A Formal Approach to Vague Expressions with Indexicals

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Abstract. In this paper, we offer a formal approach to the scantily investigated problem of vague expressions with indexicals, in particular including the spatial indexical 'here' and the temporal indexical 'now'. We present two versions of an adaptive fuzzy logic extended with an indexical, formally expressed by a modifier as a function that applies to predicative formulas. In the first version, such an operator is applied to non-vague predicates. The modified formulas may have a fuzzy truth value and fit into a Sorites paradox. We use adaptive fuzzy logics as a reasoning tool to address such a paradox. The modifier enables us to offer an adequate explication of the dynamic reasoning process. In the second version, a different result is obtained for an indexical applied to a formula with a possibly vague predicate, where the resulting modified formula has a crisp value and does not add up to a Sorites paradox.

Keywords: Indexicals, Vagueness, Adaptive Logics.

1 Introduction

Vagueness and indexicality are both thoroughly studied phenomena of natural language. 'Vagueness' or 'fuzziness' shall here be understood as the lack of clearly defined boundaries of a linguistic expression, and the presence of borderline cases.[1] Representative examples are the vague predicates 'red', 'bald', and 'tall'. Vague predicates can give rise to a Sorites paradox:[2] given a first red object a_n that clearly has the predicate R, in a series of implicative sentences starting from Ra_n, one is inclined to infer Ra_{n+1} for an object that is only a little less red, but clearly $\neg Ra_{n+i}$ for some large i. So each separate premise of this argument taken separately is true, but the conclusion turns out to be false:

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[1] See e.g. [15], [11].

[2] The Sorites paradox is a very ancient problem that has multiple instances and several formulations, and a few possible solutions. For an overview, see [5].

$$\frac{\begin{array}{c} Ra_n, \\ \text{and if } Ra_n \text{ then } Ra_{n+1}, \\ \text{and if } Ra_{n+1} \text{ then } Ra_{n+2}, \\ \vdots \\ \text{and if } Ra_{n+i-1} \text{ then } Ra_{n+i} \end{array}}{Ra_{n+i}}$$

A similar phenomenon appears to occur in expressions that use certain indexicals. Notable research in indexicality has been done by Kaplan [7]. Indexicals are linguistic expressions whose referent can be different in every context. Kaplan's standard list of indexicals includes the personal pronouns 'I', 'my', 'you', 'he', 'his', 'she' and 'it', the demonstrative pronouns 'this' and 'that', the adverbs 'here', 'now', 'tomorrow', and 'yesterday', and the adjectives 'actual' and 'present' ([7], p. 489). Indexicals are said to be 'vague' in case they have a vague reference, namely if the boundaries of what they refer to are not clearly determined.

Not all indexicals can have a vague reference, hence a classification regarding the vagueness of indexicals should be drawn here. In the ordinary use, the indexicals that are personal or demonstrative pronouns do not have a vague reference, because they refer to a person or an object that has clearly defined boundaries in the real world. Also, the adverbs 'tomorrow' and 'yesterday' generally do not have a vague reference, because the boundaries of the days they refer to are conventionally determined.[3] The adverbs 'here' and 'now', on the contrary, *may* have a vague reference. Analogous to the above formal argument, a series of implicative sentences in which the reference of these vague indexicals changes every time by a small step, can add up to a Sorites paradox. Let us clarify this with an example. When a speaker uses the vague indexical 'here' to refer to his own location, the reference is not vague, and it is completely true that he is e.g. 'standing here'. But the reference turns out to be vague when the speaker refers to the persons standing in a line next to him, starting with the closest one and proceeding with those standing further and further away from him:[4]

I am standing here,
and if I am standing here, then Tom is too,
and if Tom is standing here, then Sally is too,
$\vdots$
and then person$_i$ is too.

[3] One might notice here that this does not account for idiomatic expressions. For example, saying that 'yesterday's dreams are gone' is the same as saying that 'the dreams of the past are gone'.

[4] This is an example from [2], by which it is shown - contra [12] - that the indexical 'here' can admit of varied interpretation in verb phrase ellipsis. Verb phrase ellipsis, by allowing to recover the meaning of both the omitted verb phrase and the indexical, is here used to express reference shifting of the latter.

Throughout the series, a shift occurs between the use of the indexical 'here' and the point where the indexical 'here' can no longer be applied. An analogous reasoning can be constructed with the indexical 'now'. For example, when one throws down a row of domino blocks, and uses the indexical 'now' to refer to the moment of falling down of the dominos. When one pushes the first domino and says 'the first domino is falling now', this is completely true. But put into a series of implicative sentences, 'now' becomes vague:

> The first domino is falling now,
> and if the first domino is falling now, then the second is too,
> and if the second domino is falling now, then the third is too,
> ⋮
> and then domino$_i$ is too.

'Now' is often assumed to refer to the very moment of the utterance. In the first sentence, this is indeed the case, so there the reference is not vague. But in successive uses, where 'now' refers each time to distinct but almost indistinguishable moments (n and $n+1$), its reference becomes vague. When a series of such implicative sentences is constructed, a paradox arises.

Note that in the above cases, a gradual *shift* of reference is at hand, and not an *extension* of reference. That is, the reference of 'here' or 'now' can also gradually change in a series of sentences in which one reference always includes the previous reference. For example, when the speaker uses the indexical 'here' to refer to his own location and that of the first person close to him, and then to refer to his own location and that of the two persons closest to him, et cetera. This reasoning does not add up to a paradox.[5]

Our approach does not contradict Kaplan's 'Direct Reference' view of indexicals, but nuances it. Following [7], the 'character' of an indexical fixes its reference in certain a context. So the character is a rule that determines its correct application. We agree with Kaplan that, for instance, the character of 'here' makes sure that the indexical refers to the location of the speaker. But we argue to take this case as the default, and want to show how 'here' can also have a vague behaviour in certain cases.

