive logics.

[12] It can be shown that $\Gamma \vdash_{\mathbf{CL}} \bot$ iff there is a finite $\Delta \subset \Omega$ such that $\Gamma \vdash_{\mathbf{CLuN}} Dab(\Delta)$. So, both expressions may be taken to define that $\Gamma$ is inconsistent.

[13] This is the oldest paper on adaptive logics, but it appeared in a book that took ten years to come out.

[14] Only RC introduces non-empty conditions. In other words, as long as RC is not applied, the condition of every line is $\emptyset$.

RU If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B$ and each of $A_1, \ldots, A_n$ occur in the proof on lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) $B$, (iii) $i_1, \ldots, i_n$, (iv) RU, and (v) $\Delta_1 \cup \ldots \cup \Delta_n$.

RC If $A_1, \ldots, A_n \vdash_{\mathbf{CLuN}} B \vee Dab(\Theta)$ and each of $A_1, \ldots, A_n$ occur in the proof on lines $i_1, \ldots, i_n$ that have conditions $\Delta_1, \ldots, \Delta_n$ respectively, one may add a line comprising the following elements: (i) an appropriate line number, (ii) $B$, (iii) $i_1, \ldots, i_n$, (iv) RC, and (v) $\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta$.

Where "$A \quad \Delta$" abbreviates that $A$ occurs in the proof on the condition $\Delta$, the rules may be phrased more transparently as follows:

PREM If $A \in \Gamma$:
$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:
$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \vee Dab(\Theta)$:
$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

While the deduction rules enable one to add lines to the proof, the marking definition, which depends on the strategy, determines which lines are "in" and which lines are "out". For the Reliability strategy, we first need to define the set $U_s(\Gamma)$ of formulas that are unreliable at a stage $s$ of a proof. Let $Dab(\Delta)$ be a *minimal Dab-formula* at stage $s$ of the proof iff, at that stage, $Dab(\Delta)$ has been derived on the condition $\emptyset$ *and* there is no $\Delta' \subset \Delta$ for which $Dab(\Delta')$ has been derived on the condition $\emptyset$.[15] Let $U_s(\Gamma) =_{df} \{A \mid A \in \Delta$ for some minimal *Dab*-formula $Dab(\Delta)$ at stage $s$ of the proof$\}$.

**Definition 1.** *Where $\Delta$ is the condition of line $i$, line $i$ is marked at stage $s$ iff $\Delta \cap U_s(\Gamma) \neq \emptyset$. (Marking definition for Reliability)*

Lines that are unmarked at one stage may be marked at the next, and vice versa. Finally, I list the definitions that concern final derivability—the definitions are identical for all adaptive logics.

[15] The minimal *Dab*-formulas that occur in a proof at a stage should not be confused with minimal *Dab*-consequences of the set of premises. At a stage $s$, a new minimal *Dab*-formula may be derived, and the effect may be that a *Dab*-formula that was minimal at stage $s-1$ is not minimal at stage $s$. Whether some formula is a minimal *Dab*-consequence of the premises is obviously independent of the stage of a proof from those premises.

**Definition 2.** *$A$ is* finally derived *from $\Gamma$ on line $i$ of a proof at stage $s$ iff $A$ is derived on line $i$, line $i$ is not marked at stage $s$, and any extension of the proof in which line $i$ is marked may be further extended in such a way that line $i$ is unmarked.*

**Definition 3.** *$\Gamma \vdash_{\mathbf{AL}} A$ ($A$ is* finally **AL**-derivable *from $\Gamma$) iff $A$ is finally derived on a line of a proof from $\Gamma$.*

Remark that by "a proof" I mean (here and elsewhere) a sequence of lines that is obtained by applying certain instructions. In the present context, this means that each line in the sequence is obtained by applying a deduction rule and that the marking definition was applied. Here is a very simple dynamic proof.

| | | | | |
|---|---|---|---|---|
| 1 | $(p \wedge q) \wedge t$ | – | PREM | $\emptyset$ |
| 2 | $\sim p \vee r$ | – | PREM | $\emptyset$ |
| 3 | $\sim q \vee s$ | – | PREM | $\emptyset$ |
| 4 | $\sim p \vee \sim q$ | – | PREM | $\emptyset$ |
| 5 | $t \supset \sim p$ | – | PREM | $\emptyset$ |
| 6 | $r$ | 1, 2 | RC | $\{p \wedge \sim p\}$ $\boxed{9}$ |
| 7 | $s$ | 1, 3 | RC | $\{q \wedge \sim q\}$ |
| 8 | $(p \wedge \sim p) \vee (q \wedge \sim q)$ | 1, 4 | RU | $\emptyset$ |
| 9 | $p \wedge \sim p$ | 1, 5 | RU | $\emptyset$ |

Up to stage 7 of the proof, all lines are unmarked. At stage 8, lines 6 and 7 are marked because $U_8(\Gamma) = \{p \wedge \sim p, q \wedge \sim q\}$. At stage 9, only line 6 is marked because $U_9(\Gamma) = \{p \wedge \sim p\}$. It is easily seen that, if 1–5 are the only premises, then the marks will remain unchanged in all extensions of the proof. So, $r$ is not a final consequence of $\Gamma$ whereas $s$ is a final consequence of $\Gamma$.

