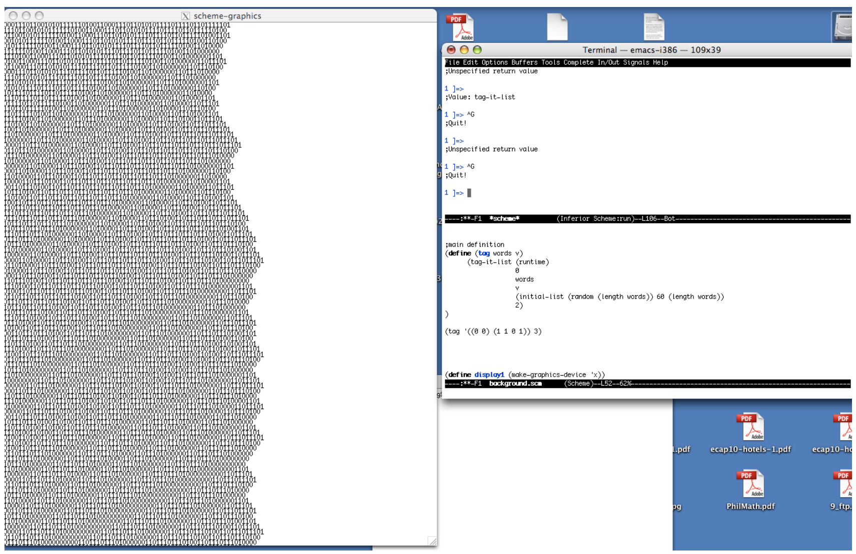
Generating, solving and the mathematics of Homo Sapiens. Emil Post's views on computation



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Some publicity first...

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Introduction

Topic What is a computation? Post's version***s*** of the Church-Turing thesis (CTT), the history that underlies them and the philosophy that results from them.

Motivation

- *If* one accepts CTT one still does not know the universe of the computable, but accepts the CTT limit
- Rise of the electronic, general-purpose computer has extended the scope of the computable (theoretical, practical and 'disciplinary') and makes this limitation 'real/concrete'

$\Rightarrow$ Significance of understanding and exploring the double-face of CTT $\rightarrow$ the non-computable?

One approach? Digging into the historical roots of CTT

1. Church-Turing thesis

What is the Church-Turing thesis?

⇒What was it about?

	Identification	Vague notion	Formal device
Church:	definition	eff. calculability	λ-def. & gen. rec. functions
Turing:	definition	computability	Turing machines

⇒Why?

- Context of mathematical logic, *NOT* computer science (20s and 30s)
- Motivation: "[T]he contemporary practice of mathematics, using as it does heuristic methods, only makes sense because of this undecidability. When the undecidability fails then mathematics, as we now understand it, will cease to exist; in its place there will be a mechanical prescription for deciding whether a given sentence is provable or not" (Von Neumann, 1927)

Why Turing rules!

Church's thesis "We now define the notion [...] of an **effectively calculable** function of positive integers by identifying it with the notion of a **recursive function** of positive integers (or of a **λ-definable** function of positive integers.)"

Turing's thesis According to my definition, a number is computable if its decimal expansion can be written down by **a machine**"

⇒ "*[I]t was Turing alone who [...] gave the first convincing formal definition of a computable function*" (Soare, 2007). Why?

- **Church's 'approach'**: Thesis *after* a thorough analysis of λ-calculus and recursive functions (bottom-up)
- **Turing's main question:** "The real question at issue is: What are the possible processes which can be carried out in computing a number?" (Turing, 1936) – from intuition to formalism; analysis of such processes results in TM-concept (top-down)

⇒ Turing: intuitively appealing TMs (the direct appeal to intuition)

2. Post's two theses

Two theses, two sides

Post′s Thesis I ⇒ *Post′s thesis II*

Normal systems | *Formulation I*

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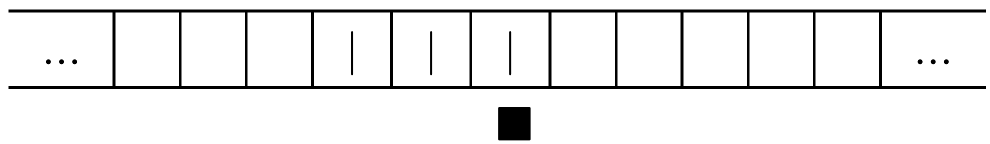
produces ⇒ 

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Generated sets ⇒ *Solvability*

I **1921** *To prove the existence of* **absolutely** *insolvable problems (e.g. halting problem)...*

