Two Adaptive Logics of Norm-Propositions

Mathieu Beirlaen, Christian Straßer

Centre for Logic and Philosophy of Science, Ghent University, Blandijnberg 2, 9000 Ghent, Belgium.
Phone: ++ 32 9 264 39 79
Fax: ++ 32 9 264 41 87

Abstract

We present two defeasible logics of norm-propositions (statements about norms) that (i) consistently allow for the possibility of normative gaps and normative conflicts, and (ii) map each premise set to a sufficiently rich consequence set. In order to meet (i), we define the logic **LNP**, a conflict- and gap-tolerant logic of norm-propositions capable of formalizing both normative conflicts and normative gaps within the object language. Next, we strengthen **LNP** within the adaptive logic framework for non-monotonic reasoning in order to meet (ii). This results in the adaptive logics $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$, which interpret a given set of premises in such a way that normative conflicts and normative gaps are avoided 'whenever possible'. $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$ are equipped with a preferential semantics and a dynamic proof theory.

Keywords: deontic logic, normative gaps, normative conflicts, adaptive logic, defeasible reasoning

*Corresponding author: Mathieu Beirlaen
Email addresses: `mathieu.beirlaen@UGent.be` (Mathieu Beirlaen), `christian.strasser@UGent.be` (Christian Straßer)

1. Introduction

1.1. Normative conflicts and normative gaps

Ideally, sets of norms issued by agents, authorities, legislators, etc. are both consistent and complete. In our everyday practice, however, such sets often contain norm-conflicts and norm-gaps. A norm-conflict occurs when two or more norms are issued that are mutually unsatisfiable. The existence of such conflicts is motivated as follows by Alchourrón and Bulygin:

> Even one and the same authority may command that p and that *not* p at the same time, especially when a great number of norms are enacted on the same occasion. This happens when the legislature enacts a very extensive statute, e.g. a Civil Code, that usually contains four to six thousand dispositions. All of them are regarded as promulgated at the same time, by the same authority, so that there is no wonder that they sometimes contain a certain amount of explicit or implicit contradictions [3, pp. 112-113].

Norm-conflicts do not always consist of conflicting commands or obligations. They also arise where both an obligation to do something and a (positive) permission not to do it are promulgated [1, 3, 12, 46].

The logics presented in this paper should not just be able to adequately deal with normative conflicts but also with normative gaps. We say that a set of norms contains a *normative gap* with respect to a A if A is neither positively permitted nor forbidden nor obliged. For a defense of the existence of normative gaps, see e.g. [2, Chapters 7,8], [13].

Note that the formulation refers to *positive permissions* (also, *strong permissions*), i.e. permissions that are either explicitly stated as such, or permissions that are derivable from other explicitly stated permissions or obligations. This is to be distinguished from so-called *weak* or *negative permissions*: A is weakly permitted in case A is not forbidden. Would we replace "positive permission" by "weak permission" in the definition of normative gaps then the concept would be vacuous since each A is either forbidden or not forbidden (and hence, weakly permitted).

The practical use of the distinction between positive and negative permission can be illustrated by means of the legal principle *nullum crimen sine lege*. According to this principle anything which is not forbidden is permitted. Alternatively, the principle states that a negative permission to do A implies a positive permission to do A. Typically, the nullum crimen principle is understood as a rule of closure permitting all the actions not prohibited by penal law [2, pp. 142-143]. We return to this principle in Section 2.1.

We will in the remainder of the paper tacitly assume that in case A is obliged then A is positively permitted. In this case, there is a normative gap with respect to A iff A is neither positively permitted nor forbidden.

Another way to think about normative gaps is in terms of normative determination: A is *normatively determined* if and only if A is either positively permitted or forbidden, which is to say that there is no normative gap with respect to A.[1] We say that a set of norms is *normatively complete* if all of its norms are normatively determined, i.e. if there are no gaps with respect to any of its norms. From the existence of incomplete legal systems, Bulygin concludes that legal gaps are perfectly possible:

> It is not true that all legal systems are necessarily complete. The problem of completeness is an empirical, contingent, question, whose truth depends on the contents of the system. So legal gaps due to the silence of the law ... are perfectly possible [13, p. 28].

1.2. Norm-propositions and their formal representation

In ordinary language, normative sentences exhibit a characteristic ambiguity. The very same words may be used to enunciate a norm (give a prescription) and to make a normative statement (description) [44, pp. 104-106]. In deontic logic, it is important to carefully distinguish between this prescriptive and descriptive use of norms.

When interpreted prescriptively, a formula of the form $\ulcorner \mathsf{O}A \urcorner$ means something like "you ought to do $\ulcorner A \urcorner$", or "it ought to be that $\ulcorner A \urcorner$", and a formula of the form $\ulcorner \mathsf{P}A \urcorner$ means something like "you may do $\ulcorner A \urcorner$", or "it is permitted that $\ulcorner A \urcorner$".[2] When interpreted descriptively, a formula of the form $\ulcorner \mathsf{O}A \urcorner$ [$\ulcorner \mathsf{P}A \urcorner$] means something like "there is a norm to the effect that $\ulcorner A \urcorner$ is obligatory [permitted]". Following [44], we take the term *norm* to denote the prescriptive, and *norm-proposition* to denote the descriptive interpretation of normative statements.[3]

According to Alchourrón and Bulygin [1, 2, 3], any perceived harmony between norms and norm-propositions in deontic logic is merely apparent. Instead of using the same calculus of deontic logic for reasoning with both norms and norm-propositions, we need two separate logics: a logic of norms and a logic of norm-propositions. This paper is concerned with the characterization of a logic of norm-propositions.

In formal language normative conflicts are expressed by formulas such as $\ulcorner \mathsf{O}A \wedge \mathsf{O}\ \textit{not}\ A \urcorner$ in case two obligations conflict, and $\ulcorner \mathsf{O}A \wedge \mathsf{P}\ \textit{not}\ A \urcorner$ in case an obligation conflicts with a permission. We call a conflict of the former kind an OO-*conflict*, and a conflict of the latter kind an OP-*conflict*.

Normative gaps occur if neither $\ulcorner \mathsf{P}A \urcorner$ nor $\ulcorner \mathsf{O}\ \textit{not}\ A \urcorner$ is the case. A full formal characterization of normative gaps is presented after the definition of our formal language. As pointed out above, the permission in question is a strong permission. Weak permissions may be simply defined as the modal dual to O:

[1] The notion of normative determination is adopted from [45].

[2] Until our formal language is defined, we use brackets "$\ulcorner$" and "$\urcorner$" for denoting formulas.

[3] Von Wright [44] and Åqvist [5] cite Ingemar Hedenius as the first philosopher to note the distinction between norms and norm-propositions. According to Hedenius, norms are "genuine", and norm-propositions are "spurious" deontic sentences [22]. The distinction between norms and norm-propositions was later also drawn – among others – by Wedberg [47], Stenius [38], Alchourrón[1], and Hansson [20] (see also [5]).

by $\ulcorner$*not* O *not* $A\urcorner$. The latter expresses that "there is no norm to the effect that $\ulcorner$*not* $A\urcorner$ is obliged" and hence it expresses the descriptive meaning of a weak permission. However, we need an independent permission operator P in order to express strong permissions. From $\ulcorner\mathsf{P}A\urcorner$ we cannot infer $\ulcorner$*not* O *not* $A\urcorner$ due to the possible existence of an OP-conflict. Similarly we cannot, vice versa, infer $\ulcorner\mathsf{P}A\urcorner$ from $\ulcorner$*not* O *not* $A\urcorner$ since, despite the absence of a norm that expresses that $\ulcorner$*not* $A\urcorner$ is obliged, $\ulcorner A\urcorner$ may not be positively permitted.[4]

1.3. The structure of the paper

This paper is structured as follows. In Section 2 we define the logic **LNP** of which the syntax is sufficiently expressive to formalize all properties peculiar to norm-propositions without having to resort to the meta-language. Inside the scope of its deontic operators, **LNP** makes use of a paraconsistent and paracomplete negation connective for dealing with normative conflicts and normative gaps.

As a result of the weakness of this negation connective, **LNP** is not powerful enough for capturing many intuitive normative inferences. We deal with this problem in Section 3, where we strengthen **LNP** within the adaptive logics framework for non-monotonic reasoning. This results in two adaptive logics which interpret a given premise set 'as consistently and as completely as possible'.

In Section 4 we equip the logics defined in Section 3 with a proof theory, and prove some further meta-theoretical results. In Section 5, we compare our logics to other approaches taken up in the literature on norm-propositions and on conflicting norms.

2. A negation-weakened foundation: the logic LNP

2.1. Syntax

In the setting of norm-propositions, negation behaves differently depending on whether it occurs inside or outside the scope of an operator O or P. Outside the scope of a deontic operator, negation behaves classically. A formula $\ulcorner$*not* $\mathsf{O}p\urcorner$ is read as "it is *not* the case that there is a norm to the effect that p is obligatory". Under this reading, $\ulcorner$*not* $\mathsf{O}p\urcorner$ is incompatible with $\ulcorner\mathsf{O}p\urcorner$: $\ulcorner\mathsf{O}p\urcorner$ and $\ulcorner$*not* $\mathsf{O}p\urcorner$ cannot both be the case.

Things change when we turn to negations inside the scope of O or P. Here, both $\ulcorner\mathsf{O}p\urcorner$ and $\ulcorner\mathsf{O}$ *not* $p\urcorner$ are verified by the same set of norm-propositions if this set contains an OO-conflict with respect to p. Moreover, neither $\ulcorner\mathsf{P}p\urcorner$ nor $\ulcorner\mathsf{O}$ *not* $p\urcorner$ are verified by a given set of norm-propositions that contains a normative gap with respect to p. Given the standard laws of distribution for O and P, this means that – inside the scope of O or P– both the consistency and

[4]See [2, 44] for further arguments against the equivalence of $\ulcorner\mathsf{P}A\urcorner$ and $\ulcorner$*not* O *not* $A\urcorner$ in a descriptive setting.

the completeness constraint for negation fail in some instances: $\ulcorner \mathsf{P}(p \wedge \textit{ not } p)\urcorner$ is true in case of a normative conflict, and $\ulcorner \mathsf{O}(p \vee \textit{ not } p)\urcorner$ is false in case of a normative gap.

The logic **LNP** is defined in such a way that it respects this distinction: outside the scope of a deontic operator, only the classical negation connective "$\neg$" occurs. Inside the scope of a deontic operator, **LNP** makes use of the connective "$\sim$", which is a paraconsistent and paracomplete "negation" connective, i.e. it invalidates both $\ulcorner (A \wedge {\sim}A) \supset B\urcorner$ (*Ex Contradictione Quodlibet*) and $\ulcorner A \vee {\sim}A\urcorner$ (*Excluded Middle*).[5]

Where $\mathcal{W}^a = \{p, q, r, \ldots\}$ is a denumerable set of atomic propositions, we define $\mathcal{W}^\neg$ as the $\langle \neg, \vee, \wedge, \supset, \equiv\rangle$-closure of $\mathcal{W}^a$, and $\mathcal{W}^\sim$ as the $\langle \sim, \vee, \wedge, \supset, \equiv\rangle$-closure of $\mathcal{W}^a$. Let:
$\mathcal{W}^{\mathsf{O}} := \mathsf{O}\langle \mathcal{W}^\sim\rangle \mid \mathsf{P}\langle \mathcal{W}^\sim\rangle \mid \neg\langle \mathcal{W}^{\mathsf{O}}\rangle \mid \langle \mathcal{W}^{\mathsf{O}}\rangle \vee \langle \mathcal{W}^{\mathsf{O}}\rangle \mid \langle \mathcal{W}^{\mathsf{O}}\rangle \wedge \langle \mathcal{W}^{\mathsf{O}}\rangle \mid \langle \mathcal{W}^{\mathsf{O}}\rangle \supset \langle \mathcal{W}^{\mathsf{O}}\rangle \mid \langle \mathcal{W}^{\mathsf{O}}\rangle \equiv \langle \mathcal{W}^{\mathsf{O}}\rangle$
Then the set $\mathcal{W}$ of well-formed formulas of **LNP** is defined as the $\langle \neg, \vee, \wedge, \supset, \equiv\rangle$-closure of $\mathcal{W}^\neg \cup \mathcal{W}^{\mathsf{O}}$.
Since the denotation of formulas is no longer ambiguous now that our language $\mathcal{W}$ is defined, we skip the $\ulcorner\urcorner$-marks in the remainder of the paper. For future reference, we also define the set $\mathcal{W}^l = \{A, {\sim}A \mid A \in \mathcal{W}^a\}$ of $\sim$-literals.

Both normative conflicts and normative gaps are expressible in the object language $\mathcal{W}$. A normative conflict occurs relating to a formula $A \in \mathcal{W}^\sim$ whenever we can derive one of $\mathsf{O}A \wedge \mathsf{O}{\sim}A$ or $\mathsf{O}A \wedge \mathsf{P}{\sim}A$. A normative gap occurs relating to A whenever we can derive $\neg\mathsf{P}A \wedge \neg\mathsf{O}{\sim}A$, i.e. whenever there is no norm to the effect that A is permitted or forbidden.

The P-operator functions as an operator for positive permission. A proposition A is said to be negatively permitted if there is no obligation to the contrary, i.e. if $\neg\mathsf{O}{\sim}A$. The nullum crimen principle can be formalized as an axiom schema:

(NC) $\quad \neg\mathsf{O}{\sim}A \supset \mathsf{P}A$

Clearly, (NC) a priori excludes the possibility of normative gaps. That is why it is invalidated by any gap-tolerant logic of norm-propositions.

2.2. Semantics

LNP is characterizable within a Kripke-style semantics with a set of worlds or points W and a designated or 'actual' world $w_0 \in W$. In w_0, negation is defined classically by means of the connective "$\neg$". In the other worlds, negation is defined by the paraconsistent and paracomplete connective "$\sim$".[6]

[5] "$\sim$" as defined below is actually a "dummy" connective rather than a negation connective: it has no properties at all, except that it validates de Morgan's laws. However, in sections 3 and 4 we show that "$\sim$" functions as a negation connective in the adaptive extensions of the logic **LNP**.

[6] The semantic clauses for accessible worlds are inspired by those for (the propositional fragment of) Batens' paraconsistent and paracomplete logic **CLoNs**, a variation on the paraconsistent logic **CLuNs** as found in e.g. [8]. **CLoNs** is defined in Section Appendix A of the Appendix.

An **LNP**-model is a tuple $\langle W, w_0, R, v_0, v\rangle$, where $R = \{w_0\} \times (W \setminus \{w_0\})$ is a serial accessibility relation, and $v_0 : \mathcal{W}^a \to \{0,1\}$ and $v : \mathcal{W}^l \times (W \setminus \{w_0\}) \to \{0,1\}$ are assignment functions. v_0 assigns truth-values to atomic propositions. Since all logical connectives (including negation) behave classically in this world, truth values for complex formulas can be defined in terms of a valuation function in the usual way. The situation is slightly different for other worlds. In the latter, the $\sim$-connective does not behave classically and truth values are assigned to all $\sim$-literals, i.e. all atomic propositions p and their $\sim$-negation $\sim p$.
Let $w \in W, w' \in W \setminus \{w_0\}$. Then the valuation $v_M : (\mathcal{W} \times \{w_0\}) \cup (\mathcal{W}^{\sim} \times W \setminus \{w_0\}) \to \{0,1\}$, associated with the model M, is defined by

(C_0)	where $A \in \mathcal{W}^a, v_M(A, w_0) = 1$ iff $v_0(A) = 1$
(C_l)	where $A \in \mathcal{W}^l, v_M(A, w') = 1$ iff $v(A, w') = 1$
(C$\neg$)	$v_M(\neg A, w_0) = 1$ iff $v_M(A, w_0) = 0$
(C$\sim\sim$)	$v_M(\sim\sim A, w') = 1$ iff $v_M(A, w') = 1$
(C$\sim\supset$)	$v_M(\sim(A \supset B), w') = 1$ iff $v_M(A \wedge \sim B, w') = 1$
(C$\sim\wedge$)	$v_M(\sim(A \wedge B), w') = 1$ iff $v_M(\sim A \vee \sim B, w') = 1$
(C$\sim\vee$)	$v_M(\sim(A \vee B), w') = 1$ iff $v_M(\sim A \wedge \sim B, w') = 1$
(C$\sim\equiv$)	$v_M(\sim(A \equiv B), w') = 1$ iff $v_M((A \vee B) \wedge (\sim A \vee \sim B), w') = 1$
(C$\supset$)	$v_M(A \supset B, w) = 1$ iff $v_M(A, w) = 0$ or $v_M(B, w) = 1$
(C$\wedge$)	$v_M(A \wedge B, w) = 1$ iff $v_M(A, w) = v_M(B, w) = 1$
(C$\vee$)	$v_M(A \vee B, w) = 1$ iff $v_M(A, w) = 1$ or $v_M(B, w) = 1$
(C$\equiv$)	$v_M(A \equiv B, w) = 1$ iff $v_M(A, w) = v_M(B, w)$
(CO)	$v_M(\mathsf{O}A, w_0) = 1$ iff $v_M(A, w') = 1$ for every w' such that Rw_0w'
(CP)	$v_M(\mathsf{P}A, w_0) = 1$ iff $v_M(A, w') = 1$ for some w' such that Rw_0w'

(C_0) and (C_l) simply take over the values of the assignment functions v_0 and v respectively. (C$\neg$) determines truth values for the classical negation connective "$\neg$" in w_0. (C$\sim\sim$)-(C$\sim\equiv$) guarantee that de Morgan's laws hold for "$\sim$" in accessible worlds. (C$\supset$)-(C$\equiv$) determine truth values for the other classical connectives $\supset, \wedge, \vee$, and $\equiv$ in all worlds, and (CO) and (CP) define the deontic operators O and P in the usual way.
A semantic consequence relation for **LNP** is defined in terms of truth preservation at the actual world. An **LNP**-model M verifies A ($M \Vdash A$) iff $v_M(A, w_0) = 1$. Where $\Gamma \subseteq \mathcal{W}$, M is an **LNP**-model of Γ iff M is an **LNP**-model and $M \Vdash A$ for all $A \in \Gamma$. $\vDash_{\mathbf{LNP}} A$ iff all **LNP**-models verify A, and $\Gamma \vDash_{\mathbf{LNP}} A$ iff all **LNP**-models of Γ verify A.

