A modal language for contextual computations

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Abstract

In this paper, we present a modal language for contextual computing, corresponding to the fragment of constructive KT with necessity and possibility operators. We interpret absolute and contextual computations as different modes of verifying the truth of propositions. The semantics of the language $\mathcal{L}^{cc}$ interprets absolute computations by a direct verification function valid in every state; contextual computations are interpreted in terms of a verification function valid under *unverified* information. Modalities are used to express extensions of contexts in order to define local and global validity. This semantics has a (weak) monotonicity property, depending on satisfaction of processes in contexts. In the corresponding axiomatic system $cKT_{\Box\Diamond}$ a restricted version of the deduction theorem for globally valid formulas holds, soundness and completeness are proven and decidability is shown to hold for the necessitation fragment of the language by a restricted finite model property.

1 Background and Motivation

The relation between modal logic $S4$ and intuitionistic logic is notoriously given via the interpretation of necessity as provability. Constructive $S4$ has been explored in the form of both Kripke and categorical semantics. Less considered in the literature are modal translations of the *contextual* notion of derivability, known from natural deduction calculi and type theories.[1] Even less so are languages defining *both* local and global validity relations, roughly corresponding to the idea of derivability from undischarged and discharged assumptions.

Semantic research in modelling contexts from AI relies on the very same background.[2] Contextual computing can be seen as the algorithmic interpretation of reasoning under contexts and its modal version in a constructive format is particularly apt for applications-oriented research. Various Kripke semantics

[1] See e.g. [14], [27], [19].
[2] See e.g. [15].

for constructive intuitionistic modal logics from K to $S4$ exist.[3] Among these, a weaker format to accomodate the notion of context is given by the calculus CK in [16], which presents a possible-world semantics sound and complete with respect to the natural deduction interpretation given in [5].

Syntactically, contextual modal type theories for programming languages and further research in linguistics and hardware verification have been pursued.[4] In the present paper, a constructive reading of computational processes is extended to contextual validity by means of modalities. This provides a general model of computations in context, with the latter intended to express background knowledge.

Among intuitionistic modal logics, one standard interpretation of the concept of truth is given in [33] by the schema: *"A is true if and only if it is possible that it is verified that A"*.[5] The resulting formal system is based on an intuitionistic language extended with modalities $\Diamond$ and $\Box$ and with a knowledge operator K, such that the previous schema is formally translated as:

$$(*) \ \vdash A \leftrightarrow \Diamond KA$$

In [1] the standard ways to define intuitionistic modalities semantically are given as follows:

$\Box_1$ $\mathcal{M}, w \vDash \Box\phi$ iff $\forall v(wRv \rightarrow \mathcal{M}, v \vDash \phi)$

$\Box_2$ $\mathcal{M}, w \vDash \Box\phi$ iff $\forall v(wRv \rightarrow \forall u(vRu \rightarrow \mathcal{M}, v \vDash \phi))$

$\Diamond_1$ $\mathcal{M}, w \vDash \Diamond\phi$ iff $\exists v(wRv \wedge \mathcal{M}, v \vDash \phi)$

$\Diamond_2$ $\mathcal{M}, w \vDash \Diamond\phi$ iff $\forall v(wRv \rightarrow \exists u(vRu \wedge \mathcal{M}, u \vDash \phi)$

where $\mathcal{M}$ is a model, w an element in the set of worlds, R the appropriate accessibility relation and ϕ a formula. The contextual interpretation given in the present paper significantly differs from any of the above given standard definitions. Necessity and possibility are here linked respectively to absolute and contextual truth via computations:[6]

1. *"A is true" is necessary if and only if A is globally verified*
2. *"A is true" is possible if and only if A is locally verified*

[3] For an overview of the various systems of intuitionistic modal logics introduced since the middle of the Sixties, see [25]. For the constructive translations, see for example [20], [32], [4], [3], [2].

[4] See e.g. [17] and the bibliography in [2]. Modalities for type-theories focus on the computability of the underlying λ-calculus, which in turn can be interpreted as the corresponding programming language, see [6].

[5] [33], p. 65.

[6] This definition has moreover the advantage of simplifying the language so that it does not require an extra knowledge operator. See also [21] and [28] for a discussion on the application of modal operators to the judgemental form "A is true".

These definitions translate modalities via appropriate validity relations. In particular, it is possible to understand global verification as truth under no condition,

$$(**)\ \Box(A\ true) \Leftrightarrow ((\emptyset)A)$$

and local verification as truth under conditions:

$$(***)\ \Diamond(A\ true) \Leftrightarrow ((\Gamma)A)$$

This requires to give an appropriate interpretation of the content in Γ. We will understand the judgement $\Diamond(A\ true)$ as 'A is true in the context of unverified information'.

We present a standard verificationist semantics for non-modal formulas extended to a modal language that contains both empty and non-empty contexts of unverified information. The two protocols are dubbed respectively 'verificational' ($\mathcal{L}^{ver}$) and 'contextual' ($\mathcal{L}^{ctx}$). Formulas verified in an empty context induce truth; formulas valid under open variables induce contextual truth. Modalities express validity under extensions of contexts. The language is then completed by an appropriate axiomatic calculus, corresponding to a version of the modal logic KT including also appropriate axioms for the possibility operator and restricting necessitation and weakening to a subset of the formulas of the language.

The structure of the paper is as follows. In section 2, we introduce the semantics that interprets logical connectives for categorical judgements and extend it to knowability in a context. A main theorem is stated for the relation to standard constructive semantics. In section 3, we present the corresponding axiomatic calculus and prove a restricted form of the deduction theorem, soundness and completeness, and characterize the maximal decidable fragment of the language in view of a restricted version of the semantics. In the conclusion, we refer to applications for which this semantics appears natural and mention related work.