This still leaves the adjectives 'actual' and 'present' from Kaplan's standard list. 'Actual' and 'present' differ from 'here' and 'now' as their reference is typically vague: one can not precisely pinpoint the boundaries of the period of time that these indexicals refer to. In this sense, the indexicals 'actual' and 'present' behave like locative terms as 'close, 'below', et cetera, which are also generally subject to vagueness. It is possible to construct a Sorites series with 'actual' or 'present', or with a vague spatial term. But since the indexical or spatial term is

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[5] Compare this to [9], where a distinction is made between 'now' and 'here' as 'automatic' indexicals (referring to respectively the time and place of the utterance) and as 'discretionary' indexicals (referring to an interval of time or a region of space, that contains respectively the time and place of the utterance, and of which the size depends on the speaker's intention).

fuzzy by itself, the reference is vague in each expression of the series, and nothing special is at hand here.[6]

In our approach, the vague reference of indexicals will be addressed by means of adaptive fuzzy logics. We develop upon the adaptive fuzzy logics for vague predicates presented in [14]. Adaptive fuzzy logics are an optimal tool to explicate the reasoning process instantiated by a Sorites paradox. Instead of limiting valid repeated applications of Modus Ponens, in an adaptive fuzzy logic one uses classical logic until fuzziness-related problems arise. One allows for the local failing of Modus Ponens applications for expressions that turn out to be fuzzy, and for unrestricted Modus Ponens applications for expressions that turn out to be non-fuzzy. In our logic, an indexical will be formally represented by a modifier in the language, varying on the notion of modifier as a function that maps from properties to properties (see the notion of 'property modification' [8]): an indexical m is applied to atomic formulas πa, where π is a unary, either non-vague or vague predicate, and a is an individual constant.

Indexicals have previously been treated as modifiers. A well-know treatment of 'now' as a temporal modifier is in [6], where it is shown that 'now' does not always occur 'vacuously' (i.e. does not always refer to the moment of utterance) in the scope of another temporal modifier, as in 'I learned last week that there would now be an earthquake' ([6], p. 229).[7] We agree with Kamp that 'now' is not vacuous in the scope of another temporal modifier, and that it can be said to be vacuous in a sentence like 'it is (now) raining'; but we do not agree with his assumption that 'now' always refers to the moment of utterance. It is precisely our goal to show how the reference of 'now' can deviate from this default since it can be vague, and that a similar approach accounts for 'here'.

We proceed as follows. First we will set out the standard format, the proof theory, and the semantics of adaptive logics. In section 3, we will present four adaptive fuzzy logics for indexicals generating (possibly) fuzzy predications. Similarly, in section 4 we will use adaptive fuzzy logics to address indexicals applied

[6] Note that indexicals belong to another word category than spatial terms, in that they are speaker-dependent (the reference of 'here' is determined by the location of the speaker, while the reference of 'close' is not) and in that they do not ask for an object ('here' is clear in itself, while 'close' is a relative term). Also, some expressions containing an indexical are always true and the indexical can be omitted, e.g. 'I am here' and 'we live now' (no matter the utterer and the time and place of utterance). The indexicals can then said to be 'vacuous' or 'redundant' (cf. [13], [6]). This does not hold for relative terms.

[7] This example shows the need for a twofold approach for the interpretation of 'now': the indexical is determined by one invariant feature (the context of utterance) and one variant feature (shifted by modal operators), so the Montagovian 'index' (representing the context of use) is needed twice. See the notion of 'double-indexing' in [7]: one indexical is related to both the context of utterance and the circumstance of evaluation. The example of a Sorites paradox with 'here' from [2] introduced above, is shown to bear a structural resemblance with the idea of 'double-indexing' in [3]: the content of 'here' is determined by both the invariant location of the speaker and its shifting elided occurrences.

to (possibly) vague predicates. In the conclusion we will mention further research issues.

2 Adaptive Fuzzy Logic

2.1 Standard Format

Definition 1. *An adaptive logic (***AL***) consists of the following elements:*

1. *A lower limit logic (***LLL***): a monotonic, reflexive, transitive and compact logic which has a characteristic semantics (with no trivial models).*
2. *A* set of abnormalities*: a set of formulas Ω characterised by a (possibly restricted) logical form that is* **LLL***-contingent and contains at least one logical symbol.*
3. *A* strategy*: the strategy determines how to cope with conditional derivations given a set of derived abnormalities. The most important strategies are* Reliability *and* Minimal Abnormality*.*

The standard format ensures soundness and completeness, and other important meta-theoretical properties. Many of these can be found in [1].

