Epistemic Modalities

Giuseppe Primiero

ABSTRACT. I present an analysis of the notion of epistemic modalities, based on an appropriate interpretation of two basic constructivist issues: verification and epistemic agency. Starting from an historical analysis of conditions for judgments, I analyze first the reading of necessity with respect to apodictic judgements, and then that of possibility with respect to hypothetical judgement. The analyis results in a formal treatment of rules for judgemental modal operators, whose aim is to preserve epistemic states corresponding to verified and unverified assumptions in contexts. In the conclusion, further tracks of research are indicated for designing a semantic framework and defining multi-agents systems.

1 Introduction

Since Brouwer's dissertation *Over de grondslagen der Wiskunde*,[1] followed by Heyting's formalization of intuitionistic logic[2] and the later Constructivist perspective,[3] logical anti-realism has evolved from a non-classical setting for the formalization of mathematics to a more extended and demanding conceptual framework for formal logical languages. Constructivism, in particular, has strengthened the philosophical orientation of intuitionistic mathematics and has reformulated some of the most important intuitions behind anti-realism. Nowadays, two issues can be located at the heart of the philosophical analysis that constructivism has brought forth in logic: *verificationism* and *epistemic agency*.[4] The importance of these topics for logic and formal epistemology in general is confirmed by the great deal of attention they are recently claiming in different areas of philosophical and mathematical logic, in game and decision theories and in artificial intelligence and social choice theory. Their combination represents probably the very core of Constructivism intended as a philosophical framework, and the

[1] Brouwer (1907), republished in Brouwer (1975).
[2] See e.g. Heyting (1956).
[3] For a systematic introduction, see Troelstra and van Dalen (1988).
[4] For a recent analysis of the realism and anti-realism debate which focuses on these two issues among others, see Brock and Mares (2007).

aim of the present contribution is to uncover some connections between a philosophical analysis of constructive logic and knowledge. In particular, I shall give some new insights in the interpretation of epistemic attitudes by knowing subjects in terms of constructive modalities.

The nature of proofs as ideals for the definition of truth, the reinterpretation of truth-values as proof-conditions and the reading of logical connectives in terms of introduction and elimination rules, have all been crucial steps in the development of the intuitionistic/constructivist approach to the philosophical analysis of logic. In particular, the issue of *verifications*, introduced already by Frege in the treatment of identity procedures for canonical definitions, was dealt with by the Positivist in terms of observation protocols at the beginning of the 1920's,[5], and later by Dummett's theory of proof-conditions.[6] In the anti-realistic perspective, the verification principle 'truth = existence of a proof' requires a solid philosophical basis: What does it mean to prove or verify a given propositional content? Under which *conditions* does such a verification become acceptable? Are degrees of certainty allowed within the definition of truth as possession of a proof? The formulation of appropriate answers needs to be given in terms not only of the formal rules that allow the constructively acceptable derivation of a theorem; It rather concerns also the level of assertions of truth for such propositional contents ('proposition A is true'), and therefore it requires an appropriate analysis of assertion conditions for judgements. In the following, this last topic plays a main role in view of the analysis of modal judgements.

On the other hand, the role of the *epistemic agent* for assertions of knowledge can be traced back to Kant's epistemology. Brouwer's theory of two-ity, based on the method of the creating subject, constituted the philosophical background for the intuitionistic reformulation of the method of proof by contradiction.[7] The issue of the knowing subject has recently been restored according to the constructivist point of view in terms of the 'first-person perspective' approach to the judgement-based process of knowledge acquisition. In view of the recent developments of modal and epistemic logics, where the role of the epistemic actor is explicitly formulated in the language,[8] it would be obvious to expect from the anti-realistic approach (and thus from the constructivist perspective in particular) to play a major role in this constantly growing track of research. This means to allow an explicit

[5]See e.g. Ayer (1959).

[6]See Dummett (1977), Dummett (1993).

[7]See van Atten (2008).

[8]The explicit formulation of an agent-operator transforms standard formulas in the form $K_a A$, where K is the epistemic/modal operator; Index a stands for the knowing agent; A stands for the propositional content to which knowledge is directed.

analysis of the agent's attitude towards the content of the knowledge act ('proposition A is true/proven by agent a'), and to reconsider in this direction the previously mentioned analysis of assertion conditions. What does it mean for an agent to prove a given propositional content, i.e. under which conditions a content is proven by an agent? Which degrees of provability are admitted in the context of an agent-based judgemental theory of knowledge? How to deal with the *communication* of these knowledge contents, i.e. how to avoid a solipsistic deviation of such a theory of knowledge?[9] The formal and conceptual analyses offered by frameworks such as Constructive Type Theory (CTT) and the Logic of Proofs (LP) are particularly apt to play this role.[10]

In connection with intuitionistic logic and the structure of proofs, the role of modalities has been recently explored in a large number of research areas.[11] Nonetheless, despite their milestone character in the explanation of knowledge attitudes since the debate on the role of hypothetical judgement in intuitionistic logic from Brouwer (1907), and even though the analysis of epistemic acts has become essential for the cognitive act and the meaning of mood,[12] no consistent explanation of epistemic modalities exists which has a non-purely arithmetical interpretation. The main aim of this paper is hence to focus on an epistemic interpretation of modalities and to analyze them formally from a first-person perspective of judgemental knowledge processes. This shall be done especially in view of the nature of *construc-*

[9]Thanks to Catarina Duthil Novaes for suggesting an explicit mention of the problem of solipsism in the present framework of modal judgemental constructive knowledge.

[10]Martin-Löf's Type Theory, first introduced in Martin-Löf (1975), has been revised and reformulated in the strongly predicative format of Constructive Type Theory in Martin-Löf (1998). For comparison with Artemov's Logic of Proof, I shall refer in particular to Artemov (1994) and Artemov (2001). The philosophical and foundational perspective offered by Constructive Type Theory has been investigated at length in a number of aspects. B.G. Sundholm has explored a great number of issues that constitute the basis for the present contribution. In the following I will hit upon themes such as the relation between proof-conditions and truth-makers, considered in Sundholm (1994); the formulation of knowledge judgements from the first-person perspective, as in Sundholm (1997); the relation between the constructive and the classical notions of inference, Sundholm (1998); the issue of identity of assertion conditions, Sundholm (1999). In Sundholm (2003), he has offered a constructive semantics for propositions and judgements which spells out the notion of epistemic necessity: I will, in the following, make an extensive comparative analysis of this work, in order to present a more complete account of epistemic modalities.

[11]Among other works, of the greatest importance are the sequential approach from Sambin and Valentini (1982), the general formulation of an intuitionistic modal logic in Bierman and de Paiva (2000), the interpretation under the Curry-Howard correspondence in Bellin et al. (2001) and the natural deduction reading of contextual reasoning in de Paiva (2003).