*The convention on $\bot$.* As promised, I now discuss the convention that the language contains $\bot$ and hence that classical negation can be defined within the language, viz. by $\neg A =_{df} A \supset \bot$. In a sense then, **CLuN** is an extension of **CL**. It has the full inferential power of **CL**, $\neg$ functioning as the **CL**-negation, and moreover contains the paraconsistent negation $\sim$. In the original application context, mentioned in the second paragraph of this section, the premises belong to the $\bot$-free and $\neg$-free fragment of the language—of course, there are different application contexts as well. However, even in the original application context the presence of $\neg$ is useful: it greatly simplifies metatheoretic proofs and technical matters in general, and in no way hampers the limitations imposed by the application context.[16] As will appear in Section 4, the presence of $\neg$ also greatly simplifies the goal directed proof procedure that will serve as a criterion for final derivability.

[16] By present lights, it is harmless as well as useful, for all adaptive logics, to extend the language and the lower limit logic in such a way that all classical connectives belong to the lower limit logic. This holds even if these connectives do not occur in the premises or in the conclusions a user is interested in—see [5] for an example.

## 3 Goal Directed Proofs for Classical Logic

In this section I merely present an example: a goal directed proof for $p \supset (q \wedge s), \neg(q \vee r) \vdash_{\mathbf{CL}} \neg p$.[17] As the proof is simple, I skip the rules as well as the heuristic instructions—these are spelled out in [13]—and merely offer some comments.

The first step introduces the main goal:

| | | | | |
|---|---|---|---|---|
| 1 | $\neg p$ | | GOAL | $\{\neg p\}$ |

This step is meant to remind one that one is looking for the formula that occurs in the condition, viz. $\neg p$. Remark that the purpose served by a condition in goal directed proofs is very different from the one in dynamic proofs—it is 'prospective' rather than 'defeasible'. In view of the condition of line 1, one introduces a premise from which $\neg p$ may be obtained, and next analyses the premise:

| | | | | |
|---|---|---|---|---|
| 2 | $p \supset (q \wedge s)$ | | PREM | $\emptyset$ |
| 3 | $\neg p$ | 2 | $\supset$E | $\{\neg(q \wedge s)\}$ |

Line 3 illustrates a formula analysing rule: in view of 2, one would have $\neg p$ if one had $\neg(q \wedge s)$. As $\neg(q \wedge s)$ cannot be obtained by analysing a premise, one applies a condition analysing rule to $\neg(q \wedge s)$:

| | | | | |
|---|---|---|---|---|
| 4 | $\neg p$ | 3 | C$\neg\wedge$E | $\{\neg q\}$ |

The following steps require no comment:

| | | | | |
|---|---|---|---|---|
| 5 | $\neg(q \vee r)$ | | PREM | $\emptyset$ |
| 6 | $\neg q$ | 5 | $\neg\vee$E | $\emptyset$ |
| 7 | $\neg p$ | 4, 6 | Trans | $\emptyset$ |

As the main goal is obtained on the empty condition at line 7, the proof is completed.

It is easily seen that, in a proof for $\Gamma \vdash_{\mathbf{CL}} A$, a formula $B$ is derivable on the condition $\Delta$ just in case $\Gamma \cup \Delta \vdash_{\mathbf{CL}} B$.

Some lines are marked in goal-directed proofs for **CL**. Unlike what was the case in the previous section, these marks indicate that one should not try to derive the members of the condition of marked lines. More details are presented in the next section, where these marks will be called D-marks because they relate to derivability—A-marks will relate to the adaptive character of the logic.

## 4 Goal Directed Proofs for ACLuN1

Goal directed proofs for **ACLuN1** have lines that contain *two* conditions:

| | | | | | |
|---|---|---|---|---|---|
| $i$ | $A$ | ... | ... | $\Delta$ | $\Theta$ |

[17] In order to avoid useless complications, I write classical negation as $\neg$ even in the context of **CL**.

The first, $\Delta$, is called the D-condition. This is the condition that also occurs in goal directed proofs for **CL**; it contains the formulas that one needs to derive in order to obtain $A$. The second condition, $\Theta$, is called the A-condition; it contains the abnormalities that should not belong to $U(\Gamma)$ in order for $A$ to be derivable from the premises. The occurrence of the above line $i$ in a proof from $\Gamma$ warrants that $\Gamma \cup \Delta \vdash_{\mathbf{CLuN}} A \vee Dab(\Theta)$. In order to show that $\Gamma \vdash_{\mathbf{ACLuN1}} G$ one needs a line like the displayed one at which $A = G$, $\Delta = \emptyset$, and $\Theta \cap U(\Gamma) = \emptyset$.

To facilitate the exposition, I shall write $A_{\Delta,\Theta}$ to denote that $A$ has been derived on the D-condition $\Delta$ and on the A-condition $\Theta$.

Let us first consider the plot. A goal directed **ACLuN1**-proof for $A_1, \dots ,$ $A_n \vdash G$ will consist of three phases. In the first phase, one tries to obtain $G_{\emptyset,\Theta}$ for some $\Theta$—this phase starts by an application of the Goal rule. If this succeeds, one moves on to phase 2 and tries to obtain $Dab(\Theta)_{\emptyset,\Lambda}$ for some $\Lambda$—this phase starts by an application of the A-Goal rule. If this succeeds, one moves on to phase 3 and tries to obtain $Dab(\Lambda)_{\emptyset,\emptyset}$—this phase starts by an application of the X-Goal rule. If, in phase 3, the X-Goal is reached or the procedure stops, one returns to phase 2; if the procedure stops in phase 2, one returns to phase 1. In phase 1, there are two subphases: 1A and 1B; subphase 1B is introduced by the first application of EFQ. Details are given below.