2.2 Proof Theory

The adaptive proof theory consists of a set of inference rules (determined by the **LLL** and the set of abnormalities Ω) and a marking definition (determined by the set of abnormalities Ω and the chosen strategy). An **AL**-proof is a chain of stages composed by lines in which for each two subsequent stages it holds that the first is an extension of the second. In an annotated **AL**-proof, a line consists of five elements: (1) a line number i, (2) a formula A, (3) a justification rule, (4) a condition consisting of a set of abnormalities $\Theta \subset \Omega$, and (5) a $\surd$ or nothing when the line is marked respectively unmarked. A *Dab-formula* $Dab(\Upsilon)$ is the disjunction of the members of a finite $\Upsilon \subset \Omega$. In an adaptive proof, three types of generic rules are used:

PREM If $A \in \Gamma$
$$\frac{\ldots \quad \ldots}{A \quad \emptyset}$$

RU If $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} B$
$$\begin{array}{ll} A_1 & \Upsilon_1 \\ \ldots & \ldots \\ A_n & \Upsilon_n \\ \hline B & \Upsilon_1 \cup \ldots \cup \Upsilon_n \end{array}$$

RC If $\{A_1, \ldots, A_n\} \vdash_{\mathbf{LLL}} B \vee Dab(\Theta)$
$$\begin{array}{ll} A_1 & \Upsilon_1 \\ \ldots & \ldots \\ A_n & \Upsilon_n \\ \hline B & \Upsilon_1 \cup \ldots \cup \Upsilon_n \cup \Theta \end{array}$$

The rule RU is used in case of an unconditional derivation. The rule RC, meanwhile, is used in case B is derived from $A_1, ..., A_n$ on the condition Θ.

Definition 2 (Marking definition for Reliability).
Where Υ is the condition of line i, line i is marked at stage s if and only if $\Upsilon \cap U_s(\Gamma) \neq \emptyset$.

Definition 3 (Marking definition for Minimal Abnormality).
Where A is the formula and Υ is the condition of line i, line i is marked at stage s if and only if

1. *there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Upsilon = \emptyset$, or*
2. *for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which A is derived on a condition Θ for which $\varphi \cap \Theta = \emptyset$.*

Following the Minimal Abnormality Strategy, the models are selected that verify as little abnormalities as possible. Following the Reliability Strategy, one falsifies any instance of abnormal formulas.

2.3 Semantics

Semantically, an **AL** selects the **LLL**-models of the premises that are as normal as possible. What 'as normal as possible' means, depends on the chosen strategy. We define the following:

Definition 4. *$Dab(\Upsilon)$ is a* minimal Dab-consequence *of Γ if and only if $\Gamma \vDash_{\mathbf{LLL}} Dab(\Upsilon)$ and, for all $\Upsilon' \subset \Upsilon$, $\Gamma \nvDash_{\mathbf{LLL}} Dab(\Upsilon')$.*

Definition 5. *Where $Dab(\Upsilon_1)$, $Dab(\Upsilon_2)$, $\ldots$ are the minimal Dab-consequences of Γ, let $U(\Gamma) =_{df} \Upsilon_1 \cup \Upsilon_2 \cup \ldots$*

Definition 6. *Where M is an* **LLL***-model, $Ab(M) =_{df} \{A \in \Omega \mid M \models A\}$.*

Definition 7. *A* **LLL***-model M of Γ is a* reliable model *if $Ab(M) \subseteq U(\Gamma)$.*

Definition 8. *A is a* reliable semantic consequence *of Γ, in symbols $\Gamma \vDash_{\mathbf{AL}^r} A$, if all reliable models of Γ verify A.*

Definition 9. *A* **LLL***-model M of Γ is a* minimally abnormal model *if there is no* **LLL***-model M' of Γ for which $Ab(M') \subset Ab(M)$.*

Definition 10. *A is a* minimal abnormal semantic consequence *of Γ, in symbols $\Gamma \vDash_{\mathbf{AL}^m} A$, if all minimally abnormal models of Γ verify A.*

3 Vague Indexicals and Non-vague Predicates

3.1 LLL: The Fuzzy Logic $\mathcal{M}$Ł$_\mathbf{n}$

The logic Ł$_\mathbf{n}$ used in [14] is the standard Łukasiewicz fuzzy logic Ł with a lower acceptability threshold. The fuzzy logic $\mathcal{M}$Ł$_\mathbf{n}$ extends the language of Ł$_\mathbf{n}$ with a modifier and forms the **LLL** of the adaptive fuzzy logics that we will define. In the examples of a Sorites paradox with a vague indexical given in section 1, the expressions are of the form $(Here(Standing(Tom)))$ and $(Now(Falling(Domino_n)))$. They contain three elements: an element in the domain (formally represented by a constant), a proper non-vague predicate, and a vague indexical. It is assumed that an expression πa_n has a crisp truth value.[8] 'Standing here' or 'falling now' should not be understood as an atomic predicate, because this would forbid one to explain the logical and semantical behaviour of the indexical 'here' or 'now' taken alone. The predication πa for $(Standing(Tom))$ or $(Falling(Domino_n))$ has the same evaluation throughout the Sorites series. But when the modifier 'here' or 'now' is applied, the formula $m[\pi a_n]$ may map to a fuzzy truth value. In sum, 'standing' or 'falling' is treated as a non-vague predicate applied to constants (individuals or objects), and 'here' or 'now' as a separate operator, namely a function that maps a non-vague predicate to a (possibly fuzzy) application of it. In this way, the conditional steps in the Sorites series are formalised as: $m[\pi a_n] \rightarrow m[\pi a_{n+1}]$. The validity of such a conditional is preserved even for cases where the consequent is a little less true than the antecedent, up to a difference that is established by the threshold.