2 A semantics for contextual computation

The language $\mathcal{L}_{cc}$ for contextual computing is given by two languages: $\mathcal{L}^{ver}$ is called the 'verificational' protocol and it is a fragment of $\mathcal{L}^{ctx}$, the 'contextual' protocol. In $\mathcal{L}_{cc}$ absolute and contextual computations are defined as modes of verifying the truth of proposition A:

1. in $\mathcal{L}^{ver}$, 'A is true' is defined by a globally valid verification of A.
2. in $\mathcal{L}^{ctx}$, 'A is true' is defined by verification in the context of an unverified A'.

In our interpretation of (i), truth corresponds to verification; by our interpretation of (ii), a verification of A holds assuming (i.e. in the context where) A' holds; verification of A' is considered as not directly accessible.

2.1 Absolute Computations

A language $\mathcal{L}^{ver}$ for absolute computations is defined by formulas built in a standard way from a finitely enumerable set of propositional atoms $\mathcal{P} = \{\phi, \psi, \dots\}$, the propositional constants $\top, \bot$, unary and binary propositional operators $\neg, \wedge, \vee, \supset$; variables for propositional formulas are referred to as $\mathcal{W} = \{A, B, \dots\}$:

$$\mathcal{L}^{ver} := \phi \mid \top \mid \bot \mid \neg A \mid A \wedge B \mid A \vee B \mid A \supset B.$$

K_i denotes a knowledge state with a set of indices $\mathbb{I} = \{i, j, k, \dots\}$ totally ordered under a $\leq$ relation holding among states. A knowledge state collects propositional atoms and formulas (possibly a singleton) that are evaluated in a model at that state. A knowledge set at some index n, $\mathbb{K} = \{K_1, \dots, K_n \mid 1 \leq n \in \mathbb{I}\}$ is a finite collection of indexed knowledge states up to n, closed under logical consequence. When needed, we shall refer to $\geq$ as the reverse order of $\leq$.

Definition 1. *A model of $\mathcal{L}^{ver}$ is a tuple $M^{ver} = \{\mathbb{K}, \leq, v\}$, where $\mathbb{K}$ is a nonempty set ranging over $\{K_i, K_j, \dots\}$; $\leq$ is a reflexive and transitive ordering relation over members of $\mathbb{K}$; v is a verification function $v : \mathbb{K} \mapsto 2^{\mathcal{W}}$.*

We shall call $M^{ver}(K_i)$ a verificationist model of knowledge state K_i if and only if all formulas of K_i are true in it; that a formula A is true in $M^{ver}(K_i)$ corresponds to the existence of a function that verifies A in K_i. The intended meaning of such verification function is that of an effective procedure that makes the truth of A explicit at that state. The verification function corresponds to an inductively generated satisfaction relation at a state and, in the case of the negation operator, defined over related members of $\mathbb{K}$:

$C1^{ver}$ $K_i \not\vDash \bot$ and $K_i \vDash \top$;

$C2^{ver}$ for all ϕ, $K_i \vDash \phi$ iff $(\phi, v(K_i))$;

$C3^{ver}$ $K_i \vDash A \vee B$ iff $K_i \vDash A$ or $K_i \vDash B$;

$C4^{ver}$ $K_i \vDash A \wedge B$ iff $K_i \vDash A$ and $K_i \vDash B$;

$C5^{ver}$ $K_i \vDash A \supset B$ iff $K_i \vDash A$ implies $K_i \vDash B$;

$C6^{ver}$ $K_i \vDash \neg A$ iff $\forall K_j \geq K_i$, it holds $K_j \vDash A \supset \bot$.

The satisfaction relation $K_i \vDash A$ reads as follows: *"A is verified in the state K_i by a verification v"*. $C1^{ver}$ declares consistency of K_i's (contradictions are not admitted); $C2^{ver}$ gives the base case of satisfaction by verification: a propositional letter ϕ is true in K_i iff there is a verification function for ϕ in K_i; $C3^{ver} - C5^{ver}$ are standardly defined for binary connectives; $A \supset B$ is satisfied in K_i if and only if a verification process for A at K_i gives a verification process for B at K_i. For the negation function $C6^{ver}$: a construction of $\neg A$ implies the implication from A to the falsity in all the accessible knowledge states.[7] $\mathcal{M}(K_i)$ denotes the set of all K_i-models. When required for clarity, we shall use the abbreviation $v_{M^{ver}}$ to refer to a verification function in a model of $\mathcal{L}^{ver}$.

Monotonicity as for standard intuitionistic models follows from the heredity condition (proven by induction on formulas):

Lemma 1 (Monotonicity)**.** *For every $\phi \in \mathcal{P}$, if $K_i \vDash \phi$ and $K_i \leq K_j$ then $K_j \vDash \phi$.*

A standard notion of validity of formulas as verification in all states holds:

Definition 2. $\vDash^{ver} A$ *iff* $K_i \vDash A$ *for every* $K_i \in \mathbb{K}$.

A formula is satisfiable if there is a knowledge state and a verification function in it that satisfies it. A formula A is a logical consequence of a set of formulae $A_1, \dots, A_n$ if for every $M^{ver}(K_i)$ such that $K_i \vDash A_i$ for every $A_{i \in \{1,\dots,n\}}$, then $K_i \vDash A$.

2.2 Contextual Computations

The extension to the language $\mathcal{L}^{ctx}$ is obtained by introducing an appropriate notion of contextual verification, simulating truth under contents that are not directly computable in $\mathcal{L}^{ver}$, but are considered admissible.[8] A contextual verification is therefore given by a verification function as in Definition 1 for a knowledge state in which some contents are taken as valid without verification.

