From Problem Solving to the Teaching of Algebra: the Genesis of the Algebra Textbook

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1 Introduction

Euler's *Vollständige Anleitung zur Algebra* is not only the most popular textbook on elementary algebra, with the exception of Euclid's *Elements* it is the most widely printed book on mathematics (Truesdell, 1972). The book was published in two volumes by the Academy of Sciences in St-Peterburg in 1770. It was translated into Russian (1768-9), Dutch (1773), French (1774), Latin (1790), English (1797, 1822) and Greek (1800). The popular German edition from Reclam Verlag sold no less than 108,000 copies between 1883 and 1943 (Reich, 1992). After 240 years this algebra textbook is still in print today, available in several languages and editions. Such enormous success poses some questions. What makes a book on algebra a good textbook? Hundreds of books on algebra were published before Euler's. Why did this book became a standard for any textbook on mathematics? We will provide an answer to these questions from a historical and epistemological viewpoint. We will do so by tracing the changing meaning of an algebraic problem in books published before Euler and the role of rhetoric in textbooks.

2 Euler's problems

Euler's selection of problems for his *Algebra* displays a great familiarity with the typical recreational and practical problems of Renaissance and sixteenth-century algebra books. A detailed study into the sources of Euler revealed that he copied most of his problems from Christoff Rudolff's *Coss* which was first published in 1525 and reissued in 1553 by Michael Stifel (Heeffer 2007). Why would Euler originate his popular textbook on algebra on a book published 250 years before? Part of the motivation could be sentimental. It appears that Euler was taught mathematics by his father using Stifel's edition of the *Coss*, and the young Euler spent several years studying the problems from the book. However, the inspiration can be explained by the changes in the function of a problem in treatises on algebra. The following section distinguishes several phases of this process.

3 Six stages in the development of algebra textbooks

3.1 Medieval problems as *questiones*

One of the first Latin problem collections found in the Western world is attributed to Alcuin of York under the title *Propositiones ad Acuendos Juvenes* or *Problems to Sharpen the Youth*. The text is dated of around 800 and consist of 53 numbered problems with their solution. The rhetorical structure of these problems is that of a dialogue between a master and his students and is typical for the function of *quaestiones* since antiquity. Rhyme and cadence in riddles and stories provided mnemonic aids and facilitated the oral tradition of problem solving. Medieval students were required to calculate the solution to problems mentally and to memorize rules and examples. The solution depends on precepts which provide no explanation, but are easy to remember rules for solving similar problems.

3.2 Problem solving in the abbaco tradition

While the medieval tradition of riddles or problems with standard recipes was carried through to sixteenth-century arithmetic books, a new tradition of algebraic problem solving emerged in Renaissance Italy. The *Catalogue* by Warren van Egmond (1980) provides ample evidence of a continuous thriving of algebraic practice from the fourteenth till the sixteenth centuries. Over two hundred manuscripts present us with a rare insight in the practice of teaching the basics of arithmetic and algebra to sons of merchants in the abacus schools of major towns in Renaissance Italy. The more skilled of these abacus masters drafted treatises on algebraic problem solving in the vernacular. These consist typically of a short introduction on the basic operations on polynomials and the rules for solving problems (resolving equations). The larger part of these treatises is devoted to the algebraic solution of problems. We can state that the algebraic practice of the abacus tradition *is* the rhetorical formulation of problems using an unknown

3.3 The beginning of algebraic theory

By the end of fifteenth century we observe a change in the rhetorical structure of algebra treatises. While the solution to problems still remains the major focus of the texts, authors pay more attention to the introductory part. While a typical abacus text on algebra was limited to thirty or forty *carta*, the new treatises easily fill hundred folio's. Two trends contribute to more comprehensive approach: the use of the algorism as a rhetorical basis for an introductory theory and the extraction of general principles from practice. An example of the first is the *Practica Arithmeticae* of Cardano (1539). Cardano begins his book with the numeration of whole numbers, fractions, and surds (irrational numbers) as in the algorisms. He then adds *de numeratione denominationum* placing expressions in an unknown in the same league with other numbers, which is completely new. In doing so he shows that the expansion of the number concept has progressed to the point of accepting polynomial expressions as one of the four basic types of numbers. For a second trend in the amplification of an introductory theory in algebraic treatises we can turn to Pacioli. In his Summa of 1494, Pacioli has chosen to present some typical derivations as general rules which are later applied to solve problems in a clear and concise way. The restructuring of existing material and the shift in rhetoric marks an important evolution in the development of sixteenth-century textbooks on algebra. Pacioli raised the testimonies of algebraic problem solving from the abacus masters to the next level of scientific discourse, the textbook. When writing the *Summa*, Pacioli had already almost twenty years of experience in teaching mathematics at several universities. His restructuring of abacus problem solving methods is undoubtedly inspired by this teaching experience.