The language schema of the **LLL** is defined as in [14], except that the set of unary predicate symbols $\mathcal{P} = \{P, Q, R\}$ consists only of non-vague predicates for this version of the adaptive logic. Let the finite set of constant symbols be $\mathcal{C} = \{a_1, a_2, \ldots\}$, and the set of variable symbols be $\mathcal{V} = \{x, y, z\}$. The language $\mathcal{F}$ of open and closed formulas consists of atomic formulas $\pi\alpha$, where $\pi \in \mathcal{P}$ and $\alpha \in \mathcal{C} \cup \mathcal{V}$, and is closed under the unary connectives and operators $\neg$, Δ, and $\mathbb{F}$, $\sim_i$ ($i \in \mathbb{N}$), the binary connectives $\&$, $\wedge$, $\vee$, $\rightarrow$, and $\veebar$ and the quantifiers $\forall$ and $\exists$ in the standard first-order way. The set of closed formulas $\mathcal{W}$ is also defined in the usual way. The connectives are defined as follows (A and B are used as metavariables for predicative formulas of the from πa) :

(D1) $A \veebar B \quad =_{df} \neg A \rightarrow B$
(D2) $A \& B \quad =_{df} \neg(\neg A \veebar \neg B)$
(D3) $A \wedge B \quad =_{df} A \& (A \rightarrow B)$
(D4) $A \vee B \quad =_{df} ((A \rightarrow B) \rightarrow B) \& ((B \rightarrow A) \rightarrow A)$
(D5) $\sim_i A \quad =_{df} \neg \Delta(A \veebar A^i)$, where $i \in \mathbb{N}$
(D6) $\mathbb{F} A \quad =_{df} \neg \Delta \neg A \wedge \neg \Delta A.$
(D7) $\forall \alpha A(\alpha) =_{df} \bigwedge \{A(\alpha) | \alpha \in \mathcal{C}\}$
(D8) $\exists \alpha A \quad =_{df} \neg \forall \alpha \neg A$

3.2 Axiomatisation

For the proof theory, we start by axiomatising the logic Ł, for which the Rose-Rosser axioms are used [10] (corresponding to some **CL**-axioms):[9]

[8] Crispness is the opposite of vagueness. While a vague expression has a truth value lying between 0 and 1, a crisp expression has either the truth value 0 or 1.

[9] Notice that this corresponds to the set of axioms valid for all non-modified formulas in the **LLL** $\mathcal{M}$Ł.

(A1) $(A \to B) \to ((B \to C) \to (A \to C))$
(A2) $A \to (B \to A)$
(A3) $((A \to B) \to B) \to ((B \to A) \to A)$
(A4) $(\neg A \to \neg B) \to (B \to A)$

We define a set of extra axioms in order to fix the meaning of the Baaz' Δ-operator (cf. [4]) that can be intuitively read as 'it is completely true that' (and behaves axiomatically as a $\Box$-operator):

(AΔ1) $\Delta A \vee \neg \Delta A$
(AΔ2) $\Delta(A \vee B) \to (\Delta A \vee \Delta B)$
(AΔ3) $\Delta A \to A$
(AΔ4) $\Delta A \to \Delta\Delta A$
(AΔ5) $\Delta(A \to B) \to (\Delta A \to \Delta B)$

The following axioms define how the modified formulas with a non-vague predicate behave proof-theoretically:

(A1) $A \to mA$
(A2) $m[A \vee B] \to (m[A] \vee m[B])$

The first axiom corresponds to a possibility or local version of Axiom T: it says that for any true formula, there is at least one application of the modifier that is true for it. The second axiom allows distribution over disjunction: if for some application of the modifier a true disjunction holds, at any of the disjuncts the application must be true. Finally, the rules of Ł are:

(MP) From A and $A \to B$ derive B
(NEC) From A derive ΔA

Modus Ponens and the necessitation rule are valid in Ł. As we will see, unrestricted applications of these rules in $\mathcal{M}$Ł are problematic and for this reason, the adaptive strategies on $\mathcal{M}$Ł with a prefixed acceptability threshold will be deployed.

3.3 Semantics

A $\mathcal{M}$Ł-model is a pair $M = \langle D, v \rangle$ whenever the domain D is a finite set of the same cardinality as the finite set of constant symbols $\mathcal{C}$; v maps the constants to the elements of D and the modified predicates to possibly fuzzy subsets of D:

(i) $v\colon \mathcal{C} \mapsto D$ is a one-to-one mapping
(ii) $v\colon \mathcal{P} \mapsto (D \mapsto \{0,1\})$
(iii) $v\colon \mathcal{M} \mapsto (\mathcal{P} \mapsto (D \mapsto [0,1]))$

where $\mathcal{M} = \{m\}$. So the non-modified formula has a truth value of either 0 or 1, i.e. is not fuzzy. And the modified formula has a truth value between 0 and 1, including 0 and 1, i.e. is possibly fuzzy. The valuation function $v_M\colon \mathcal{W} \mapsto [0,1]$, determined by M, is defined by the following conditions:

(S1) $v_M(\pi\alpha) = v(\pi)\langle v(\alpha)\rangle$
(S2) $v_M(A \to B) = \min(1, (1 - v_M(A)) + v_M(B))$
(S3) $v_M(\neg A) = 1 - v_M(A)$
(S4) $v_M(\Delta A) = 1$ if $v_M(A) = 1$ and $v_M(\Delta A) = 0$ if $v_M(A) \neq 1$
(S5) $v_M(m[A]) = 0$ if $v_M(A) = 0$, $v_M(m[A]) = 1$ if $v_M(\Delta A) = 1$, and $v_M(m[A]) \in\ ]0, 1[$ otherwise

where $\to$ stands for the classical implication, and Δ enables one to express non-fuzzy applications of the modified formulas. Intuitively, the fifth condition states that e.g. if someone is not 'standing', then he is not 'standing here'. And if it is completely true that he is 'standing', then there is a value of 'here' for which it is true that he is 'standing here'. And in all other cases, 'standing here' has a fuzzy evaluation. Semantic evaluations for the other operators can be defined accordingly, as e.g.: $v_M(\mathbb{F}m[A]) = 1$ if $v_M(m[A]) =]0, 1[$ and $v_M(\mathbb{F}m[A]) = 0$ otherwise.