The first step is therefore the definition of such notion of unverified but admissible content. We will denote in the following a finite set of variables by $\mathcal{V} = \{x_1, x_2, \dots\}$:

Definition 3. *For any $K_i \in \mathbb{K}$, an informational context $\Gamma : \mathcal{V} \mapsto \mathcal{W}$ for K_i consists of a finite set of injective functions $\gamma_1, \gamma_2, \dots, \gamma_n$ such that $\gamma_i := x_i \mapsto A_i$. We then say that the truth of A_i is admissible in K_i if $K_h \nvDash \neg A_i$ for all $K_h \leq K_i$.*

By an informational context we refer therefore to a (set of) mapping(s) from variables to propositional contents; each such function is a place holders for a missing verification; by the last condition in the definition above, a formula A introduced in a context is *admissible* for a knowledge state in the sense that its negation is not validated at the current knowledge state. Notice that the construction of a context does not forbid inadmissible formulas to be selected. When

[7] One enters here the debate on the constructive treatment of negation, and in particular the largely discussed standard BHK-interpretation, according to which a proof of $\neg A$ is a function that converts each proof of A into a proof of $\perp$. See [30], [26] and the overview given in [31]. The standard treatment of constructive negation, that requires knowledge of hypothetical proofs and what a proof of an absurdity is, has been reformulated by introducing the primitive notion of *disproof* in [13], leading to the interpretation of Nelson's constructive logic with negation but without *ex-contradictione quodlibet*, see [18].

[8] In the syntactic model designed in [22] for a type-theoretical language with open assumptions, this property corresponds to the formulation of type contructions that not necessarily β-reduce by admissibility of non-contradictory constructions.

a context Γ is defined for a knowledge state K_i, we shall refer to the resulting state with contextual information by $(\Gamma)K_i$. By a function γ the inductive step is given for the extension from $\mathcal{L}^{ver}$ to $\mathcal{L}^{ctx}$ by taking the empty informational context $\Gamma = \emptyset$ and a non-empty one $\Gamma' = \{\gamma\}$. Non-empty contexts Γ, Γ' are ordered according to an inclusion relation $\subseteq$, indexed by the novel functions added by the new context: $\Gamma \subseteq_{\gamma_{n+1}} \Gamma'$ iff $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ for propositions $A_1, \ldots A_n$ and $\Gamma' = \{\Gamma \cup \gamma_{n+1}\}$ such that $\gamma_{n+1} := x_{n+1} \mapsto A_{n+1}$ for a new proposition A_{n+1} and a fresh variable x_{n+1}. Extension of contexts determines the pre-order on $\mathbb{K}$: when $(\Gamma)K_i$, $\Gamma \subseteq_{\gamma_{n+1}} \Gamma'$ and $(\Gamma')K_j$, $(\Gamma)K_i \leq^{\Gamma \cup \gamma_{n+1}} (\Gamma')K_j$.[9]

The grammar of $\mathcal{L}^{ctx}$ is given as follows:

$$\mathcal{L}^{ctx} := \phi \mid \top \mid \bot \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A$$

The notion of a model for $\mathcal{L}^{ctx}$ is formulated by modifying the previous definition of a model for $\mathcal{L}^{ver}$ with the newly defined order among knowledge states:

Definition 4. *A model for $\mathcal{L}^{ctx}$ is a tuple $M^{ctx} = \{\mathbb{K}, \leq^{\gamma}, v\}$, where $\mathbb{K}$ is a nonempty set ranging over $\{(\Gamma)K_i, (\Gamma')K_j, \ldots\}$; $\leq^{\gamma}$ is a reflexive ordering relation over members of $\mathbb{K}$ such that if $\Gamma(K_i)$ and $\Gamma \subseteq_{\gamma'} \Gamma'$, then $(\Gamma)K_i \leq^{\gamma'} (\Gamma')K_j$; v is a verification function $v : \mathbb{K} \mapsto 2^{\mathcal{W}}$.*

We shall call $M^{ctx}(K_i)$ a contextual model of knowledge state K_i if and only if all the members of K_i are true in it; that a formula A is true in $M^{ctx}(K_i)$ corresponds to the existence of a function that verifies A in $(\Gamma)K_i$. A new inductively generated satisfaction relation to evaluate formulas in $\mathcal{L}^{ctx}$ holds, denoted as $K_i \vDash^{\Gamma} A$, which reads: *"A is verified in K_i on the basis of information Γ"*. The definitional clauses for connectives of the language are given below with the extension to $\Box$ for global validity over all extensions to accessible knowledge states and $\Diamond$ for the counterpart local validity:

$C1^{ctx}$ $K_i \vDash^{\Gamma} \phi$ iff $(\phi, v((\Gamma)K_i))$;

$C2^{ctx}$ $K_i \vDash^{\Gamma} \bot$ iff $(A, v((\Gamma)K_i))$ and $K_h \vDash^{\emptyset} \neg A$, for some $K_h \leq^{\gamma} K_i$;[10]

$C3^{ctx}$ $K_i \vDash^{\Gamma} A \vee B$ iff $K_i \vDash^{\Gamma} A$ or $K_i \vDash^{\Gamma} B$;

$C4^{ctx}$ $K_i \vDash^{\Gamma} A \wedge B$ iff $K_i \vDash^{\Gamma} A$ and $K_i \vDash^{\Gamma} B$;

$C5^{ctx}$ $K_i \vDash^{\Gamma} A \supset B$ iff $K_i \vDash^{\Gamma} A$ implies $K_i \vDash^{\Gamma} B$;

$C6^{ctx}$ $K_i \vDash^{\Gamma} \Box A$ iff for all $(\Gamma')K_j \geq^{\gamma} (\Gamma)K_i$, it holds $K_j \vDash^{\Gamma \cup \gamma} A$;

$C7^{ctx}$ $K_i \vDash^{\Gamma} \Diamond A$ iff there is a $(\Gamma')K_j \geq^{\gamma} (\Gamma)K_i$ such that $K_j \vDash^{\Gamma \cup \gamma} A$.

[9] Notice that in Definition 3, the old ordering $\leq$ is referred to only in view of the necessary condition over states with *empty* contexts preceding a state with a non-empty one.