2.3. Axiomatization and meta-theory

Inside the scope of O and P, we want to allow for the consistent possibility of contradictions and gaps. In order to do so, we make use of the propositional fragment of the logic **CLoNs** (cfr. footnote 6). **CLoNs** is defined by adding de Morgan's laws for "$\sim$" to the positive fragment of classical (propositional) logic **CL**:[7]

[7] An axiomatization of the positive fragment of **CL** (i.e. **CL** without negation) is contained in Section Appendix A of the Appendix.

(A$\sim\sim$)	$\sim\sim A \equiv A$
(A$\sim\supset$)	$\sim(A \supset B) \equiv (A \wedge \sim B)$
(A$\sim\wedge$)	$\sim(A \wedge B) \equiv (\sim A \vee \sim B)$
(A$\sim\vee$)	$\sim(A \vee B) \equiv (\sim A \wedge \sim B)$
(A$\sim\equiv$)	$\sim(A \equiv B) \equiv ((A \vee B) \wedge (\sim A \vee \sim B))$

Except for de Morgan's laws, "$\sim$" has no properties at all. The logic **LNP** is fully axiomatized by **CL** (with the classical negation connective "$\neg$") plus:

(K)	$\mathsf{O}(A \supset B) \supset (\mathsf{O}A \supset \mathsf{O}B)$
(D)	$\mathsf{O}A \supset \mathsf{P}A$
(NEC$^{\sim}$)	If $\vdash_{\mathbf{CLoNs}} A$ then $\vdash \mathsf{O}A$
(KP)	$\mathsf{O}(A \supset B) \supset (\mathsf{P}A \supset \mathsf{P}B)$
(OD)	$\mathsf{O}(A \vee B) \supset (\mathsf{O}A \vee \mathsf{P}B)$
(PD)	$P(A \vee B) \supset (\mathsf{P}A \vee \mathsf{P}B)$

We write $\Gamma \vdash_{\mathbf{LNP}} A$ to denote that A is **LNP**-derivable from Γ, and $\vdash_{\mathbf{LNP}} A$ to denote that A is **LNP**-derivable from the empty premise set.

LNP resembles **SDL** in the sense that it contains (K), (D), and a necessitation rule. However, it is non-standard in the sense that its necessitation rule (NEC$^{\sim}$) is defined in terms of theoremhood in **CLoNs** instead of theoremhood in **CL**. Moreover, in **LNP** the permission operator P is not definable in terms of the obligation operator O. Instead, the P-operator is characterized by the axiom schemata (KP), (OD), and (PD), all of which also hold in **SDL**.
The axiom schemata (O-AND) and (P-AND) are derivable in **LNP** (their derivability is shown in Fact 1 in Section Appendix B of the Appendix):

(O-AND)	$\mathsf{O}A, \mathsf{O}B \vdash_{\mathbf{LNP}} \mathsf{O}(A \wedge B)$
(P-AND)	$\mathsf{O}A, \mathsf{P}B \vdash_{\mathbf{LNP}} \mathsf{P}(A \wedge B)$

Theorem 1. *If $\Gamma \vdash_{\mathbf{LNP}} A$, then $\Gamma \vDash_{\mathbf{LNP}} A$. (Soundness of* **LNP***)*

Theorem 2. *If $\Gamma \vDash_{\mathbf{LNP}} A$, then $\Gamma \vdash_{\mathbf{LNP}} A$. (Strong Completeness of* **LNP***)*

Proofs for Theorem 1 and Theorem 2 are contained in Section Appendix C of the Appendix.

2.4. Discussion

LNP allows for the consistent possibility of normative conflicts and normative gaps, and invalidates deontic explosion:

$$\mathsf{O}p \wedge \mathsf{O}{\sim}p \nvdash_{\mathbf{LNP}} \mathsf{O}q \tag{1}$$

$$\mathsf{O}p \wedge \mathsf{P}{\sim}p \nvdash_{\mathbf{LNP}} \mathsf{O}q \tag{2}$$

$$\neg\mathsf{P}p \wedge \neg\mathsf{O}{\sim}p \nvdash_{\mathbf{LNP}} \mathsf{O}q \tag{3}$$

In accordance with the discussion in Section 1.2, the following interdependencies between the O- and P-operators are invalid in **LNP**:

$$\mathsf{P}p \nvdash_{\mathbf{LNP}} \neg\mathsf{O}{\sim}p \tag{4}$$

$$\neg \mathsf{P}p \nvdash_{\mathbf{LNP}} \mathsf{O}{\sim}p \tag{5}$$
$$\mathsf{O}p \nvdash_{\mathbf{LNP}} \neg \mathsf{P}{\sim}p \tag{6}$$
$$\neg \mathsf{O}p \nvdash_{\mathbf{LNP}} \mathsf{P}{\sim}p \tag{7}$$

(4)-(7) correspond to the characterization of the P-operator as an operator for positive permission. (4) fails in the presence of an OP-conflict $\mathsf{P}p \wedge \mathsf{O}{\sim}p$. (5) fails in the presence of a gap $\neg\mathsf{P}p \wedge \neg\mathsf{O}{\sim}p$. (6) fails in the presence of a conflict $\mathsf{O}p \wedge \mathsf{P}{\sim}p$, and (7) fails in the presence of a gap $\neg\mathsf{P}{\sim}p \wedge \neg\mathsf{O}p$.

The conflict- and gap-tolerance of **LNP**, as well as the non-interdefinability of its O- and P-operators, all depend crucially on the paraconsistency and paracompleteness of the "$\sim$"-connective. However, the very weak characterization of "$\sim$"also causes the **LNP**-invalidity of the following inferences:

$$\mathsf{O}(p \vee q), \mathsf{O}{\sim}q \nvdash_{\mathbf{LNP}} \mathsf{O}p \tag{8}$$
$$\mathsf{O}(p \vee q), \mathsf{O}({\sim}p \vee q) \nvdash_{\mathbf{LNP}} \mathsf{O}q \tag{9}$$
$$\mathsf{O}(p \supset q), \mathsf{O}{\sim}q \nvdash_{\mathbf{LNP}} \mathsf{O}{\sim}p \tag{10}$$

Indeed, except for de Morgan's laws **LNP** invalidates all classically valid inferences that somehow depend on the properties of the $\sim$-connective, e.g. the Disjunctive Syllogism or Contraposition rules. (8) is invalid because the possibility of an OO-conflict $\mathsf{O}q \wedge \mathsf{O}{\sim}q$ cannot be excluded. In that case, $\mathsf{O}p$ need not follow from the premises $\mathsf{O}(p \vee q)$ and $\mathsf{O}{\sim}q$. Likewise, (9) is invalid since $\mathsf{O}q$ need not follow from $\mathsf{O}(p \vee q)$ and $\mathsf{O}({\sim}p \vee q)$ in the presence of an OO-conflict $\mathsf{O}p \wedge \mathsf{O}{\sim}p$.

(10) fails (i) in case of a normative conflict relating to q *or* (ii) in case of a normative gap relating to p. Suppose that $\mathsf{O}(p \supset q)$ and $\mathsf{O}{\sim}q$ are true at the actual world. Then $p \supset q$ and ${\sim}q$ are true at all accessible worlds. In case (i), both q and ${\sim}q$ are true in at least one accessible world. In this world, $p \supset q$ is automatically true in view of (C$\supset$), and ${\sim}p$ need not be true. In case ${\sim}p$ is false at an accessible world, we have a model in which $\mathsf{O}{\sim}p$ is false at the actual world. In case (ii), both p and ${\sim}p$ will be false in at least one accessible world. Again we have a model in which $\mathsf{O}{\sim}p$ is false at the actual world.

For similar reasons all of the following 'variants' of (8)-(10) are invalid in **LNP**:

$$\mathsf{O}(p \vee q), \mathsf{P}{\sim}q \nvdash_{\mathbf{LNP}} \mathsf{P}p \tag{11}$$
$$\mathsf{P}(p \vee q), \mathsf{O}{\sim}q \nvdash_{\mathbf{LNP}} \mathsf{P}p \tag{12}$$
$$\mathsf{O}(p \vee q), \mathsf{P}({\sim}p \vee q) \nvdash_{\mathbf{LNP}} \mathsf{P}q \tag{13}$$
$$\mathsf{P}(p \vee q), \mathsf{O}({\sim}p \vee q) \nvdash_{\mathbf{LNP}} \mathsf{P}q \tag{14}$$
$$\mathsf{O}(p \supset q), \mathsf{P}{\sim}q \nvdash_{\mathbf{LNP}} \mathsf{P}{\sim}p \tag{15}$$
$$\mathsf{P}(p \supset q), \mathsf{O}{\sim}q \nvdash_{\mathbf{LNP}} \mathsf{P}{\sim}p \tag{16}$$
$$\mathsf{O}(p \supset q) \nvdash_{\mathbf{LNP}} \mathsf{O}({\sim}q \supset {\sim}p) \tag{17}$$
$$\mathsf{P}(p \supset q) \nvdash_{\mathbf{LNP}} \mathsf{P}({\sim}q \supset {\sim}p) \tag{18}$$

In spite of the rationale behind their invalidity (i.e. the possibility of normative conflicts/gaps), all of (8)-(17) have some intuitive appeal. In real life, we tend to *assume* that norms behave consistently and that propositions are normatively regulated. Normative conflicts and normative gaps are *anomalies*. We *rely* on inferences like (8)-(17) in our everyday reasoning processes, albeit in a *defeasible* way.

It seems then, that **LNP** is too weak to account for our normative reasoning. Inferences like (8)-(17) should only be blocked once we can reasonably assume that one of the norm-propositions needed in the inference behaves abnormally, i.e. that there might be a conflict or gap relating to this norm-proposition. Note that this reasoning process is non-monotonic: new premises may provide the information that there is a conflict or gap relating to some norm-proposition that was previously deemed to behave normally. Consider, for instance, the inference from $\mathsf{O}(p \vee q)$ and $\mathsf{O}{\sim}p$ to $\mathsf{O}q$. This inference is intuitive assuming that there is no normative conflict relating to p. If, however, we obtain the new information that there *is* a normative conflict relating to p, then the inference should be blocked, since we do not want to rely on conflicted norm-propositions in deriving new information.

In the next section, we strengthen **LNP** in a non-monotonic fashion in order to overcome the problems mentioned here, and to make formally precise the idea of 'assuming' norm-propositions to behave 'normally'.

3. Two adaptive extensions

For any $A \in \mathcal{W}^a$, the classical negation connective "$\neg$" satisfies the following semantic conditions at the actual world:

(†) If $v_M(A, w_0) = 1$, then $v_M(\neg A, w_0) = 0$,
(‡) If $v_M(A, w_0) = 0$, then $v_M(\neg A, w_0) = 1$.

(†) guarantees the consistency of A: A and $\neg A$ cannot both be true at w_0. (‡) imposes a completeness condition on A: at least one of A and $\neg A$ is true at w_0.

As is clear from the **LNP**-semantics, (†) and (‡) fail for "$\sim$" at accessible worlds. Instead of (†) and (‡), only the weaker conditions (†') and (‡') hold for "$\sim$" at a world $w \in W \setminus \{w_0\}$:

(†') If $v_M(A, w) = 1$, then $v_M({\sim}A, w) = 0$ *or* $v_M(A \wedge {\sim}A, w) = 1$,
(‡') If $v_M(A, w) = 0$, then $v_M({\sim}A, w) = 1$ *or* $v_M(A \vee {\sim}A, w) = 0$.

In view of the semantic clauses for **LNP** it is easily checked that whenever a normative conflict occurs relating to a proposition p, the formula $p \wedge {\sim}p$ is true at some accessible world. In case of an OP-conflict $\mathsf{O}p \wedge \mathsf{P}{\sim}p$ or $\mathsf{O}{\sim}p \wedge \mathsf{P}p$, this follows in view of (CO), (CP), and (C$\wedge$). In case of an OO-conflict $\mathsf{O}p \wedge \mathsf{O}{\sim}p$, it follows in view of (CO), (C$\wedge$) and the seriality of the accessibility relation.

In a similar fashion, we can check that whenever a normative gap occurs relating to p, the formula $p \vee {\sim}p$ is false at some accessible world. Suppose, for instance, that $\neg\mathsf{O}p \wedge \neg\mathsf{P}{\sim}p$ is true at w_0. Then by (C$\neg$), both $\mathsf{O}p$ and $\mathsf{P}{\sim}p$ are false at w_0. By (CO), there is a world w such that Rw_0w and $v(p,w) = 0$. By (CP), $\sim p$ too is false at this world: $v(\sim p, w) = 0$. By (C$\vee$), $v(p \vee \sim p, w) = 0$.

Normative conflicts create *truth-value gluts*, whereas normative gaps create *truth-value gaps* at accessible worlds. Suppose now that we label such gluts and gaps as *abnormal*, and that we try to interpret our worlds *as normally as possible*. Then, in view of (†') and (‡'), normal behavior corresponds to the satisfaction of the consistency and completeness demands (†) and (‡) for "$\sim$" at accessible worlds.

The adaptive logics $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$ defined in this section exploit the above idea in making the assumption that norm-propositions behave 'normally' unless and until we find out that they are involved in some normative conflict or gap.

3.1. Semantic characterization of $\mathbf{LNP^r}$ *and* $\mathbf{LNP^m}$

Adaptive logics in the so-called *standard format* from [6] are characterized as triples, consisting of:

(1) A lower limit logic (LLL): a compact, reflexive, transitive, and monotonic logic that contains **CL** and has a characteristic semantics.
(2) A set of abnormalities Ω: a set of formulas, characterized by a (possibly restricted) logical form F; or a union of such sets.
(3) An adaptive strategy: reliability or minimal abnormality

The LLL of an adaptive logic (AL) in standard format is its monotonic base; everything derivable by means of the LLL is derivable by means of the AL. The AL extends the LLL by interpreting abnormalities as false "as much as possible". The formal disambiguation of this idea is relative to the adaptive strategy used by the AL. At the moment, two adaptive strategies are included in the standard format: the reliability strategy and the minimal abnormality strategy.

We use $\mathbf{LNP^x}$ as a generic name for the logics $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$. The former uses the reliability strategy, whereas the latter uses minimal abnormality (hence the superscripts r and m). Both of these logics have **LNP** as their LLL. Moreover, $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$ share the same set of abnormalites $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{\mathsf{P}(A \wedge \sim A) \mid A \in \mathcal{W}^a\}$ and $\Omega_2 = \{\neg\mathsf{O}(A \vee \sim A) \mid A \in \mathcal{W}^a\}$.

Ω_1 is the set of atomic gluts true at some accessible world. Note that, in view of the validity of de Morgan's laws for "$\sim$", more complex gluts can be reduced to (disjunctions of) atomic gluts by the LLL, e.g. if $v_M((p\vee q)\wedge\sim(p\vee q), w) = 1$, then ($v_M(p \wedge \sim p, w) = 1$ or $v_M(q \wedge \sim q, w) = 1$). Consequently, whenever some **LNP**-model verifies an OO- or OP-conflict, it also validates an abnormality in the set Ω_1.