[12]As it has been recently considered in van der Schaar (2007) for CTT.

tive modalities, their role for epistemic agency and the analysis of assertion conditions for judgements.[13] I will argue that an epistemic interpretation of modalities is given by a crucial attitude towards knowledge contents, by analysing their assertion conditions. This analysis will not be propositional, in the form '(Agent a knows that) A is necessary/possible;' rather, it will be given with respect to the related judgemental formulation: '(Agent a necessarily/possibly knows that) 'A is true.' In this latter form, modalities require an analysis that considers the judgement at hand both in its apodictic and hypothetical form. This allows to reformulate their constructive interpretation by proof-object and assertion conditions. The here introduced analysis allows moreover for an extension in terms of prioritized structures and multi-agents languages. The implications of these latter issues shall only be sketched towards the end of this contribution.

2 From Constructions to Conditions

The notion of construction – as the formal counterpart to truth-value for connectives in a classical logic setting – has been considered systematically at least since Heyting's work, and its analysis has received a new impulse with Kreisel's interpretation.[14] This led to the translation of the definition of meaning based on alethic conditions to one based on epistemic conditions.

A crucial step in the formulation of epistemic conditions for propositional connectives is represented by the interpretation of hypothetical judgements. The peculiar nature of such form of judgement was obvious already to Brouwer[15] and it can be taken as the basis for his explanation of the relation between intuitionistic logic and (mathematical) knowledge. In the third part of his dissertation, titled "Wiskunde en Logica," Brouwer deals explicitly with the seemingly innate hypothetical nature of logical reasoning, the one where logic seems to proceed ahead of mathematics:

> "There is a special case, where the combination of syllogism has a different nature, that appears to resemble the usual logical figures, and which really seems to presuppose the hypothetical judgement from logic. This occurs when a construction is defined through some relation in a construction, without being directly evident how to provide it. It seems one assumes here that the sought was constructed, and a chain of hypothetical judgements derives from the assumptions."[16]

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[13]Such explanation refers mainly to the epistemic analysis of dependent judgements for CTT introduced in Primiero (2008); it is crucially based on the description of the role of assumptions and presuppositions, following Primiero (2004), and it focuses explicitly on an agent-based definition of information, as shown in Primiero (2007). The proper semantic interpretation of this notion of information is extensively analyzed in Primiero (2009).

[14]See Heyting (1956), Kreisel (1962); see Sundholm (1983) for an overview.

[15]See Brouwer (1907).

[16]Brouwer (1907), pp.124-125. The explanation of the very different interpretations of

The kind of construction involved by hypothetical judgements is central to the understanding of the intuitionistic inference relation. The reading of the implication sign from a BHK-style semantics reflects this very same difficulty. Let us mention two standard interpretations:

> Kreisel (1962): The implication $p \rightarrow q$ can be asserted, if and only if we possess a construction r, which, joined to any construction proving p (supposing the latter be effected), would automatically effect a construction proving q;
>
> van Dalen (1979): A proof p of $A \rightarrow B$ is a construction which assigns to each proof q of A a proof $p(q)$ [p, provided that q] of B, plus a verification that p indeed satisfies these conditions.

According to these standard readings, the satisfaction of the epistemic condition on a proof p for the connective $\rightarrow$ applied to the couple of propositional contents A and B requires two separate operations to be performed: the formulation of a construction a for A, along with a construction b for B, the latter being deduced from a in terms of explicitly or implicitly given tautologies.[17] The latter condition on tautologies requires that any step from the first construction to the latter can be performed according to definitions and logical laws.

Sundholm (1983) has argued for the basic distinction between the explanation of an implicational relation and the process of constructing such a proof-object: The hypothetical method – one of the allowed construction methods in intuitionistic logic – is the corresponding abstracted process. Under Sundholm's reasoning, the explanation provided by the formula 'for all q: q proves $A \Rightarrow p(q)$ proves B,' cannot itself be regarded as a mathematical object. This is essentially due to the distinction between the dynamic process that is the act of proving, and the resulting mathematical object that is a proof. Hence, in particular for the construction stating that 'A is a proposition,' the assertion condition cannot be propositional itself.[18]

The way out of this impasse is represented by the constructive distinction between act of knowledge and its content: To demonstrate the truth of a proposition A one needs to carry out the construction a which corresponds to a proof-object for A, which will in turn allow to state the judgement 'proposition A is true.' This sets the basic constructive distinction between proposition and judgement. In turn, also the analysis of conditions is extended: *Proof-conditions* formulated for propositional contents are reconsidered as *assertion conditions* for judgements. I aim to show in the following

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the nature of hypothetical judgements in the history of intuitionistic and constructive logic is presented in a quite detailed and fascinating way in van Atten (in press); my personal thanks to Mark van Atten for providing me with a preprint.

[17] This is called the α-interpretation in van Atten (in press).

[18] See Sundholm (1983), pp.161-165. As I have recalled in my Primiero (2008), this reflects the Russellian distinction between *knowing-that* and *knowing-how*.

that the former are explicitely identified with the latter only in the case of *categorical judgements*, and in the case of *hypothetical judgements* only under the explicit requirement of closed constructions. This is shown starting from the basic distinction between implication among propositional contents and inference between judgements.

The just mentioned basic distinction between categorical and hypothetical judgement is fully endorsed by Martin-Löf's Type Theory.[19] The formulation of propositional contents A, B for the judgment '$A \rightarrow B$ true,' and justified in terms of constructions, provides the following analysis of the implicational relation:

> **Proof Conditions-interpretation**: A proof p of '$A \rightarrow B$ true' is given as the pair of proof-objects $< a, b >$, such that one obtains a formal object of a function type $f =< a, b >$, which is the construction for the implicational relation $f : (A \rightarrow B)$.

The ground distinction between this standard constructive implicational relation and the inferential hypothetical relation is expressed by the resulting switch from formal constructions to assertion conditions. Whereas by the implicational relation one obtains a categorical object of the function type $f : (A \rightarrow B)$, corresponding to a categorical judgement satisfied by the ordered pair of constructions $< a, b >$; In the assertion conditions interpretation of the inferential relation, one requires a dependent object which represents a new functional relation of the form $f : (x : A)B$; The inferential relation is therefore justified by a formal construction for B whose formulation depends on condition $x : A$.

The construction of such formal object $f : (x : A)B$, is given by the implicational relation abstracted with respect to all possible instances of the construction a:

$$(1) \quad \frac{x : A \vdash b : B}{\lambda((x)b) : A \rightarrow B.}$$

The object $f : (x : A)B$ denotes a dependent function type, that is the type that contains *all* the functions with domain A and range B such that $f(a) : B$ for all a of type A. The values satisfying this function define the hypothetical judgement, or logical consequence, 'If A is true, then B is true:'[20]

> **Assertion Conditions-interpretation**: In order to establish 'A true $\Rightarrow$ B true,' one requires that the satisfaction of the conditions that make the proposition A true, can be transformed constructively into the satisfaction of the conditions that make the proposition B true.