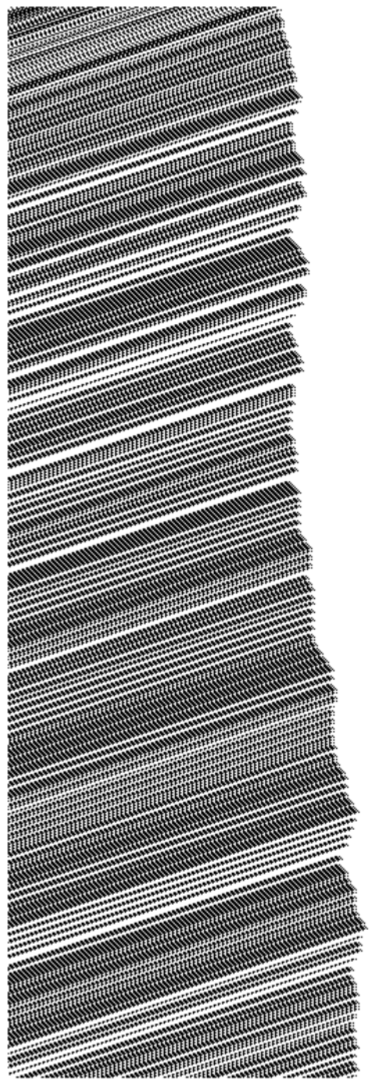
Post's thesis I For every set of sequences for which a process can be set-up to generate it there is also a normal system which will generate it.

II **1936** *To improve on Church's thesis (making it intuitively appealing)...*

Post's thesis II Solvability of a problem in the intuitive sense coincides with solvability by formulation 1

⇒ *Where do these two logically equivalent formulations come from? Why two theses?*

Thesis I: Generating sequences and limits of the computable



Post's radical formalism as a method to study math (Post's programme)

⇒ Various documents: (PhD, *Account of an anticipation*, *Note on a fundamental problem in postulate theory*)

⇒ **Goal?** "*[T]o obtain theorems about all [mathematical] assertions*"

⇒ **Approach?** Development of a "*general form of symbolic logic*" as an "*instrument of generalization*" characterized by the "*method of combinatory iteration*" which "*eschews all interpretation*" – modeling (processes of) symbolic logic (∼ Lewis' "mathematics without meaning"):

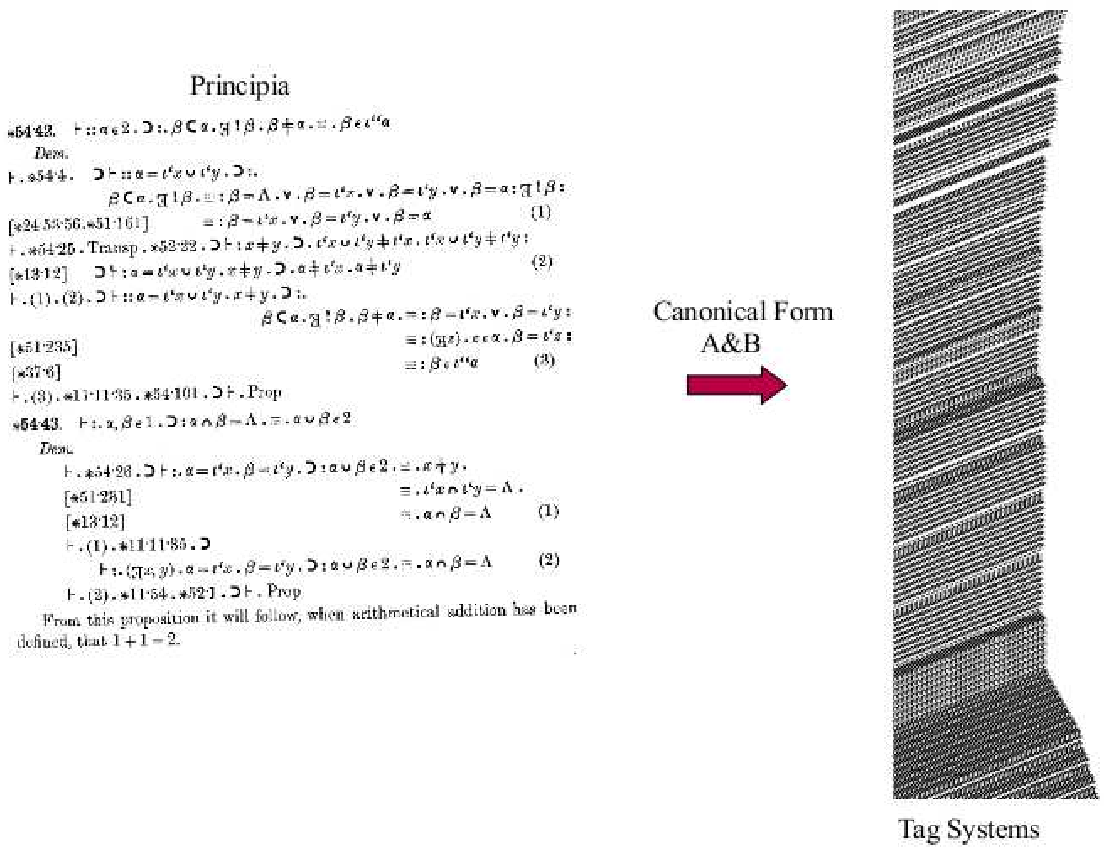
> [T]he method of combinatory iteration **completely neglects [...] meaning**, and considers the entire system purely from the symbolic standpoint as one in which both the enunciations and assertions are groups of symbols or symbol-complexes [....] and where these symbol assertions are obtained by starting with certain initial assertions and repeatedly applying certain rules for obtaining new symbol-assertions from old.