Four kinds of rules govern a proof for $\Gamma \vdash_{\mathbf{ACLuN1}} G$. The following rules introduce premises or start new phases or subphases of the proof. A-Goal and X-Goal are identical but are used in different contexts.

| | |
|---|---|
| Prem | If $A \in \Gamma$, introduce $A_{\emptyset,\emptyset}$. |
| Goal | Introduce $G_{\{G\},\emptyset}$. |
| A-Goal | If $\Delta \subseteq \Omega$, introduce $Dab(\Delta)_{\{Dab(\Delta)\},\emptyset}$. |
| X-Goal | If $\Delta \subseteq \Omega$, introduce $Dab(\Delta)_{\{Dab(\Delta)\},\emptyset}$. |
| EFQ | If $A \in \Gamma$, introduce $G_{\{\neg A\},\emptyset}$. |

Formula analysing rules (two formulas below the line indicate variants):

$$\supset\text{E}\ \frac{(A \supset B)_{\Delta,\Theta}}{B_{\Delta\cup\{A\},\Theta} \quad \neg A_{\Delta\cup\{\neg B\},\Theta}} \qquad \neg\!\supset\text{E}\ \frac{\neg(A \supset B)_{\Delta,\Theta}}{A_{\Delta,\Theta} \quad \neg B_{\Delta,\Theta}}$$

$$\vee\text{E}\ \frac{(A \vee B)_{\Delta,\Theta}}{A_{\Delta\cup\{\neg B\},\Theta} \quad B_{\Delta\cup\{\neg A\},\Theta}} \qquad \neg\vee\text{E}\ \frac{\neg(A \vee B)_{\Delta,\Theta}}{\neg A_{\Delta,\Theta} \quad \neg B_{\Delta,\Theta}}$$

$$\wedge\text{E}\ \frac{(A \wedge B)_{\Delta,\Theta}}{A_{\Delta,\Theta} \quad B_{\Delta,\Theta}} \qquad \neg\wedge\text{E}\ \frac{\neg(A \wedge B)_{\Delta,\Theta}}{(\neg A \vee \neg B)_{\Delta,\Theta}}$$

$$\equiv\text{E}\ \frac{(A \equiv B)_{\Delta,\Theta}}{(A \supset B)_{\Delta,\Theta} \quad (B \supset A)_{\Delta,\Theta}} \qquad \neg\!\equiv\text{E}\ \frac{\neg(A \equiv B)_{\Delta,\Theta}}{(A \vee B)_{\Delta,\Theta} \quad (\neg A \vee \neg B)_{\Delta,\Theta}}$$

$$\sim\text{E}\ \frac{\sim A_{\Delta,\Theta}}{\neg A_{\Delta,\Theta\cup\{A\wedge\sim A\}}} \qquad \neg\!\sim\text{E}\ \frac{\neg\sim A_{\Delta,\Theta}}{A_{\Delta,\Theta}}$$

$$\neg\neg\text{E}\ \frac{\neg\neg A_{\Delta,\Theta}}{A_{\Delta,\Theta}}$$

Condition analysing rules:

$$\text{C}{\supset}\text{E}\ \frac{A_{\Delta\cup\{B\supset C\},\Theta}}{A_{\Delta\cup\{\neg B\},\Theta}\quad A_{\Delta\cup\{C\},\Theta}} \qquad \text{C}\neg{\supset}\text{E}\ \frac{A_{\Delta\cup\{\neg(B\supset C)\},\Theta}}{A_{\Delta\cup\{B,\neg C\},\Theta}}$$

$$\text{C}{\vee}\text{E}\ \frac{A_{\Delta\cup\{B\vee C\},\Theta}}{A_{\Delta\cup\{B\},\Theta}\quad A_{\Delta\cup\{C\},\Theta}} \qquad \text{C}\neg{\vee}\text{E}\ \frac{A_{\Delta\cup\{\neg(B\vee C)\},\Theta}}{A_{\Delta\cup\{\neg B,\neg C\},\Theta}}$$

$$\text{C}{\wedge}\text{E}\ \frac{A_{\Delta\cup\{B\wedge C\},\Theta}}{A_{\Delta\cup\{B,C\},\Theta}} \qquad \text{C}\neg{\wedge}\text{E}\ \frac{A_{\Delta\cup\{\neg(B\wedge C)\},\Theta}}{A_{\Delta\cup\{\neg B\},\Theta}\quad A_{\Delta\cup\{\neg C\},\Theta}}$$

$$\text{C}{\equiv}\text{E}\ \frac{A_{\Delta\cup\{B\equiv C\},\Theta}}{A_{\Delta\cup\{B,C\},\Theta}\quad A_{\Delta\cup\{\neg B,\neg C\},\Theta}} \qquad \text{C}\neg{\equiv}\text{E}\ \frac{A_{\Delta\cup\{\neg(B\equiv C)\},\Theta}}{A_{\Delta\cup\{\neg B,C\},\Theta}\quad A_{\Delta\cup\{B,\neg C\},\Theta}}$$

$$\text{C}{\sim}\text{E}\ \frac{A_{\Delta\cup\{\sim B\},\Theta}}{A_{\Delta\cup\{\neg B\},\Theta}} \qquad \text{C}\neg{\sim}\text{E}\ \frac{A_{\Delta\cup\{\neg\sim B\},\Theta}}{A_{\Delta\cup\{B\},\Theta\cup\{B\wedge\sim B\}}}$$

$$\text{C}\neg\neg\text{E}\ \frac{A_{\Delta\cup\{\neg\neg B\},\Theta}}{A_{\Delta\cup\{B\},\Theta}}$$

