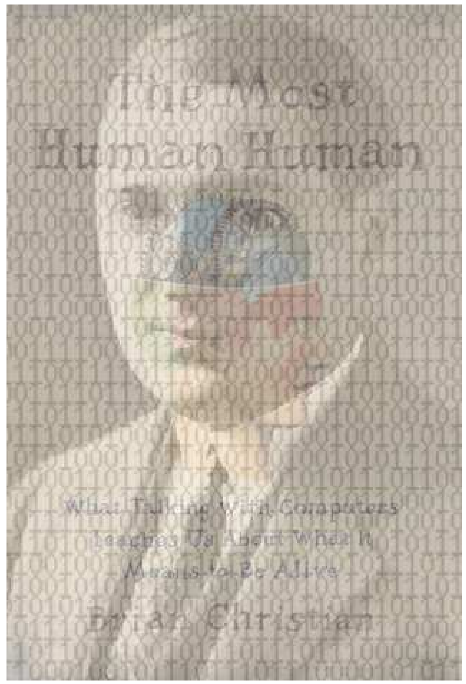
When the bubble of symbolic logic finally burst.
Emil Post's formalism(s)



Liesbeth De Mol
Centre for Logic and Philosophy of Science
Universiteit Gent, Belgium
elizabeth.demol@ugent.be

First, some publicity.....

Turing in Context II: Historical and Contemporary research in Logic, computing machinery and AI

10-12 October 2012

http://www.computing-conference.ugent.be/tic2

Brussels

Keynotes: S. Barry Cooper, Leo Corry, Daniel Dennett, Marie Hicks, Maurice Margenstern, Elvira Mayordomo, Alexandra Shlapentokh, Rineke Verbrugge

Introduction

Topic Post's formalism(s) – show how it resulted in two different versions of the Church-Turing thesis (CTT) + in its own 'destruction' (the bubble)

Motivation

– (Turing year ;-))

– *If* one accepts CTT one still does not know the universe of the computable, but accepts the CTT limit

– Rise of the electronic, general-purpose computer has extended the scope of the computable (theoretical, practical and 'disciplinary') and makes this limitation 'real/concrete'

⇒ Significance of understanding and exploring the double-face of CTT → the non-computable?

One approach? Digging into the historical roots of CTT

1. Church-Turing thesis

What is the Church-Turing thesis?

⇒What was it about?

	Identification	**Vague notion**	**Formal device**
Church:	definition	eff. calculability	λ-def. & gen. rec. functions
Turing:	definition	computability	Turing machines

⇒Why?

- Context of mathematical logic, *NOT* computer science (20s and 30s)
- Motivation: "[T]he contemporary practice of mathematics, using as it does heuristic methods, only makes sense because of this undecidability. When the undecidability fails then mathematics, as we now understand it, will cease to exist; in its place there will be a mechanical prescription for deciding whether a given sentence is provable or not" (Von Neumann, 1927)

Why Turing rules!

⇒ "*[I]t was Turing alone who [...] gave the first convincing formal definition of a computable function*" (Soare, 2007). Why?

- **Church's 'approach'**: Thesis *after* a thorough analysis of λ-calculus and recursive functions (bottom-up)
- **Turing's main question:** "The real question at issue is: What are the possible processes which can be carried out in computing a number?" (Turing, 1936) – from intuition to formalism; analysis of such processes results in TM-concept (top-down)

⇒ Turing: intuitively appealing TMs (the direct appeal to intuition)