[10] The formula $K_h \vDash^{\emptyset} \neg A$ corresponds to an evaluation $(A \supset \bot, v_{M^{ver}}(K_i))$, for all $K_i \geq K_h$. To reduce it to a corresponding evaluation in M^{ctx}, we simply take $\Gamma = \emptyset$.

The clauses from $\mathcal{L}^{ver}$ are here reformulated for $\vDash^{\Gamma\cup\gamma}$, with eventually any of Γ or γ possibly empty. $C1^{ctx}$ says that ϕ is satisfied in a contextual model for K_i if there is a verification of ϕ in K_i with the informational context Γ; by $C2^{ctx}$ a knowledge state K_i with information Γ is inconsistent if the consistency requirement from Definition 3 is not satisfied; $C3^{ctx} - C4^{ctx}$ are standard compositional clauses; $C5^{ctx}$ says that $A \supset B$ is satisfied in an informational model of K_i if and only if A is verified in $(\Gamma)K_i$, then so is B. Modal formulas express the conditions for validating a formula under extensions of a context Γ: by $C6^{ctx}$, if A is verified by any non-empty extension of a (possibly empty) context, then $\Box A$ is valid (because it can be verified under any context); by $C7^{ctx}$, if A is verified by some non-empty extension of some context Γ' extending the current one, then $\Diamond A$ is valid, i.e. there is some γ on whose basis A is validated (this formulation does not force consistency of the contextual extension).

Transitivity no longer holds in general, in view of the non-monotonic nature of the contextual models; symmetry cannot be validated as it would always require a verification of an assumption (which is not always available) and an assumption on a construction (which is not always needed). In the following we refer to a frame of the language $\mathcal{L}^{ctx}$ as a structure including $\mathbb{K}$ and an accessibility relation R among models M^{ctx} of its elements. Properties of such a frame correspond to properties of the order relation $\leq^{\gamma}$ among states $\{(\Gamma)K_i, (\Gamma')K_j, \dots\} \in \mathbb{K}$. In the following, to simplify the notation, we avoid the explicit signature of contexts before knowledge states and the notation on models that identifies them as contextual ones.

Theorem 1 (Reflexivity)**.** *Every frame $\mathcal{F} : \langle \mathbb{K}, R\rangle$ for a M^{ctx} model is reflexive.*

Proof. If $\mathcal{F}$ is reflexive, it means that for every $K_i \in \mathbb{K}$, there is a R such that $M(K_i)\ R\ M(K_i)$. We consider the modal cases.

If $K_i \vDash \Box A$, then $K_i \vDash A$, standardly proven on the construction of R in the canonical model that satisfies axiom T (see e.g. [11, p.120]).

Similarly, the construction of R in the canonical model for $T_\Diamond$ means that if $K_i \vDash A$ then $K_i \vDash \Diamond A$. Now suppose that if $K_i \vDash A$ then $K_i \nvDash \Diamond A$; if so, then there is a γ such that $K_i \vDash^{\gamma} \neg\Diamond A$; then $K_i \vDash \Box\neg A$, which means that for all γ's $K_i \vdash^{\gamma} \neg A$ and therefore $(\neg A, v_{M^{ver}}(K_i))$ so that $K_i \nvDash A$, contrary to the hypothesis. $\Box$

Theorem 2 (Non-transitivity)**.** *Every frame $\mathcal{F} : \langle \mathbb{K}, R\rangle$ for a M^{ctx} model is non-transitive.*

Proof. If $\mathcal{F}$ is transitive, then for every $K_i, K_j, K_k \in \mathbb{K}$: if $M(K_i)\ R\ M(K_j)$ and $M(K_j)\ R'\ M(K_k)$, then $M(K_i)\ R''\ M(K_k)$. In other words, if $K_i \vDash^{\Gamma} A$, then *for all* γ, γ' and K_j, K_k such that $K_i \leq^{\gamma} K_j$ and $K_j \leq^{\gamma'} K_k$, it holds $K_k \vDash^{\Gamma\cup\gamma\cup\gamma'} A$. The latter, means actually that $K_i \vDash^{\Gamma} \Box A$. Let now M be a model based on $\mathcal{F}$ such that $K_i \vDash^{\Gamma} A$, and for some $K_i \leq^{\gamma} K_j$ it holds $K_j \vDash^{\Gamma\cup\gamma} \neg A$: this satisfies $K_k \vDash^{\Gamma} \Diamond\neg A$ for some $K_j \leq^{\gamma} K_k$. Such M is not transitive, and neither can $\mathcal{F}$ be.

$\Box$

Validity for $\mathcal{L}^{ctx}$ is contextually restricted:

Definition 5. $\models^{\Gamma} A$ *if and only if for all* $K_i \in \mathbb{K}$, $K_i \models^{\Gamma} A$.

The *Monotonicity Lemma*,[11] as in Definition 1 is not easily extended to $\mathcal{L}^{ctx}$. Hereditariness for modal frames is usually obtained either by assuming the standard definition of the possibility operator, and requiring the condition that if $K \models \Diamond A$ and $K \leq K'$, then $K' \models \Diamond A$; or by imposing universal quantification over the pre-order successors in the definitional clause of $\Diamond$.[12] In $\mathcal{L}^{ctx}$, we restrict the first clause and explicitely disregard the second solution: hereditariness of knowledge states holds only with the additional requirement that consistency holds at the current state and it is preserved by every informational function extending contexts:[13]

Lemma 2 (Contextual Monotonicity for $\mathcal{L}^{ctx}$)**.** *If* $K_i \models^{\Gamma} \top$, *and* $\forall K_j \geq^{\gamma} K_i$ *holds* $K_j \models^{\Gamma \cup \Gamma'} \top$, *if* $K_i \models^{\Gamma} \phi$ *then* $K_j \models^{\Gamma'} \phi$.