In view of the **LNP**-semantics, $p \vee \sim p$ is false at some accessible world whenever $\neg\mathsf{O}(p \vee \sim p)$ is true at the actual world. Thus Ω_2 is the set of atomic gaps true at some accessible world. Again, complex instances of gaps are **LNP**-reducible to a (disjunction of) atomic gap(s), e.g. if $v_M((p\vee q)\vee\sim(p\vee q), w) = 0$, then ($v_M(p \vee \sim p, w) = 0$ or $v_M(q \vee \sim q, w) = 0$). Hence whenever some **LNP**-model verifies a normative gap, it also validates an abnormality in the set Ω_2.

For any atomic proposition p, the Ω_2-abnormality $\neg\mathsf{O}(p \vee {\sim}p)$ expresses that there is an accessible world in which *neither* p nor ${\sim}p$ is verified, whereas the Ω_1-abnormality $\mathsf{P}(p \wedge {\sim}p)$ expresses that there is an accessible world in which *both* p and ${\sim}p$ are verified. Thus, in **LNP** both gluts and gaps in accessible worlds constitute abnormalities. In view of the discussion at the beginning of this section, this means that both normative conflicts and normative gaps constitute abnormalities in **LNP**.

Semantically, adaptive logics proceed by selecting a subset of their LLL-models.[8] This selection makes use of the *abnormal part* of an **LNP**-model, i.e. the set of all abnormalities verified by it. The abnormal part of an **LNP**-model M is defined as $Ab(M) = \{A \in \Omega \mid M \Vdash_{\mathbf{LNP}} A\}$.

The minimal abnormality strategy selects all **LNP**-models of a premise set Γ which have a *minimal* abnormal part (with respect to set-inclusion).

Definition 1. *An* **LNP***-model M of Γ is* minimally abnormal *iff there is no* **LNP***-model M' of Γ such that $Ab(M') \subset Ab(M)$.*

The semantic consequence relation of the logic $\mathbf{LNP^m}$ is defined by selecting the minimally abnormal **LNP**-models:

Definition 2. $\Gamma \models_{\mathbf{LNP^m}} A$ *iff A is verified by all minimally abnormal* **LNP***-models of Γ.*

Before we can define the semantic consequence relation for $\mathbf{LNP^r}$, we need some more terminology. Where Δ is a finite, non-empty set of abnormalities, the disjunction $\bigvee \Delta$ is called a *Dab-formula* and is written as $Dab(\Delta)$. A *Dab*-formula $Dab(\Delta)$ is a *Dab-consequence* of Γ if it is **LNP**-derivable from Γ; $Dab(\Delta)$ is a *minimal Dab-consequence* of Γ if it is a *Dab*-consequence of Γ and there is no Δ' such that $Dab(\Delta')$ is a *Dab*-consequence of Γ and $\Delta' \subset \Delta$.

The set of formulas that are *unreliable* with respect to Γ, denoted by $U(\Gamma)$, is defined by

Definition 3. *Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-consequences of Γ, $U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$ is the set of formulas that are* unreliable *with respect to Γ.*

Where $Ab(M)$ is defined as above, we can now select the reliable models and define the semantic consequence relation for $\mathbf{LNP^r}$:

Definition 4. *An* **LNP***-model M of Γ is* reliable *iff $Ab(M) \subseteq U(\Gamma)$.*

Definition 5. $\Gamma \models_{\mathbf{LNP^r}} A$ *iff A is verified by all reliable models of Γ.*

The fact that the set of $\mathbf{LNP^x}$-models of Γ is a subset of the set of **LNP**-models of Γ immediately ensures that $\mathbf{LNP^x}$ strengthens **LNP**.

---

[8] Besides adaptive logics many other formal frameworks make use of semantic selections, e.g. [27, 37].

Theorem 3. *If* $\Gamma \models_{\mathbf{LNP}} A$*, then* $\Gamma \models_{\mathbf{LNP}^{\mathbf{x}}} A$*.*

Where $\mathcal{M}_{\Gamma}^{\mathbf{LNP}}, \mathcal{M}_{\Gamma}^{m}$, and $\mathcal{M}_{\Gamma}^{r}$ denote the set of **LNP**-models, minimally abnormal **LNP**-models, resp. reliable **LNP**-models of Γ, we also know that:

Theorem 4. *(Strong Reassurance.) If* $M \in \mathcal{M}_{\Gamma}^{\mathbf{LNP}} - \mathcal{M}_{\Gamma}^{m}$*, then there is a* $M' \in \mathcal{M}_{\Gamma}^{m}$ *such that* $Ab(M') \subset Ab(M)$*. If* $M \in \mathcal{M}_{\Gamma}^{\mathbf{LNP}} - \mathcal{M}_{\Gamma}^{r}$*, then there is a* $M' \in \mathcal{M}_{\Gamma}^{r}$ *such that* $Ab(M') \subset Ab(M)$*.*

Theorem 4 is shown generically for adaptive logics in standard format as Corollary 1 in [6].

3.2. Some illustrations

Example 1. Let $\Gamma_1 = \{\mathsf{O}p, \mathsf{O}(\sim p \vee q)\}$. Then, for all **LNP**-models M of Γ_1, $M, w_0 \models \mathsf{O}p$ and $M, w_0 \models \mathsf{O}(\sim p \vee q)$. By (C$\mathsf{O}$), $M, w \models p$ and $M, w \models \sim p \vee q$ for all worlds w such that Rw_0w. The possible truth values for $p, \sim p, q$, and $\sim q$ at accessible worlds in M are depicted in Table 1a. Let $R(w_0)$ abbreviate the set of worlds $w \in W \setminus \{w_0\}$ such that Rw_0w. Then each $w \in R(w_0)$ is of one of types (1)-(6).

Table 1: Accessible worlds for Γ_1 and Γ_3

(a) Accessible worlds for Γ_1

w	p	$\sim p$	q	$\sim q$
(1)	1	0	1	0
(2)	1	0	1	1
(3)	1	1	0	0
(4)	1	1	0	1
(5)	1	1	1	0
(6)	1	1	1	1

(b) Accessible worlds for Γ_3

w	p	$\sim p$	q	$\sim q$
(1)	0	0	0	1
(2)	0	0	1	1
(3)	0	1	0	1
(4)	0	1	1	1
(5)	1	0	1	1
(6)	1	1	1	1

If at least one $w \in R(w_0)$ is of one of types (3)-(6), then, by (C$\wedge$) and (CP), $M, w_0 \models \mathsf{P}(p \wedge \sim p)$, and $\mathsf{P}(p \wedge \sim p) \in Ab(M)$. Similarly, if at least one $w \in R(w_0)$ is of type (2) or type (6), then $\mathsf{P}(q \wedge \sim q) \in Ab(M)$. Moreover, if some $w \in R(w_0)$ is of type (3), then, by (C$\vee$), (CO) and (C$\neg$), $M, w_0 \models \neg\mathsf{O}(q \vee \sim q)$, and $\neg\mathsf{O}(q \vee \sim q) \in Ab(M)$.

If, however, all worlds $w \in R(w_0)$ are of type (1), then M verifies no abnormalities relating to p or q. In view of Definition 1, only models for which all worlds $w \in R(w_0)$ are of type (1) qualify as minimally abnormal **LNP**-models of Γ_1. Note that, for all type (1)-worlds $w \in R(w_0)$, $M, w \models q$. By (CO), $M, w_0 \models \mathsf{O}q$. By Definition 2, $\Gamma_1 \models_{\mathbf{LNP}^{\mathbf{m}}} \mathsf{O}q$.

Since Γ_1 has **LNP**-models M of which all accessible worlds $w \in R(w_0)$ are such that, for all $A \in \mathcal{W}^a$, $M, w \not\models A \wedge \sim A$ and $M, w \models A \vee \sim A$, we can conclude that Γ_1 has **LNP**-models M such that $Ab(M) = \emptyset$. It follows that Γ_1 has no minimal *Dab*-consequences. In view of Definition 3, $U(\Gamma_1) = \emptyset$. By Definition 4, $Ab(M) = \emptyset$ for all reliable **LNP**-models M of Γ_1. Again, only models for which all worlds $w \in R(w_0)$ are of type (1) qualify as reliable **LNP**-models of Γ_1. By Definition 5, $\Gamma_1 \models_{\mathbf{LNP^r}} \mathsf{O}q$.

Example 2. Let $\Gamma_2 = \{\mathsf{O}p, \mathsf{O}({\sim}p \vee q), \mathsf{O}{\sim}p\}$. It is easily checked that $\Gamma_2 \models_{\mathbf{LNP}} \mathsf{P}(p \wedge {\sim}p)$. Consequently, all **LNP**-models verify this abnormality, including the minimally abnormal and reliable ones. Hence all accessible worlds in all **LNP**-models of Γ_2 are of one of types (3)-(6) in Table 1a. Since $\mathsf{P}(p \wedge {\sim}p)$ is the only *Dab*-consequence of Γ_2, the selected $\mathbf{LNP^x}$-models for both strategies are those which verify exactly this abnormality, i.e. models of which all accessible worlds are of type (4) or (5). In all of these models, $p, {\sim}p \vee q$, and ${\sim}p$ are true at all accessible worlds. Since q need not be true at some of these worlds, Γ_2 has $\mathbf{LNP^x}$-models in which $\mathsf{O}q$ is false. Hence $\Gamma_2 \not\models_{\mathbf{LNP^x}} \mathsf{O}q$.

Note that Examples 1 and 2 illustrate the non-monotonicity of $\mathbf{LNP^x}$: adding the premise $\mathsf{O}{\sim}p$ to Γ_1 blocks the derivation of $\mathsf{O}q$.

Example 3. Let $\Gamma_3 = \{\mathsf{O}(p \supset q), \mathsf{O}{\sim}q\}$, and let M be an **LNP**-model of Γ_3. The possible truth values for $p, {\sim}p, q$, and ${\sim}q$ at accessible worlds in M are depicted in Table 1b.

If at least one $w \in R(w_0)$ is of one of types (1) or (2), then $\neg\mathsf{O}(p \vee {\sim}p) \in Ab(M)$. If at least one $w \in R(w_0)$ is of one of types (2), (4), (5) or (6), then $\mathsf{P}(q \wedge {\sim}q) \in Ab(M)$. Only if all $w \in R(w_0)$ are of type (3) it is possible that $Ab(M) = \emptyset$. In view of Definition 1, only models of which all $w \in R(w_0)$ are of type (3) qualify as minimally abnormal models. But then $M, w_0 \models \mathsf{O}{\sim}p$, and, by Definition 2, $\Gamma_3 \models_{\mathbf{LNP^m}} \mathsf{O}{\sim}p$. It is safely left to the reader to check that, in view of Definitions 4 and 5, $\Gamma_3 \models_{\mathbf{LNP^r}} \mathsf{O}{\sim}p$.

Example 4. Let $\Gamma_4 = \{\mathsf{O}(p \wedge q), \mathsf{O}({\sim}(p \vee q) \vee r), \mathsf{P}({\sim}p \vee {\sim}q)\}$, and let M be an **LNP**-model of Γ_4. By (CO) we know that, for all $w \in R(w_0)$ in M, $M, w \models p \wedge q$ and $M, w \models {\sim}(p \vee q) \vee r$. Hence every $w \in R(w_0)$ is of one of types (1)-(10) depicted in Table 2.

w	p	${\sim}p$	q	${\sim}q$	r	${\sim}r$
(1)	1	0	1	0	1	0
(2)	1	0	1	0	1	1
(3)	1	0	1	1	1	0
(4)	1	0	1	1	1	1
(5)	1	1	1	0	1	0
(6)	1	1	1	0	1	1
(7)	1	1	1	1	0	0
(8)	1	1	1	1	0	1
(9)	1	1	1	1	1	0
(10)	1	1	1	1	1	1

Table 2: Accessible worlds for Γ_4

By (CP), we also know that there is at least one world w such that $w \in R(w_0)$ and $M, w \models {\sim}p \vee {\sim}q$. Thus, w cannot be of type (1) or type (2). If w is of type

(3), then $\mathsf{P}(q \wedge {\sim}q) \in Ab(M)$. If w is of type (5), then $\mathsf{P}(p \wedge {\sim}p) \in Ab(M)$. It is easily checked that if w is of type (4), (6), (7), (8), (9), or (10), then either $\{\mathsf{P}(p \wedge {\sim}p)\} \subset Ab(M)$ or $\{\mathsf{P}(q \wedge {\sim}q)\} \subset Ab(M)$.

In general, it follows by Definition 1 that M only qualifies as a minimally abnormal **LNP**-model of Γ_4 if either w is of type (3) and all $w' \in R(w_0) \setminus \{w\}$ are of type (1) or type (3), or w is of type (5) and all $w' \in R(w_0) \setminus \{w\}$ are of type (1) or type (5). Hence if M is minimally abnormal, then all accessible worlds in M are of type (1), type (3), or type (5). But then, by (CO), $M, w_0 \models \mathsf{O}r$ and, by Definition 2, $\Gamma_4 \models_{\mathbf{LNP^m}} \mathsf{O}r$.

Since at least one accessible world w in M is of types (3)-(10), it follows by (C$\wedge$), (CP), and (C$\vee$) that $M, w_0 \models \mathsf{P}(p \wedge {\sim}p) \vee \mathsf{P}(q \wedge {\sim}q)$. Since $M, w_0 \not\models \mathsf{P}(p \wedge {\sim}p)$ and $M, w_0 \not\models \mathsf{P}(q \wedge {\sim}q)$, it follows that $\mathsf{P}(p \wedge {\sim}p) \vee \mathsf{P}(q \wedge {\sim}q)$ is a minimal *Dab*-consequence of Γ_4. Thus, by Definition 3, $\mathsf{P}(p \wedge {\sim}p), \mathsf{P}(q \wedge {\sim}q) \in U(\Gamma_4)$.

Suppose now that all $w \in R(w_0)$ are of type (8), and that, for all $A \in \mathcal{W}^a \setminus \{p, q, r\}$, $M, w \not\models A \wedge {\sim}A$ and $M, w \models A \vee {\sim}A$. Then it is easily verified that the only abnormalities verified by M are $\mathsf{P}(p \wedge {\sim}p)$ and $\mathsf{P}(q \wedge {\sim}q)$. Thus, $Ab(M) \subseteq U(\Gamma_4)$. By Definition 4, M is reliable. However, $M, w_0 \not\models \mathsf{O}r$. Thus, by Definition 5, $\Gamma_4 \not\models_{\mathbf{LNP^r}} \mathsf{O}r$.

Example 4 illustrates that there are premise sets $\Gamma \subseteq \mathcal{W}$ and formulas $A \in \mathcal{W}$ such that $\Gamma \not\models_{\mathbf{LNP^r}} A$ and $\Gamma \models_{\mathbf{LNP^m}} A$. The inverse does not hold: in Section 4.3 we show that in general the logic $\mathbf{LNP^m}$ is strictly stronger than $\mathbf{LNP^r}$.

4. Proof theory and meta-theory for $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$

In Section 4.1, we define the proof theory for $\mathbf{LNP^x}$ in a generic way. An illustration is provided in Section 4.2. Further (meta-)theoretical properties of $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$ are stated in Section 4.3.

4.1. Proof theory

A line in an annotated $\mathbf{LNP^x}$-proof consists of a line number, a formula, a justification and a condition. The justification consists of a (possibly empty) list of line numbers (from which the formula is derived) and of the name of a rule. The presence of a condition is part of what makes an adaptive proof *dynamic*. The dynamics of these proofs is controlled by attaching conditions to derived formulas and by a marking definition. The rules determine which lines (consisting of the four aforementioned elements) may be added to a given proof. The effect of the marking definition is that, at every stage[9] of the proof, certain lines may be marked whereas others are unmarked. Whether or not a line is marked depends only on the condition of the line and on the minimal *Dab*-formulas (cfr. Definition 6) that have been derived in the proof.

[9] A stage of a proof is a sequence of lines and a proof is a sequence of stages. Every proof starts off with stage 1. Adding a line to a proof by applying one of the rules of inference brings the proof to its next stage, which is the sequence of all lines written so far.

The rules of inference of $\mathbf{LNP^x}$ reduce to three generic rules. Where Γ is the set of premises, and where

$$A \quad \Delta$$

abbreviates that A occurs in the proof on the condition Δ, the inference rules are given by

PREM If $A \in \Gamma$:

$$\begin{array}{ll} \vdots & \vdots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LNP}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LNP}} B \vee Dab(\Theta)$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \vdots & \vdots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

The premise rule PREM simply states that, at any line of a proof, a premise may be introduced on the empty condition. What the unconditional rule RU comes to is that, whenever $A_1, \ldots, A_n \vdash_{\mathbf{LNP}} B$ and $A_1, \ldots, A_n$ occur in the proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then B may be added to the proof on the condition $\Delta_1 \cup \ldots \cup \Delta_n$.