3.4 Designated Values and Acceptability Threshold

In Ł the only designated value is 1. However, it is more natural to use an interval of designated values $[\frac{n}{n+1}, 1]$, for some natural number $n > 1$. We call $\frac{n}{n+1}$ the acceptability threshold as it is the threshold for determining whether a formula is true enough, i.e. acceptable, or not. Let n be such a number. The logic $\mathcal{M}$Ł$_\mathbf{n}$ has the same language, the same models and the same truth-functionality for the connectives as $\mathcal{M}$Ł. Obviously, $\mathcal{M}$Ł$_\mathbf{n}$ defines another semantic consequence relation (cf. [14], p. 1878):[10]

Definition 12. *A formula A is a semantic consequence of Γ in $\mathcal{M}$Ł$_\mathbf{n}$, in symbols $\Gamma \vDash_{\mathcal{M}Ł_\mathbf{n}} A$, if $v_M(A) \in [\frac{n}{n+1}, 1]$ for every model M in which $v_M(B) \in [\frac{n}{n+1}, 1]$ for every $B \in \Gamma$.*

[10] The resulting systems are not axiomatisable in the strict sense, because there are obviously applications of Modus Ponens that are no longer valid. In ([14], p. 1878) a special translation tr_n from Ł$_\mathbf{n}$ to Ł is proposed:

Definition 11. $\mathrm{tr}_n(\Gamma \vDash_{Ł_\mathbf{n}} A) =_{df} \{B \veebar B^n | B \in \Gamma\} \vDash_{Ł} A \veebar A^n$

It says that if a formula A is a semantic consequence in Ł$_\mathbf{n}$ of a set of premises Γ, then $\neg A \to A^n$ is a consequence of $\neg B \to B^n$ for every $B \in \Gamma$. Hence, if the formula is invalid in general, then one can make it valid at the given acceptability threshold n. The translation allows to define a proof theory for Ł$_\mathbf{n}$ in view of the following theorem

Theorem 1. $\Gamma \vDash_{Ł_\mathbf{n}} A$ *if and only if* $\mathrm{tr}_n(\Gamma \vDash_{Ł_\mathbf{n}} A)$

which is immediately valid for formulas preceded by Δ. The translation can be adapted to $\mathcal{M}$Ł$_\mathbf{n}$, which is a suitable **LLL**: it is monotonic, reflexive, transitive as well as compact, and the connectives $\sim_n$ and $\vee$ have a **CL**-meaning.

3.5 Strategies and Abnormalities

The acceptability threshold determines which applications of the Modus Ponens rule are not valid, but generate *abnormalities*, such that for some a_i, $\neg\Delta m[\pi a_i]$ and $\neg\Delta\neg m[\pi a_i]$. By selecting an appropriate *Strategy* (*Reliability* or *Minimal Abnormality*), one proceeds in the semantics by selecting the models that either eliminate all the abnormal cases, or precisely identify which abnormality is valid, hence indicate the shift point between crisp evaluations; in the proof theory one proceeds by establishing which derivable formulas are finally retained.

There are two options to conceive of the vagueness of an indexical. These two ways are reflected by the two sets of abnormalities defined in [14] that we here adapt to conceive of the fuzziness of the modifier:

- The first option states that a modifier should be interpreted as abnormal when there is at least one application of the modifier that turns out to be fuzzy. That is, suppose we have expressions of the form $m[\pi a_i]$. For some $i = \{1 \ldots n-1\}$ the expression is not fuzzy, in other words $\Delta m[\pi a_i]$. Also, for some $j = \{n+1 \ldots n+m\}$ the expression is not fuzzy, i.e. $\Delta\neg m[\pi a_j]$. But for some intermediary $m[\pi a_n]$, the application becomes fuzzy. When i may be interpreted as normal, i.e. bivalent, classical logic (**CL**) remains the way to go. This first option holds for the logics $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{mg}}$ and $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{rg}}$. In these two logics the set of abnormalities is the same. Let $\exists(A)$ denote the existential closure of A and Ω the set of abnormalities. Then the set Ω is defined as follows.

$$\Omega_g = \{(\exists\alpha)\mathbb{F}m[\pi\alpha] | m \in \mathcal{M}; \pi \in \mathcal{P}; \alpha \in \mathcal{V}\} \qquad (1)$$

  Read in an informal way, the abnormalities of the Soritical argument express that:

  1. There is a (set of) person$_n$, standing in the group of people around the speaker, for whom it is both *not* completely true that he is standing *not* here, and *not* completely true that he is standing here.
  2. There is a (set of) domino$_n$, with $n > 1$, for which it is both *not* completely true that it is falling *not* now, and *not* completely true that it is falling now.