[19] See Martin-Löf (1987) and Primiero (2008), especially chapter 2, section 2.

[20] Such an explanation corresponds to what has been formulated in van Atten (in press) as the β-interpretation.

The explicit requirement on the constructive transformation of satisfiable conditions is similar to that of the previous implicational relation (definitions plus logical laws).

This distinction reflects the switch from construction (*proof*) to process-of-construction (*proving*), according to the requirement from Sundholm (1983). Hence, assertion conditions for $f : (A \rightarrow B)$ rely on the ordered pairs of constructions $a : A$ and $b : B$ defining f, provided that A and B are both of the type of propositions: $A, B : prop$ (the latter represent the due presuppositions allowing for the required constructions); The last condition on type-introductions expresses type-predicability, which corresponds to *knowability* in view of the definition of truth as *knowledge*.[21] On the other hand, the formal expression $f : (x : A)B$ requires two separate such conditions: the first is the type declaration for B ($B : prop$), allowed by the formulation of a construction $b : B$; The second condition (on which such construction b depends) is the assumption ($x : A$) that declares that a construction of A is given. In other words, provided the *verification* of the assumption on the truth of A, it is possible to formulate the construction b. Also in this case a principle of knowability is at hand for the conditions on constructions.[22] It is at this second stage that an important step occurs in the constructive perspective: The introduction of the assumption $x : A$ and the generalization thereof is allowed by term introduction, i.e., by the presence of some a which can be substituted for the place-holder x; This means in other words that the assertion conditions interpretation relies ultimately on the proof conditions of the corresponding implicational relation. This interpretation allows therefore the generalization on the application of identical proof objects $a_1, \ldots, a_n$ for the ordered pair $< a_i, b >$ that build all the valid implicational relations $A \rightarrow B$. On the other hand, this reading clearly misses the aim of providing an interpretation for hypothetical reasoning under simple assumption on the formulation of such constructions a_i, that is for the pure reason of expressing what one *should* know, in order the conclusion to be inferred.

There is therefore an important distinction in the shift from constructions to assertion conditions. On what Martin-Löf calls the *conceptual order*, this shows knowability to be presupposed by actual knowledge: This principle, which might result trivial for categorical judgements, reveals the requirement of an explicit *verification procedure* in the case of hypothetical judgements. It opens therefore the up-to-now scarcely explored issue of

[21]In Primiero (2004) I have suggested the use of the notion of *meaningfulness* as an appropriate counterpart of the standard verificationist definition of *meaning* for type-introductions.

[22]For more on this and the connection to the notion of function see Primiero (2008), pp.47-54.

derivations under open assumptions in the construcitve setting. It is intuitive to translate this epistemic relation in terms of provability and knowledge and to show their connection to the notion of logical necessity. In view of the proposed account, it seems appropriate to reconsider this issue from the perspective of the epistemic interpretation of modalities, including both necessity and possibility. In particular, the reading of epistemic modalities in terms of assertion conditions proceeds in two directions:

1. it preserves the usual reading of necessity satisfying the constructive interpretation of the provability predicate (semantically satisfied in a modal language **S4** for factual truth);
2. it provides an embedded reading of possibility that interprets the agent-based perspective of assertion conditions in a weaker frame for open assumptions.

I shall investigate the resulting notion of constructive knowledge under an interpretation that does not consider necessity as arithmetical provability, rather from the perspective of agent-based justified knowledge, where modalities are intended as expressing epistemic attitudes.

3 Constructive modalities as epistemic attitudes

The explanation of the notions of construction (*process of proving*) and proof (*object that shows what is proved*) in the previous section leads to the distinction between proof-conditions for propositions and assertion conditions for judgements: First one goes through the process of satisfying all the conditions needed for proving a proposition A, and then one obtains the object that allows one to assert that 'A is true.' In the case of a categorical judgement, the notion of assertion condition simply reduces to that of proof-object for the propositional content at hand; On the other hand, whereas the standard constructive interpretation of dependent (hypothetical) judgements obtains the same by requiring the substitution of closed construction for assumptions, the extension to conditions becomes enlightening when looking at a formulation of dependency from open assumptions. This last step is essential to provide an interpretation of modalities as epistemic attitudes for the constructive framework.

Martin-Löf (1996) introduces an analysis of the notion of hypothetical judgement based on the comparison with Gentzen style sequents, considering the required presuppositions for the formulation of antecedents (or hypotheses) $(A_1, \ldots, A_n)$. That 'A_1 is true' is a judgement presupposes, according to such explanation, that A_1 is a proposition; and that 'A_2 is true' is a judgement under the assumption that A_1 holds, presupposes that A_2 is a proposition and that A_1 is true, and so on up to A_n:

$$
\begin{gathered}
< prop : type > \\
(A_1 : prop) A_1 \; true \\
(A_1 : true) A_2 : prop \\
\vdots \\
(A_1 : true, \ldots, A_{n-1} : true) A_n : true.
\end{gathered}
$$

This shows the intuitionistic formulation of hypothetical judgements based on presuppositions and satisfied assumptions. An interpretation of modalities as epistemic attitudes for the constructive framework – extending the usual interpretation of the necessity operator with a special reading for the possibility operator – aims at illustrating the connected epistemic values of proof-objects and (open) assumptions.

The first step in formulating the connection between epistemic attitudes and constructive modalities is obviously the direct translation of intuitionistic truth as classical provability in terms of necessity, as introduced by Gödel's modal calculus of provability.[23] As recollected in Artemov (2001), Gödel's introduction of the modal reading of intuitionistic provability establishes that a intuitionistically derivable formula F implies a formula $t(F)$ with proof-term t derivable in **S4** such that each subformula of F is boxed:

$$\vdash_{Int} F \Rightarrow \quad \vdash_{S4} t(F) \mid \forall A \subseteq F, \vdash_{S4} \Box A.$$

The inverse was established by McKinsey and Tarski (1948). Nonetheless, the intended semantics of the provability operator $Provable(F)$ with respect to **S4** and for $\Box F$ were to be found different. Gödel intended the provability predicate to denote the form 'x is a code of a proof of a formula having a code y' for a theory containing Peano Arithmetic (PA), such that a translation is possible where $\Box$ means 'it is provable in PA.' But the problem arises with the non-constructive nature of the existential quantifier, which implies that the reflection principle $Provable(F) \rightarrow F$ is not derivable. The appropriate semantics for $\Box F$ is then given in Artemov (2001) as the derivability of the explicit version of the provability predicate $Provable(n, F) \rightarrow F$ for any natural number n.[24] The corresponding modal logic of the arithmetical provability predicate $Provable(F)$ was given in Solovay (1976). This has an identical counterpart in the formulation '$Proof(A)$ exists' which defines constructively 'proposition A is true:' 'a is a proof-object for A' $(a : A)$ justifies A *true*.