If $A_1, \ldots, A_n \vdash_{\mathbf{LNP}} B \vee Dab(\Theta)$ and $A_1, \ldots, A_n$ occur in a proof on the conditions $\Delta_1, \ldots, \Delta_n$ respectively, then, by the conditional rule RC, we can infer B on the condition $\Delta_1 \cup \ldots \cup \Delta_n \cup \Theta$. RC is the only rule that allows for the introduction of new conditions in an adaptive proof.

The marking definition for the reliability strategy proceeds in terms of the *minimal Dab-formulas* derived at a stage of the proof:

Definition 6. *$Dab(\Delta)$ is a* minimal Dab-formula *at stage s iff, at stage s, $Dab(\Delta)$ is derived on the condition $\emptyset$, and no $Dab(\Delta')$ with $\Delta' \subset \Delta$ is derived on the condition $\emptyset$.*

Definition 7. *Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal Dab-formulas derived at stage s, $U_s(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots$.*

Definition 8. *Marking for reliability. Where Δ is the condition of line i, line i is marked at stage s iff $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

The marking definition for minimal abnormality requires some more terminology. A *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ is a set that contains one element out of each member of Σ. A *minimal choice set* of Σ is a choice set of Σ of which no proper subset is a choice set of Σ. Where $Dab(\Delta_1), Dab(\Delta_2), \ldots$ are the minimal *Dab*-formulas that are derived at stage s, $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \Delta_2, \ldots\}$.

Definition 9. *Marking for minimal abnormality. Where $A \in \mathcal{W}$ is derived at line i of a proof from Γ on a condition Δ, line i is marked at stage s iff*
(i) there is no $\Delta' \in \Phi_s(\Gamma)$ such that $\Delta' \cap \Delta = \emptyset$, or
(ii) for some $\Delta' \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\Delta' \cap \Theta = \emptyset$.

For both strategies, the marking proceeds in terms of the minimal *Dab*-formulas that are derived at a certain stage. It is clear that marking is a dynamic matter: a line may be unmarked at a stage s, marked at a later stage s' and again unmarked at an even later stage s''. This is why a more stable notion of derivability is needed:

Definition 10. *A is* finally derived *from Γ at line i of a proof at finite stage s iff (i) A is the second element of line i, (ii) line i is not marked at stage s, and (iii) every extension of the proof in which line i is marked may be further extended in such a way that line i is unmarked.*

The derivability relation of $\mathbf{LNP^x}$ is defined with respect to the notion of final derivability:

Definition 11. $\Gamma \vdash_{\mathbf{LNP^x}} A$ *(A is finally $\mathbf{LNP^x}$-derivable from Γ) iff A is finally derived at a line of a $\mathbf{LNP^x}$-proof from Γ.*

Theorem 5 below is proved generically for adaptive logics in standard format as Theorem 16 in [6]:

Theorem 5. *(Proof Invariance.) If $\Gamma \vdash_{\mathbf{LNP^x}} A$, then every finite $\mathbf{LNP^x}$-proof from Γ can be extended in such a way that A is finally derived in it.*

Any adaptive logic in standard format is sound and complete with respect to its semantics:

Theorem 6. $\Gamma \vdash_{\mathbf{LNP^x}} A$ *iff* $\Gamma \vDash_{\mathbf{LNP^x}} A$.

Since $\mathbf{LNP^x}$ is defined within the standard format for adaptive logics, the proof of Theorem 6 follows immediately by Corollary 2 and Theorem 9 from [6].

4.2. Illustration

In this section, we illustrate the proof theory by means of the premise set Γ_4 from Section 3.2. We start a $\mathbf{LNP^r}$-proof for Γ_4 by entering the premises via the premise introduction rule:

1	$\mathsf{O}(p \wedge q)$	PREM	$\emptyset$

2	$\mathsf{O}(\sim(p \vee q) \vee r)$	PREM	$\emptyset$
3	$\mathsf{P}(\sim p \vee \sim q)$	PREM	$\emptyset$

Since $\mathsf{O}(p \wedge q) \vdash_{\mathbf{LNP}} \mathsf{O}p$, we can derive $\mathsf{O}p$ by applying RU to line 1:

4	$\mathsf{O}p$	1; RU	$\emptyset$

By (O-AND), we can aggregate lines 2 and 4. Hence line 5 follows via RU:

5	$\mathsf{O}((\sim(p \vee q) \vee r) \wedge p)$	2,4; RU	$\emptyset$

As $((\sim(p \vee q) \vee r) \wedge p) \supset (r \vee (p \wedge \sim p))$ is a **CLoNs**-theorem, line 6 follows from line 5 by (NEC$^{\sim}$) and (K). Thus, via RU:

6	$\mathsf{O}(r \vee (p \wedge \sim p))$	5; RU	$\emptyset$
7	$\mathsf{O}r \vee \mathsf{P}(p \wedge \sim p)$	6; RU	$\emptyset$
8	$\mathsf{O}r$	7; RC	$\{\mathsf{P}(p \wedge \sim p)\}$

Line 7 is obtained by applying (OD) to line 6. At line 8, the abnormality derived in disjunction with $\mathsf{O}r$ at line 7 is moved to the condition by means of an application of RC. At stage 8 of the proof, $\mathsf{O}r$ is considered derived. Intuitively, line 8 can be interpreted as "$\mathsf{O}r$ is derived on the assumption that $\mathsf{P}(p \wedge \sim p)$ is false".

In the proof above, lines 4-7 serve a purely explanatory purpose; their aim is to show that $\mathsf{O}r \vee \mathsf{P}(p \wedge \sim p)$ is a **LNP**-consequence of lines 1 and 2. However, since indeed $\mathsf{O}(p \wedge q), \mathsf{O}(\sim(p \vee q) \vee r) \vdash_{\mathbf{LNP}} \mathsf{O}r \vee \mathsf{P}(p \wedge \sim p)$, we could have skipped these lines and applied RC immediately to lines 1 and 2.

In a fashion analogous to the derivation of $\mathsf{O}r \vee \mathsf{P}(p \wedge \sim p)$ above, we can show that $\mathsf{O}(p \wedge q), \mathsf{O}(\sim(p \vee q) \vee r) \vdash_{\mathbf{LNP}} \mathsf{O}r \vee \mathsf{P}(q \wedge \sim q)$. Hence we can apply RC immediately to lines 1 and 2:

9	$\mathsf{O}r$	1,2; RC	$\{\mathsf{P}(q \wedge \sim q)\}$

Consider now the premises at lines 1 and 3. By (P-AND), we can derive $\mathsf{P}((p \wedge q) \wedge (\sim p \vee \sim q))$. Since $\vdash_{\mathbf{CLoNs}} ((p \wedge q) \wedge (\sim p \vee \sim q)) \supset ((p \wedge \sim p) \vee (q \wedge \sim q))$, it follows by (NEC$^{\sim}$) and (KP) that $\mathsf{P}((p \wedge \sim p) \vee (q \wedge \sim q))$. By (PD), we obtain $\mathsf{P}(p \wedge \sim p) \vee \mathsf{P}(q \wedge \sim q)$. Thus we can continue the proof as follows (we repeat the proof from line 8 on):

8	$\mathsf{O}r$	6; RC	$\{\mathsf{P}(p \wedge \sim p)\}\checkmark^{10}$
9	$\mathsf{O}r$	1,2; RC	$\{\mathsf{P}(q \wedge \sim q)\}\checkmark^{10}$
10	$\mathsf{P}(p \wedge \sim p) \vee \mathsf{P}(q \wedge \sim q)$	1,3; RU	$\emptyset$

The formula $\mathsf{P}(p \wedge \sim p) \vee \mathsf{P}(q \wedge \sim q)$ is a minimal *Dab*-formula at stage 10. Both its disjuncts are in the set of unreliable formulas at this stage: $U_{10}(\Gamma_4) = \{\mathsf{P}(p \wedge \sim p), \mathsf{P}(q \wedge \sim q)\}$. In view of Definition 8, this causes the marking of lines 8 and 9 at stage 10 (hence the checkmark sign). Consequently, the formula $\mathsf{O}r$ is no longer considered derived at stage 10 of the proof. Moreover, since there is no way to extend the proof from Γ_4 in such a way that $\mathsf{O}r$ is derived on an unmarked line, we know by Definitions 10 and 11 that $\Gamma_4 \nvdash_{\mathbf{LNP^r}} \mathsf{O}r$.

Suppose now that the above proof is a $\mathbf{LNP^m}$-proof from Γ_4. We repeat the proof from line 8 on:

8	$\mathsf{O}r$	7; RC	$\{\mathsf{P}(p \wedge {\sim}p)\}$
9	$\mathsf{O}r$	6; RC	$\{\mathsf{P}(q \wedge {\sim}q)\}$
10	$\mathsf{P}(p \wedge {\sim}p) \vee \mathsf{P}(q \wedge {\sim}q)$	1,3; RU	$\emptyset$

In the $\mathbf{LNP^m}$-proof from Γ_4, the set $\Phi_{10}(\Gamma_4)$ of minimal choice sets of Γ_4 at stage 10 consists of the sets $\{\mathsf{P}(p \wedge {\sim}p)\}$ and $\{\mathsf{P}(q \wedge {\sim}q)\}$. In view of Definition 9, lines 8 and 9 remain unmarked. Since there is no way to extend the proof in such a way that these lines become marked, it follows by Definitions 10 and 11 that $\Gamma_4 \vdash_{\mathbf{LNP^m}} \mathsf{O}r$.

The different behavior of the logics $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$ for this example is explained by considering the intuitions behind the reliability and minimal abnormality strategies. According to the reliability strategy, a formula is deemed 'suspicious' and is subsequently marked whenever it is derived on a line of which the condition intersects with some disjunct of a minimal *Dab*-formula. The minimal abnormality strategy is a tad less cautious. In this example, the latter strategy takes only one of the disjuncts $\mathsf{P}(p \wedge {\sim}p)$ and $\mathsf{P}(q \wedge {\sim}q)$ to be true, although of course we do not know which one. If, on the one hand, $\mathsf{P}(p \wedge {\sim}p)$ is true, then $\mathsf{P}(q \wedge {\sim}q)$ is safely considered false. Hence $\mathsf{O}r$ is derivable in view of line 9. If, on the other hand, $\mathsf{P}(q \wedge {\sim}q)$ is true, then $\mathsf{P}(p \wedge {\sim}p)$ is safely considered false. Hence $\mathsf{O}r$ is derivable in view of line 8.

4.3. Further meta-theoretical properties

Formulating adaptive logics in the standard format has the advantage that a rich meta-theory is immediately available for this format. Generic proofs for Theorems 7-9 below can be found in [6].

In case no *Dab*-formulas are derivable from a premise set by means of the lower limit logic, it is safe to consider all abnormalities as false. As a consequence, the adaptive logic will then yield the same consequence set as the logic that interprets all abnormalities as false (or equivalently, as the logic that fully validates the inference rules whose application the adaptive logic only allows conditionally). This logic is called the upper limit logic (ULL) of an adaptive logic. The ULL of $\mathbf{LNP^x}$ is obtained by adding to $\mathbf{LNP}$ the axiom schemas (U_1) and (U_2), which trivialize all members of Ω_1 and Ω_2 respectively:

(U_1) $\mathsf{P}(A \wedge {\sim}A) \supset B$
(U_2) $\neg\mathsf{O}(A \vee {\sim}A) \supset B$

ULL is related to **LNP** as set out by the *Derivability Adjustment Theorem*:

Theorem 7. *$\Gamma \vdash_{\mathbf{ULL}} A$ iff (there is a $\Delta \subseteq \Omega$ for which $\Gamma \vdash_{\mathbf{LNP}} A \vee Dab(\Delta)$ or $\Gamma \vdash_{\mathbf{LNP}} A$).*

The set of *Dab*-consequences derivable from the premise set determines the amount to which the $\mathbf{LNP^x}$-consequence set will resemble the **ULL**-consequence set. This is why adaptive logicians say that $\mathbf{LNP^x}$ *adapts* itself to a premise set. $\mathbf{LNP^x}$ will always be at least as strong as **LNP** and maximally as strong as **ULL**:

Theorem 8. $Cn_{\mathbf{LNP}}(\Gamma) \subseteq Cn_{\mathbf{LNP^{x}}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.

In view of Theorem 11 from [6], it follows immediately that:

Corollary 1. $Cn_{\mathbf{LNP}}(\Gamma) \subseteq Cn_{\mathbf{LNP^{r}}}(\Gamma) \subseteq Cn_{\mathbf{LNP^{m}}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.

If Γ is *normal*, i.e. if Γ has no *Dab*-consequences, then we can even prove a stronger result:

Theorem 9. *If Γ is normal, then $Cn_{\mathbf{LNP^{x}}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.*

The reader may have noticed that **ULL** trivializes both gluts and gaps at accessible worlds, thus promoting "$\sim$" to a fully classical negation connective. It should come as no surprise then, that **ULL** is just **SDL** in disguise. Where $\Gamma \subseteq \mathcal{W}$, define $\Gamma^{\neg}$ by replacing every $A \in \Gamma$ by $\pi(A)$, where $\pi(A)$ is the result of replacing every occurrence of "$\sim$" in A by "$\neg$". Then:

Theorem 10. $\Gamma \vdash_{\mathbf{ULL}} A$ *iff* $\Gamma^{\neg} \vdash_{\mathbf{SDL}} \pi(A)$.

A proof outline for Theorem 10 is contained in Section Appendix D of the Appendix.

5. Related work

5.1. Alchourrón and Bulygin

In [1, 2, 3, 4], Alchourrón and Bulygin present a logic of norm-propositions that is built 'on top' of a logic of norms.[10] A norm-proposition "there exists a norm to the effect that A is permitted" is formalized as $\mathsf{NP}A$, where the operator N behaves like a quantifier over the norm $\mathsf{P}A$. The latter formula (without N) is read simply as "A is permitted".

Alchourrón and Bulygin's logic of norms is just **SDL**. Their logic of norm-propositions **NL** extends **SDL** by adding to it the axiom schema (NK) and the rule (NRM):

(NK)	$\mathsf{N}(A \supset B) \supset (\mathsf{N}A \supset \mathsf{N}B)$
(NRM)	If $\vdash A \supset B$ then $\vdash \mathsf{N}A \supset \mathsf{N}B$

In **NL**, OO-conflicts are formulas of the form $\mathsf{NO}A \wedge \mathsf{NO}\neg A$. Similarly, OP-conflicts are formulas of the form $\mathsf{NO}A \wedge \mathsf{NP}\neg A$. As opposed to normative conflicts, normative gaps cannot be expressed in the object language of **NL**. Instead, Alchourrón and Bulygin define a normative gap as a situation in which, for some **CL**-formula A we cannot derive $\mathsf{NP}A$ nor $\mathsf{NO}\neg A$, i.e. $\nvdash_{\mathbf{NL}}$ $\mathsf{NP}A \vee$

[10] Alchourrón and Bulygin's logic of norm-propositions is inspired by Rescher's assertion logic from [35].

$\mathsf{NO}\neg A$. Normative conflicts and gaps are treated consistently in **NL**. Where A and B are well-formed **NL**-formulas:[11]

$$\mathsf{NO}A \wedge \mathsf{NO}\neg A \nvdash_{\mathbf{NL}} B \tag{19}$$

$$\mathsf{NO}A \wedge \mathsf{NP}\neg A \nvdash_{\mathbf{NL}} B \tag{20}$$

$$\nvdash_{\mathbf{NL}} \mathsf{NP}A \vee \mathsf{NO}\neg A \tag{21}$$

However, the following variants of deontic explosion are valid in **NL**:

$$\mathsf{NO}A \wedge \mathsf{NO}\neg A \vdash_{\mathbf{NL}} \mathsf{NO}B \tag{22}$$

$$\mathsf{NO}A \wedge \mathsf{NP}\neg A \vdash_{\mathbf{NL}} \mathsf{NO}B \tag{23}$$

With Alchourrón, Bulygin, and von Wright, we agree that "experience seems to testify that mutually contradictory norms may co-exist within one and the same legal order – and also that there are a good many "gaps" in any such order" [46, p. 32]. But if conflicting normative propositions indeed often coexist within a normative order, then deontic explosion should be avoided by any logic of normative propositions. No judge will agree that a normative order containing one or more conflicts contains norms to the effect that anything whatsoever is obligatory. Hence (22) and (23) cause serious problems for **NL**.