3.4 Algebra as a model for method and demonstration

The two decades following Cardano's *Practica Arithmeticae* were the most productive in the development towards a symbolic algebra. Cardano (1545) himself secured his fame by publishing the rules for solving the cubic equation in his *Ars Magna* and introduced operations with two equations. In Germany, Michael Stifel (1544) produces his *Arithmetica Integra* which serves as a model of clarity and method for many authors during the following two centuries. Stifel also provided significant improvements in algebraic symbolism, which have been essential during the sixteenth century. In France, Jacques Peletier (1554) published the first French work entirely devoted to algebra, heralding a new wave of French. The process of representing a problem in a symbolic mode and applying the rules of algebra to arrive at a certain solution, have reinforced the belief in a *mathesis universalis*. Such a universal *mathesis* not only allows us to address numerical problems but possibly allows us to solve all problems which we can formulate. The thought originates within the Ramist tradition as part of a broader philosophical discussion on the function and method of mathematics, but the term turns up first in the writings of Adriaan Van Roomen (1597). The idea will flourish in the seventeenth century with Descartes and Leibniz. A *mathesis universalis* is inseparably connected with the newly invented symbolism.

3.5 The generalization of problems to propositions

Many of the textbooks of mid-sixteenth century contained hundreds of problems often of similar types which were intentionally dispersed throughout the book. It is evident that someone who can solve the general case, can solve all individual problems belonging to that case. The path to further generalization was realized as a change in the concept of an equation to a general structure which can be approached under different circumstances. It was Viète who initiated the shift from the solution of problems to the study of the structure of equations and transformations of equations is his *In Artem Analyticem Isagoge* (1591). This method of generalization is continued and completed by Clavius (1608) and Jacques de Billy (1643). The general recipes of Clavius, formulated as canons by de Billy, become propositions and *lemata* themselves which are used as justifications of steps in further propositions. Algebraic reasoning built on references to previous proven propositions and *lemata* changes the rhetoric of problem solving to that of concise rhetoric of justification based on the logic of syllogisms.

3.6 Towards an axiomatization of algebra

The rhetoric of the algebra textbooks in the second half of the seventeenth century clearly shifts through the adoption of the Euclidian style of demonstration. Algebra starts from some simple facts which can be formulated as axioms. All other knowledge about algebraic theorems can be derived from these axioms by deduction. John Wallis introduces the term ‘axioms’ in relation to algebra in an early work, called *Mathesis Universalis*, included in his *Operum mathematicorum* (1657, 85).

4 The function of problems

The examination of algebra textbooks from the point of view of the changing rhetoric of problems, provides us with some interesting insights. Different ways of presenting problems have played a crucial role in the transformation of early abacus manuscripts on algebra into the typical eighteenth-century textbook. While algebra consisted originally of problem solving only, an expansion through the amalgamation of medieval algorisms with abacus texts was the first step towards the modern textbook. Pacioli's appropriation of abacus texts in his *Summa* initiated an important restructuring of algebraic derivations into a theoretical introduction and its application in problem solving. The extension of the number concept and the treatment of operations on irrational binomials and polynomials by Cardano set a new standard for algebra textbooks by his *Practica Arithmeticae*. Humanists such as Ramus and Peletier were inspired by the developments within rhetoric to restructure algebra books and paid more attention to the art of demonstration in algebraic derivations. The emergence of symbolic algebra in the mid-sixteenth century contributed to the idea of a *mathesis universalis*, as a normative discipline for arriving at certain knowledge. By the end of the sixteenth-century the change of focus to the study of the structure of equations led to a more general formulation of problems. The solutions to general problems yielded theorems, propositions and canons, which constituted an extensive body of algebraic knowledge. The rhetoric of seventeenth-century textbooks adopted the Euclidian style of demonstration to provide more rigor in demonstration. The algebra textbooks of the eighteenth century abandoned the constructive role of problems in producing mathematical knowledge. Instead, problems were used only for illustration and for practicing the algebraic language. Recreational problems from the Renaissance, which disappeared from books for almost two centuries, acquired the new function of exercises in transforming problems into equations. Euler's *Algebra* is the textbook intended for self-study, *par excellence*, which revives many older problems. This new established role of problems in algebra text books explains why Euler found in Rudolff's *Coss* a suitable repository of examples.

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