A generalized consequence relation $\models^{\Gamma}$ can now be defined for global and local assumptions (see e.g. [7]) in a unified frame. We shall call *global* a context Γ that contains all formulae that are themselves valid in *any* extension of a (possibly empty) context:

Definition 6. $\Box\Gamma$ *is called a global context for* K_i *iff for all* $\gamma_i := x_i \mapsto A_i$ *in* Γ *and all* $\Gamma' \supseteq \Gamma$ *it holds* $\models^{\Gamma'} A_i$.

We shall call *local* a context Γ that contains some (but not necessarily all) formulas verifiable in the extension of *some* (possibly empty or global) context:

Definition 7. $\Diamond\Gamma$ *is called a local context for* K_i *iff for some* $\gamma_i := x_i \mapsto A_i$ *in* Γ, *there is a* $\Gamma' \supseteq \Gamma$ *such that* $\models^{\Gamma'} A_i$.

Let us consider our language $\mathcal{L}^{ctx}$ *restricted* to the set of formulas $\mathcal{L}^{glob}$: $\{A \mid \models^{\Box\Gamma} A\}$; $\models_{\mathcal{L}^{glob}}$ will be therefore the consequence relation construed by the satisfaction clauses of $\mathcal{L}^{ctx}$ with only global contexts;

$C1^{glob}$ $K_i \models^{\Box\Gamma} \phi$ iff for every γ, it holds $K_i \models^{\Gamma \cup \gamma} \phi$;

$C2^{glob}$ $K_i \models^{\Box\Gamma} \top$;

$C3^{glob}$ $K_i \models^{\Box\Gamma} A \vee B$ iff $K_i \models^{\Box\Gamma} A$ or $K_i \models^{\Box\Gamma} B$;

$C4^{glob}$ $K_i \models^{\Box\Gamma} A \wedge B$ iff $K_i \models^{\Box\Gamma} A$ and $K_i \models^{\Box\Gamma} B$;

$C5^{glob}$ $K_i \models^{\Box\Gamma} A \supset B$ iff $K_i \models^{\Box\Gamma} A$ implies $K_i \models^{\Box\Gamma} B$.

[11] See [25], p.22.

[12] In the corresponding syntactic translation, the latter solution has the well-known effect of eliminating the axiom for distribution of $\Diamond$ over $\vee$ and the one for the impossibility of absurdum. This is the way to admit inconsistency in [16].

[13] As by clause $C2^{ctx}$, not every K_i is necessarily consistent in view of a context Γ.

We denote by $\models^{\Box\Gamma} A$ a semantic consequence of every K_i with global context $\Box\Gamma$. By definition, the form of the semantic consequence of a global context will be $\Box A$.[14] Notice that the condition on validity from a global context is satisfied also by a formula satisfied under the empty context in $\mathcal{L}^{ctx}$. An axiomatization of $\mathcal{L}^{glob}$ corresponds to that of $S4$, with distribution of $\Box$ over implication, axiom T, iteration and the Necessitation Rule. A model $\mathcal{M}^{glob}$ is in turn a model of $\mathcal{L}^{ctx}$ where all formulas of the knowledge state considered are formulas of $\mathcal{L}^{glob}$ and they are all true in it.

Definition 8. *A model for global contextual knowledge is a tuple* $M^{glob} = \{\mathbb{K}, \leq^{\Box\gamma}, v\}$*, where* $\mathbb{K}$ *is a nonempty set ranging over* $\{(\Box\Gamma)K_i, (\Box\Gamma')K_j, \dots\}$*;* $\leq^{\Box\gamma}$ *is a reflexive and transitive ordering relation over members of* $\mathbb{K}$ *such that if* $\Box\Gamma \subseteq_{\gamma'} \Box\Gamma'$*, then* $(\Box\Gamma)K_i \leq^{\gamma'} (\Box\Gamma')K_j$*;* v *is a verification function* $v : \mathbb{K} \mapsto 2^{\mathcal{W}}$*.*

Correspondingly, in the frame $\mathcal{F}$ the accessibility relation R on $\mathbb{K}$ will satisfy reflexivity and transitivity.

Theorem 3. *For every* $A \in \mathcal{W}$*,* $\models^{\Box\Gamma} A$ *iff* $\models_{S4} A$

Proof. By induction on A, relying on the fact that A is intuitionistically derivable if for every propositional atom ϕ in A it holds $\models_{S4} \Box\phi$ and on the obvious inclusion of $\mathcal{L}^{glob}$ in a language of modal intuitionistic logic, given that it corresponds to $\mathcal{L}^{ver}$ extended by an intuitionistically admissible $\Box$-operator. The only interesting step is in the contextual construction:

- Left-Right: $\models^{\Box\Gamma} A \Rightarrow \models_{S4} A$:
  - for atomic $\models^{\emptyset} A$, by Definition 6 it follows $\models^{\Box\Gamma} \Box A$ and so $\models_{S4} A$;
  - construction by connectives is standardly preserved;
  - for $\models^{\Box\Gamma} A$, infer $\models \bigwedge\Gamma \supset A$; then $\Box\phi$ for every $\phi \in \bigwedge\Gamma \supset A$ by Definition 6; so $\models_{S4} A$.
- Right-Left: $\models_{S4} A \Rightarrow \models^{\Box\Gamma} A$.
  - If valid in $S4$, every propositional atom $\phi \in A$ can be prefixed by a $\Box$; then $\models^{\emptyset} A$ or $\models^{\Box\Gamma} A$.