(22) and (23) follow by applications of (NRM) and (NK) to the **SDL**-theorems $\vdash \mathsf{O}A \supset (\mathsf{O}{\sim}A \supset \mathsf{O}B)$ and $\vdash \mathsf{O}A \supset (\mathsf{P}{\sim}A \supset \mathsf{O}B)$ respectively. This led von Wright to questioning the presupposition of **SDL** by **NL** [46, footnote 2].

As opposed to **NL**, $\mathbf{LNP^x}$ is not built 'on top' of the **CL**-based logic **SDL**. Although **LNP** contains full **CL**, its 'deontic' formulas make use of the much weaker logic **CLoNs** inside the scope of the O- and P-operator. This way, $\mathbf{LNP^x}$ avoids deontic explosion.

Interestingly, Alchourrón and Bulygin point out that under the assumptions of consistency and completeness, the logic of norm-propositions is 'isomorphic' to **SDL**: if we dismiss the possibility of normative conflicts and normative gaps, the differences between both logics disappear [1, 4]. In Section 4.3 we proved this isomorphism for $\mathbf{LNP^x}$ by showing that for normal (consistent and complete) premise sets, $\mathbf{LNP^x}$ is just as strong as **SDL**.

5.2. Input/output logic

In input/output logic (I/O logic), norms are represented as ordered pairs of formulas (a, x), where each coordinate of a pair is a **CL**-formula.[12] The

[11] Alchourrón and Bulygin allow for iterated/nested deontic and normative operators. Nothing in principle prevents the occurrence of such nestings in $\mathbf{LNP^x}$. This requires some modifications of the language $\mathcal{W}$ and of the sets Ω_1 and Ω_2 such that e.g. $\mathsf{PP}(p \wedge {\sim}p)$ is also considered an abnormality.

[12] The framework of I/O logic was initially developed for dealing with conditional norms. We do not discuss its merits as a conditional logic here. Instead, we focus on issues related to conflict- and gap-tolerance. For a discussion of the representation of conditional norms in I/O logic, see [48].

body of such a pair constitutes an input consisting of some condition or factual situation. The head constitutes an output representing what the norm tells us to be desirable/obligatory/permitted in that situation. A normative order or system is a set G of input/output pairs. G is seen as a 'transformation device' in which **CL** functions as its 'secretarial assistant' [31, p. 2].

In [28], Makinson and van der Torre define various operations of the form $out(G, A)$ for making up the output A of G. In [29], the authors add constraints to these systems for dealing with contrary-to-duty scenarios and conflicting norms. In [30], the framework is extended for dealing with permissions. Constrained I/O logics make use of maximally consistent subsets. In doing so, they avoid explosion when dealing with conflicting conditional obligations, even if e.g. the norms (a, x) and $(a, \neg x)$ tell us that both x and $\neg x$ are obligatory under the same circumstances.

The treatment of obligation-permission conflicts by constrained I/O logics is less straightforward. In [39], Stolpe noted that the constrained systems deontically explode when facing a conflict between an obligation (a, x) and a positive permission $(a, \neg x)$.[13] Stolpe's solution to this problem is to treat positive permissions as *derogations*: "a positive permission suspends, annuls or obstructs a covering prohibition, thereby generating a corresponding set of liberties" [39, p. 99].

Stolpe's solution creates an asymmetry between obligations and permissions. In obligation-obligation conflicts, both norms may still be of equal importance. In obligation-permission conflicts however, the permission always overrides the obligations it is in conflict with. Although certainly of interest in legal contexts, where the concept of derogation is a very important one, we doubt that *all* obligation-permission conflicts can be dealt with in this way.

In the literature on I/O logic, normative gaps are left unmentioned. However, it seems possible to model gaps in this framework. For instance, we could say that there is a normative gap relating to proposition x in circumstances a if neither the obligations to do x or $\neg x$, nor the positive permissions to do x or $\neg x$ are in the output of a given set of norms. One drawback seems to be that, whichever I/O operation we pick, both the obligation to do $x \vee \neg x$ and the positive permission to do $x \vee \neg x$ will always be in the output set. This is due to the closure of the output set under classical logic. Furthermore, as with Alchourrón and Bulygin's approach, normative gaps cannot be modeled at the object level in I/O logic.

Another difference between I/O logic and $\mathbf{LNP^x}$ is that for I/O operations the input is restricted to simple norm-bases, i.e. sets of input-output pairs. More complex formulas such as disjunctions between norms or negated norms cannot be fed into the system. $\mathbf{LNP^x}$ is more flexible in this sense, since it can easily deal with premise sets containing formulas such as $\neg \mathsf{O}p$, $\mathsf{O}q \vee \mathsf{P}r$, etc.

[13] Translated to the I/O setting, deontic explosion ensues from a given input if – under certain circumstances invoked by the input – everything becomes obligatory in the output.

5.3. **SDL***-weakened modal logics*

The logics **LNP** and $\mathbf{LNP^x}$ are **SDL**-weakened conflict-tolerant deontic logics. In the literature on conflicting norms in deontic logic, many different weakenings of **SDL** have been proposed for dealing with conflicting norms.

The most popular approach for tolerating normative conflicts is to weaken **SDL** by invalidating the aggregation principle (O-AND), e.g. [15, 36, 49, 16]. Non-monotonic systems that validate only certain applications of (O-AND) were presented in [25, 23, 19, 32, 33].[14]

An alternative approach for weakening **SDL** suggested by Goble in [17, 18] restricts the modal inheritance principle (RM):

(RM) If $\vdash A \supset B$ then $\vdash \mathsf{O}A \supset \mathsf{O}B$

The result is a family of monotonic **SDL**-weakened systems capable of consistently accommodating OO-conflicts. Non-monotonic adaptive extensions of this logic were presented in [40, 42].

A third way of weakening **SDL** consists in giving up the *Ex Contradictione Quodlibet* schema:

(ECQ) $(A \wedge \neg A) \supset B$

The result is a paraconsistent deontic logic. This approach was taken up in [14, 34]. A non-monotonic adaptive paraconsistent deontic logic is presented in [11, 10]. The systems defined in this paper are 'paraconsistent' in the sense that they restrict (ECQ) for the negation connective "$\sim$".

None of the **SDL**-weakened systems mentioned in this section were designed for reasoning in the presence of normative gaps. Consequently, these logics do not provide a satisfactory treatment of normative gaps. Moreover, in systems in which a permission operator is characterized, the latter is always treated as the dual of the obligation-operator, i.e. $\mathsf{P}A \equiv \neg\mathsf{O}\neg A$ is valid.

5.4. **SDL***-strengthened modal logics*

In [26], Kooi and Tamminga deal with conflicting norms by enriching **SDL** so as to be capable of distinguishing between various sources and interest groups in view of which norms arise. Moreover, following [24] they equip their system with modal *stit*-operators for dealing with the difficult notion of (moral) agency. Similarly, we could try to deal with conflicting norms by imposing a preference ordering on our obligations and permissions, e.g. [21].

Such extensions are very successful in increasing the expressive power of **SDL**, but they are unable to consistently allow for *all* normative conflicts. Remember from Section 1.1 that conflicts may arise between norms promulgated at the same time, by the same authority. It is not difficult to see how we could

[14] Although the (constrained) I/O logics from Section 5.2 are presented in a non-modal framework, these too are non-monotonic non-aggregative systems in the sense that for two obligations (a, x) and (a, y) fed to the logic, the output does not always contain $x \wedge y$, e.g. if x and y are logically incompatible.

extend this type of reasoning to norms of the same hierarchical foot, addressed at the same group of people etc. so that in the end we need a logic that invalidates at least some **SDL**-theorems if we want to deal with all instances of normative conflicts.

A weakness of the systems devised in this paper is that they are not very expressive. Relativizing the deontic operators to individual/group operators is relatively straightforward (this can be done by defining a set of operators O^I and P^I where I is an index set representing some (group of) agent(s)), but it is less clear how to extend $\mathbf{LNP^x}$ in a way that it satisfactorily treats e.g. conditional norms or the notion of agency.

Moreover, the **SDL**-strengthened logics discussed here point to a trade-off between complexity and degrees of conflict-tolerance in dealing with conflicts in deontic logic. On the one hand, many normative conflicts can be dealt with in a monotonic way by making explicit the different sources they arise from or the different levels of priority attributed to them. On the other hand, if we want our logic to be *fully* conflict-tolerant we need to weaken **SDL** and allow only for the *defeasible* validity of some of its inferences. The latter option is technically more involving and computationally more complex.[15]

6. Conclusion and outlook

We presented two non-monotonic logics for reasoning with norm-propositions in the presence of normative conflicts and normative gaps. The logics $\mathbf{LNP^r}$ and $\mathbf{LNP^m}$ interpret a given premise set 'as consistently and as completely as possible'. $\mathbf{LNP^r}$ uses a slightly more cautious strategy than $\mathbf{LNP^m}$.

$\mathbf{LNP^r}$ and $\mathbf{LNP^m}$ are equipped with a well-defined semantics and proof theory. Due to their characterization within the standard framework for adaptive logics, soundness and completeness properties are guaranteed, as are many of their meta-theoretical properties.

As opposed to other systems devised for dealing with norm-propositions, the logics defined in this paper make use of a formal language in which all necessary distinctions can be made already at the object level. This is realized by making use of a classical negation connective outside, and a paraconsistent and paracomplete negation connective inside the scope of the operators O and P.

Two possible drawbacks of the non-classical **SDL**-weakened approach taken up here are that (i) due to their non-monotonicity the resulting logics are highly complex from the computational point of view; and (ii) the systems defined in this paper are not very expressive. In response to (i), we conjecture that in order to model actual human reasoning, a complex logic is what we need (see e.g. [9]). In response to (ii), we point to some existing work on more expressive adaptive deontic logics [10, 41, 40] and hope to provide more results in the future.

[15] For some results and a discussion on the computational complexity of adaptive logics, see [9, 43].

APPENDIX

Appendix A. CLoNs and the positive fragment of CL

Syntactically, the positive fragment of **CL** is defined by Modus Ponens (MP) and the following axiom schemata:

(A⊃1) $A \supset (B \supset A)$
(A⊃2) $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
(A⊃3) $((A \supset B) \supset A) \supset A$
(A∧1) $(A \wedge B) \supset A$
(A∧2) $(A \wedge B) \supset B$
(A∧3) $A \supset (B \supset (A \wedge B))$
(A∨1) $A \supset (A \vee B)$
(A∨2) $B \supset (A \vee B)$
(A∨3) $(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$
(A≡1) $(A \equiv B) \supset (A \supset B)$
(A≡2) $(A \equiv B) \supset (B \supset A)$
(A≡3) $(A \supset B) \supset ((B \supset A) \supset (A \equiv B))$

In order to obtain full **CL**, all one has to do is to add the axiom schemata (A¬1) and (A¬2) below to its positive fragment.

(A¬1) $(A \supset \neg A) \supset \neg A$
(A¬2) $A \supset (\neg A \supset B)$

CLoNs is defined by adding to the positive fragment of **CL** the axiom schemata (A∼∼)-(A∼≡) from Section 2.3. **CLoNs**-derivability for a formula $A \in \mathcal{W}^{\sim}$ from a set of formulas $\Gamma \subseteq \mathcal{W}^{\sim}$ is denoted by $\Gamma \vdash_{\mathbf{CLoNs}} A$. If $A \in \mathcal{W}^{\sim}$ is **CLoNs**-derivable from the empty set, then $\vdash_{\mathbf{CLoNs}} A$.

Semantically, **CLoNs**-models are associated with an assignment function $v : \mathcal{W}^l \to \{0, 1\}$. The valuation function v_M for the **CLoNs**-model M is defined by adjusting clauses (C_l) and (C∼∼)-(C≡) so that references to accessible worlds are dropped, e.g.

(C_l) where $A \in \mathcal{W}^l, v_M(A) = 1$ iff $v(A) = 1$
(C∼∼) $v_M(\sim\sim A) = 1$ iff $v_M(A) = 1$

and so on. A **CLoNs**-model M verifies A ($M \Vdash A$) iff $v_M(A) = 1$. Where $\Gamma \subseteq \mathcal{W}^{\sim}$, M is a **CLoNs**-model of Γ iff M is a **CLoNs**-model and $M \Vdash A$ for all $A \in \Gamma$. $\vDash_{\mathbf{CLoNs}} A$ iff all **CLoNs**-models verify A, and $\Gamma \vDash_{\mathbf{CLoNs}} A$ iff all **CLoNs**-models of Γ verify A.

Appendix B. Some Facts about LNP and CLoNs

The following theorems will come in handy for the proof of Theorem 2. Let in the remainder $\mathbf{L} \in \{\mathbf{CLoNs}, \mathbf{LNP}\}$:

Theorem 11. **L** *is reflexive, transitive and monotonic.*[16]

[16] Where $\mathrm{Cn}_{\mathbf{L}}(\Gamma)$ denotes the consequence set of some premise set Γ for **L**, **L** is reflexive iff, for all premise sets Γ, $\Gamma \subseteq \mathrm{Cn}_{\mathbf{L}}(\Gamma)$; it is transitive iff, for all sets of wffs Γ and Γ', if $\Gamma' \subseteq \mathrm{Cn}_{\mathbf{L}}(\Gamma)$ then $\mathrm{Cn}_{\mathbf{L}}(\Gamma \cup \Gamma') \subseteq \mathrm{Cn}_{\mathbf{L}}(\Gamma)$; and it is monotonic iff, for all sets of wffs Γ and Γ', $\mathrm{Cn}_{\mathbf{L}}(\Gamma) \subseteq \mathrm{Cn}_{\mathbf{L}}(\Gamma \cup \Gamma')$.

Theorem 12. $\mathbf{L}$ *is compact (if* $\Gamma \vdash_{\mathbf{L}} A$ *then* $\Gamma' \vdash_{\mathbf{L}} A$ *for some finite* $\Gamma' \subseteq \Gamma$*).*

Theorem 13. *If* $\Gamma \vdash_{\mathbf{L}} B$ *and* $A \in \Gamma$*, then* $\Gamma - \{A\} \vdash_{\mathbf{L}} A \supset B$ *(Generalized Deduction Theorem for* $\mathbf{L}$*).*

The proofs of Theorems 11 – 13 are straightforward and safely left to the reader.

Fact 1. *(i)* $\mathsf{O}A, \mathsf{O}B \vdash_{\mathbf{LNP}} \mathsf{O}(A \wedge B)$
(ii) $\mathsf{O}A, \mathsf{P}B \vdash_{\mathbf{LNP}} \mathsf{P}(A \wedge B)$
(iii) $\vdash_{\mathbf{LNP}} (\mathsf{O}A \wedge \mathsf{O}B) \supset \mathsf{O}(A \wedge B)$
(iv) $\vdash_{\mathbf{LNP}} (\mathsf{O}A \wedge \mathsf{P}B) \supset \mathsf{P}(A \wedge B)$
(v) $\vdash_{\mathbf{LNP}} \mathsf{P}(A \supset A)$
(vi) If $\vdash_{\mathbf{CLoNs}} A' \supset A$ *then* $A \supset B \vdash_{\mathbf{CLoNs}} A' \supset B$*.*
(vii) If $\vdash_{\mathbf{CLoNs}} B \supset B'$ *then* $A \supset B \vdash_{\mathbf{CLoNs}} A \supset B'$*.*
(viii) $\vdash_{\mathbf{CLoNs}} (A \vee (A \supset B)) \equiv (((A \vee (A \supset B)) \supset B) \supset B)$
(ix) $\vdash_{\mathbf{DP}} (A \supset (A \supset B)) \supset (A \supset B)$
(x) $\vdash_{\mathbf{CLoNs}} A \vee (A \supset B)$*.*

Proof. Ad(i). Suppose $\mathsf{O}A$ and $\mathsf{O}B$. By (A$\wedge$3), $\vdash_{\mathbf{CLoNs}} A \supset (B \supset (A \wedge B))$. By (NEC$^{\sim}$), it follows that $\vdash_{\mathbf{LNP}} \mathsf{O}(A \supset (B \supset (A \wedge B)))$. By (K), $\vdash_{\mathbf{LNP}} \mathsf{O}A \supset \mathsf{O}(B \supset (A \wedge B))$. By (MP), $\mathsf{O}(B \supset (A \wedge B))$. By (K), $\mathsf{O}B \supset \mathsf{O}(A \wedge B)$. By (MP), $\mathsf{O}(A \wedge B)$.

Ad(ii). Suppose $\mathsf{O}A$ and $\mathsf{P}B$. By (A$\wedge$3), $\vdash_{\mathbf{CLoNs}} A \supset (B \supset (A \wedge B))$. By (NEC$^{\sim}$), $\vdash_{\mathbf{LNP}} \mathsf{O}(A \supset (B \supset (A \wedge B)))$. By (K), $\vdash_{\mathbf{LNP}} \mathsf{O}A \supset \mathsf{O}(B \supset (A \wedge B))$. By (MP), $\mathsf{O}(B \supset (A \wedge B))$. By (KP), $\mathsf{P}B \supset \mathsf{P}(A \wedge B)$. By (MP), $\mathsf{P}(A \wedge B)$.

