Two type-theoretical approaches to privative modification

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November 20, 2009 - LENLS 6 – Tokyo

Outline

Procedural Semantics

- Programming languages:
  - Denotational semantics: semantics is exhausted by terms denoting extensional entities like sets, individuals and truth-values; the meaning (if any) of a term is its denotation;
  - Procedural semantics: the meaning of a term is one or more computational steps whose product is the term's denotation;
- Two approaches to procedural semantics:
  - Realism: Tichý's Transparent Intensional Logic
  - Idealism: Martin-Löf's Constructive Type Theory

Property Modification

- Property modification:
  - with M a modifier and F a property, (MF) is the result of the procedure of applying the function M to the argument F;

- A full semantic theory of modification must be able to account for the following variants:
  - Subsective: $(M'F)a \therefore Fa$;
  - Intersective: $(M''F)a \therefore M^* a \wedge Fa$;
  - Modal/intensional: $(M'''F)a \therefore Fa \vee \neg Fa$;
  - Privative: $(M''''F)a \therefore \neg Fa$.

Subsective vs. Privative Modification

"a is a prime number"

- given a set of (natural) numbers, the modification of the property of being a number generates the subset of those numbers that have the additional property of being prime numbers;

Conjecture

Subsection exhausts modification for mathematical language.

Subsective vs. Privative Modification (II)

“b is a forged banknote”

- if a privative modifier *M* is applied to a property *F*, then the result is a function whose value is always an empty set of *F*’s;

Open Problem

The problem of positive characterization of privation: what do banknotes and forged banknotes have in common?

Modification and Procedural Semantics

Common features of procedural semantics:

1. a notion of construction;
2. a functional language;
3. type theory;
4. interpreted syntax.

CTT constructions

Predication: necessary and sufficient conditions for a judgement of the form F *type*

- Categorical predication: $f : F$

  f is an element of the set F (or a proof f of proposition F);

- Identity predication: $f = f' : F$

  f and f' are equal elements in F (equivalent proofs)

- Hypothetical predication: F' *type*$[x : F]$

  F' is a type provided there is a construction for F (functional abstraction).

TIL constructions

- Composition: $[X_0 X_1 \ldots X_n]$
  X_0 is a construction of a function, $X_1, \ldots, X_n$ constructions of its arguments and [ ] the procedure of functional application;
- Closure: $[\lambda x_1 \ldots x_n Y]$
  $x_1, \ldots, x_n$ construct arguments, Y constructs values of a function and $[\lambda x_1 \ldots x_n Y]$ is the procedure of functional abstraction.

CTT as a functional language

- Propositional function F' *type*$[x : F]$: is the predication of a type F' depending on some predication holding for type F;
- Subsective modification $M(F)$: treated by functional abstraction producing subset formation $\{x : F \mid M(x)\}$ (extensional): for every element in the set F taken as argument, it returns a function $M(x)$;

Privative Modification

$M(f)$: *takes as arguments elements in F and ranges over functions from the basic type F to the empty set of F's.*

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TIL as a functional language

- Ramified type hierarchy where each entity receives a type:
  - ground types (o-truth values, ι-individuals, τ-reals doubling as times, ω-possible worlds),
  - functional types by induction over ground types
  - constructions of order $n+1$ constructing constructions of order n;

Privative Modification

$(MF)a$: functional application of M to a property F; the extensionalization of (MF) is predicated of an individual a.

Privative Subset Formation Rule

Privation: Given $x : F$ as input of a function M, $M(x)$ returns the empty set of f's as its output:

$$\frac{F\ set \qquad M(x)[F : El(\{\})\ set; x : El(\{\}); El(F(x))]}{\{x : F \mid M(x)\}}$$

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Identity: For any equivalent set taken as argument of the modifier function, the same empty set is obtained:

$$\frac{F\ set \qquad F = F'\ set \qquad M(x)[F = F' : El(\{\})\ set; x : El(\{\}); El(F = F'(x))]]}{\{x : F = F' \mid M(x)\}}$$

Introduction and Elimination Rules

Introduction rule provides an appropriate construction of a set F of privatively modified individuals:

$$\frac{f : F \qquad m : M(f)[F : El(\{\})\ set; f : El(\{\}); El(F(f))]}{f : \{x : F \mid M(x)\}}$$

$$\frac{f = f' : F \qquad m : M(f)[F : El(\{\})\ set; f : El(\{\}); El(F(f))]}{f = f' : \{x : F \mid M(x)\}}$$

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Elimination rule specifies how to extract a modified individual from its corresponding set:

$$\frac{f : \{x : F \mid M(x)[\Delta]\} \qquad f'(x) : M'(x)[x : F, m : M(x)]}{f'(f) : M'(f)}$$

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Iteration of Modifiers

The construction of a (well-made (forged banknote)) is of the following form:

$$\frac{\dfrac{\textit{banknote set} \qquad \textit{forged}(x)[\Delta]}{\{x : \textit{banknote} \mid \textit{forged}(x)\}} \qquad \textit{well} - \textit{made}(x)[x : \textit{banknote} \mid \textit{forged}(x)]}{\{x : \textit{banknote} \mid \textit{well} - \textit{made} \times \textit{forged}(x)\}}$$

The construction of a ((well-made forged) banknote) is an illegitimate one:

$$\frac{\textit{banknote set} \qquad \textit{well} - \textit{made}(x)[x : \textit{banknote}] \times \textit{forged}(x)[\Delta]}{\{x : \textit{banknote} \mid \textit{well} - \textit{made}(x) \wedge \textit{forged}(x)[\Delta]\}}$$

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TIL Constructions of Modified Properties

- Predication as application of extensionalized property to individual:

$$\lambda w \lambda t \, [property_{wt} \; a]$$

- Composition of a modified property:

$$[modifier \; property]$$

- Predication of a modified property:

$$\lambda w \lambda t \, [[modifier \; property]_{wt} \; a]$$

Iteration

- Predication of a modified modified property:

$$\lambda w \lambda t\ [[modifier'\ [modifier\ property]]_{wt}\ a]$$

Examples:

- "*a* is a burned forged banknote"
- "*a* is a well-made forged banknote".

Requisites of Privation

- The essence of the property F is the set of properties p such that p is a *requisite* of F:

$$[essence\ F] = \lambda p\,[Req\ p\ F]$$

- Definition of the *requisite* relation:

$$[Req\ YX] = \forall w \forall t\,[\forall x\,[[True_{wt} \lambda w \lambda t\,[X_{wt} x]] \rightarrow [True_{wt} \lambda w \lambda t\,[Y_{wt} x]]]]$$

Requisites of Privation (cont.)

- The property *not being a banknote* is a requisite of the property *being a forged banknote*:

$$[Req\ \lambda w \lambda t\ \neg[banknote_{wt}\ x][forged\ banknote]]$$

- This Composition is equivalent to the following Composition:

$$\forall w \forall t\ [\forall x\ [[forged\ banknote]_{wt}\ x] \rightarrow [\neg[banknote_{wt}\ x]]]$$

- No forged banknote is a banknote and some non-banknotes are forged banknotes:

$$\forall w \forall t\ [[[All\ [forged\ banknote]_{wt}][\lambda x\ \neg[banknote_{wt}\ x]]] \wedge [[Some\ [\lambda x\ \neg[banknote_{wt}\ x]]]\ \lambda x\ [[forged\ banknote]_{wt}\ x]]]$$

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Conclusions

1. CTT construes privation as dependent typing under condition of a typed empty set;
2. TIL produces a modified property by the functional application of a modifier to a property; the resulting modified property is extensionalized for predication.