[23] Gödel (1933).

[24] It also provides a complete axiom system for a classical propositional logic with the additional axiom 't is a proof of T' $(t : T)$, and the classical BHK semantics for **Int** is formulated.

Considering the more recent approach to modal languages to formulate properties of epistemic subjects involved in intelligent processes of knowledge acquisition and exchange, and provided the innate nature of constructivism to deal with the notion of knowledge and judgement from a first person perspective, it seems reasonable to require a reading of justifications (and in turn of modalities) not only in view of arithmetical provability,[25] but also as epistemic attitudes of knowing agents. Such a non-standard interpretation underlines the distinction between the pure provability of a given propositional content A and the description of conditions under which an agent can prove/has proven A. This is the very aim of the present and following sections. In such context, the mentioned switch from formulas to judgements, leading to the reading of assertion conditions along with proof-conditions, plays an important role for the epistemic interpretation of the standard possibility operator.

Sundholm (2003) presents the different interpretations of the necessity operator applied to the judgemental form 'proposition A is true.' The following different readings are provided:

1. Necessarily A is true;
2. A is necessarily true;
3. 'A is true' is necessary.

In the first form, the judgement declares the truth of the proposition $\Box A$; In the second form, it is a form of judgement, where necessity is expressed as a form of predicating the truth; In the third form, a judgement 'A is true' is declared to be necessary. Sundholm claims that modal logic accounts only for the first reading; The contentual approach required by a constructivist's (anti-formalist) perspective suggests the connection to necessity as truth in all possible worlds, and when the semantic truth-conditions are reflected by syntactic proof-terms in the constructive vein, equi-assertability (or identity of assertion conditions) allows for readings 1 and 2 to be identified. The kind of necessity declared by the third form is different, because it applies in a proper sense to a judgement. The judgement $\Box(A\ true)$, and the related possibility version, are the forms of modal judgements I will refer to throughout the rest of this paper.

The meaning of judgemental necessity $\Box(A\ true)$ needs to be interpreted in terms of assertion conditions. Provided that the conditions for having the right to express a judgement are satisfied, the corresponding notion of necessity for a judgement-candidate is that of an *apodictic judgement*:

[25] See e.g. the one given in Fitting (2005).

what is known to be so and cannot be known to be otherwise. Hence, the constructive interpretation that identifies provability, truth and knowledge (that a proposition A is true means that a proof for A is known), allows to justify the following extension:[26]

4. 'A is true' is necessary $\Rightarrow$ 'A is true' is known

$$\Box(A\ true) \Rightarrow K(A\ true).$$

Here K can be seen as a *knowledge-operator*, in the style of the mentioned systems of epistemic logic, or as an explicit operator for the knowing agent. Necessity and knowledge relate here in terms of inference under assertion conditions, rather than by equivalence of such conditions: We shall see that necessary knowledge is implied only by knowledge under an empty set of conditions.

For the previously introduced analysis of assertion conditions, the notion of judgemental knowledge needs to be understood as satisfaction of the conditions for the judgement 'A is true.' The basic condition for the truth of A is the construction a that makes it true $(a : A)$; When A presupposes further propositions to be known, these represent the context in which A is known to be true, $\Gamma = (A_1\ true, \ldots, A_n\ true)$; then the notation $(\Gamma)A\ true$ will be used. Thus, when $\Box(A\ true)$ is referred to contextually formulated knowledge, one needs to give explicit satisfaction procedures for each $x_i : A_i \in \Gamma$, in line with the constructivist requirement. We can see this as providing the condition for the reduction to the implicational relation $\bigwedge A_i \rightarrow A$, so that it corresponds to knowledge for which no further contextual conditions are needed $(\Gamma = \emptyset)$:

5. 'A is true' is necessary $\Leftrightarrow$ Agent K knows that A, for any knowledge state agent K is in

$$\Box(A\ true) \Leftrightarrow K((\emptyset)A\ true).$$

The latter represents a crucial step: 'alethic'(/modal-theoretical) necessity as truth in all possible worlds corresponds directly to the 'epistemic'(/proof-

[26] Sundholm (2003). In the following, when the symbol $\Rightarrow$ occurs among (modal) judgements ('A is true'), such symbol is not to be intended as the propositional connective of implication, rather as a meta-theoretical sign of inferential assertability. It can be read as follows: If the conditions to assert X are satisfied, then the conditions to assert Y must be satisfied as well (where here X and Y are meta-variable for judgements). Correspondingly, the bidirectional arrow $\Leftrightarrow$ expresses identity of assertion conditions for the judgements on the two sides. Thanks to Maria van der Schaar for having pointed out to me the possible confusion.

theoretical) satisfaction of proof-conditions, including satisfaction of assumptions, which means derivation from premises.[27]

The corresponding interpretation of a judgemental possibility operator can now be provided. Let us consider the propositional equivalence $\Box A \leftrightarrow \neg\Diamond\neg A$, then the following is formulated in Sundholm (2003):

6. 'A is true' is possible $\Leftrightarrow$ 'A is false' is not known

$$\Diamond(A\ true) \Leftrightarrow \neg\Box(\neg A\ true).$$

This equivalence is based on the fact that the right-hand side formula expresses that it is not known that 'A is false,' which obviously does not allow constructively to say that 'A is true.' But this use of the possibility operator does not give yet a corresponding interpretation on the syntax of judgements, which also shows how under this reading the duality on the two modal operators is partially lost. One can obtain such translation based on the previous remark on assertion conditions as contextual knowledge, by referring to conditions needed for knowledge that can be satisfied, but not necessarily are:

7. Agent K knows that A, for some knowledge state Γ agent K is in

$$\Diamond(A\ true) \Rightarrow K((\Gamma)A\ true).$$

For this to make a difference with respect to the reading of the necessity operator, we need to interpret the context Γ as the set of data or information on which the construction of A depends, without the reduction to a proof-bject for the implication relation being guaranteed. In other words, one needs to keep the reasoning at the level of assertion-conditions rather than bring it at the level of proof-objects. Only with Γ empty this formula will then reduce to the conditions for $\Box(A\ true)$, which provides again the translation to the dual operator. Otherwise, it means that truth is preserved under certain knowledge states in which the agent can formulate the appropriate conditions that need to be satisfied (where $\Gamma = (A_1\ true, \ldots, A_n\ true), n \geq 1$),