Ad(iii)-(iv). Immediate in view of (i),(ii), and Theorem 13.

Ad(v). Since $A \supset A$ is a theorem of the positive fragment of **CL**, it is also a **CLoNs**-theorem. By (NEC$^{\sim}$), $\vdash_{\mathbf{LNP}} \mathsf{O}(A \supset A)$. By (D), $\vdash_{\mathbf{LNP}} \mathsf{P}(A \supset A)$.

Ad (vi): Suppose $\vdash_{\mathbf{CLoNs}} A' \supset A$. By (A$\supset$2), $\vdash_{\mathbf{CLoNs}} (A' \supset (A \supset B)) \supset ((A' \supset A) \supset (A' \supset B))$. By (A$\supset$1) and (MP), $A \supset B \vdash_{\mathbf{DP}} A' \supset (A \supset B)$. The rest follows by multiple applications of (MP).

Ad (vii): The proof is similar and left to the reader.

Ad (viii): *Left-to-right*: By (MP), $(A \vee (A \supset B)) \supset B, A \vee (A \supset B) \vdash_{\mathbf{CLoNs}} B$. The rest follows by Theorem 13. *Right-to-left*: By (A$\supset$1), (†) $\vdash_{\mathbf{CLoNs}} B \supset (A \supset B)$. By (A$\vee$2), (‡) $\vdash_{\mathbf{CLoNs}} (A \supset B) \supset (A \vee (A \supset B))$. Altogether, by (†), (‡), (vii) and (MP), $\vdash_{\mathbf{CLoNs}} B \supset (A \vee (A \supset B))$. Hence, by (vii), $\vdash_{\mathbf{CLoNs}} ((A \vee (A \supset B) \supset B) \supset B) \supset ((A \vee (A \supset B) \supset B) \supset (A \vee (A \supset B)))$. By (A$\supset$3), $\vdash_{\mathbf{CLoNs}} ((A \vee (A \supset B) \supset B) \supset (A \vee (A \supset B))) \supset (A \vee (A \supset B))$. Hence, again by (vii), $((A \vee (A \supset B) \supset B) \supset B) \supset (A \vee (A \supset B))$.

Ad (ix): By (MP), $A, A \supset (A \supset B) \vdash_{\mathbf{CLoNs}} A \supset B$. By (MP), $A, A \supset (A \supset B) \vdash_{\mathbf{CLoNs}} B$. By Theorem 13, $A \supset (A \supset B) \vdash_{\mathbf{CLoNs}} A \supset B$, $\vdash_{\mathbf{CLoNs}} (A \supset (A \supset B)) \supset (A \supset B)$.

Ad (x): By (A$\vee$1), $\vdash_{\mathbf{CLoNs}} A \supset (A \vee (A \supset B))$. By (vi), $\vdash_{\mathbf{CLoNs}} (A \vee (A \supset B)) \supset B \vdash_{\mathbf{CLoNs}} A \supset B$. By Theorem 13, $\vdash_{\mathbf{CLoNs}} ((A \vee (A \supset B)) \supset B) \supset (A \supset B)$. By (A$\vee$2), $\vdash_{\mathbf{CLoNs}} (A \supset B) \supset (A \vee (A \supset B))$. Hence, by (vii), $\vdash_{\mathbf{CLoNs}} ((A \vee (A \supset B)) \supset B) \supset (A \vee (A \supset B))$. By (viii), $\vdash_{\mathbf{CLoNs}} (A \vee (A \supset B)) \equiv (((A \vee (A \supset B)) \supset B) \supset B)$. Thus, by (vii), $\vdash_{\mathbf{CLoNs}} ((A \vee (A \supset B)) \supset B) \supset (((A \vee (A \supset B)) \supset B) \supset B)$. By (vix) and (MP), $\vdash_{\mathbf{CLoNs}} ((A \vee (A \supset B)) \supset B) \supset B$. By (viii), (A$\equiv$2), and (MP), $\vdash_{\mathbf{DP}} A \vee (A \supset B)$. □

Appendix C. Proofs of Theorems 1 and 2

In order to simplify the notation in the following meta-proofs we define $R(w) = \{w' \mid Rww'\}$.

Proof of Theorem 1. Let in the following $M = \langle W, w_0, R, v_0, v\rangle$ be an **LNP**-model.

It is easy to check that all **CL**-axiom schemata hold at w_0 in M due to (C_0), (C¬), and (C⊃)-(C≡). Similarly, (†) where $w \in W \setminus \{w_0\}$, all **CLoNs**-axiom schemata hold at w in M due to (C_l) and (C∼∼)-(C≡).

Ad (NEC$^\sim$). Let $\models_{\mathbf{CLoNs}} A$. By (C$\mathsf{O}$), (†) and the definition of R, $v_M(\mathsf{O}A, w_0) = 1$.

Ad (K). Suppose $M \Vdash \mathsf{O}(A \supset B)$. By (C$\mathsf{O}$) and (C⊃), for all $w \in R(w_0)$, $v_M(A, w) = 0$ or $v_M(B, w) = 1$. Suppose $M \Vdash \mathsf{O}A$, then for all $w \in R(w_0)$, $v_M(A, w) = 1$. Hence, for all $w \in R(w_0)$, $v_M(B, w) = 1$. Thus by (CO), $M \Vdash \mathsf{O}B$. Hence, by (C⊃), $M \Vdash \mathsf{O}A \supset \mathsf{O}B$. Altogether, by (C⊃), $M \Vdash \mathsf{O}(A \supset B) \supset (\mathsf{O}A \supset \mathsf{O}B)$.

Ad (D). Suppose $M \Vdash \mathsf{O}A$. Hence for all $w \in R(w_0)$, $v_M(A, w) = 1$ (by (CO)). By the seriality of R, there is a $w \in R(w_0)$ for which $v_M(A, w) = 1$. By (CP), $M \Vdash \mathsf{P}A$. By (C⊃), $M \Vdash \mathsf{O}A \supset \mathsf{P}A$.

Ad (KP). Suppose $M \Vdash \mathsf{O}(A \supset B)$. By (C$\mathsf{O}$) and (C⊃), (‡) for all $w \in R(w_0)$, $v_M(A, w) = 0$ or $v_M(B, w) = 1$. Suppose $M \Vdash \mathsf{P}A$. Then, by (CP) there is a $w \in R(w_0)$ for which $v_M(A, w) = 1$. Hence, by (‡), there is a $w \in R(w_0)$ such that $v_M(B, w) = 1$. Thus, by (CP) $M \Vdash \mathsf{P}B$ and, by (C⊃), $M \Vdash \mathsf{P}A \supset \mathsf{P}B$. Altogether, by (C⊃), $M \Vdash \mathsf{O}(A \supset B) \supset (\mathsf{P}A \supset \mathsf{P}B)$.

Ad (OD). Suppose $M \Vdash \mathsf{O}(A \vee B)$. By (C$\mathsf{O}$) and (C∨), (⋆) for all $w \in R(w_0)$, $v_M(A, w) = 1$ or $v_M(B, w) = 1$. Suppose $M \nVdash \mathsf{P}B$. By (CP): for all $w \in R(w_0)$, $v_M(B, w) = 0$. By (⋆), for all $w \in R(w_0)$, $v_M(A, w) = 1$. Thus, by (CO), $M \Vdash \mathsf{O}A$. Hence, by (C∨), $M \Vdash \mathsf{O}A \vee \mathsf{P}B$. Altogether, by (C⊃), $M \Vdash \mathsf{O}(A \vee B) \supset (\mathsf{O}A \vee \mathsf{P}B)$.

Ad (PD). This is similar to the previous case and is left to the reader.

We now know that all axiom schemata and rules of **LNP** are semantically valid. That $\Gamma \vdash_{\mathbf{LNP}} A$ implies $\Gamma \models_{\mathbf{LNP}} A$ can now be shown via the usual induction on the length of the proof of A. This is safely left to the reader. □

Let in the remainder W_c be the **LNP**-deductively closed and maximally **LNP**-non-trivial subsets of $\mathcal{W}$.[17] Moreover, let $W_c^\sim$ be the **CLoNs**-deductively closed subsets Γ of $\mathcal{W}^\sim$ where Γ is prime, i.e. for each $A \vee B \in \Gamma$ either $A \in \Gamma$ or $B \in \Gamma$.

For the completeness proof of **LNP**, we make use of the following lemmas.[18]

Lemma 1. *If $\Delta \in W_c$, then Δ is prime.*

Proof. Suppose that, for a $\Delta \in W_c$, $A \vee B \in \Delta$ and $A \notin \Delta$ and $B \notin \Delta$. Then, since Δ is maximally **LNP**-non-trivial, $\Delta \cup \{A\}$ is trivial and $\Delta \cup \{B\}$ is trivial. Then, for any $C \in \mathcal{W}$, $\Delta \cup \{A\} \vdash_{\mathbf{LNP}} C$ and $\Delta \cup \{B\} \vdash_{\mathbf{LNP}} C$. Then, by Theorem 13, $\Delta \vdash_{\mathbf{LNP}} A \supset C$ and $\Delta \vdash_{\mathbf{LNP}} B \supset C$. But then, by (MP) and (A∨3), $\Delta \vdash_{\mathbf{LNP}} (A \vee B) \supset C$. Since $A \vee B \in \Delta$, since by (MP) $\Delta \vdash_{\mathbf{LNP}} C$, and since Δ is **LNP**-deductively closed, $C \in \Delta$. This contradicts the supposition. Hence if $A \vee B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$. The other direction is shown in a similar way by means of (A∨1) and (A∨2). This is left to the reader. □

Where $\Gamma \in W_c$ and $A \in \mathcal{W}^\sim$, we will use the following abbreviations: $\Gamma_O = \{B \mid \mathsf{O}B \in \Gamma\}$, $\Gamma_O^A = \Gamma_O \cup \{A\}$, $\Gamma_P = \{B \mid \mathsf{P}B \notin \Gamma\}$, ${}^\vee\Gamma_P = \{\bigvee_I B_i \mid B_i \in \Gamma_P\}$ and ${}^\vee\Gamma_P^B = \{\bigvee_I B_i \mid B_i \in \Gamma_P \cup \{B\}\}$.

[17] Where $\mathcal{W}^{\mathbf{L}}$ is the set of wffs of **L**, Γ is **L**-trivial iff $\mathrm{Cn}_{\mathbf{L}}(\Gamma) = \mathcal{W}^{\mathbf{L}}$, Γ is **L**-deductively closed iff $\mathrm{Cn}_{\mathbf{L}}(\Gamma) = \Gamma$, and Γ is maximally **L**-non-trivial iff it is **L**-non-trivial and all supersets $\Gamma' \supset \Gamma$ are **L**-trivial.

[18] The proof of Lemma 4 is inspired by the proof of Lemma 1.7.1 from [7].

Lemma 2. *Let $\Gamma \in W_c$. (i) If $C \in \mathrm{Cn}_{\mathbf{CLoNs}}(\Gamma_O)$ then $\mathsf{O}C \in \Gamma$. (ii) Where $\mathsf{P}A \in \Gamma$, if $C \in \mathrm{Cn}_{\mathbf{CLoNs}}(\Gamma_O^A)$ then $\mathsf{P}C \in \Gamma$.*

Proof. Ad (i): Suppose that $\Gamma_O \vdash_{\mathbf{CLoNs}} C$. Then $\Gamma' \vdash_{\mathbf{CLoNs}} C$ for some finite $\Gamma' \subseteq \Gamma_O$ (given the compactness of **CLoNs**). Hence, $\vdash_{\mathbf{CLoNs}} \big(\bigwedge \Gamma'\big) \supset C$ by Theorem 13. Thus, $\vdash_{\mathbf{LNP}} \mathsf{O}\big(\big(\bigwedge \Gamma'\big) \supset C\big)$ by (NEC$^{\sim}$). By (K), $\vdash_{\mathbf{CLoNs}} \mathsf{O} \bigwedge \Gamma' \supset \mathsf{O}C$. By the deductive closure of Γ, the fact that $\Gamma' \subseteq \Gamma$ and Fact 1 (i), $\mathsf{O} \bigwedge \Gamma' \in \Gamma$. By (MP), $\mathsf{O}C \in \Gamma$.

Ad (ii): Suppose that $\Gamma_O^A \vdash_{\mathbf{CLoNs}} C$. Then $\Gamma' \vdash_{\mathbf{CLoNs}} C$ for some finite $\Gamma' \subseteq \Gamma_O^A$ (given the compactness of **CLoNs**). Then $\Gamma' \cup \{A\} \vdash_{\mathbf{CLoNs}} C$ by the monotonicity of **CLoNs**. Then $\vdash_{\mathbf{CLoNs}} \big(\bigwedge \Gamma' \wedge A\big) \supset C$ by Theorem 13. Then $\vdash_{\mathbf{LNP}} \mathsf{O}\big(\big(\bigwedge \Gamma' \wedge A\big) \supset C\big)$ by (NEC$^{\sim}$). Then $\vdash_{\mathbf{LNP}} \mathsf{P}\big(\bigwedge \Gamma' \wedge A\big) \supset \mathsf{P}C$ by (KP) and (MP). By the supposition, $\{\mathsf{O}B \mid B \in \Gamma'\} \subseteq \Gamma$ and $\mathsf{P}A \in \Gamma$. Given the deductive closure of Γ and $\vdash_{\mathbf{LNP}} (\mathsf{O}(\bigwedge \Gamma') \wedge \mathsf{P}A) \supset \mathsf{P}(\bigwedge \Gamma' \wedge A)$ (which follows from Fact 1 (ii)), it follows that $\mathsf{P}\big(\bigwedge \Gamma' \wedge A\big) \in \Gamma$. Hence $\mathsf{P}C \in \Gamma$, since Γ is deductively closed and $\vdash_{\mathbf{LNP}} \mathsf{P}\big(\bigwedge \Gamma' \wedge A\big) \supset \mathsf{P}C$. $\square$

Lemma 3. *Let $\Gamma \in W_c$. (i) Where $\mathsf{P}A \in \Gamma$, ${}^{\vee}\Gamma_P \cap \mathrm{Cn}_{\mathbf{CLoNs}}(\Gamma_O^A) = \emptyset$. (ii) Where $B \notin \Gamma_O$, ${}^{\vee}\Gamma_P^B \cap \mathrm{Cn}_{\mathbf{CLoNs}}(\Gamma_O) = \emptyset$.*

Proof. Ad (i): Let $C = \bigvee_I C_i$ where $C_i \in \Gamma_P$. Suppose $C \in \mathrm{Cn}_{\mathbf{CLoNs}}(\Gamma_O^A)$ then by Lemma 2 (ii), $\mathsf{P} \bigvee_I C_i \in \Gamma$. Hence, by (PD), $\bigvee_I \mathsf{P}C_i \in \Gamma$. Hence, since Γ is prime, there is an $i \in I$ for which $\mathsf{P}C_i \in \Gamma$ and hence $C_i \notin \Gamma_P$,—a contradiction.