⇒ **1920-21**: Deciding the "*finiteness problem*" for first-order logic

> "*Since Principia was intended to formalize all of existing mathematics, Post was proposing no less than to find a single algorithm for all of mathematics.*" (Davis, 1994)

Account of an anticipation: towards the reversal of Post's programme

Simplification through generalization:



Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion

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$\cancel{001}\underbrace{101110111010000\textcolor{blue}{00}}_{A_0} \Rightarrow$ Periodicity!

- $\Rightarrow$ Definition of a *class* of symbolic logics according to a form
- $\Rightarrow$ Very much in the spirit of the method of combinatory iteration – pure symbol manipulators without meaning. Symbolization?
- $\Rightarrow$ Study of two decision problems (finiteness problems) for tag systems: the halting and reachability problem starting from the simplest case to the more 'complex' ones ($\mu = 1, 2, 3, ..., v = 1, 2, 3...$ – unpublished manuscript)

The frustrating problem of "Tag" and the reversal of Post's programme

⇒ **Exploring tag systems: pencil-and-paper computations and "observations"**

- "Observation" of three classes of behavior: periodicity, halt, unbounded growth.
- Three decidable classes ($v = 1$;$\mu = 1$; $\mu = v = 2$) (Wang, 1963; De Mol, 2010) – the proof involved "*considerable labor*"
- Infinite class with $\mu = 2, v = 3$: "intractable" (Minsky, 1967; De Mol, 2011)
- Infinite class with $\mu > 2, v = 2$: a zoo of TS of "bewildering complexity"

⇒ *Principia* vs. Lewis-like Abstract form ("mathematics without meaning") → shift to an analysis of the behavior → limitations of Lewis' ideal mathematics

⇒ **The reversal** "[T]he general problem of "tag" appeared hopeless, and with it our entire program of the solution of finiteness problems. This *frustration* [my emphasis], however, was largely based on the assumption that "tag" was but a minor, if essential, stepping stone in this wider program." (Post,1965)

After nine months of tagging....

⇒ Development of two more forms: *canonical form C* (Post production systems) and *Normal form:*

$$\begin{array}{ccc} g_i P_i & 1101P_i & \cancel{1101}110111010000000 \\ & \textit{produces} & \\ P_i g_{i'} & P_i 001 & 110111010000000001 \end{array}$$

⇒ Insight that apparent simplicity does not imply 'real' simplicity: Proof of *"the most beautiful theorem in mathematics"* (Minsky, 1961)