We need two more rules to obtain a complete system. The derivable rule EM0 and the permissible rule IC greatly simplify the proof procedure.

$$\text{Trans}\ \frac{\begin{array}{c}A_{\Delta\cup\{B\},\Theta}\\ B_{\Delta',\Theta'}\end{array}}{A_{\Delta\cup\Delta',\Theta\cup\Theta'}} \qquad \text{EM}\ \frac{\begin{array}{c}A_{\Delta\cup\{B\},\Theta}\\ A_{\Delta'\cup\{\neg B\},\Theta'}\end{array}}{A_{\Delta\cup\Delta',\Theta\cup\Theta'}}$$

$$\text{EM0}\ \frac{A_{\Delta\cup\{\neg A\},\Theta}}{A_{\Delta,\Theta}} \qquad \text{IC}\ \frac{Dab(\Lambda\cup\Lambda')_{\Delta,\Theta\cup\Lambda'}}{Dab(\Lambda\cup\Lambda')_{\Delta,\Theta}}$$

Each phase of the proof starts by applying a goal rule. All further steps proceed in view of D-conditions of unmarked lines, or in view of A-conditions of unmarked lines—see the procedure below. Premises are introduced and formulas analysed iff an element of a D-condition is a *positive part* of the added formula.

That $A$ *is a positive part of* $B$ is defined as follows:

(i) $A$ is a positive part of each of the following: $A$, $A \wedge B$, $B \wedge A$, $A \vee B$, $B \vee A$, $B \supset A$, $A \equiv B$, and $B \equiv A$;
(ii) $A$ is a negative part of $\neg A$, $\sim A$, $A \supset B$, $A \equiv B$, and $B \equiv A$;
(iii) if $A$ is a negative part of $B$, then $\neg A$ and $\sim A$ are positive parts of $B$.
(iv) if $A$ is a positive part of $B$ and $B$ is a positive part of $C$, then $A$ is a positive part of $C$;
(v) if $A$ is a positive part of $B$ and $B$ is a negative part of $C$, then $A$ is a negative part of $C$;
(vi) if $A$ is a negative part of $B$ and $B$ is a positive part of $C$, then $A$ is a negative part of $C$;
(vii) if $A$ is a negative part of $B$ and $B$ is a negative part of $C$, then $A$ is a positive part of $C$.

The efficiency of phase 3 and phase 1A is increased by defining, for those phases, $A$ as a positive part of $\neg{\sim}A$ and by dropping "${\sim}A$" from clauses (ii) and (iii).

A-marking (marking in view of the A-conditions, providing from the adaptive character of the logic) is taken over by the procedure below. D-marking (marking in view of D-conditions) is governed by the following definition.

**Definition 4.** *Where $A_{\Delta,\Theta}$ occurs in the proof at line $i$, line $i$ is D-marked iff one of the following conditions is fulfilled:*

1. *line $i$ is not an application of a goal rule and $A \in \Delta$,*
2. *line $i$ is not an application of a goal rule and, for some $\Delta' \subset \Delta$ and $\Theta' \subseteq \Theta$, $A_{\Delta',\Theta'}$ occurs in the proof,*
3. *no application of EFQ occurs in the proof and $B, \neg B \in \Delta$ for some $B$,*
4. *no application of EFQ occurs in the proof and, for some $B \in \Delta$, $\neg B_{\emptyset,\emptyset}$ occurs in the proof at an unmarked line.*

If 1 is the case, the condition is circular; if 2 is the case, some (set theoretically) weaker condition is sufficient to obtain $A$. In the other two cases, line $i$ indicates a search path that can only be successful if the premises are $\neg$-inconsistent. Although it is not necessary to mark such search paths, it turns out more efficient to postpone them to phase 1B.

I shall first present a rough outline of the procedure and next shall offer some comments on fine tuning.

*The procedure.* The proof procedure for $\Gamma \vdash_{\mathbf{ACLuN1}} G$ consists of three phases—I shall disregard infinite $\Gamma$. The procedure starts in phase 1, may move to phases 2 and 3, and returns to phase 1. During phases 2 and 3, a line may be A-marked (marked in view of its A-condition). A phase stops if no lines can be added in view of conditions introduced during that phase.

*Phase 1.* Aim: to derive $G_{\emptyset,\Theta}$ for some $\Theta$. There are three cases:
(1.1) $G_{\emptyset,\emptyset}$ is derived. Then $\Gamma \vdash_{\mathbf{ACLuN1}} G$.
(1.2) $G_{\emptyset,\Theta}$ is derived, say at line $i$. The procedure moves to phase 2 and later returns to phase 1. There are two cases:
  (1.2.1) line $i$ is not A-marked: $\Gamma \vdash_{\mathbf{ACLuN1}} G$.
  (1.2.2) line $i$ is A-marked: go on (aim: derive $G_{\emptyset,\Theta'}$ for some $\Theta' \not\supseteq \Theta$).
(1.3) The procedure stops and $G_{\emptyset,\Theta}$ is not derived on an unmarked line for any $\Theta$. Then $\Gamma \nvdash_{\mathbf{ACLuN1}} G$.