Ad (ii): Let $C = \bigvee_I C_i$ where $C_i \in \Gamma_P \cup \{B\}$. Suppose $C \in \mathrm{Cn}_{\mathbf{CLoNs}}(\Gamma_O)$. By Lemma 2 (i), $\mathsf{O} \bigvee_I C_i \in \Gamma$. Assume that all $C_i \in \Gamma_P$. By (D), $\mathsf{P} \bigvee_I C_i \in \Gamma$. By (PD), $\bigvee_I \mathsf{P}C_i \in \Gamma$. Hence, since Γ is prime, there is an $i \in I$ such that $\mathsf{P}C_i \in \Gamma$ and hence $C_i \notin \Gamma_P$,—a contradiction. Hence there is a non-empty $J \subseteq I$ such that for each $j \in J$, $C_j = B$. Hence, by (OD), $\mathsf{O}B \vee \mathsf{P} \bigvee_{I \setminus J} C_i$. Thus, by (PD), $\mathsf{O}B \vee \bigvee_{I \setminus J} \mathsf{P}C_i$. Since $B \notin \Gamma_O$ and since Γ is prime, there is an $i \in I \setminus J$ such that $\mathsf{P}C_i \in \Gamma$ and hence $C_i \notin \Gamma_P$,—a contradiction. $\square$

Lemma 4. *Let $\Gamma \in W_c$.*

1. *Where $\mathsf{P}A \in \Gamma$, there is a $\Delta \subseteq \mathcal{W}^{\sim}$ for which (i) $\Gamma_O^A \subseteq \Delta$, (ii) ${}^{\vee}\Gamma_P \cap \Delta = \emptyset$, and (iii) $\Delta \in W_c^{\sim}$.*
2. *Where $B \notin \Gamma_O$, there is a $\Delta \subseteq \mathcal{W}^{\sim}$ for which (i) $\Gamma_O \subseteq \Delta$, (ii) ${}^{\vee}\Gamma_P^B \cap \Delta = \emptyset$, and (iii) $\Delta \in W_c^{\sim}$.*

Proof. Let $\langle \mathbf{\Gamma_O}, \mathbf{\Gamma_P} \rangle \in \{\langle \Gamma_O^A, {}^{\vee}\Gamma_P \rangle, \langle \Gamma_O, {}^{\vee}\Gamma_P^B \rangle\}$. Where $\langle B_1, B_2, \ldots \rangle$ is a list of the members of $\mathcal{W}^{\sim}$, define $\Delta_0 = \mathrm{Cn}_{\mathbf{CLoNs}}(\mathbf{\Gamma_O})$ and $\Delta = \Delta_0 \cup \Delta_1 \cup \ldots$ where

$$\Delta_{i+1} = \begin{cases} \mathrm{Cn}_{\mathbf{CLoNs}}(\Delta_i \cup \{B_{i+1}\}) & \text{if } \mathbf{\Gamma_P} \cap \mathrm{Cn}_{\mathbf{CLoNs}}(\Delta_i \cup \{B_{i+1}\}) = \emptyset \\ \Delta_i & \text{otherwise} \end{cases}$$

Ad (i): this holds by the construction and the reflexivity of **CLoNs**.

Ad (ii): By Lemma 3 $\Delta_0 \cap \mathbf{\Gamma_P} = \emptyset$. The rest follows by the construction.

Ad (iii): We first show that Δ is **CLoNs**-deductively closed. Suppose there is a $B_i \notin \Delta$ such that $\Delta \vdash_{\mathbf{CLoNs}} B_i$. Then, by the construction of Δ, there is a $D \in \mathbf{\Gamma_P}$ such that $\Delta \cup \{B_i\} \vdash_{\mathbf{CLoNs}} D$ and hence by Theorem 13, $\Delta \vdash_{\mathbf{CLoNs}} B_i \supset D$. However, by (MP) also $\Delta \vdash_{\mathbf{CLoNs}} D$. By the compactness of **CLoNs** there is a Δ_i for which $\Delta_i \vdash_{\mathbf{CLoNs}} D$. By the construction $\Delta_i = \mathrm{Cn}_{\mathbf{CLoNs}}(\Delta_i)$ and whence $D \in \Delta_i$. Hence, $D \in \Delta$,—a contradiction with (ii).

We now show that Δ is prime. Suppose $A_1 \vee A_2 \in \Delta$. Assume $A_1, A_2 \notin \Delta$. Hence, by the construction of Δ, $\Delta \cup \{A_1\} \vdash_{\mathbf{CLoNs}} D_1$ and $\Delta \cup \{A_2\} \vdash_{\mathbf{CLoNs}} D_2$ for some $D_1, D_2 \in \mathbf{\Gamma_P}$. By Theorem 13, $\Delta \vdash_{\mathbf{CLoNs}} A_1 \supset D_1$ and $\Delta \vdash_{\mathbf{CLoNs}} A_2 \supset D_2$. By some simple propositional manipulations, $\Delta \vdash_{\mathbf{CLoNs}} (A_1 \vee A_2) \supset (D_1 \vee D_2)$. By (MP), $\Delta \vdash_{\mathbf{CLoNs}} D_1 \vee D_2$ and hence $D_1 \vee D_2 \in \Delta$. However, by the definition of $\mathbf{\Gamma_P}$, $D_1 \vee D_2 \in \mathbf{\Gamma_P}$,—a contradiction with (ii). $\square$

Definition 12. *The binary relation* $\mathsf{R} \subseteq \left(W_c \times W_c^{\sim}\right)$ *is defined as follows:* $\mathsf{R}\Gamma\Delta$ *iff the following two conditions are met*

(a) if $\mathsf{O}A \in \Gamma$ *then* $A \in \Delta$*, and*

(b) if $A \in \Delta$ *then* $\mathsf{P}A \in \Gamma$*.*

In view of the definition of R, the following holds:

Lemma 5. *Where* $\Gamma \in W_c$*,* $\mathsf{P}A \in \Gamma$ *iff there is a* $\Delta \in W_c^{\sim}$ *such that* $\mathsf{R}\Gamma\Delta$ *and* $A \in \Delta$*.*

Proof. Left-right: Suppose $\mathsf{P}A \in \Gamma$. Then, by Lemma 4.1, there is a $\Delta \subseteq \mathcal{W}^{\sim}$ such that (i) $\Gamma_O^A \subseteq \Delta$, (ii) for all $C \in \Gamma_P$, $C \notin \Delta$, (iii) $\Delta \in W_c^{\sim}$. We now show that $\mathsf{R}\Gamma\Delta$. Ad (a): if, for some D, $\mathsf{O}D \in \Gamma$ then $D \in \Gamma_O^A$, hence $D \in \Delta$ by (i). Ad (b): suppose $\mathsf{P}E \notin \Gamma$ for some $E \in \mathcal{W}^{\sim}$. Then $E \in \Gamma_P$, hence $E \notin \Delta$ by (ii).

Right-left: Follows directly by Definition 12. $\square$

Lemma 6. *For every* $\Gamma \in W_c$*, there is a* $\Delta \in W_c^{\sim}$ *such that* $\mathsf{R}\Gamma\Delta$ *(i.e.* R *is serial).*

Proof. By Fact 1 (v), $\vdash_{\mathbf{LNP}} \mathsf{P}(A \supset A)$. Hence, $\mathsf{P}(A \supset A) \in \Gamma$ for every $\Gamma \in W_c$. But then, by Lemma 5, there is a $\Delta \in W_c^{\sim}$ such that $\mathsf{R}\Gamma\Delta$ and $A \supset A \in \Delta$. Hence R is serial as required. $\square$

Lemma 7. *Where* $\Gamma \in W_c$*,* $\mathsf{O}A \in \Gamma$ *iff, for all* $\Delta \in W_c^{\sim}$ *such that* $\mathsf{R}\Gamma\Delta$*,* $A \in \Delta$*.*

Proof. Left-right: This is an immediate consequence of Definition 12.

Right-left: Suppose $\mathsf{O}A \notin \Gamma$. Hence, $A \notin \Gamma_O$. By Lemma 4.2, there is a $\Delta \subseteq \mathcal{W}^{\sim}$ for which (i) $\Gamma_O \subseteq \Delta$, (ii) $(\Gamma_P \cup \{A\}) \cap \Delta = \emptyset$, and (iii) $\Delta \in W_c^{\sim}$. We now show that $\mathsf{R}\Gamma\Delta$. Ad (a): if, for some D, $\mathsf{O}D \in \Gamma$ then $D \in \Gamma_O$, hence $D \in \Delta$ by (i). Ad (b): suppose $\mathsf{P}E \notin \Gamma$ for some $E \in \mathcal{W}^{\sim}$. Then $E \in \Gamma_P$, hence $E \notin \Delta$ by (ii). $\square$

Lemma 8. *Where* $\Delta \in W_c$*, there is an* **LNP***-model* M *such that* $M \Vdash A$ *for all* $A \in \Delta$ *and* $M \nVdash A$ *for all* $A \in \mathcal{W} \setminus \Delta$*.*

Proof. Let $\Delta \in W_c$. We construct an **LNP**-model $M = \langle \{\Delta\} \cup W_c^{\sim}, w_0, \mathsf{R}, v_0, v\rangle$ such that:

(i) $w_0 = \Delta$
(ii) For all $A \in \mathcal{W}^a$, $v_0(A) = 1$ iff $A \in w_0$
(iii) For all $A \in \mathcal{W}^l$ and all $w \in W_c^{\sim}, v(A, w) = 1$ iff $A \in w$

By Lemma 6, R is serial. We now show that:

(*) (a) for all $A \in \mathcal{W}$, $v_M(A, w_0) = 1$ iff $A \in w_0$,
(b) for all $A \in \mathcal{W}^{\sim}$ and all $w \in W_c^{\sim}$, $v_M(A, w) = 1$ iff $A \in w$.

The proof proceeds as usual by an induction on the complexity of A. Let $w \in \{w_0\} \cup W_c^{\sim}$, and $A \in \mathcal{W}^a$. If $w = w_0$, then, by (ii), $v_0(A) = 1$ iff $A \in w_0$. By (C_0), it follows that $v_M(A, w) = 1$ iff $A \in w$. If $w \neq w_0$, then, by (iii), $v(A, w) = 1$ iff $A \in w$. By (C_l), it follows that $v_M(A, w) = 1$ iff $A \in w$. Hence, for all $w \in \{w_0\} \cup W_c^{\sim}$, $v_M(A, w) = 1$ iff $A \in w$ and (*) is valid for all $A \in \mathcal{W}^a$.

Depending on the logical form of A, we distinguish 8 cases (6 for the connectives $\sim, \neg, \vee, \wedge, \supset, \equiv$, and 2 for the modal operators O and P) and show for each of them that $v_M(A, w) = 1$ iff $A \in w$.

Case 1. Let $w \in W_c^{\sim}$. We show that $v_M(\sim A, w) = 1$ iff $\sim A \in w$. Either $\sim A \in \mathcal{W}^l$, or A has one of the forms $\sim B, B \vee C, B \wedge C, B \supset C$, or $B \equiv C$ (note that, since $w \neq w_0$, A cannot have the form $\mathsf{O}B$ or $\mathsf{P}B$).

If $\sim A \in \mathcal{W}^l$, then, by (C$_l$), $v_M(\sim A, w) = 1$ iff $v(\sim A, w) = 1$. By (iii), it follows that $v_M(\sim A, w) = 1$ iff $\sim A \in w$.

If A has the form $\sim B$, then, by (C$\sim\sim$), $v_M(\sim\sim B, w) = 1$ iff $v_M(B, w) = 1$. By the induction hypothesis, $v_M(\sim\sim B, w) = 1$ iff $B \in w$. By (A$\sim\sim$), $v_M(\sim A, w) = 1$ iff $\sim A \in w$.

If A has the form $B \vee C$, then, by (C$\sim\vee$), $v_M(\sim(B \vee C), w) = 1$ iff $v_M(\sim B \wedge \sim C, w) = 1$ iff [by (C$\wedge$)] $v_M(\sim B, w) = 1$ and $v_M(\sim C, w) = 1$ iff [by the induction hypothesis] $\sim B \in w$ and $\sim C \in w$ iff [by (A$\wedge$3)] $\sim B \wedge \sim C \in w$ iff [by (A$\sim\vee$)] $\sim A \in w$.

The cases where A is of one of the forms $B \wedge C, B \supset C$, or $B \equiv C$ are similar and left to the reader.

Case 2. Let $w = w_0$. Suppose $v_M(\neg A, w) = 1$. By (C$\neg$), $v_M(A, w) = 0$. By the induction hypothesis, $A \notin w$. Then, since w is maximally **LNP**-non-trivial, $w \cup \{A\}$ is **LNP**-trivial and $w \cup \{A\} \vdash_{\mathbf{LNP}} \neg A$. By Theorem 13, it follows that $w \vdash_{\mathbf{LNP}} A \supset \neg A$. Then, since w is **LNP**-deductively closed, $A \supset \neg A \in w$ and, by (A$\neg$1) and (MP), $\neg A \in w$.

Suppose $\neg A \in w$. We show via reductio that $A \notin w$. Suppose thus that $A \in w$. Then, by (A$\neg$2), (MP), and since w is **LNP**-deductively closed, $B \in w$ for any $B \in \mathcal{W}$. This contradicts the non-triviality of w, hence $A \notin w$. But then, by the induction hypothesis $v_M(A, w) = 0$ and, by (C$\neg$), $v_M(\neg A, w) = 1$.

Case 3. Let $w \in \{w_0\} \cup W_c^{\sim}$. Suppose $v_M(A \vee B, w) = 1$. Then, by (C$\vee$), $v_M(A, w) = 1$ or $v_M(B, w) = 1$. By the induction hypothesis, $A \in w$ or $B \in w$. Hence, by (A$\vee$1), (A$\vee$2), (MP), and the fact that w is **LNP**-(in case $w = w_0$)/**CLoNs**-(in case $w \in W_c^{\sim}$)-deductively closed, $A \vee B \in w$.

Suppose $A \vee B \in w$. If $w = w_0$, then, by the definition of $W_c^{\sim}$, $A \in w$ or $B \in w$. If $w \neq w_0$, then, by Lemma 1, $A \in w$ or $B \in w$. By the induction hypothesis, $v_M(A, w) = 1$ or $v_M(B, w) = 1$. Hence, by (C$\vee$), $v_M(A \vee B, w) = 1$.

Case 4. Let $w \in \{w_0\} \cup W_c^{\sim}$. Suppose $v_M(A \supset B, w) = 1$. Then by (C$\supset$), $v_M(A, w) = 0$ or $v_M(B, w) = 1$. By the induction hypothesis, $A \notin w$ or $B \in w$. Let now $w \in W_c^{\sim}$. If $A \notin w$, then, since $\vdash_{\mathbf{CLoNs}} A \vee (A \supset B)$ by Fact 1 (x) and since w is prime, also $A \supset B \in w$. If $B \in w$, then since by (A$\supset$1) $\vdash_{\mathbf{CLoNs}} B \supset (A \supset B)$ and by (MP), also $A \supset B \in w$. The same argument applies to $w = w_0$ since also $\vdash_{\mathbf{LNP}} A \vee (A \supset B)$, and (A$\supset$1) and (MP) are also valid in **LNP**.

Suppose $A \supset B \in w$. By (MP), if $A \in w$ then $B \in w$. By the induction hypothesis, if $v_M(A, w) = 1$ then $v_M(B, w) = 1$. Hence, by (C$\supset$), $v_M(A \supset B, w) = 1$.

The proof for the other classical connectives (cases 4-6) is similar and left to the reader. We proceed with the cases for O and P.

Case 7. Let $w = w_0$. By Lemma 7, $\mathsf{O}A \in w_0$ iff $A \in w$ for all w such that $\mathsf{R}w_0w$. Hence, by the induction hypothesis, $\mathsf{O}A \in w_0$ iff $v_M(A, w) = 1$ for all w such that $\mathsf{R}w_0w$. But then, by (CO), $\mathsf{O}A \in w_0$ iff $v_M(\mathsf{O}A, w_0) = 1$.

Case 8. Let $w = w_0$. By Lemma 5, $\mathsf{P}A \in w_0$ iff $A \in w$ for some w such that $\mathsf{R}w_0w$. Hence, by the induction hypothesis, $\mathsf{P}A \in w_0$ iff $v_M(A, w) = 1$ for some w such that $\mathsf{R}w_0w$. But then, by (CP), $\mathsf{P}A \in w_0$ iff $v_M(\mathsf{P}A, w_0) = 1$.

The rest follows by (i) and (*). $\square$

Lemma 9. *Let $\Gamma \subseteq \mathcal{W}$ and $\Gamma \nvdash_{\mathbf{LNP}} A$. There is a $\Delta \subseteq \mathcal{W}$ such that (i) $\Gamma \subseteq \Delta$, (ii) $A \notin \Delta$, and (iii) $\Delta \in W_c$.*

Proof. Where $\langle B_1, B_2, \ldots \rangle$ is a list of the members of $\mathcal{W}$, define $\Delta_0 = \mathrm{Cn}_{\mathbf{LNP}}(\Gamma)$ and $\Delta = \Delta_0 \cup \Delta_1 \cup \ldots$ where

$$\Delta_{i+1} = \begin{cases} \mathrm{Cn}_{\mathbf{DP}}(\Delta_i \cup \{B_{i+1}\}) & \text{if } A \notin \mathrm{Cn}_{\mathbf{LNP}}(\Delta_i \cup \{B_{i+1}\}) \\ \Delta_i & \text{else} \end{cases}$$

Ad (i): This holds by the construction of Δ and the reflexivity of **LNP**.

Ad (ii): This holds by the construction and since $A \notin \mathrm{Cn}_{\mathbf{LNP}}(\Gamma)$.

Ad (iii): Assume that $B \notin \Delta$ and $\Delta \vdash_{\mathbf{LNP}} B$. Hence, by the construction of Δ, $\Delta \cup \{B\} \vdash_{\mathbf{LNP}} A$ and whence by Theorem 13, $\Delta \vdash_{\mathbf{LNP}} B \supset A$. But then by (MP), $\Delta \vdash_{\mathbf{LNP}} A$. Thus, by the compactness of **LNP** and since each $\Delta_i = \mathrm{Cn}_{\mathbf{LNP}}(\Delta_i)$, there is a Δ_i such that $A \in \Delta_i$,—a contradiction to (ii).