[27]This interpretation appears to me slightly more effective then the one given in Pfenning and Davies (2001). In this latter work, the context of hypotheses corresponds to a description of the knowledge of a given world; A valid judgement A is one of the form *'A is true in a world about which we know nothing.'* This interpretation reflects the knowing subject's attitude towards the apodictic judgment. Nonetheless, it is difficult to understand the formulation of the required demonstration in such unknown world. It seems more intuitive to understand emptiness of a context as indifference with respect to the required conditions; Then it follows that the agent *knows everything needed* for the due construction.

but without expressing the appropriate constructions. This leads directly to a treatment of derivations from open assumptions.[28]

In Pfenning and Davies (2001), the knowledge that A is possibly true corresponds to the existence of a world in which A is true but nothing else is known; this allows to assume that A is true and that (any) C is possible. This let to draw conclusions on the *possibility* of the propositional content C. In section 4, I shall show how appropriate structural rules can be formulated for a judgemental possibility operator that preserves the meaning of conditions for hypothetical judgements.[29] Under such interpretation, one focuses therefore on the different epistemic conditions at the basis of categorical and dependent judgements, justifying the conceptual distinction between *unverified* and *verified assumptions* within the process of proving. An agent who is always able to verify the assumptions in proof-procedures,

[28]From this conceptual justification we introduce the role of the possibility operator for judgements. The syntactic justification obviously requires a system in which terms for the type of propositions can be given both as proper constructions and as assumptions. This clearly would weaken the construcitve nature of the system at least for the module of the language that allows truth to be predicated of non properly proved (assumed) contents. This aim, only sketched in a later section, lies outside of the scope of the present contribution and is addressed in complementary research.

[29]The interpretation of modalities in the form of propositions 5 and 7 presents the immediate and intuitive correspondence also with an intuitionistic model for Kripke semantics, requiring a tuple $\langle W, \leq, R, v\rangle$ where R is the usual accessibility relation over the ordered set of worlds $W, \leq$, with worlds being intended as epistemic states, and the function v evaluating necessity and possibility formulas as accessibility respectively in all and in some orderly accessible world. Under the epistemic reading, assertion conditions are reduced to the accessibility relations on epistemic states for the formulas K_aA (agent a knows A) and B_aA (agent a believes A), where epistemic states substitute 'ontological worlds' and the semantic definition on K/B-operators are dictated by the different clauses for defining respectively proven and assumed truths. This gives also a new impulse on the distinction between knowledge and belief, started with Hintikka (1962), where the system $S4$ is chosen to express both knowledge/belief of a propositional content as compatibility with previous knowledge/belief; Compatibility amounts in turn to nothing else than accessibility by an epistemic relation on worlds. Even stronger systems have been proposed: in van der Hoek (1996) the system $S4.3$; in Fagin et al. (1995) and van Ditmarsch et al. (2006) the system $S5$ to include negative introspection. In general, in epistemic logics the meaning of necessity and possibility for epistemic states has been variously interpreted: Sometimes one speaks in terms of the distinction between knowledge and belief; or corresponding notions of hard/soft information are called upon; or one analyzes the persistence of the given contents in possible epistemic alternatives. In turn, knowledge is sometimes interpreted as a strong notion that requires alternatively truth, correctness or verification on propositional contents; whereas belief asks for some sort of individual, weak confirmation attached to contents. The standard interpretation of accessibility on propositional contents is preserved in the judgemental interpretation from Pfenning and Davies (2001), where modal propositional operators are still present, and no full explanation of the corresponding epistemic attitudes for the first-person perspective approach is required. Such explanation is here obtained by an analysis of the verification principle of truth defining knowledge under conditions.

is an ideal knower; A real knower shall often be in the condition of formulating contents only assuming their conditions being satisfied (e.g., because so a reliable source says), but in practice being unable to provide appropriate verifications.[30] These are the different epistemic attitudes analyzed by modal judgements. I shall in the following unfold this interpretation and show how this allows to formulate appropriate definitions of constructive epistemic attitudes.

4 The meaning of satisfied conditions

The aim of the present section is to complete the process of translating the meaning of epistemic attitudes into constructive modalities. To formulate such an interpretation, I shall focus on the notion of conditions on constructions for propositional contents (as suggested by the act/object of knowledge distinction); This will provide an appropriate syntactic reading for judgements of the form $\Box(A\ true)$ and $\Diamond(A\ true)$.

Let us start again from the judgemental reading of the necessity operator suggested in Sundholm (2003):

4. 'A is true' is necessary $\Rightarrow$ 'A is true' is known.

Let us recall that this translation is based on the identity between $A\ true$ and $Proof(A)$. The corresponding epistemic attitude for constructive necessity is given by the apodictic judgement in which the truth of A is asserted under an empty context of conditions $(\Gamma = \emptyset)A\ true$, where one has reduced to implication (and thus also to an empty context of assumptions) the case of hypothetical reasoning. This means that conditions for the truth of the propositional content are already satisfied by the formulation of the judgement, which in turn provides an appropriate proof-object.

In correspondence with the previous reading in 5, I shall now focus on the relation between possibility and conditions via knowledge (rather than necessity and conditions): If necessity corresponds to knowledge under no extra conditions (than proof), possibility shall be related to knowledge under *some* assertion conditions to be satisfied. In both cases, we have a relation among knowledge and conditions, and the epistemic attitude of knowing

[30] The issue of reliability of sources is a very important one in this context. Obviously, e.g. in a mathematical context, researchers rely on each other's knowledge very often, for example in assuming the content of a given theorem proven by someone else. The generalization here presented, is based on the import of the first-person perspective principle, where the ideal knower may be a given epistemic agent or the whole of an epistemic community. In each case, it is relevant to stress the different values and roles that contents have in the process of knowledge. A next stage in this research is the analysis of prioritized structures of information sources, by which it is possible to formalize the relation of dependency between the knowing agent and his sources in a particular relevance order.

agents in the first person perspective can be introduced, so that K is no longer a knowledge-operator, rather an agent-operator:

5′. 'A is true' is known $\Leftrightarrow$ Agent K has satisfied all conditions to know that A is true.