*Phase 2.* $G_{\emptyset,\Theta}$ was derived in phase 1 for some $\Theta$, say at line $i$. Phase 2 starts by applying A-Goal in order to add $Dab(\Theta)_{\{Dab(\Theta)\},\emptyset}$. Aim: to derive $Dab(\Theta)_{\emptyset,\Lambda}$ for some $\Lambda$ ($\subseteq \Omega$). There are three cases:
(2.1) $Dab(\Theta)_{\emptyset,\emptyset}$ is derived: line $i$ is A-marked; the procedure returns to phase 1.
(2.2) $Dab(\Theta)_{\emptyset,\Lambda}$ is derived for some $\Lambda$, say at line $j$. The procedure moves to phase 3 and later returns to phase 2. There are two cases:

(2.2.1) line $j$ is A-marked: go on (aim: derive $Dab(\Theta)_{\emptyset,\Lambda'}$ for some $\Lambda' \not\supseteq \Lambda$).

(2.2.2) line $j$ is not A-marked: line $i$ is A-marked; the procedure returns to phase 1.

(2.3) $Dab(\Theta)_{\emptyset,\Lambda}$ is not derived for any $\Lambda$ when phase 2 stops. Line $i$ is not A-marked and the procedure returns to phase 1.

*Phase 3.* $G_{\emptyset,\Theta}$ was derived in phase 1 for some $\Theta$, say at line $i$, and $Dab(\Theta)_{\emptyset,\Lambda}$ was derived in phase 2 for some $\Lambda$, say at line $j$. Phase 3 starts by applying X-Goal in order to add $Dab(\Lambda)_{\{Dab(\Lambda)\},\emptyset}$. Aim: to derive $Dab(\Lambda)_{\emptyset,\emptyset}$—all lines added in phase 3 should have the A-condition $\emptyset$. There are two cases:

(3.1) $Dab(\Lambda)_{\emptyset,\emptyset}$ is derived: line $j$ is A-marked; the procedure returns to phase 2.

(3.2) Phase 3 stops without $Dab(\Lambda)_{\emptyset,\emptyset}$ being derived: line $j$ is not A-marked; the procedure returns to phase 2.

*Some fine tuning.* I shall start with some comments that concern the procedure itself, and next offer some comments that pertain to the efficiency of the proofs.

The order in which one tries to apply rules is spelled out in [13]. The idea is: first apply rules in order to obtain the goal of the current phase in a strictly goal directed way, viz. by a sequence of applications of formula analysing rules, condition analysing rules, Trans, EM0, and IC. Next, one tries to obtain the goal by combining the former rules with applications of EM and Trans.

EFQ is never applied in phase 2 or 3. EFQ is only useful if the premises are inconsistent. This is justified by the following consideration. EFQ can only be successfully applied in a proof for $\Gamma \vdash G$ if $\Gamma$ is $\neg$-inconsistent. In that case, $G_{\emptyset,\emptyset}$ is derivable from the premises and will be derived in phase 1B. Deriving any $Dab$-formula from $\Gamma$ by applying EFQ (possibly combined with other rules) is a useless detour.

Moreover, EFQ is only applied in phase 1 at points where no other rule can be applied and, from that point on—that is in subphase 1B—one adds only lines with an empty A-condition to the proof, and hence never moves on to phase 2. The reason for this is obvious: if the main goal can only be obtained by EFQ, then it is derivable by the lower limit logic, viz. **CLuN**, and hence there is no point in deriving it on some A-condition.

I now describe an apparently rather efficient way of proceeding; it is nearly identical for the three phases. Let me start with some general instructions. First, one never applies a formula analysing rule on a formula that does not have a premise in its path. Such steps are provably complications only. Moreover, no line is added to the proof if it would at once be marked.

At each point after line 1 has been written, one first tries to apply EM0, EM and Trans provided this leads to a line being marked.

If this fails, one proceeds in a strictly goal directed way. More particularly, one acts in view of the first formula in the last unmarked condition (of the current phase). If this formula cannot be obtained from the premises, then obtaining the other members of the same condition is useless anyway. If no step is possible in view of the first formula in the last unmarked condition of the current phase—this means that this formula is a dead end—one acts in view of the first formula in the next-to-last unmarked condition of the current phase, and so on.

If it is possible to act in view of the first formula of an unmarked condition of the current phase, one applies the rules in the following order—remember what was said about positive parts. First one tries to apply a formula analysing rule on a formula that occurs on an unmarked line. Next, one tries to introduce a premise. Finally one applies a condition analysing rule (to the formula in view of which one proceeds).

If the goal of the current phase cannot be obtained by strictly goal directed moves, one also applies Trans in order to obtain the goal on all possible (unmarked) conditions, and next one applies EM to unmarked lines that have the current goal as their second element.[18] One returns to strictly goal directed moves as soon as possible.

Only if all this fails, one applies EFQ in phase 1 and, as said before, from there on only adds lines with an empty A-condition.

*Some comments on the metatheory.* The procedure is provably an algorithm for $\Gamma \vdash_{\mathbf{ACLuN1}} A$. In [13] this is proved for **CL**. That proof can easily be transformed to show that the rules from the present section are sound and complete with respect to **CLuN** in the following sense (for finite $\Gamma$):

(1) If $\Gamma \vdash_{\mathbf{CLuN}} G$, then $G_{\emptyset,\emptyset}$ is derived in the dynamic proof for $\Gamma \vdash_{\mathbf{CLuN}} G$.
If $\Gamma \nvdash_{\mathbf{CLuN}} G$, then the dynamic proof for $\Gamma \vdash_{\mathbf{CLuN}} G$ stops.