$\Box$

The corresponding local version of semantic consequence is formulated as follows:

Definition 9. *For every* $A \in \mathcal{W}$*,* $K_i \models^{\Diamond\Gamma} A$ *iff for some* γ *it holds* $K_i \models^{\Gamma\cup\gamma} A$*. We denote by* $\models^{\Diamond\Gamma} A$ *a semantic consequence of every* K_i *with local context* $\Diamond\Gamma$*.*

[14] From a syntactic point of view, this interpretation deals with the necessitation of undischarged assumptions; by consequence from a global context, necessitation is preserved only for assumptions *closed under substitution*. See [2] for the corresponding formulation of the $\Box$-Introduction Rule in Natural Deduction.

AXIOMS	
Axioms of IPL	
$K_\Box$	$\Box(A \supset B) \supset (\Box A \supset \Box B)$
$K_\Diamond$	$\Diamond(A \supset B) \supset (\Diamond A \supset \Diamond B)$
$T_\Box$	$\Box A \supset A$
$T_\Diamond$	$A \supset \Diamond A$
RULES	
Modus Ponens	
Uniform Substitution	
Nec^{glob}	$\vdash^{\Box\Gamma} A \Rightarrow\ \Box\Gamma \vdash \Box A$
$Weak^{glob}$	$\Box\Gamma \vdash A \Rightarrow\ \Box\Gamma, \Gamma' \vdash A$

Figure 1: The system $cKT_{\Box\Diamond}$

Consequence from a local context $\vDash^{\Diamond\Gamma}$ is required to preserve the $\Diamond$ operator.[15]

3 The Calculus $cKT_{\Box\Diamond}$

In the following we design the calculus $cKT_{\Box\Diamond}$, sound with respect to the semantic local/global consequence relation from the previous section. It amounts to a fragment of the standard constructive $S4$.[16] $cKT_{\Box\Diamond}$ includes the possibility version of axiom T, which expresses the fact that a derivable formula is possibly true; standardly, the necessity version of this axiom says that a necessarily true formula is derivable. The Necessitation rule ($\vdash A \Rightarrow \vdash \Box A$) is not admissible for derivability relations instantiated by a non-empty local context $\Diamond\Gamma$, i.e. where our language is generalized to accommodate the locally valid derivability relation. On the other hand, Necessitation still holds under a global context, by a rule we call Nec^{glob}; also weakening holds in the same form by a rule called $Weak^{glob}$. These rules also have appropriate counterparts with $\Gamma = \{\emptyset\}$. Possibility is unrestricted, so that the axiom $\neg\Diamond\bot$ is no longer validated; it is possible to define the evaluation function for $\Diamond A$ for any A, except when A is valid in a global state in which $\Box(\neg A \supset \bot)$ is validated. tThere is therefore a conceptual priority of consistent states over those where inconsistency are admissible so that there must be at least one consistent state, and there can be one or more inconsistent ones. Figure 1 presents the basic axiomatization of $cKT_{\Box\Diamond}$.

Derivability from a global context for $cKT_{\Box\Diamond}$ is easily defined:

Theorem 4 (Derivability from a global context)**.** $\Box\Gamma \vdash \Box A$, *iff* $\emptyset \vdash \bigwedge \Gamma \supset A$.

[15] See again [2] for the equivalent remark on the $\Diamond$-Elimination Rule for Natural Deduction, with the additional requirement that the extended context be empty or global in order to reflect closure under substitution.

[16] See [2] for the language of $CS4$. A natural deduction formulation of our calculus corresponds to the set of rules for the type-theoretical system introduced in [22], which includes appropriate introduction and elimination rules for both $\Box$ and $\Diamond$.

Proof. By induction on the construction of any formula $A \in \Gamma$:

- Left-Right: By $T_\Box$ and appropriate applications of $\supset$ axioms;
- Right-Left: By Nec^{Glob} and appropriate applications of $\supset$ axioms.

$\Box$

This notion of derivability has a counterpart in a necessitation version of the Deduction Theorem:[17]

Theorem 5 ((Global) Deduction Theorem)**.** *If* $\Gamma, \Box A \vdash B$ *then* $\Gamma \vdash \Box(A \supset B)$.

Proof. By induction.

1. if $B \equiv A$:
   - $\Box(A \supset A)$ by Axiom $A \supset A$ and Nec^{glob};
   - $\Gamma \vdash \Box(A \supset B)$ by $Weak^{glob}$ and Sub.
2. if $B \in \Gamma$:
   - $\Gamma \vdash B$;
   - $\vdash B \supset (A \supset B)$ (Axiom);
   - $\vdash \Box(B \supset (A \supset B))$ by Nec^{glob};
   - $\vdash \Box B \supset \Box(A \supset B)$ by $K_\Box$;
   - $\Gamma \vdash \Box(A \supset B)$ by MP and $Weak^{glob}$.
3. if B is an axiom: as above;
4. if the last rule is Nec^{glob}:
   - if $\vdash^{\Box A} B$, then $\Box A \vdash \Box B$ by Nec^{glob};
   - $A \supset (B \supset A)$ (Axiom);
   - $\vdash \Box(A \supset B)$ by MP.
5. if the last rule is MP and none of the premises is obtained by Nec^{glob}:
   - A is among the assumptions in Δ for $\Gamma = \{\Delta, \Delta'\}$

$$\frac{\Delta, A \vdash C \qquad \Delta' \vdash C \supset B}{\Delta, \Delta', A \vdash B}$$

   - $\Delta \vdash \Box(A \supset C)$ by hypothesis step;
   - $(B \supset C) \supset ((A \supset B) \supset (A \supset C))$ (Axiom);

[17] For the validity of the Deduction Theorem in Modal Logic see [7] and [10]. The present version differs from the usual version with global assumptions as the interpretation of the $\Box$ operator is intended as expressing assertion conditions.

- $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$ (Axiom);
- $\Box(A \supset C) \supset (\Box(C \supset B) \supset \Box(A \supset B))$ by MP;
- $\Delta \vdash \Box(C \supset B) \supset \Box(A \supset B)$ by MP;
- $\Gamma \vdash \Box(A \supset B)$ again by MP and $Weak^{glob}$.