Under this reading, necessity is reflected as *actual knowledge* in the following syntactic representation (where J is the judgement stating 'A is true' and Γ a set of assumptions $x_n : A_n$):[31]

$$(2) \quad \Box - Rule \qquad \frac{\Gamma \vdash J}{\Box\Gamma \vdash \Box J}$$

By this rule, one accounts for judgements whose conditions have been verified, therefore allowing provability of the conclusion. It says that, given the derivability of the judgement 'A is true' under the list of (minimal) conditions expressed by Γ, the satisfaction of proof conditions for each element in Γ (interpreted as substitutions of proof-variables: $([x_1/a_1] : A_1, \ldots [x_n/a_n] : A_n)$) allows to formulate proof condition a such that it makes A known.[32] This $\Box$-Rule is justified in a different way than the necessitation rule introduced in Pfenning and Davies (2001). In the latter, an introduction rule for (contextual) boxed formulas has the following form:[33]

$$(3) \quad \frac{\Delta; \cdot \vdash J}{\Delta; \Gamma \vdash \Box J}$$

it means that propositional necessity is implied by judgmental validity: If J is justified by any context, then it is necessarily justified. The corresponding elimination rule allows to use a judgement J as an assumption in context once $\Box J$ has been derived:

$$(4) \quad \frac{\Delta; \Gamma \vdash \Box J \qquad \Delta, J, \Gamma \vdash J_1}{\Delta; \Gamma \vdash J_1}$$

which means to extend contexts in terms of proved formulas. Obviously, this is valid also in our interpretation: By the previous $\Box$-Rule in equation 2, the derivation of the formula $\Box J$ means that related judgmental assertion

[31] See Bellin et al. (2001).

[32] For the problem of composition of boxed formulas see de Paiva (2003).

[33] In the following, the original notation from Pfenning and Davies (2001) has been modified to conform to ours. J is equivalent to A *true* expressed on empty context, from which one can infer A *valid*, i.e. judgmental necessity; their $\Box J$ is equivalent to $(\Box A)$ *true* and it refers to the propositional form of necessity – whereas we always look at judgements of the form $\Box(A\ true)$. The notation '$\cdot$' refers to an empty context.

conditions (in contexts) have all been boxed (proved), therefore the content of J can be safely used in a context of assumptions.[34] The extension of context by new formulas still preserves derivability of $\Box J$ if its minimal conditions have been satisfied, provided that no contradictory extension can be allowed of the initial context.

An important issue at this stage is the iteration of the necessity operator, which is usually interpreted as positive introspection by axiom 4 in epistemic logic, and as exponentiation in the arithmetical interpretation of provability. According to the previous analysis introducing the first-person perspective on knowing acts, one can easily substitute justified truth by satisfaction of categorical constructions by means of the necessity operator and to extend it by adding explicitly an index that expresses the knowing agent:

$4'$. $\Box(A\ true) \Leftrightarrow \Box_K(A\ true)$.

This formulation says that 'A is true' is necessary if and only if there is an agent K satisfying all the due conditions in order to formulate a proof for A. Provided $\Box(A\ true)$ is justified by a formula $a\!:\!A$, the identity with the operator $\Box_K$ allows the reduction to the standard possible iteration of modalities. That is, the following derivation is sound:

1	$a\!:\!A$	PREM
2	$A\ true$	verification principle of truth
3	$\Box(A\ true)$	(5)
4	$K(A\ true)$	(4)
5	$\Box_K(A\ true)$	$(4')$
6	$\Box_K\Box(A\ true)$	$(4, 5)$
7	$\Box_K\Box_K(A\ true)$	$(4')$

The formula $\Box_K\Box(A\ true)$, as the basic formulation for the iteration of modalities in the perspective of epistemic attitudes, says that 'Agent K knows that she has a proof object for A.' The formula $4'$ allows to reduce this to the usual positive introspection, being existence of a proof-object admissible only if an agent K is able to formulate it. Even though the reduction is admissible for proven contents (necessity judgements), the meta-theoretical structure is crucial with respect to the iteration of the $\Diamond$-operator: When switching to knowledge of the conditions that make a hypothetical judgement true, one cannot express them – as in the previous

[34] In the following section I shall introduce hypotheses by a formal rule in the style of Pfenning and Davies (2001) but with the basic difference of formulating it both for the $\Box$ and the $\Diamond$ version (reflecting the verified/unverified distinction).

case – by iteration of $\Box_K$, which implies direct satisfaction (as a provability operator). From the present perspective, if one wants to preserve the different attitudes expressed as 'knowing that' and 'assuming that' (otherwise also expressible as: 'receiveing the information that'), it is crucial to express the difference between conditions for knowing and conditions for the possibility of a propositional content.[35]

As I hope to make clear in what follows, there are two sorts of *desiderata* that one wants to satisfy when the interpretation of necessity is connected to the verification procedure of assertion conditions:

- to make the necessitation of the consequent dependent on the verification of the assumptions; This is formally obtained by the introduction rule for $\Box J$;
- consequently, to provide a $\Diamond$-introduction rule as an appropriate counterpart of the $\Box$-elimination rule: Not only one wants to be able to extend contexts by valid judgments, rather one also wants to make explicit extensions of contexts by *unverified assumptions*, preserving their epistemic value in the conclusion.

5 Interpreting possibility as satisfiable conditions

In correspondence with the list $1 - 3$ of interpretations of necessity given in Sundholm (2003), the following list of meanings for possibility are formulated:

1′. Possibly A is true;

2′. A is possibly true;

3′. 'A is true' is possible.

As in the previous analysis, let us focus on the judgemental form 3′, to show how it corresponds to a reading in terms of assertion conditions. The proper counterpart to the notion of *apodictic judgement* – covering also the notion of judgement formulated under satisfied conditions – is obviously the one of *hypothetical judgement* with open assumptions: what is known to be

[35]This argument is particularly relevant in view of the development of a multi-agent system, by which one wants to formulate knowledge of a propositional content possessed by one agent and transmitted to another (which might not possess the corresponding verification object). This point is obviously linked to the already mentioned problem of solipsism and to the resulting epistemic attitude of 'becoming informed,' see Primiero (2009). More on this in the concluding section. Thanks to Bjørn Jespersen, for urging an exposition of the problem of introspection at this particular stage of the discussion on epistemic modalities.

so, but can be known to be otherwise; In particular, it can be otherwise if its conditions are not satisfied. The final aim is therefore to explicitate the appropriate epistemic attitude formulated by the constructive interpretation of hypothetical judgement under the formulation of the possibility modality.

The sentence in $5'$ from the previous section, implicitly expressing the condition on the formula '$Proof(A)$ exists' as the categorical judgement of the form 'a is a proof for A' $(a : A)$, turns explicitly as follows in the interpretation of truth under assumptions:

$5''$. 'A is true' is known, provided knowledge of contents $(A_1, \dots, A_n) \Leftrightarrow$ Agent K knows that A is true, *if* K satisfies conditions $(A_1, \dots, A_n)$.