(2) $A_{\emptyset,\Theta}$ is derivable in the dynamic proof for $\Gamma \vdash_{\mathbf{CLuN}} G$ iff $\Gamma \vdash_{\mathbf{CLuN}} A \vee Dab(\Theta)$.

Given this, it is easily seen that the procedure is sound and complete with respect to **ACLuN1**.

An essential point concerns phase 2. Suppose that $G_{\emptyset,\Theta}$ is derived at line $i$ for some $\Theta$, and that $Dab(\Theta)_{\emptyset,\Lambda}$ is derived for some $\Lambda$ at line $j$. It follows that $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Theta \cup \Lambda)$. If $Dab(\Lambda)_{\emptyset,\emptyset}$ is derived in phase 3, then $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Lambda)$, and hence $Dab(\Theta \cup \Lambda)$ is not a minimal $Dab$-consequence of $\Gamma$. So, $\Theta \cap U(\Gamma) = \emptyset$ iff the following holds for all $\Lambda$: if $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Theta \cup \Lambda)$, then $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Lambda)$. This condition comes to: if $Dab(\Theta)_{\emptyset,\Lambda}$ is derivable, then $Dab(\Lambda)_{\emptyset,\emptyset}$ is derivable.

Precisely this is checked in phase 2: the procedure returns to phase 1 with line $i$ not A-marked iff it holds for all $\Lambda$ that $Dab(\Lambda)_{\emptyset,\emptyset}$ is derivable whenever $Dab(\Theta)_{\emptyset,\Lambda}$ is derivable. So, if the procedure returns to phase 1 with line $i$ not A-marked, then $\Theta \cap U(\Gamma) = \emptyset$ and hence $G$ is finally derived at line $i$.

It is equally easy to see that line $i$ is marked just in case $\Theta \cap U(\Gamma) \neq \emptyset$. If, for some $\Lambda$, $Dab(\Theta)_{\emptyset,\Lambda}$ is derivable whereas $Dab(\Lambda)_{\emptyset,\emptyset}$ is not derivable, then

[18] Apparently this application of EM is useless, but our proof in [13] that the procedure is complete relies on it.

$\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Theta \cup \Lambda)$ whereas $\Gamma \nvdash_{\mathbf{ACLuN1}} Dab(\Lambda)$. It follows that $\Theta \cap U(\Gamma) \neq \emptyset$.[19]

*Some examples.* Let us start with two simple examples. Consider first a goal directed proof for $\sim p \vee r, p \wedge \sim q, q \vdash_{\mathbf{ACLuN1}} r$ :

| | | | | | |
|---|---|---|---|---|---|
| 1 | $r$ | | Goal | $\{r\}$ | $\emptyset$ |
| 2 | $\sim p \vee r$ | | Prem | $\emptyset$ | $\emptyset$ |
| 3 | $r$ | 2 | $\vee$E | $\{\neg\sim p\}$ | $\emptyset$ |
| 4 | $r$ | 3 | C$\neg\sim$E | $\{p\}$ | $\{p \wedge \sim p\}$ |
| 5 | $p \wedge \sim q$ | | Prem | $\emptyset$ | $\emptyset$ |
| 6 | $p$ | 5 | $\wedge$E | $\emptyset$ | $\emptyset$ |
| 7 | $r$ | 4, 6 | Trans | $\emptyset$ | $\{p \wedge \sim p\}$ |
| 8 | $p \wedge \sim p$ | | A-Goal | $\{p \wedge \sim p\}$ | $\emptyset$ |
| 9 | $p \wedge \sim p$ | 8 | C$\wedge$E | $\{p \wedge \sim p\}$ | $\emptyset$ |
| 10 | $p \wedge \sim p$ | 6, 9 | Trans | $\{\sim p\}$ | $\emptyset$ |
| 11 | $\sim p$ | 2 | $\vee$E | $\{\neg r\}$ | $\emptyset$ |
| 12 | $p \wedge \sim p$ | 10 | C$\sim$E | $\{\neg p\}$ | $\emptyset$ |
| 13 | $\neg p$ | 11 | $\sim$E | $\{\neg r\}$ | $\{p \wedge \sim p\}$ |
| 14 | $p \wedge \sim p$ | 12, 13 | Trans | $\{\neg r\}$ | $\{p \wedge \sim p\}$ |
| 15 | $p \wedge \sim p$ | 14 | IC | $\{\neg r\}$ | $\emptyset$ |

The proof is successful: at line 7 $r$ is derived on the empty D-condition and on the A-condition $\{p \wedge \sim p\}$, and in phase 2 $p \wedge \sim p$ turns out not to be derivable on any A-condition. The situation is similar whenever $G_{\emptyset,\Theta}$ is derivable and $Dab(\Theta)_{\emptyset,\Lambda}$ is not derivable for any $\Lambda$. Remark that this always obtains if the premise set is $\sim$-consistent.