2. Post's two theses/formalisms

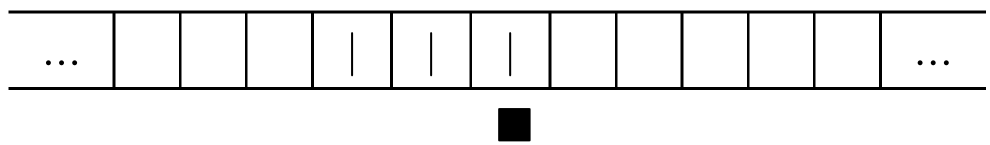
Two theses, two sides

Post′s Thesis I ⇒ *Post′s thesis II*

Normal systems — *Formulation I*

~~1101~~11011101000000

produces ⇒

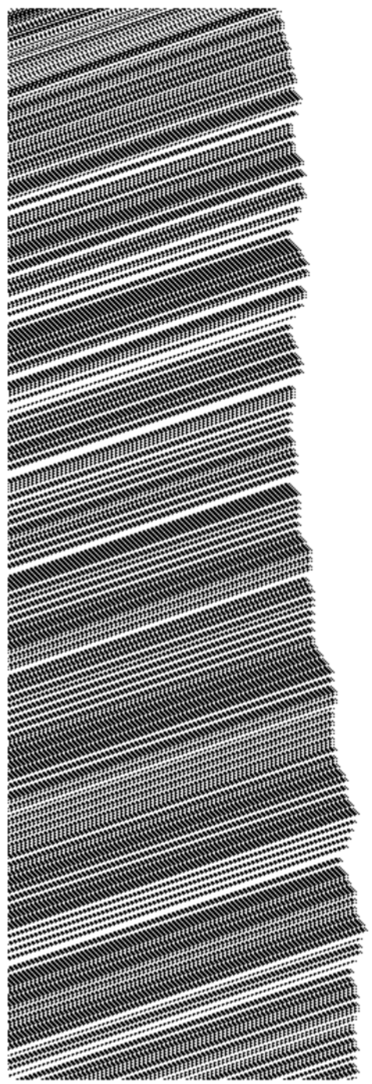


1101110100000001

Generated sets ⇒ *Solvability*

⇒ *Where do these two logically equivalent formulations come from? Why two theses?*

Thesis I: Generating sequences and limits of the computable



Post's radical formalism as a method to study math (Post's programme)

⇒ Various documents: (PhD, *Account of an anticipation*, *Note on a fundamental problem in postulate theory*)

⇒ **Approach?** Development of a "*general form of symbolic logic*" as an "*instrument of generalization*" characterized by the "*method of combinatory iteration*" which "*eschews all interpretation*" – modeling (processes of) symbolic logic (∼ Lewis' "mathematics without meaning"):

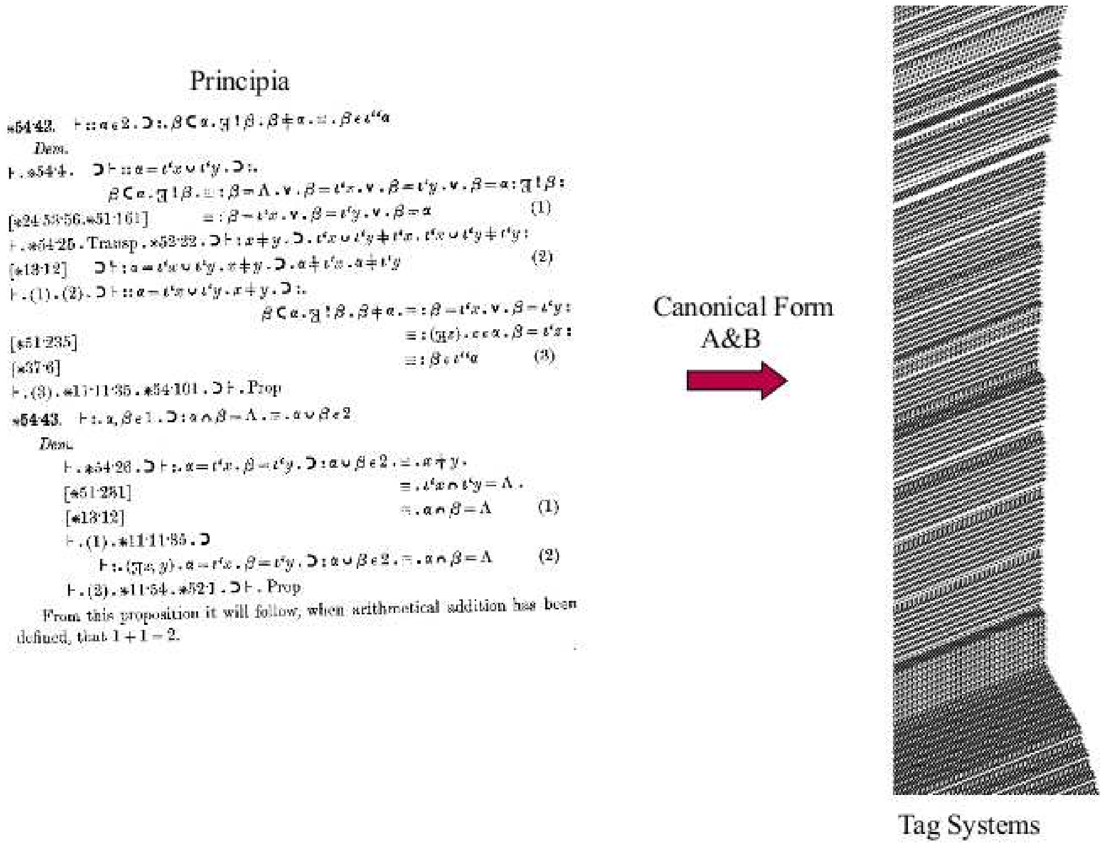
> [T]he method of combinatory iteration **completely neglects [...] meaning**, and considers the entire system purely from the symbolic standpoint as one in which both the enunciations and assertions are groups of symbols or symbol-complexes [....] and where these symbol assertions are obtained by starting with certain initial assertions and repeatedly applying certain rules for obtaining new symbol-assertions from old.