- A is among the assumptions in Δ' for $\Gamma = \{\Delta, \Delta'\}$

$$\frac{\Delta \vdash C \qquad \Delta', A \vdash C \supset B}{\Delta, \Delta', A \vdash B}$$

- $\Delta' \vdash \Box(A \supset (C \supset B))$ by hypothesis step;
- $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$ (Axiom);
- $\Delta' \vdash \Box(C \supset (A \supset B))$ by MP;
- $\Gamma \vdash \Box(A \supset B)$ by $K_{\Box}$.

6. The last rule is MP and one of the premises is obtained by Nec^{glob}:

- Case 1:

$$\dfrac{\dfrac{\dfrac{\vdash C}{\Delta, A \vdash \Box C}\,Nec^{glob}, Weak^{glob} \qquad \Delta' \vdash \Box C \supset B}{\Delta, \Delta' \vdash B} \qquad \dfrac{\vdash B \supset (A \supset B)}{\vdash \Box B \supset (A \supset B)}\,Nec^{glob}}{\Gamma \vdash \Box(A \supset B)}$$

- Case 2:

$$\dfrac{\dfrac{\dfrac{\vdash C}{\Delta \vdash \Box C}\,Nec^{glob}, Weak^{glob} \qquad \Delta', A \vdash \Box C \supset B}{\Gamma, A \vdash B} \qquad \dfrac{\Delta' \vdash \Box A \supset \Box(C \supset B)}{\Delta' \vdash \Box C \supset \Box(A \supset B)}\,Nec^{glob}}{\Gamma \vdash \Box(A \supset B)}$$

□

Local derivability takes the following form:

Theorem 6 (Derivability from a local context)**.** *$\Diamond\Gamma \vdash \Diamond A$, iff there exists some finite $\Gamma' \subseteq \Gamma$ such that $\Gamma \vdash \bigwedge \Gamma' \supset A$.*

Proof. By induction, assuming by definition that for at least one $A \in \Gamma$ it does not hold $\vdash^{\emptyset} A$. Then proceed by contradiction on the result of Theorem 4 applying at least once $T_{\Diamond}$. □

The characterization of $cKT_{\Box\Diamond}$ with respect to $\mathcal{L}^{cc}$ is provided by the following theorem:[18]

[18] In general, as for the theorem here stated, the consequence relation sign should always be indexed as $\vDash_{\mathcal{L}^{cc}}$, with an appropriate characterization of the nature of context used, as in $\vDash^{\emptyset}$ or $\vDash^{\Diamond\Gamma}$. In order to simplify the notation, where possible we will in general skip the former index.

Theorem 7. *For every set of formulae* Γ *and formula* A*, it holds* $\Gamma \vdash_{cKT_{\Box,\Diamond}} A$ *iff either* $\models^{\emptyset} \bigwedge \Gamma \supset A$*, or* $\models^{\Box\Gamma} A$*, or* $\models^{\Diamond\Gamma} A$*.*

Proof. Soundness.

1. $\Gamma \vdash_{cKT_{\Box,\Diamond}} A$ implies $\models \bigwedge \Gamma \supset A$ is not problematic at all; the proof simply goes on the length of the derivations starting from $\Gamma = \{\emptyset\}$, where it concerns only the preservation of validity of axioms of IPL and the related inference rules; for Γ non-empty it shows a reduction to the implication $\supset$ from the conjunction of formulae in Γ.

2. For $\Gamma \vdash_{cKT_{\Box,\Diamond}} A$ implies $\models^{\Box\Gamma} A$ one has only to show that modal axioms for $\Box$ together with Nec^{glob} preserve validity. Axioms $K_\Box$ and $T_\Box$ are valid because of the frame condition on the transitivity and reflexivity of $\leq^\gamma$ with global assumptions. Nec^{glob} allows the further reduction to boxed formulas.

3. $\Gamma \vdash_{cKT_{\Box,\Diamond}} A$ implies $\models^{\Diamond\Gamma} A$ is the non-trivial part of the proof. It holds by the cases applying derivability from a local context given in Theorem 6, by which one makes sure that $\Diamond\Gamma \vdash_{cKT_{\Box,\Diamond}} \Diamond A$ means that $\exists\Gamma'$ such that $\Gamma \vdash_{cKT_{\Box,\Diamond}} \bigwedge \Gamma' \supset A$. If Γ cannot be exhausted, axiom $T_\Diamond$ is invoked: then for the corresponding model it holds $K_i \models^{\Gamma} \Diamond A$. The reduction to case 2. happens by application of Nec^{glob} on *every* subset $\Gamma' \subseteq \Gamma$, up to exhaustion of extensions $\models^{\Diamond\Gamma,\emptyset} \Diamond A$. For all non modal formulas in Γ, a straightforward reduction to case 1. applies.

Completeness. The proof for the cases of non-modal formulas is straightforward and can happen entirely with respect to M^{ver} models with a standard model existence theorem. The case of modal formulae can be distinguished for derivability from boxed formulas (or without assumptions) and derivability from locally valid assumptions ($\Diamond$-prefixed formulas). The former case is straightforward by the use of the following lemma:[19]

Lemma 3. *For any set of formulas* Γ *and global context* $\Box\Gamma$

1. *If* $\models^{\Box\Gamma \subseteq_\gamma \Box\Gamma'} A$ *then* $\models^{\Box(\Gamma\cup\Gamma')} \Box A$*;*

2. *If* $\Box(\Gamma \cup \Gamma') \vdash_{cKT_{\Box\Diamond}} \Box A$ *then* $\Gamma, \Gamma' \vdash_{cKT_{\Box\Diamond}} A$*.*

Proof. This proof is entirely similar to the one for CK,[20] adapted for the $\Box$ operator and with respect to the contextual extension function and the use of axiom $T_\Box$ and Nec^{glob}.