Next, consider the goal directed proof for $\sim p, p \vee q, p \vdash_{\mathbf{ACLuN1}} q$ :

| | | | | | |
|---|---|---|---|---|---|
| 1 | $q$ | | Goal | $\{q\}$ | $\emptyset$ |
| 2 | $p \vee q$ | | Prem | $\emptyset$ | $\emptyset$ |
| 3 | $q$ | 2 | $\vee$E | $\{\neg p\}$ | $\emptyset$ |
| 4 | $\sim p$ | | Prem | $\emptyset$ | $\emptyset$ |
| 5 | $\neg p$ | 4 | $\sim$E | $\emptyset$ | $\{p \wedge \sim p\}$ |
| 6 | $q$ | 3, 5 | Trans | $\emptyset$ | $\{p \wedge \sim p\}$ |
| 7 | $p \wedge \sim p$ | | A-Goal | $\{p \wedge \sim p\}$ | $\emptyset$ |
| 8 | $p \wedge \sim p$ | 7 | C$\wedge$E | $\{p, \sim p\}$ | $\emptyset$ |
| 9 | $p \wedge \sim p$ | 4, 8 | Trans | $\{p\}$ | $\emptyset$ |
| 10 | $p$ | 2 | $\vee$E | $\{\neg q\}$ | $\emptyset$ |
| 11 | $p$ | | Prem | $\emptyset$ | $\emptyset$ |
| 12 | $p \wedge \sim p$ | 9, 11 | Trans | $\emptyset$ | $\emptyset$ |
| 13 | $q$ | | EFQ | $\{\neg(p \vee q)\}$ | $\emptyset$ |
| 14 | $q$ | | EFQ | $\{\neg\sim p\}$ | $\emptyset$ |

[19] Indeed, if $\Gamma \vdash_{\mathbf{ACLuN1}} Dab(\Theta \cup \Lambda)$ and $\Gamma \nvdash_{\mathbf{ACLuN1}} Dab(\Lambda)$, there is a non-empty $\Theta' \subseteq \Theta$ and a (possibly empty) $\Lambda' \subseteq \Lambda$ such that $Dab(\Theta' \cup \Lambda')$ is a minimal *Dab*-consequence of $\Gamma$.

After $q_{\{p\wedge\sim p\},\emptyset}$ is derived at line 6, $p \wedge \sim p_{\emptyset,\emptyset}$ turns out to be derivable (line 12). The procedure then sets out to derive $q$ in a different way, which fails. Neither variant of C$\vee$E is applied to the condition of line 13 because the resulting line would at once be marked. C$\neg\sim$E is not applied to the condition of line 14 because doing so would introduce a non-empty A-condition.

Finally, let us consider the goal directed proof for $p, \sim p \vee s, r \supset t, \sim p \vee q, \sim q \vdash_{\mathbf{ACLuN1}} s$:

| | | | | | |
|---|---|---|---|---|---|
| 1 | $s$ | | Goal | $\{s\}$ | $\emptyset$ |
| 2 | $\sim p \vee s$ | | Prem | $\emptyset$ | $\emptyset$ |
| 3 | $s$ | 2 | $\vee$E | $\{\neg\sim p\}$ | $\emptyset$ |
| 4 | $s$ | 3 | C$\neg\sim$E | $\{p\}$ | $\{p \wedge \sim p\}$ |
| 5 | $p$ | | Prem | $\emptyset$ | $\emptyset$ |
| 6 | $s$ | 4, 5 | Trans | $\emptyset$ | $\{p \wedge \sim p\}$ |
| 7 | $p \wedge \sim p$ | | A-Goal | $\{p \wedge \sim p\}$ | $\emptyset$ |
| 8 | $p \wedge \sim p$ | 7 | C$\wedge$E | $\{p, \sim p\}$ | $\emptyset$ |
| 9 | $p \wedge \sim p$ | 8, 5 | Trans | $\{\sim p\}$ | $\emptyset$ |
| 10 | $\sim p$ | 2 | $\vee$E | $\{\neg s\}$ | $\emptyset$ |
| 11 | $\sim p \vee q$ | | Prem | $\emptyset$ | $\emptyset$ |
| 12 | $\sim p$ | 11 | $\vee$E | $\{\neg q\}$ | $\emptyset$ |
| 13 | $\sim q$ | | Prem | $\emptyset$ | $\emptyset$ |
| 14 | $\neg q$ | 13 | $\sim$E | $\emptyset$ | $\{q \wedge \sim q\}$ |
| 15 | $\sim p$ | 12, 14 | Trans | $\emptyset$ | $\{q \wedge \sim q\}$ |
| 16 | $p \wedge \sim p$ | 9, 15 | Trans | $\emptyset$ | $\{q \wedge \sim q\}$ |
| 17 | $q \wedge \sim q$ | | X-Goal | $\{q \wedge \sim q\}$ | $\emptyset$ |
| 18 | $q \wedge \sim q$ | 17 | C$\wedge$E | $\{q, \sim q\}$ | $\emptyset$ |
| 19 | $q \wedge \sim q$ | 13, 18 | Trans | $\{q\}$ | $\emptyset$ |
| 20 | $q$ | 11 | $\vee$E | $\{\neg\sim p\}$ | $\emptyset$ |

Here phase 3 stops, $\neg\sim p$ not being **CLuN**-derivable from the premises. Line 16 is not A-marked and the procedure returns to phase 2; there line 6 is A-marked and the procedure returns to phase 1. The procedure there aims at deriving $s_{\emptyset,\Theta}$ in phase 1 for some $\Theta \not\supseteq \{p \wedge \sim p\}$, which fails.

It is instructive to study the procedure and consider the different states in which it may stop in phase 1.

A computer programme that implements the procedure is available—the above proofs are produced by it. The programme will be used for presenting further examples during the lecture and will be on the internet before this paper appears—`http://logica.rug.ac.be/dirk/`. The data file that goes with the programme contains a set of instructive example exercises.