The case of $5''$ refers (explicitly) to a hypothetical judgement of the form 'provided that (all and only) constructions a_1 for A_1 up to a_n for A_n are satisfied, a is a proof for A.' In this last reading, one implicitly refers to a minimality property on the conditions needed to satisfy the verification of A; More to the point, deviating from the strictly constructive meaning of hypothetical judgements, one allows here the formulation of construction a assuming that constructions $a_1, \dots, a_n$ are given. The constructive interdefinability of possibility and necessity is correspondingly translated as follows:

$6'$. $\Diamond(A\ true) \Leftrightarrow$ in some minimal world the conditions for $A\ true$ are satisfied

$$\Diamond(A\ true) \Leftrightarrow \exists\Gamma(\neg\Box((\Gamma)\neg A\ true)).$$

The previous means that, considering conditions for hypothetical judgements, there is a list of assumptions $\Gamma = (A_1, \dots, A_n)$ such that if these are verified, no pair composed with them and a construction a can be formulated such that $\neg A$ holds true. Consider that, according to this translation of possibility under assertion conditions and for the forthcoming analysis, our $\Diamond A$ collapses into the standard possibility operator of modal logic only when $\Gamma = \emptyset$ (that is only for the case of categorical apodictic judgement).

The corresponding positive reading under assertability conditions is as follows:

$7'$. If (all and only) conditions $\Gamma = (A_1, \dots, A_n)$ are satisfied, a proof can be formulated such that agent K knows that A

$$\exists\Gamma, s.t.\ (\Gamma)a : A \text{ and } K(\Gamma) \Leftrightarrow K(A\ true).$$

The explanation of assertion conditions for possibility finally amounts to:

8. 'A is true' is knowable $\Leftrightarrow$ Conditions for A are satisfiable

$$\Diamond(A\ \mathit{true}) \Leftrightarrow \exists\Gamma, s.t.\ (\Gamma)a : A \text{ and } \Diamond(\Gamma).$$

The latter in turn means that a list of assumptions can be formulated such that its knowledge makes A true, and thus A is *coherently assertable* in a hypothetical context (again, only in the case $\Gamma = \emptyset$ the previous reduces to the case of the apodictic judgement). Under the first person perspective interpretation and the reading of conditions on constructions for judgements, knowability of 'A is true' means that A can be satisfied, and therefore it can be used to satisfy conditions on further constructions, as explained in the conceptual order among constructions and conditions in the previous section.

The definition given in Pfenning and Davies (2001) of possibility judgments valid under assumptions is again a useful starting point for a better understanding of our notion of possible knowledge. Their definition preserves possibility under validity (which is intuitive, because it allows verified formulas to be formulated in hypothetical contexts), and it defines possibility with necessity as a substitution principle. The calculus in Pfenning and Davies (2001) defines a propositional operator ($\Diamond$) along with a judgemental predicate (*poss*):

- If $\Gamma \vdash A\ \mathit{true}$ then $\Gamma \vdash A\ \mathit{poss}$;
- If $\Gamma \vdash A\ \mathit{poss}$ and $A\ \mathit{true} \vdash C\ \mathit{poss}$ then $\Gamma \vdash C\ \mathit{poss}$.

In this definition, the *poss* predicate is judgemental in the same sense the *true* predicate is judgemental, that is, it forms a judgement $A\ \mathit{poss}$. Its interpretation based on necessity is obtained by allowing assumptions about validity (where, again, validity is interpreted as necessity):

- If $\Delta; \Gamma \vdash A\ \mathit{poss}$ and $\Delta; A\ \mathit{true} \vdash C\ \mathit{poss}$, then $\Delta; \Gamma \vdash C\ \mathit{poss}$.

In the present context I focus instead on the judgemental possibility intended as possibility applied to a judgement: $\Diamond(A\ \mathit{true})$. The main property in common among the two analyses is that assumptions of validity (i.e. formulas $A\ \mathit{true}$ in contexts) are extended in order to derive further possible contents (formulas $C\ \mathit{poss}$ in the analysis from Pfenning and Davies (2001); formulas of the form $\Diamond(A\ \mathit{true})$ in the present context[36]). This desirable

[36]It is not my aim here to investigate to which extent the judgemental formulation $C\ \mathit{poss}$ is relevantly different from the propositional $\Diamond C$ in Pfenning and Davies (2001).

property is complementary to the $\Box$-elimination rule in the previous analysis of the necessitation procedure. The validity of assumptions considered in Pfenning and Davies (2001) corresponds to our explicit formulation of *verified assumptions*, namely the derivation-tree that goes from Γ to $\Box\Gamma$, where $\Gamma = (x_1 : A_1, \ldots, x_n : A_n)$. The extension that preserves *unverified assumptions*, inducing possibility on the conclusion, is given as follows (as in the previous rule, J stands for a formula of the form $A\ true$):

$$(5) \quad \Diamond - Rule \qquad \frac{\Gamma, J_1 \vdash J_2}{\Box\Gamma, \Diamond J_1 \vdash \Diamond J_2.}$$

This reading extends the previous interpretation of necessity as proof-conditions to the assertability conditions of hypothetical judgements: It preserves the formulation of knowledge contents epistemically weaker than strictly proved ones. Possibility expresses thus – in the present context – the knowability of contents under *assumption of appropriate assertion conditions*.[37] This second rule accounts therefore for the *transmission* of contents in the knowledge frame without explicit proof/verification, by referring to the epistemic modality expressed by the $\Diamond$-operator.[38] This formulation has the corresponding agent-based abbreviation of the formula at 7:

$$7''. \ \Diamond(A\ true) \Leftrightarrow \Diamond_K(A\ true)$$

saying that 'A is true' is possible if and only if there is an agent K who can formulate the due conditions needed to assert that A is true. This condition expresses the ability to tell what would be needed in order to know that A (without the verification procedure required by the corresponding necessitation attitude); The object of knowledge for the agent's epistemic state is obviously given by the formulation of construction a under conditions in Γ.[39]

[37] This rule is introduced also for the calculus presented in Bellin et al. (2001). The modal analysis presented in Primiero (2009) extends the application to the syntactic-semantic method of CTT in the same direction, and as such it provides an adequate modal reading of the constructive calculus of dependent judgements. It shows how to interpret axiom **B** as the core of **S4** in which the logic of dependent proofs can be formulated.

[38] The procedure of transmission of contents (already mentioned in relation with the problem of solipsism and introspection) is not analyzed in the present paper, but it describes in an intuitive way the various processes of communication based on reliability, trust, signed messages. An extension of the present framework on the basis of prioritized contexts shall cover this important aspect of multi-agent knowledge processes. See the final section for some further remarks.

[39] It is maybe useful to mention that the identification of the notion of dependent condition with the **B**-axiom from Primiero (2009), referred to in footnote 37, is at this

To sum up, consider the third reading of the $\Box$ operator in its application to the judgemental form ('A is true' is necessary) from the previous section. It satisfies the notion of apodictic judgement: A proposition known to be true and which cannot be otherwise, being its assertion conditions necessarily satisfied. The relation here described is of an analytical nature. The role of the $\Diamond$-rule is complementary. It refers to knowledge assertions formulated on the basis of a set of assumptions: If knowing the truth of a formula depends on the validity of a set of assumptions, then the instantiation of these assumptions constraints to the knowledge of the given formula. The epistemic value of open assumptions expresses the notion of acquired information, where contents that might not be explicitly verified are used to coherently extend a knowledge base. If the derivability of a judgement is valid under extension of its assertion conditions by a further judgement, then the inferred judgement becomes dependent on the verification of the new assumptions. The relation here described is of a synthetic nature.