- According to the second option to conceive of the fuzziness of the modifier, individual applications of the modifier m may be (ab)normal. This implies that in some cases $m[\pi a_i]$ would be fuzzy, while in other cases it might not be, and it leads to a more fine-grained solution. So for some application to a given individual, the intended meaning of 'here' is preserved, while in other applications, it is no longer valid. This second option holds for the logics $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{ml}}$ and $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{rl}}$, and results in a different set of abnormalities:

$$\Omega_l = \{\exists(\mathbb{F}m[\pi\alpha]) | m \in \mathcal{M}; \pi \in \mathcal{P}; \alpha \in \mathcal{V} \cup \mathcal{C}\} \qquad (2)$$

  Read in an informal way, these abnormalities express that:

1. Individual applications of 'here' may turn out to be abnormal. Hence, for some persons, 'standing here' is completely true at some stage and completely not true at some other stage.
2. Individual applications of 'now' may turn out to be abnormal. Hence, for some dominos, 'falling now' may be completely true at some stage and completely not true at some other stage.

Hence, we conclude by formulating the four logics: $\mathcal{M}\text{Ł}_{\mathbf{n}}^{\mathbf{mg}} = \mathcal{M}\text{Ł}_{\mathbf{n}} + \Omega_g +$ *Minimal Abnormality*; $\mathcal{M}\text{Ł}_{\mathbf{n}}^{\mathbf{rg}} = \mathcal{M}\text{Ł}_{\mathbf{n}} + \Omega_g +$ *Reliability*; $\mathcal{M}\text{Ł}_{\mathbf{n}}^{\mathbf{ml}} = \mathcal{M}\text{Ł}_{\mathbf{n}} + \Omega_l +$ *Minimal Abnormality*; $\mathcal{M}\text{Ł}_{\mathbf{n}}^{\mathbf{rl}} = \mathcal{M}\text{Ł}_{\mathbf{n}} + \Omega_l +$ *Reliability*.

3.6 Example

We now present a concrete example of an adaptive proof for 10 instances of πa_n. The premises (which are derived on the empty condition) are the following:

1	$\Delta m[\pi a_1]$	PREM	$\emptyset$
2	$\Delta\neg m[\pi a_{10}]$	PREM	$\emptyset$
3.1	$m[\pi a_1] \to m[\pi a_2]$	PREM	$\emptyset$
3.2	$m[\pi a_2] \to m[\pi a_3]$	PREM	$\emptyset$
$\vdots$			
3.9	$m[\pi a_9] \to m[\pi a_{10}]$	PREM	$\emptyset$
4.1	$\Delta(m[\pi a_2] \to m[\pi a_1])$	PREM	$\emptyset$
4.2	$\Delta(m[\pi a_3] \to m[\pi a_2])$	PREM	$\emptyset$
$\vdots$			
4.9	$\Delta(m[\pi a_{10}] \to m[\pi a_9])$	PREM	$\emptyset$

The first premise states that the modified predicate completely holds for a_1, and the second that it is completely false for a_{10}. Premises 3.1 to 3.9 express the conditional steps: if the modified predicate holds for a given constant, it will hold for the next one (strict preservation upwards). Premises 4.1 to 4.9 state that at each stage, indexicality is fully preserved downwards (formally: for all i, $v_M(m[\pi a_{i+1}]) \leq v_M(m[\pi a_i])$).

5.1	$m[\pi a_2]$	1,3.1 RU	$\emptyset$
5.2	$m[\pi a_3]$	5.1, 3.2; RC	$\{\mathbb{F}m[\pi a_2]\}$
$\vdots$			
5.9	$m[\pi a_{10}]$	5.8, 3.9; RC	$\{\mathbb{F}m[\pi a_i] \mid i \in [2,9]\}$
6	$m[\pi a_{10}] \wedge \neg m[\pi a_{10}]$	5.9, 2; RU	$\{\mathbb{F}m[\pi a_i] \mid i \in [2,9]\}$
6'	$m[\pi a_{10}] \wedge \neg m[\pi a_{10}]$		
	$\bigvee\{\mathbb{F}m[\pi a_i] \mid i \in [2,9]\}$	1–4.9; RU	$\emptyset$
7	$\bigvee\{\mathbb{F}m[\pi a_i] \mid i \in [2,9]\}$	6'; RU	$\emptyset$

At line 5.1, it is unconditionally derived that the modified predicate holds for a$_2$. At line 5.2, we establish that $m[\pi a_3]$ holds given: the content of line 5.1, the implication $m[\pi a_2] \to m[\pi a_3]$ derived at line 3.2, and the condition that a_2 does not behave abnormally, namely that it is not the case that both $\neg\Delta m[\pi a_2]$ and $\neg\Delta\neg m[\pi a_2]$. The same inference can be performed for every πa_{4-10}, where at each stage an additional abnormality is required to be false. At line 6, we derive a contradiction from lines 2 and 5.9, preserving the condition that no abnormality is valid at any of the previous stages. At line 7, we use the mechanism that is known as *Dab intro-shortcut*: if a contradiction A is derived on a condition Θ on some line i, then $\bigvee\Theta$ may be unconditionally derived from the premise set on line $i+1$. From this moment on, the proofs of $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{mg}}$ and $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{rg}}$ on the one hand, and of $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{ml}}$ and $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{rl}}$ on the other, proceed differently.

$\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{mg}}$ and $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{rg}}$. Since a contradiction is conditionally derived at line 6, we know that the modifier m cannot be a crisp modifier: line 7 says that at least one abnormality is derivable. Given the marking definitions (see Definitions 2 and 3), all conditional lines in the proof should be marked. Only fuzzy logic can be applied here, and thus the $Ł_{\mathbf{n}}^{\mathbf{mg}}$- and $Ł_{\mathbf{n}}^{\mathbf{rg}}$-consequences are exactly the same as the $\mathcal{M}Ł_{\mathbf{n}}$-consequences.