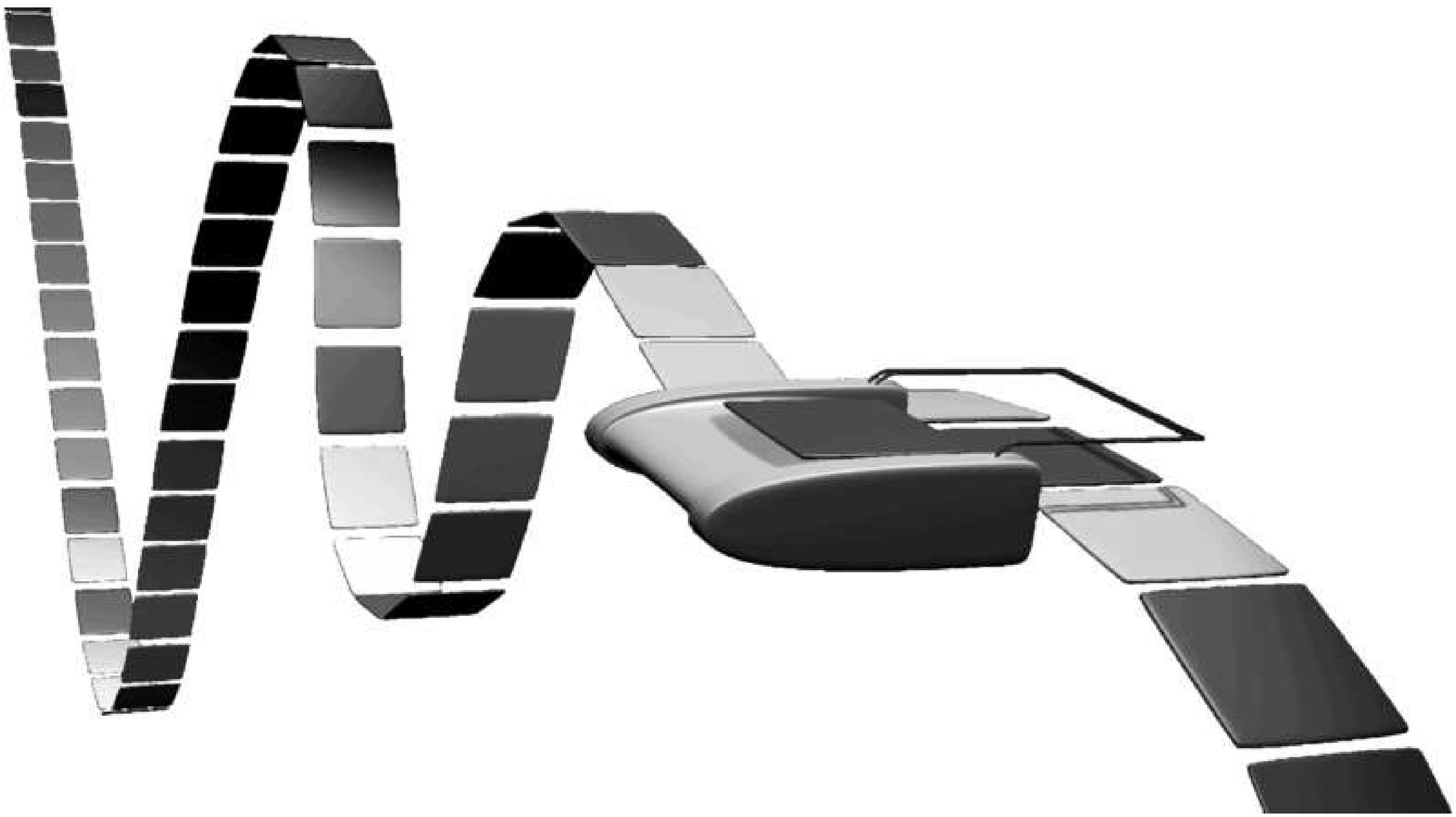
⇒ Idea that the whole PM can be reduced to normal form

> [F]or if the meager formal apparatus of our final normal systems can wipe out all of the additional vastly greater complexities of canonical form [...], the more complicated machinery of [Principia] should clearly be able to handle formulations correspondingly more complicated than itself.

⇒ Post's thesis I – anything that can be "generated" can also be "generated" by the "primitive" normal form

⇒ The finiteness problem for normal form is *absolutely* unsolvable

Thesis II: Solvability and the realm of the computable



Taking into account the human factor in generating sets....

Problem with thesis I: Post's believe in thesis I rooted in his own experiences and interaction with his forms ⇒ less convincing for people not familiar with these forms (see e.g. correspondence Church-Post: "*while it is clear that every generated set in your sense is lambda-enumerable (recursively enumerable), I can see no way of proving the converse of this, and at the moment, therefore, it seems to me possible that the notion of a generated set is less general.*" (Church to Post, June 26, 1936)

Post's analysis: "*[for the thesis to obtain its full generality] an analysis should be made of all the possible ways* **the human mind** *can set up finite processes to generate sequences.*" (∼ Turing's "What are the possible processes which can be carried out in computing a number?")

> "[E]stablishing this universality [of the characterization of generated set of sequences in terms of normal form] is not a matter for mathematical proof, but of **psychological analysis of the mental processes involved in combinatory mathematical processes**.

⇒ **Post's solution**: Identification between Solvability and Formulation 1 (almost identical to Turing machines)

… opposition against definitional character of Church's thesis

A working hypothesis "*Its purpose is not only to present a system of a certain logical potency but also, [...] of psychological fidelity*" (more than just about defining the scope of the computable – to capture all humanly possible processes!)