## 5 In Conclusion

The 'defeasible' conditions that occur in dynamic proofs of adaptive logics suggested a kind of dynamic proofs with 'prospective' conditions. This led to a specific form of goal directed proofs. Later, these goal directed proofs turned out to provide a proof procedure that forms an algorithm for final derivability at the propositional level. As remarked in Section 1, the central interest of the procedure is that it provides a criterion at the predicative level if it stops.

The dynamic proofs explicate actual reasoning. The goal directed proofs do not, but there is an algorithm for turning them into dynamic proofs (by reordering and replacing lines). So, after finding out that some formula is derivable at a stage from the premises, one may switch to the goal directed format in order to find out whether the formula is finally derivable. If a decision is reached, one may transform the result to a regular dynamic proof, if desired. After this, the proof may proceed and, if a further interesting formula is derived at a stage, one may again switch to the goal directed format to settle its final derivability.

This seems the right place to insert a comment on the original application context mentioned in Section 2. It was proved in [4] that $Dab(\Delta \cup \{A\})$ is not a minimal $Dab$-consequence of $\Gamma$ unless $\sim A$ is a subformula of some member of $\Gamma$.[20] In view of this, the goal directed proofs provide a means to locate all minimal $Dab$-consequences of finite premise sets.

Given the present standard characterization (from [8]) of flat adaptive logics, some minimal changes to the aforementioned rules will result in a goal directed procedure for any other adaptive logic. Basically, one replaces the rules that pertain to the abnormalities—in the case of **ACLuN1**, the rules containing the paraconsistent negation $\sim$.

While these replacements are straightforward, further research is required for the predicative level. Devising sensible rules is unproblematic—the relevant research was finished. However, more work is needed to improve the efficiency of the procedure and to avoid infinite loops whenever possible. It is easily seen that known techniques from tableau methods and resolution methods may easily be transposed to the goal directed proofs.[21]

## References

1. Arnon Avron and Iddo Lev. Formula-preferential systems for paraconsistent non-monotonic reasoning (an extended abstract). To appear.
2. Diderik Batens. Dynamic dialectical logics. In Graham Priest, Richard Routley, and Jean Norman, editors, *Paraconsistent Logic. Essays on the Inconsistent*, pages 187–217. Philosophia Verlag, München, 1989.
3. Diderik Batens. Blocks. The clue to dynamic aspects of logic. *Logique et Analyse*, 150–152:285–328, 1995. Appeared 1997.
4. Diderik Batens. Inconsistency-adaptive logics. In Orłowska [19], pages 445–472.
5. Diderik Batens. Zero logic adding up to classical logic. *Logical Studies*, 2:15, 1999. (Electronic Journal: `http://www.logic.ru/LogStud/02/LS2.html`).

[20] At the predicative level, $A$ may be an open formula in which case the corresponding abnormality is the existential closure of $A \wedge \sim A$. The criterion in the text has then to be modified to: $\sim B$ is a subformula of a member of $\Gamma$, where $B$ is obtained by relettering the individual variables in $A$.

[21] Unpublished papers by members of our research group are available from the internet address `http://logica.rug.ac.be/centrum/writings/`.

6. Diderik Batens. Towards the unification of inconsistency handling mechanisms. *Logic and Logical Philosophy*, 8:5–31, 2000. Appeared 2002.
7. Diderik Batens. A dynamic characterization of the pure logic of relevant implication. *Journal of Philosophical Logic*, 30:267–280, 2001.
8. Diderik Batens. A general characterization of adaptive logics. *Logique et Analyse*, in print.
9. Diderik Batens. The need for adaptive logics in epistemology. To appear.
10. Diderik Batens. A strengthening of the Rescher–Manor consequence relations. To appear.
11. Diderik Batens and Joke Meheus. A tableau method for inconsistency-adaptive logics. In Roy Dyckhoff, editor, *Automated Reasoning with Analytic Tableaux and Related Methods*, Lecture Notes in Artificial Intelligence Vol. 1847, pages 127–142. Springer, 2000.
12. Diderik Batens and Joke Meheus. Shortcuts and dynamic marking in the tableau method for adaptive logics. *Studia Logica*, 69:221–248, 2001.
13. Diderik Batens and Dagmar Provijn. Pushing the search paths in the proofs. A study in proof heuristics. *Logique et Analyse*, in print.
14. Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 1: The flat case. *Studia Logica*, 58:17–45, 1997.
15. Salem Benferhat, Didier Dubois, and Henri Prade. Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 2: The prioritized case. In Orłowska [19], pages 473–511.
16. George S. Boolos and Richard J. Jeffrey. *Computability and Logic*. Cambridge University Press, 1989. (Third edition).
17. Iddo Lev. Preferential systems for plausible non-classical reasoning. Master's thesis, Department of Computer Science, Tel-Aviv University, 2000.
18. Joke Meheus. Adaptive logic in scientific discovery: the case of Clausius. *Logique et Analyse*, 143–144:359–389, 1993. Appeared 1996.
19. Ewa Orłowska, editor. *Logic at Work. Essays Dedicated to the Memory of Helena Rasiowa*. Physica Verlag (Springer), Heidelberg, New York, 1999.
20. Nicholas Rescher. *Hypothetical Reasoning*. North-Holland, Amsterdam, 1964.
21. Nicholas Rescher and Ruth Manor. On inference from inconsistent premises. *Theory and Decision*, 1:179–217, 1970.