6 The formal system

I shall now briefly formulate the formal language for the calculus of modal dependent judgements. Standard types for propositions and formulas in contexts are introduced as axioms. Constructed formulas are standardly given by propositional closure. Judgments formulated within a *context* are considered assumptions or hypotheses are introduced in terms of elements of the set of proof-variables ($Var = \{x_1, \ldots, x_n\}$); judgments for the *prop* : *type* are justified in terms of elements of the set of proof-constants ($Con = \{a_1, \ldots, a_n\}$). In order to satisfy the requirement on non-reducibility of the former to the latter and thus to the implicational relation, one needs to restrict the truth relation for categorical judgements and to allow hypotheses to be of the type of propositions without appropriate construction being already provided. The standard judgemental grammar is then extended by using a different truth predicate when introduced by non-contradictory assumptions; finally, modal judgements are introduced:[40]

stage easily justified: $A \rightarrow \Box\Diamond A$ means that the truth of A implies that the assertion conditions for A (i.e. formally $\Diamond A$) have been entirely verified (i.e. the necessitation imposed by the $\Box$-operator).

[40]An extended version of the following formal analysis is contained in Primiero (Technical Report 1/09).

$prop : type$
$context : type$
$A : prop$
$J ::= A\ true \mid (A_1 \wedge A_2)\ true \mid (A_1 \vee A_2)\ true \mid (A_1 \supset A_2)\ true \mid (A \supset \bot)\ true;$
$a : A := A\ true$
$(\neg(A \supset \bot)) \supset x : A := A\ true^*$
$\Gamma, \Delta : context$
$\Gamma ::= (x_1 : A_1, \ldots, x_n : A_n); \Delta ::= (x_{n+1} : A_{n+1});$
$mod(J) ::= \Box J \mid \Diamond J.$

Whereas Introduction and Elimination Rules for the propositional connectives are standard, let us mention here only the formulation of rules for modal judgments. The standard hypothesis rule will be extended by rules for introducing modal judgments as verified/unverified assumptions; I shall accordingly change the nomenclature as follows:

(6) Premise Rule $\dfrac{}{\Gamma, \Box J, \Delta \vdash \Box J}$

(7) Hypothesis Rule $\dfrac{}{\Gamma, \Diamond J, \Delta \vdash \Diamond J}$

The $\Box$- and $\Diamond$-Rules are finally given as follows:

(8) $\Box$ – Rule Introduction $\dfrac{\Gamma \vdash J}{\Box\Gamma \vdash \Box J}$

(9) $\Box$ – Rule Elimination $\dfrac{\Box\Gamma \vdash \Box J}{\Gamma, \Delta \vdash J}$

(10) $\Diamond$ – Rule Introduction $\dfrac{\Gamma, J_1 \vdash J_2}{\Box\Gamma, \Diamond J_1 \vdash \Diamond J_2}$

(11) $\Diamond$ – Rule Elimination $\dfrac{\Gamma, \Delta \vdash J_1 \quad \Box\Gamma, \Diamond J_1 \vdash \Diamond J_2}{\Gamma, \Delta \vdash J_2}$

7 Sensible extensions for the epistemic attitudes framework

The analysis of epistemic possibility carried out in the present contribution lies crucially on the understanding of the notion of knowledge and the related necessitation rule. In particular, it refers to the role of assumptions and the explanation of the notion of dependent derivation. The introduction of the latter (which in CTT is completely satisfied by the formulation of dependent types with substitution rules on place-holders) requires that the implicational relation from truth to necessary truth be translated in terms of the corresponding epistemic version: From known content to necessarily known content. This relation is weakened in view of the formulation of proven contents under open assumptions.[41]

A calculus that includes categorical and dependent (open) judgements needs to be appropriately tuned with respect to the mentioned epistemic modalities. The standard constructivist view on hypothetical reasoning makes knowledge of the conclusion dependent on knowledge of the premises, thus preserving knowability.[42] In the present context, the role of assumptions is taken in a more strict sense: By the use of modalities, one is able to import in the language the distinction between *verified* premises and *unverified* assumptions, reflected by the corresponding epistemic attitudes formulated by the knowing agent. Moreover, it is well-known that dependent knowledge in this sense describes the kind of effective knowledge processes advocated e.g. by natural deduction systems and derivations by contexts. The introduced modal rules are equivalent to those in Bellin et al. (2001) for the calculus of constructive modal logic **IK**, extended by the needed hypothesis rules. In such a system, axioms for intuitionistic logic hold, plus an axiom that allows for the distribution of the $\Box$-operator on implication; On the other hand, distribution of the $\Diamond$-operator on disjunction is discarded. The further step of this research is obviously represented by the formulation of a corresponding Constructive Kripke semantics, that be sound and complete with respect to the syntactic representation here introduced. It seems safe at this stage to suggest that such semantics might be defined by a set of different accessibility relations on a subset of the worlds representing those where contents would hold, were the appropriate conditions be satisfied.[43]

A recent result in Kramer (2008) has shown the reducibility of provability to knowledge, referring to the identity between provability and "a

[41] See Hakli and Negri (2008) for the role of this distinction in the formulation of the Deduction Theorem. In Primiero (2009) I have considered the relevance of this distinction for the problem of logical omniscience.

[42] See Martin-Löf (1996), Sundholm (1997).

[43] See Primiero (Technical Report 2/09)

combination of individual knowledge (knowledge of messages), plain propositional knowledge, common knowledge (propositional knowledge shared in a community of agents) and a new kind of knowledge, namely adductive knowledge (propositional knowledge contingent on the adduction of certain individual knowledge, e.g. through oracle invocation)."[44] This shows that, in the context of information exchange for multi-agent systems, the notion of knowledge requires transfer of (signed) messages and (signed) proofs. This reflects the very same distinction here underlined between the epistemic attitudes towards proved and assumed contents. In turn, this suggests that a modal type-theoretical framework, including appropriate judgments for necessity and possibility, can be extended in view of a contextual dynamics to a multi-modal version. In this way, the role of messages is played by update dynamic operations with assumed formulas in contexts. The corresponding semantic interpretation would be designed by a set of indexed accessibility relations (for the agents) on the subset of possible worlds, on whose basis respectively distributed and common knowledge can be modelled.

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[44] Kramer (2008), p.1.

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