> [T]o mask this identification under a definition hides the fact that a fundamental discovery in the limitations of the mathematicizing power of *Homo Sapiens* has been made and blinds us to the need of its continual verification.

Post's new programme – Towards a natural law In search of wider and wider formulations and to prove that all these are logically reducible to the original formulation 1

When the bubble of symbolic logic finally burst...

Two theses, two sides: Limitations and extending the scope of the 'solvable'

Thesis I

- Insight that a simple form can be much more powerful than its apparent simplicity would suggest
- Confrontation with the limitation of the method of combinatory iteration/radical formalism – even tag system escape the finite methods of symbolic logic!

⇒ *If* thesis I true, then the ultimate human method for solving/deciding all mathematical problems in finite time does not exist. Thesis I is as much about finding a general form for symbolic logic as it is about the *limits* of that same symbolic logic

Thesis II

- To capture all the possible processes the human mind can set up to solve decision problems (Post's second programme)
- Motivation programme lies in the negative side of Thesis I and II → limitations and necessity to explore them (to see how far they reach).

Two theses, two sides: time, processes and experimentation

Thesis II

- solv*ability*, comput*ability*, calcul*ability*
- Goal: develop formal device which allows to *correctly* solve a problem after a finite number of steps, *at some point in time* (included as a formal requirement in formulation 1!)

⇒ What about modern computational situations? Thesis II still a good paradigm?

Thesis I (older models)

- No halting requirement
- Generating sequences (as general form of math) rather than solving a problem

⇒ BUT: forces attention on **computational processes** – time and computation!

⇒ More exploratory approach and significance of connection limits thesis I and II and the **complexity of the behavior** of computational processes

⇒ In this way, one of Post's historically older and less intuitively appealing 'models' are more adept to modern research with e.g. its focus on the relation between processes in nature and computation,

Thesis I-II and the mathematics of Homo Sapiens (I)

⇒ Why this insistence of Post on thesis as a hypothesis?

- Results rooted in confrontation with his own human limitations – not only those of symbolic logic ("*my wife is much worried. So I told her, really for the first time, the exact history of my mental ups and downs and worse from its first inception in trying to solve the probably unsolvable tag-problem in Princeton*')
- Post's philosophy of math:

  "I consider mathematics as a product of the human mind, not as absolute"

  [T]he finitary character of symbolic logic follows from the fact that it is "*essentially a human enterprise, and that when this is departed from, it is* then *incumbent on such a writer to add a qualifying "non-finitary"*.

Thesis I-II and the mathematics of Homo Sapiens (II)

⇒ Existence absolutely unsolvable problem

> The writer cannot overemphasize the fundamental importance to mathematics of the existence of absolutely unsolvable combinatory problems. True, with a specific criterion of solvability under consideration, say recursiveness [...], the unsolvability in question, as in the case of the famous problems of antiquity, becomes merely unsolvability by a given set of instruments. [The] fundamental new thing is that for the combinatory problems the given set of instruments is in effect the only humanly possible set.

⇒ Only relative to humans: "*the troubling thought [is suggested] have we so fathomed all our own powers as to insure our assertion of absolute unsolveability relative to us.*"

⇒ Future for symbolic logic?

> with the bubble of symbolic logic as universal logical machine finally burst, a new future dawns for it as the indispensable means for revealing and developing those limitations. For [...] Symbolic Logic may be said to be Mathematics become self-conscious.

Discussion – afterthoughts

Discussion – afterthoughts

- Value historical studies of early models? Lesson Post for me?
  - CTT is as much about the universe of computable as it is about the limits of that self-same universe, shaped by *human mathematics*
  - Interest in processes that result from generalization, formalization and symbolization without the aim of modeling something 'natural' (BUT, human math/symb.log.-model) – 'anti-simulation' ⇒ abstract computational devices that necessitate computer-assisted studies
- Physical machines and Post's work? (Burroughs company and syntactical machines; Backus, Chomsky, etc)
- Significance studying older and less 'natural' models (not intended as models of ...) + their limits
  - Study limits from the 'computable' side (bottom-up) – determines limits also for more 'natural' models (physics, biology, computer science)
  - Different enough from natural processes – allows to zoom-in on 'non-natural' aspects of computation
  - But, also several 'properties' in common (complex and erratic behavior, unpredictability, time-aspect etc)
  - Easily studied with computer (very 'simple' description)

For if symbolic logic has failed to give wings to mathematicians this study of symbolic logic opens up a new field concerned with the fundamental limitations of mathematics, more precisely the mathematics of Homo Sapiens." (Post to Church, March 24, 1936)