Suppose $B \notin \Delta$. Assume that $\neg B \notin \Delta$. By the construction of Δ and the monotonicity of **LNP**, $\Delta \cup \{\neg B\} \vdash_{\mathbf{LNP}} A$ and whence by Theorem 13, $\Delta \vdash_{\mathbf{LNP}} \neg B \supset A$. Analogously, $\Delta \vdash_{\mathbf{LNP}} B \supset A$. By (A$\vee$3), $\Delta \vdash_{\mathbf{LNP}} (B \vee \neg B) \supset A$. Since $\vdash_{\mathbf{CL}} B \vee \neg B$, also $\Delta \vdash_{\mathbf{LNP}} B \vee \neg B$. By (MP), $\Delta \vdash_{\mathbf{LNP}} A$,—a contradiction to (ii). Hence, $\neg B \in \Delta$. Thus, $\Delta \cup \{B\}$ is **CL**-trivial and hence also **LNP**-trivial. □

Proof of Theorem 2. Suppose $\Gamma \nvdash_{\mathbf{LNP}} A$. Then, by Lemma 9, there is a $\Delta \supseteq \Gamma$ such that $A \notin \Delta$ and $\Delta \in W_c$. Then, by Lemma 8, there is an **LNP**-model M such that $M \Vdash B$ for all $B \in \Gamma$ and $M \nVdash A$. Hence $\Gamma \nvDash_{\mathbf{LNP}} A$. □

Appendix D. Proof of Theorem 10

SDL is fully axiomatized by adding to **CL** the axiom schemata (K), (D), (PO), and (NEC$^\neg$):

(PO) $\quad \mathsf{P}A \equiv \neg \mathsf{O} \neg A$
(NEC$^\neg$) $\quad$ If $\vdash_{\mathbf{CL}} A$ then $\vdash \mathsf{O}A$

Lemma 10. *If $\Gamma \vdash_{\mathbf{ULL}} A$ then $\Gamma^\neg \vdash_{\mathbf{SDL}} \pi(A)$.*

Proof. It easily checked that, under the transformation given in Section 4.3, all of (K), (D), (KP), (OD), (PD), (U_1), and (U_2) are **SDL**-valid. Moreover, since **CLoNs** is a proper fragment of **CL**, (NEC$^\sim$) too is valid in **SDL** (assuming again the transformation from Section 4.3). □

Lemma 11. *If $\Gamma^\neg \vdash_{\mathbf{SDL}} \pi(A)$ then $\Gamma \vdash_{\mathbf{ULL}} A$.*

Proof. By the definition, **ULL** verifies (K) and (D). It remains to show that **ULL** verifies (i) all instances of $\mathsf{P}A \equiv \neg \mathsf{O} {\sim} A$ and (ii) the rule "If $\vdash_{\mathbf{CL}^\sim} A$ then $\vdash_{\mathbf{ULL}} \mathsf{O}A$", where $\mathbf{CL}^\sim$ is classical propositional logic with the negation symbol $\sim$.

Ad (i). *Left-Right.* By (A$\wedge$3), $\mathsf{P}A \supset (\mathsf{O}{\sim}A \supset (\mathsf{P}A \wedge \mathsf{O}{\sim}A))$. By Fact 1 (iv), $(\mathsf{P}A \wedge \mathsf{O}{\sim}A) \supset \mathsf{P}(A \wedge {\sim}A)$. Thus, by some propositional manipulations in **CL**, $\mathsf{P}A \supset (\mathsf{O}{\sim}A \supset \mathsf{P}(A \wedge {\sim}A))$, which is **CL**-equivalent to (†) $\mathsf{P}A \supset (\neg \mathsf{O}{\sim}A \vee \mathsf{P}(A \wedge {\sim}A))$. Suppose now that $\mathsf{P}A$. By (†), $\neg \mathsf{O}{\sim}A \vee \mathsf{P}(A \wedge {\sim}A)$. Moreover, by (U1), $\mathsf{P}(A \wedge {\sim}A) \supset \neg \mathsf{O}{\sim}A$. Thus, by (MP) and some simple **CL**-manipulations, we obtain $\neg \mathsf{O}{\sim}A$.
Right-left. By **CL**, $\mathsf{O}(A \vee {\sim}A) \vee \neg \mathsf{O}(A \vee {\sim}A)$. By (OD), $\mathsf{O}(A \vee {\sim}A) \supset (\mathsf{O}{\sim}A \vee \mathsf{P}A)$. Thus, by some propositional manipulations in **CL**, $(\mathsf{O}{\sim}A \vee \mathsf{P}A) \vee \neg \mathsf{O}(A \vee {\sim}A)$. The

latter formula is **CL**-equivalent to $\neg\mathsf{O}{\sim}A \supset (\mathsf{P}A \vee \neg\mathsf{O}(A \vee {\sim}A))$. Suppose now that $\neg\mathsf{O}{\sim}A$. By (MP), $\mathsf{P}A \vee \neg\mathsf{O}(A \vee {\sim}A)$. By (U2), $\neg\mathsf{O}(A \vee {\sim}A) \supset \mathsf{P}A$. Thus, by (MP) and some simple **CL**-manipulations, $\mathsf{P}A$.

Ad (ii). Note that $A \in \mathcal{W}^{\sim}$ iff $\pi(A) \in \mathcal{W}^{\neg}$. Thus, where

(A∼1): $(A \supset {\sim}A) \supset {\sim}A$,

(A∼2): $A \supset ({\sim}A \supset B)$,

it follows by the definitions of **CLoNs** and **CL** that $\vdash_{\mathbf{CLoNs}\cup\{(A\sim1),(A\sim2)\}} A$ iff $\vdash_{\mathbf{CL}} \pi(A)$. We show that (i) if $\vdash_{\mathbf{CLoNs}} A$, then $\vdash_{\mathbf{ULL}} A$, (ii) $\vdash_{\mathbf{ULL}} \mathsf{O}((A \supset {\sim}A) \supset {\sim}A)$, and (iii) $\vdash_{\mathbf{ULL}} \mathsf{O}(A \supset ({\sim}A \supset B))$.

(i) In case A is a **CLoNs**-theorem, $\mathsf{O}A$ follows immediately in view of (NEC$^{\sim}$).

(ii) $(A \vee {\sim}A) \supset ((A \supset {\sim}A) \supset {\sim}A)$ is an instance of the theorem $(A \vee B) \supset ((A \supset B) \supset B)$ of positive **CL**, thus it is a **CLoNs**-theorem. By (NEC$^{\sim}$), $\vdash_{\mathbf{ULL}} \mathsf{O}((A \vee {\sim}A) \supset ((A \supset {\sim}A) \supset {\sim}A))$. By (K), $\vdash_{\mathbf{ULL}} \mathsf{O}(A \vee {\sim}A) \supset \mathsf{O}((A \supset {\sim}A) \supset {\sim}A)$. By **CL**, (†) $\vdash_{\mathbf{ULL}} \mathsf{O}((A \supset {\sim}A) \supset {\sim}A) \vee \neg\mathsf{O}(A \vee {\sim}A)$. We know by (U2) that $\neg\mathsf{O}(A \vee {\sim}A) \supset \mathsf{O}((A \supset {\sim}A) \supset {\sim}A)$. Hence, by (†) and **CL**, $\vdash_{\mathbf{ULL}} \mathsf{O}((A \supset {\sim}A) \supset {\sim}A)$.

(iii) $(A \supset ({\sim}A \supset B)) \vee (A \wedge {\sim}A)$ is an instance of the theorem $(A \supset (B \supset C)) \vee (A \wedge B)$ of positive **CL**, thus it is a **CLoNs**-theorem. By (NEC$^{\sim}$), $\vdash_{\mathbf{ULL}} \mathsf{O}((A \supset ({\sim}A \supset B)) \vee (A \wedge {\sim}A))$. By (OD), (‡) $\vdash_{\mathbf{ULL}} \mathsf{O}(A \supset ({\sim}A \supset B)) \vee \mathsf{P}(A \wedge {\sim}A)$. We know by (U1) that $\mathsf{P}(A \wedge {\sim}A) \supset \mathsf{O}(A \supset ({\sim}A \supset B))$. Hence, by (‡) and **CL**, $\vdash_{\mathbf{ULL}} \mathsf{O}(A \supset ({\sim}A \supset B))$. □

Theorem 10 follows immediately by Lemmas 10 and 11.

References

[1] Carlos E. Alchourrón. Logic of norms and logic of normative propositions. *Logique & Analyse*, 47:242–268, 1969.

[2] Carlos E. Alchourrón and Eugenio Bulygin. *Normative Systems*. Springer-Verlag, Wien/New York, 1971.

[3] Carlos E. Alchourrón and Eugenio Bulygin. The expressive conception of norms. In Risto Hilpinen, editor, *New Studies in Deontic Logic*, pages 95–124. D. Reidel Publishing Company, Dordrecht, 1981.

[4] Carlos E. Alchourrón and Eugenio Bulygin. Von Wright on deontic logic and the philosophy of law. In P.A. Schilpp and L.E. Hahn, editors, *The Philosophy of Georg Henrik Von Wright*, pages 665–693. Open Court Publishing Company, 1989.

[5] Lennart Åqvist. Deontic logic. In Dov Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic (2nd edition)*, volume 8, pages 147–264. Kluwer Academic Publishers, 2002.

[6] Diderik Batens. A universal logic approach to adaptive logics. *Logica Universalis*, 1:221–242, 2007.

[7] Diderik Batens. *Adaptive Logics and Dynamic Proofs. Mastering the Dynamics of Reasoning, with Special Attention to Handling Inconsistency.* Forthcoming.

[8] Diderik Batens and Kristof De Clercq. A rich paraconsistent extension of full positive logic. *Logique et Analyse*, 185-188:227–257, 2004.

[9] Diderik Batens, Kristof De Clercq, Peter Verdée, and Joke Meheus. Yes fellows, most human reasoning is complex. *Synthese*, 166(1):113–131, 2009.

[10] Mathieu Beirlaen and Christian Straßer. A paraconsistent multi-agent framework for dealing with normative conflicts. In Joao Leite, Paolo Torroni, Thomas Ågotnes, Guido Boella, and Leon van der Torre, editors, *Computational Logic in Multi-Agent Systems. Proceedings of the* 12^{th} *International Workshop CLIMA XII*, volume 6814 of *Lecture Notes in Artificial Intelligence*, pages 312–329. Springer-Verlag, 2011.

[11] Mathieu Beirlaen, Christian Straßer, and Joke Meheus. An inconsistency-adaptive deontic logic for normative conflicts. *Journal of Philosophical Logic*. Forthcoming.

[12] Guido Boella and Leendert van der Torre. Permissions and obligations in hierarchical normative systems. In *Proceedings of the 9th International Conference on Artificial Intelligence and Law*, ICAIL '03, pages 109–118, New York, NY, USA, 2003. ACM.

[13] Eugenio Bulygin. On legal gaps. In P.Comanducci and R.Guastini, editors, *Ricerche Di Giurisprudenza Analitica*, pages 21–28. Giappichelli, Turin, 2003.

[14] Newton Da Costa and Walter Carnielli. On paraconsistent deontic logic. *Philosophia*, 16:293–305, 1986.

[15] Bas C. Van Fraassen. Values and the heart's command. *Journal of Philosophy*, 70:5–19, 1973.

[16] Lou Goble. Multiplex semantics for deontic logic. *Nordic Journal of Philosophical Logic*, 5(2):113–134, 2000.

[17] Lou Goble. A proposal for dealing with deontic dilemmas. In Alessio Lomuscio and Donald Nute, editors, *DEON*, volume 3065 of *Lecture Notes in Computer Science*, pages 74–113. Springer, 2004.

[18] Lou Goble. A logic for deontic dilemmas. *Journal of Applied Logic*, 3:461–483, 2005.

[19] Jörg Hansen. Conflicting imperatives and dyadic deontic logic. *Journal of Applied Logic*, 3:484–511, 2005.

[20] Bengt Hansson. An analysis of some deontic logics. *Noûs*, 3:373–398, 1969.

[21] Sven Ove Hansson. *The Structure of Values and Norms*. Cambridge University Press, 2001.

[22] Ingemar Hedenius. *Om Rätt och Moral*. Tiden, Stockholm, 1941.

[23] John Francis Horty. Moral dilemmas and nonmonotonic logic. *Journal of Philosophical Logic*, 23(1):35–66, 1994.

[24] John Francis Horty. *Agency and Deontic Logic*. Oxford University Press, 2001.

[25] John Francis Horty. Reasoning with moral conflicts. *Noûs*, 37:557–605, 2003.

[26] Barteld Kooi and Allard Tamminga. Moral conflicts between groups of agents. *Journal of Philosophical Logic*, 37:1–21, 2008.

[27] Sarit Kraus, Daniel J. Lehmann, and Menachem Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence*, 44:167–207, 1990.

[28] David Makinson and Leendert van der Torre. Input/output logics. *Journal of Philosophical Logic*, 29:383–408, 2000.

[29] David Makinson and Leendert van der Torre. Constraints for input/output logics. *Journal of Philosophical Logic*, 30:155–185, 2001.

[30] David Makinson and Leendert van der Torre. Permission from an input/output perspective. *Journal of Philosophical Logic*, 32(4):391–416, 2003.

[31] David Makinson and Leendert van der Torre. What is input/output logic? input/output logic, constraints, permissions. In Guido Boella, Leon van der Torre, and Harko Verhagen, editors, *Normative Multi-agent Systems*, number 07122 in Dagstuhl Seminar Proceedings, Dagstuhl, Germany, 2007. Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss Dagstuhl, Germany.

[32] Joke Meheus, Mathieu Beirlaen, and Frederik Van De Putte. Avoiding deontic explosion by contextually restricting aggregation. In Guido Governatori and Giovanni Sartor, editors, *DEON (10th International Conference on Deontic Logic in Computer Science)*, volume 6181 of *Lecture Notes in Artificial Intelligence*, pages 148–165. Springer, 2010.

[33] Joke Meheus, Mathieu Beirlaen, Frederik Van De Putte, and Christian Straßer. Non-adjunctive deontic logics that validate aggregation as much as possible. Under review.

[34] Graham Priest. *In Contradiction: A Study of the Transconsistent (2nd Edition)*. Oxford University Press, 2006.

[35] Nicholas Rescher. *Topics in Philosophical Logic*. Reidel, Dordrecht, 1968.

[36] Peter Schotch and Raymond Jennings. Non-Kripkean deontic logic. In Risto Hilpinen, editor, *New Studies In Deontic Logic*, pages 149–162. D. Reidel Publishing Company, Dordrecht, 1981.

[37] Yoav Shoham. A semantical approach to nonmonotonic logics. In Matthew L. Ginsberg, editor, *Readings in Nonmonotonic Reasoning*, pages 227–250. Morgan Kaufmann Publishers, 1987.

[38] Erik Stenius. The principles of a logic of normative systems. *Acta Philosophica Fennica*, 16:247–260, 1963.

[39] Audun Stolpe. A theory of permission based on the notion of derogation. *Journal of Applied Logic*, 8(1):97–113, 2010.

[40] Christian Straßer. An adaptive logic framework for conditional obligations and deontic dilemmas. *Logic and Logical Philosophy*, 19(1-2):95–128, 2010.

[41] Christian Straßer. A deontic logic framework allowing for factual detachment. *Journal of Applied Logic*, 9(1):61–80, 2011.

[42] Christian Straßer, Mathieu Beirlaen, and Joke Meheus. Tolerating deontic conflicts by adaptively restricting inheritance. Under review.

[43] Peter Verdée. Adaptive logics using the minimal abnormality strategy are π_1^1-complex. *Synthese*, 167:93–104, 2009.

[44] Georg Henrik von Wright. *Norm and Action. A Logical Enquiry*. Routledge and Kegan Paul, London, 1963.

[45] Georg Henrik von Wright. *An Essay in Deontic Logic and the General Theory of Action*. North-Holland, Amsterdam, 1968.

[46] Georg Henrik von Wright. Deontic logic: a personal view. *Ratio Juris*, 12(1):26–38, 1999.

[47] Anders Wedberg. Some problems in the logical analysis of legal science. *Theoria*, 17:246–275, 1951.

[48] Gregory Wheeler and Marco Alberti. No revision and no contraction. *Minds and Machines*, 21(3):411–430, 2011.

[49] Bernard Williams. Ethical consistency. *Proceedings of the Aristotelian Society (Supplementary Volumes)*, 39:103–124, 1965.