5.1	$m[\pi a_2]$	1, 3.1 RU	$\emptyset$	
5.2	$m[\pi a_3]$	5.1, 3.2; RC	$\{\exists x(\mathbb{F}Px)\}$	$\checkmark$
...				
5.9	$m[\pi a_{10}]$	5.8, 3.9; RC	$\{\exists x(\mathbb{F}Px)\}$	$\checkmark$
6	$m[\pi a_{10}] \wedge \neg m[\pi a_{10}]$	5.9, 2; RU	$\{\exists x(\mathbb{F}Px)\}$	$\checkmark$
7	$\{\exists x(\mathbb{F}m[\pi x])\}$	1–4.9; RU	$\emptyset$	

In this case, the modifier is fuzzy as a whole.

$\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{ml}}$ and $\mathcal{M}Ł_{\mathbf{n}}^{\mathbf{rl}}$. Let us consider the logics $\mathcal{M}Ł_{\mathbf{2}}^{\mathbf{ml}}$ and $\mathcal{M}Ł_{\mathbf{2}}^{\mathbf{rl}}$.[11] By accumulating conditions, we can derive $m[\pi a_{10}]$ on line 5.9, but this is in contradiction with line 2. Using the *Dab intro-shortcut*, $\bigvee\{\mathbb{F}m[\pi a_i] \mid i \in [2,9]\}$ is unconditionally derived from the premise set on line 7. This formula states that one $m[\pi a_i]$, where $i \in [2,9]$, should be fuzzy. The strategy (together with our chosen threshold) will allow us to resolve the impasse, leading to the marking of several lines. Because of the space limitations, we will only set out part of the proof here.[12] First one derives all the possible sets of pairs that violate the Sorites-like inference steps.

11.1	$\bigvee\{\mathbb{F}m[\pi a_2], \mathbb{F}m[\pi a_4], \mathbb{F}m[\pi a_6], \mathbb{F}m[\pi a_8]\}$	10; RU	$\emptyset$
11.2	$\bigvee\{\mathbb{F}m[\pi a_3], \mathbb{F}m[\pi a_4], \mathbb{F}m[\pi a_6], \mathbb{F}m[\pi a_8]\}$	10; RU	$\emptyset$
11.3	$\bigvee\{\mathbb{F}m[\pi a_3], \mathbb{F}m[\pi a_5], \mathbb{F}m[\pi a_7], \mathbb{F}m[\pi a_8]\}$	10; RU	$\emptyset$
11.4	$\bigvee\{\mathbb{F}m[\pi a_3], \mathbb{F}m[\pi a_5], \mathbb{F}m[\pi a_7], \mathbb{F}m[\pi a_9]\}$	10; RU	$\emptyset$
11.5	$\bigvee\{\mathbb{F}m[\pi a_3], \mathbb{F}m[\pi a_5], \mathbb{F}m[\pi a_6], \mathbb{F}m[\pi a_8]\}$	10; RU	$\emptyset$

[11] These logics have the threshold $\frac{2}{3}$. Accordingly, Modus Ponens is only conditionally applicable: the truth-degree of the 'hereness' of two adjacent persons or the 'nowness' of two adjacent dominos can differ at most by $\frac{1}{3}$ if the implication is valid from one to the other.

[12] Accordingly, line numbers will not be consecutive.

11.1-11.5 are the minimal *Dab*-consequences for the premise set. Following the Minimal Abnormality Strategy (Definition 3), one selects the least necessary number of such abnormal formulas (excluding those sets where the difference between elements is above the threshold, e.g. $\{\mathbb{F}m[\pi a_3], \mathbb{F}m[\pi a_6]\}$): $\Phi_s(\Gamma) = \{\{\mathbb{F}m[\pi a_2], \mathbb{F}m[\pi a_3]\}, \{\mathbb{F}m[\pi a_3], \mathbb{F}m[\pi a_4]\}, \{\mathbb{F}m[\pi a_4], \mathbb{F}m[\pi a_5]\}, \{\mathbb{F}m[\pi a_5], \mathbb{F}m[\pi a_6]\}, \{\mathbb{F}m[\pi a_6], \mathbb{F}m[\pi a_7]\}, \{\mathbb{F}m[\pi a_7], \mathbb{F}m[\pi a_8]\}, \{\mathbb{F}m[\pi a_8], \mathbb{F}m[\pi a_9]\}\}$. Keeping deriving formulas on conditions, one marks all those lines whose condition has a non-empty intersection with one of such minimal disjunction of abnormalities. As an example:

14.2	$\neg\mathbb{F}m[\pi a_5] \vee \neg\mathbb{F}m[\pi a_6]$	14.1; RC	$\{\mathbb{F}m[\pi a_5]\}$	$\checkmark$
15.2	$\neg\mathbb{F}m[\pi a_5] \vee \neg\mathbb{F}m[\pi a_6]$	15.1; RC	$\{\mathbb{F}m[\pi a_6]\}$	$\checkmark$

Eventually, the only derivable content that does not get marked will be $(m[\pi a_4] \vee \neg m[\pi a_4]) \wedge (m[\pi a_3] \vee \neg m[\pi a_3])$:

26	$(m[\pi a_4] \vee \neg m[\pi a_4]) \wedge$ $(m[\pi a_3] \vee \neg m[\pi a_3])$	23; RU	$\{\mathbb{F}m[\pi a_2], \mathbb{F}m[\pi a_3]\}$
27	$(m[\pi a_4] \vee \neg m[\pi a_4]) \wedge$ $(m[\pi a_3] \vee \neg m[\pi a_3])$	24; RU	$\{\mathbb{F}m[\pi a_9], \mathbb{F}m[\pi a_8], \mathbb{F}m[\pi a_7], \mathbb{F}m[\pi a_6]$ $m[\pi a_5], m[\pi a_2]\}$
28	$(m[\pi a_4] \vee \neg m[\pi a_4]) \wedge$ $(m[\pi a_3] \vee \neg m[\pi a_3])$	25; RU	$\{\mathbb{F}m[\pi a_9], \mathbb{F}m[\pi a_8], \mathbb{F}m[\pi a_7], \mathbb{F}m[\pi a_6]$ $m[\pi a_5], m[\pi a_4]\}$

This will be finally derived from the premises. So either $m[\pi]$ or $\neg m[\pi]$ holds for both a_3 and a_4. This value identifies therefore where the shift in the applicability of the indexical occurs. Applying the Reliability Strategy will induce a marking on every conditional line; hence, the consequence set will again correspond to that of $\mathcal{M}$Ł$_{\mathbf{2}}$.

4 Indexicals and Vague Predicates

In [14], the setting was said to resolve the Sorites paradox with vague predicates. We can also give an interpretation of expressions where the predicate is vague and applying the indexical makes as many as possible of those predications crisp. Expressions considered are of the form (*Here*(*Steep*(*Hill*))) and (*Now*(*Healthy*(*David*))). Let us again clarify this with an example. Suppose you are at the top of a steep hill (hill$_1$), the steepness of which decreases gradually:

> The hill$_1$ is steep,
> and if the hill$_1$ is steep, then the hill$_2$ is too,
> and if the hill$_2$ is steep, then the hill$_3$ is too,
> ⋮
> and then the hill$_n$ is too.

While the hill is clearly steep for some small n, there is some value of n for which 'steep' is vague, and the hill is clearly not steep for some large n, since you have then reached its bottom.[13] In the derivation in section 3, we generated cases where the predications are vague. We can use the same machinery to disambiguate a vague predicate by identifying the point at which the shift occurs and the predicate becomes crisp. This approach reverses the previous format: the non-modified formula (may) have a vague behaviour, and the modified formula turns out to be crisp in most cases. So the non-modified formula fits into a Sorites paradox, but the modified formula does not. We return to the original setting of [14] where predicates are vague, and we use a modifier that takes the crisp values {0,1} for bunches of fuzzy values of such predicates. We take the fuzzy logic $\mathcal{M}$Ł$_{\mathbf{n}}$ with the appropriate changes as our **LLL**. The designated values and acceptability threshold, and the strategies and abnormalities are defined as before.

4.1 Axiomatisation

(A1) $\pi a_n \to m[\pi a_n]$
(A2) $m[\pi a_n] \vee \neg m[\pi a_n]$
(A3) $m[\pi a_n \vee \pi a_{n+i}] \to (m[\pi a_n] \vee m[\pi a_{n+i}])$
(A4) $m[\pi a_n \to \pi a_{n+i}] \to (m[\pi a_n] \to m[\pi a_{n+i}])$

(A1) and (A3) correspond to the axioms above. Since the modified formula with a vague predicate has a crisp truth value, (A2) and (A4) also hold.

4.2 Semantics

The semantics of this version of the adaptive fuzzy logic is different from the one above, because the function v now maps predicates to elements in a domain, to truth values in a fuzzy domain, and it maps a modifier to predicates to elements in a domain, to truth values in a non-fuzzy domain:

(i) $v\colon \mathcal{C} \mapsto D$ is a one-to-one mapping
(ii) $v\colon \mathcal{P} \mapsto (D \mapsto [0,1])$
(iii) $v\colon \mathcal{M} \mapsto (\mathcal{P} \mapsto (D \mapsto \{0,1\}))$

The valuation function v_M is defined by conditions S1-S4 as above, but S5 differs:

(S5) $v_M(m[A]) = 0$ if $v_M(A) \in [0,1[$ and $v_M(m[A]) = 1$ if $v_M(A) \in]0,1]$

[13] The indexical 'here' attaches to an identified point (possibly detached from the location of the speaker). Similarly, a series of predications with 'now' identifies points in time (not necessarily corresponding to moments of utterance).

It states that if the non-modified formula has some value between false (included) and true (excluded), then the modified formula is false. And if the non-modified formula has some value between false (excluded) and true (included), then the modified formula is true. Intuitively, this means that the hill is not 'steep here' if it is not 'steep'; and it is 'steep here', if it is 'steep', both for a large bunch of values up to some $]n[$. At the shift point n between steep and not-steep, the hill is both 'steep here' and not 'steep here'. So in all the cases before n, the hill is 'steep here'. And in all the cases after n, the hill is not 'steep here'. So $v_M(\pi a_i = [0, n])$ induces $v_M(m[\pi a_i]) = 0$, and $v_M(\pi a_i = [n, 1])$ induces $v_M(m[\pi a_i]) = 1$. The same logics presented above will identify the values of πa_i that represent the shift point from $m[\pi a_{i-1}] = 0$ to $m[\pi a_{i+1}] = 1$. At point i, there is a truth value glut ($m[\pi a_i]$ evaluates to both 0 and 1).

5 Conclusion

This paper introduced a dynamic logical treatment of vague expressions with indexicals in a series of implicative sentences. In future research, we hope to clarify further uses of indexicals in a temporal setting and Sorites-like phenomena with distinct indexicals or distinct predicates.

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