⇒ **Goal?** "*[T]o obtain theorems about all [mathematical] assertions*"

⇒ **1920-21**: Deciding the "*finiteness problem*" for first-order logic

> "*Since Principia was intended to formalize all of existing mathematics, Post was proposing no less than to find a single algorithm for all of mathematics.*" (Davis, 1994)

Post's Formalism(s) at work: Generalization



Generalization I

	Propositional Logic	**Canonical form A**
I.	If p is an elementary proposition than so is $\sim p$ If p and q are elementary propositions than so is $p \vee q$	If $p_1, \ldots, p_{m_1}$ are elementary propositions than so is $f_1(p_1, \ldots, p_{m_1})$ $\vdots$ If $p_1, \ldots, p_{m_\mu}$ are elementary propositions than so is $f_\mu(p_1, \ldots, p_{m_\mu})$
II.	The assertion of a function involving a variable p produces the assertion of any function found from the given one by substituting for p any other variable q, or $\sim q$, or $(q \vee r)$[a]	The assertion of a function involving a variable p produces the assertion of any function found from the given one by subsituting for p any other variable q, or $f_1(q_1, \ldots, q_{m_1})$, or $f_\mu(q_1, \ldots, q_{m_\mu})$
III.	$\vdash P$ $\vdash \sim P \vee Q$	$\vdash g_{11}(P_1, ..., P_{k_1}) \ldots \vdash g_{r_1}(P_1, ..., P_{r_r})$ $\vdots$ $\vdash g_{1r_1}(P_1, ..., P_{r_1}) \ldots g_{rr_r}(P_1, ..., P_{r_r})$

[a]This corresponds to substitution Continued on next page

Table 1 – continued from previous page		
	Propositional Logic	**Canonical form A**
	produce $\vdash Q$	produce produce $\vdash g_1(P_1, ..., P_{k_1}) \ldots \vdash g_r(P_1, ..., P_{r_r})$
IV.	Postulates: $\vdash \sim (p \vee p) \vee p$ $\vdash \sim (p \vee (q \vee r)) \vee (q \vee (p \vee r))$ $\vdash \sim q \vee (p \vee q)$ $\vdash \sim (\sim q \vee r) \vee (\sim (p \vee q). \vee (p \vee r))$ $\vdash (p \vee q) \vee (q \vee p)$	Postulates: $\vdash h_1(p_1, p_2, \ldots, p_{l_1})$ $\vdash h_2(p_1, p_2, \ldots, p_{l_2})$ $\ldots$ $\ldots$ $\vdash h_\lambda(p_1, p_2, \ldots, p_{l_\lambda})$

Generalization II

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

Generalization II

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0,1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion

~~101~~110111010000001101

Generalization II

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion

~~101~~11011101000000 1101

~~110~~111010000001101 1101

Generalization II

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion
~~101~~110111010000001101
~~110~~1110100000011011101
~~111~~01000000110111011101

Generalization II

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion

~~101~~11011101000000**1101**

~~110~~1110100000011011**101**

~~111~~0100000011011101**1101**

~~010~~0000011011101110100

Generalization II

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion

~~101~~110111010000001101

~~110~~1110100000011011101

~~111~~01000000110111011101

~~010~~0000011011101110100

~~000~~001101110111010000

Generalization II

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion

~~101~~110111010000001101

~~110~~1110100000011011101

~~111~~01000000110111011101

~~010~~0000011011101110100

~~000~~001101110111010000

~~001~~$\underbrace{10111011101000000}_{A_0}$ $\Rightarrow$ Periodicity!

Generalization II

Definition of tag systems. A (relatively) famous Example

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion

~~101~~110111010000001101

~~110~~1110100000011011101

~~111~~01000000110111011101

~~010~~0000011011101110100

~~000~~001101110111010000

~~001~~$\underbrace{10111011101000000}_{A_0}$ $\Rightarrow$ Periodicity!

- $\Rightarrow$ Definition of a *class* of symbolic logics according to a form
- $\Rightarrow$ Very much in the spirit of the method of combinatory iteration – pure symbol manipulators without meaning. Symbolization?
- $\Rightarrow$ Study of two decision problems (finiteness problems) for tag systems: the halting and reachability problem starting from the simplest case to the more 'complex' ones ($\mu = 1, 2, 3, ..., v = 1, 2, 3...$ – unpublished manuscript)

The frustrating problem of "Tag" and the reversal of Post's programme

⇒ **Exploring tag systems: pencil-and-paper computations and "observations"**

- "Observation" of three classes of behavior: periodicity, halt, unbounded growth.
- Three decidable classes ($v = 1$;$\mu = 1$; $\mu = v = 2$) (Wang, 1963; De Mol, 2010) – the proof involved "*considerable labor*"
- Infinite class with $\mu = 2, v = 3$: "intractable" (Minsky, 1967; De Mol, 2011)
- Infinite class with $\mu > 2, v = 2$: a zoo of TS of "bewildering complexity"

⇒ *Principia* vs. Lewis-like Abstract form ("mathematics without meaning") → forces shift to an analysis of the behavior → limitations of Lewis' ideal mathematics

⇒ **The reversal** "[T]he general problem of "tag" appeared hopeless, and with it our entire program of the solution of finiteness problems. This *frustration* [my emphasis], however, was largely based on the assumption that "tag" was but a minor, if essential, stepping stone in this wider program." (Post,1965)

After nine months of tagging....

⇒ Development of two more forms: *canonical form C* (Post production systems) and *Normal form:*

$$g_i P_i \qquad 1101P_i \qquad \cancel{1101}11011101000000$$

$$\textit{produces}$$

$$P_i g_{i'} \qquad P_i 001 \qquad 11011101000000\color{blue}{001}$$

⇒ Insight that apparent simplicity does not imply 'real' simplicity: Proof of *"the most beautiful theorem in mathematics"* (Minsky, 1961)

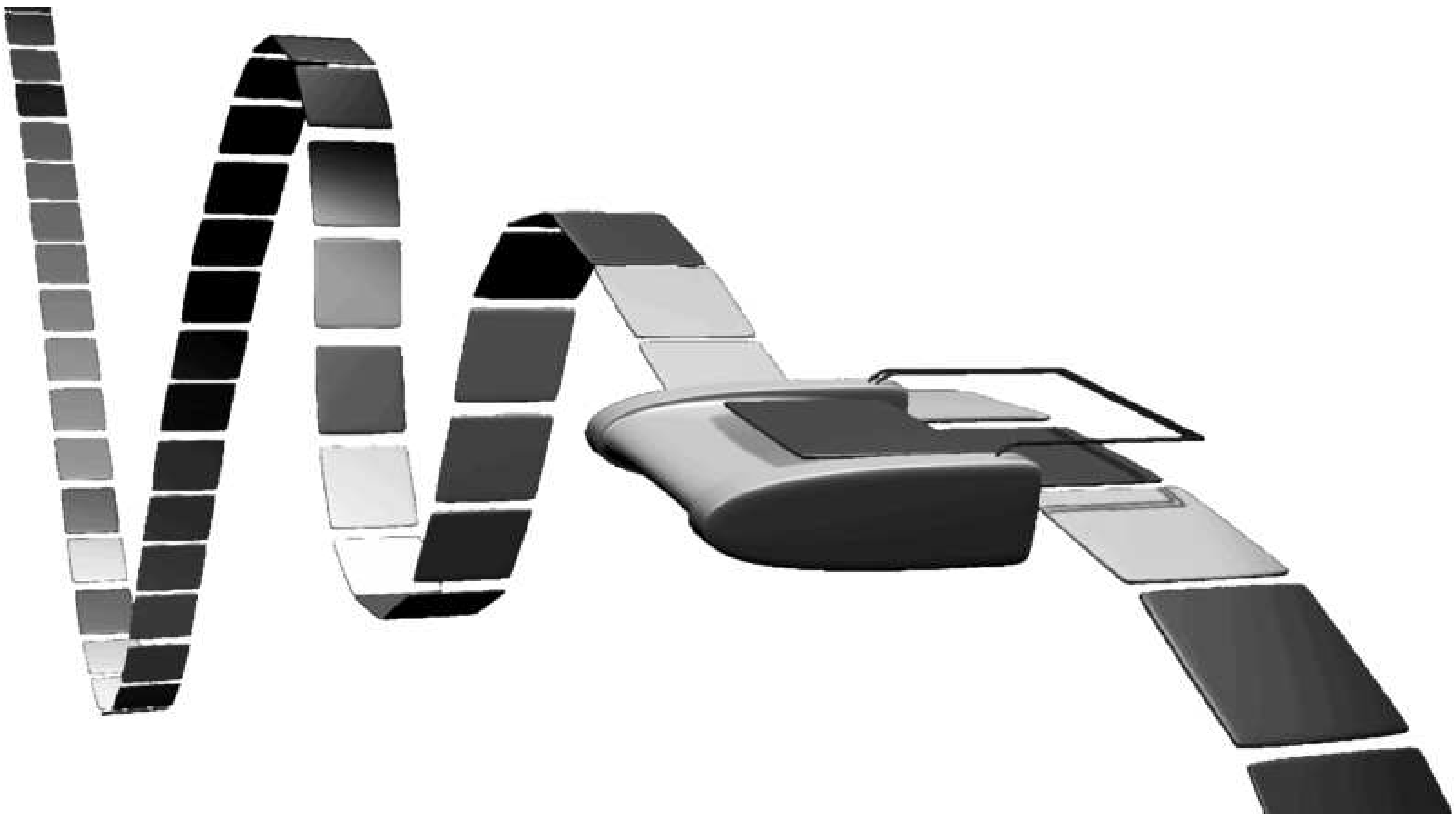
⇒ Idea that the whole PM can be reduced to normal form

> [F]or if the meager formal apparatus of our final normal systems can wipe out all of the additional vastly greater complexities of canonical form [...], the more complicated machinery of [Principia] should clearly be able to handle formulations correspondingly more complicated than itself.

⇒ Post's thesis I – anything that can be "generated" can also be "generated" by the "primitive" normal form

⇒ The finiteness problem for normal form is *absolutely* unsolvable

Thesis II: Solvability and the realm of the computable



Taking into account the human factor in generating sets....

Problem with thesis I: Post's believe in thesis I rooted in his own experiences and interaction with his forms ⇒ less convincing for people not familiar with these forms (see e.g. correspondence Church-Post: "*while it is clear that every generated set in your sense is lambda-enumerable (recursively enumerable), I can see no way of proving the converse of this, and at the moment, therefore, it seems to me possible that the notion of a generated set is less general.*" (Church to Post, June 26, 1936)

Post's analysis: "*[for the thesis to obtain its full generality] an analysis should be made of all the possible ways* **the human mind** *can set up finite processes to generate sequences.*" (∼ Turing's "What are the possible processes which can be carried out in computing a number?")

> "[E]stablishing this universality [of the characterization of generated set of sequences in terms of normal form] is not a matter for mathematical proof, but of **psychological analysis of the mental processes involved in combinatory mathematical processes**.

⇒ **Post's solution**: Identification between Solvability and Formulation 1 (almost identical to Turing machines)

... opposition against definitional character of Church's thesis

A working hypothesis "*Its purpose is not only to present a system of a certain logical potency but also, [...] of psychological fidelity*" (more than just about defining the scope of the computable – to capture all humanly possible processes!)

> [T]o mask this identification under a definition hides the fact that a fundamental discovery in the limitations of the mathematicizing power of *Homo Sapiens* has been made and blinds us to the need of its continual verification.

Post's new programme – Towards a natural law In search of wider and wider formulations and to prove that all these are logically reducible to the original formulation 1

When the bubble of symbolic logic finally burst...

Thesis I-II and the mathematics of Homo Sapiens (I)

⇒ Why this insistence of Post on thesis as a hypothesis?

- Results rooted in confrontation with his own human limitations – not only those of symbolic logic ("*my wife is much worried. So I told her, really for the first time, the exact history of my mental ups and downs and worse from its first inception in trying to solve the probably unsolvable tag-problem in Princeton*')
- Post's philosophy of math:

  > "I consider mathematics as a product of the human mind, not as absolute"
  >
  > [T]he finitary character of symbolic logic follows from the fact that it is "*essentially a human enterprise, and that when this is departed from, it is* then *incumbent on such a writer to add a qualifying "non-finitary".*

Thesis I-II and the mathematics of Homo Sapiens (II)

⇒ Existence absolutely unsolvable problem

> The writer cannot overemphasize the fundamental importance to mathematics of the existence of absolutely unsolvable combinatory problems. True, with a specific criterion of solvability under consideration, say recursiveness [...], the unsolvability in question [...] becomes merely unsolvability by a given set of instruments. [The] fundamental new thing is that for the combinatory problems **the given set of instruments is in effect the only humanly possible set.**

⇒ Only relative to humans: "*the troubling thought [is suggested] have we so fathomed all our own powers as to insure our assertion of absolute unsolveability relative to us.*"

⇒ Future for symbolic logic?

> with the bubble of symbolic logic as universal logical machine finally burst, a new future dawns for it as the indispensable means for revealing and developing those limitations. For [...] Symbolic Logic may be said to be Mathematics become self-conscious.

Discussion – afterthoughts

Discussion: Time and processes in Post's theses

Thesis II

- solv*ability*, comput*ability*, calcul*ability*
- Goal: develop formal device which allows to *correctly* solve a problem after a finite number of steps, *at some point in time* (included as a formal requirement in formulation 1!)

⇒ What about modern computational situations? Thesis II still a good paradigm?

Thesis I (older models)

- No halting requirement
- Generating sequences (as general form of math) rather than solving a problem

⇒ BUT: forces attention on **computational processes** – time and computation!

⇒ More exploratory approach and significance of connection limits thesis I and II and the **complexity of the behavior** of computational processes

⇒ In this way, one of Post's historically older and less intuitively appealing 'models' are more adept to modern research with e.g. its focus on the relation between processes in nature and computation,

Afterthoughts

- Lesson Post for me?
  - CTT is as much about the universe of computable as it is about the limits of that self-same universe, shaped by *human mathematics*
  - Interest in processes that result from generalization, formalization and symbolization without the aim of modeling something 'natural' (BUT, human math/symb.log.-model) – 'anti-simulation' ⇒ abstract computational devices that necessitate computer-assisted studies
- Significance studying older and less 'natural' models (not intended as models of ...) + their limits
  - Study limits from the 'computable' side (bottom-up) – determines limits also for more 'natural' models (physics, biology, computer science)
  - Different enough from natural processes – allows to zoom-in on 'non-natural' aspects of computation
  - But, also several 'properties' in common (complex and erratic behavior, unpredictability, time-aspect etc)
  - Easily studied with computer (very 'simple' description)

For if symbolic logic has failed to give wings to mathematicians this study of symbolic logic opens up a new field concerned with the fundamental limitations of mathematics, more precisely the mathematics of Homo Sapiens." (Post to Church, March 24, 1936)