1. Assume $\models^{\Box\Gamma \subseteq_\gamma \Box\Gamma'} A$. Given a model M in which $K_i \models \Box\Gamma$ and $K_i \models \Box\Gamma'$ hold, we must show that $K_i \models^{\Gamma,\Gamma'} \Box A$. We then need to start from the generated submodel of M at K_i that satisfies $\Box\Gamma$; this model will have all

[19]See [16], Lemma 1.
[20]See [16].

the truths derivable from Γ and in particular $K_i \vDash^{\Box\Gamma} \Gamma'$, so that by Nec^{glob} the model will also satisfy $\Box\Gamma'$. Then by definition of the $\Box$ operator, these truths are valid at all states reachable with finite extensions from the model generated at K_i, so from our assumption we can infer $K_i \vDash \Box A$.

2. Assume $\Box(\Gamma \cup \Gamma') \vdash_{cKT_{\Box,\Diamond}} \Box A$. Then, by Theorem 4, it follows that for any $\Delta \subset \Gamma$, $\emptyset \vdash_{cKT_{\Box,\Diamond}} \bigwedge \Delta \supset A$. This implies $\Delta \vdash_{cKT_{\Box,\Diamond}} A$ and so $\Gamma \vdash_{cKT_{\Box,\Diamond}} A$.

$\Box$

Using this Lemma, an appropriate Model Existence Theorem becomes obvious for all cases including $\Box$. In the following we shall focus on how to obtain completeness for the more interesting case $\vDash^{\Box\Gamma \subseteq_\gamma \Diamond\Gamma'} \Diamond A$ implies $\Gamma, \Gamma' \vdash_{cKT_{\Box,\Diamond}} A$. To this aim, we shall pursue the same strategy by using a counterpart of Lemma 3:

Lemma 4. *For any set of formulae Γ, global context $\Box\Gamma$ and local context $\Diamond\Gamma$*

1. *If $\vDash^{\Box\Gamma \subseteq_\gamma \Diamond\Gamma'} A$ then $\vDash^{\Diamond(\Gamma\cup\Gamma')} \Diamond A$;*
2. *If $\Diamond(\Gamma \cup \Gamma') \vdash_{cKT_{\Box,\Diamond}} \Diamond A$ then $\Gamma, \Gamma' \vdash_{cKT_{\Box,\Diamond}} A$.*

Proof. Adapted from the proof of Lemma 3:

1. Assume $\vDash^{\Box\Gamma \subseteq_\gamma \Diamond\Gamma'} A$. Given a model M in which $K_i \vDash \Box\Gamma$ and $K_i \vDash \Diamond\Gamma'$ hold, we must show that $K_i \vDash^{\Box\Gamma, \Diamond\Gamma'} \Diamond A$. We then need to start from the generated submodel of M at K_i that satisfies both $\Box\Gamma$ and $\Diamond\Gamma'$, which restricts the set of *all* states originally reachable from K_i by functions $\leq^\Gamma$ to those reachable at K_j by functions $\leq^\gamma \Gamma'$: whereas for any such function γ it is the case that $K_i \vDash^\gamma \Gamma$, it is not the case for all γ that $K_i \vDash^\gamma \Gamma'$. But then from our assumption not for all γ holds that $K_i \vDash^\gamma A$, and hence $K_i \vDash^{\Diamond(\Gamma\cup\Gamma')} \Diamond A$ by definition of the $\Diamond$ operator.

2. Assume $\Diamond(\Gamma \cup \Gamma') \vdash_{cKT_{\Box,\Diamond}} \Diamond A$. Then, by Theorem 6, it follows there are $\Delta \subset \Gamma$, $\Delta' \subset \Gamma'$ *and* $\phi \in \bigcup\{\Gamma, \Gamma'\}$ such that $\phi \vdash_{cKT_{\Box,\Diamond}} \bigwedge \Delta \wedge \bigwedge \Delta' \supset A$. This implies $\phi, \Gamma, \Gamma' \vdash_{cKT_{\Box,\Diamond}} A$ and so $\Gamma, \Gamma' \vdash_{cKT_{\Box,\Diamond}} A$.

$\Box$

Now an appropriate formulation of a finite model construction for this modal case will be of the form:

Theorem 8 (Model Existence for $\Diamond$)**.** *If $\Diamond\Gamma \nvdash_{cKT_{\Box,\Diamond}} \Diamond A$, then there is a model $M^{ctx} = \{\mathbb{K}, \leq^\gamma, v\}$ and a state $K_i \in \mathbb{K}$ such that $K_i \vDash \Diamond\Gamma$ and $K_i \nvDash^\Gamma \Diamond A$.*

Proof. The canonical model M that falsifies $\Diamond A$ under Γ necessarily verifies $K_j \vDash^{\Gamma \leq \gamma} \neg A$, for any function γ and any $K_j \geq^\gamma K_i$. Hence, $K_j \vDash^\Gamma \Box\neg A$, which makes $\neg A$ valid at any state reachable from K_i. Now the construction of finite model appropriate for $cKT_{\Box,\Diamond}$ including the canonical model for $\Diamond$ will have the following cases:

1. Assume $\Diamond A \in \Gamma$. Then for all $\Gamma \subseteq_{\gamma} \Gamma'$ it holds $\Diamond A \in \Gamma'$. This makes $\Box\Gamma \vdash_{cKT_{\Box,\Diamond}} \Diamond A$ consistent. If not, there is a γ' and Γ' s.t. $\Box\Gamma \subseteq_{\gamma'} \Gamma' \nvdash_{cKT_{\Box,\Diamond}} \Diamond A$ and so $v, K_j \models^{\Gamma\cup\Gamma'} \Box\neg A$ for some K_i. But then necessarily $\Gamma \not\models \Diamond A$ against the hypothesis.

2. Assume $\Diamond A \notin \Gamma$. Then for some $\Gamma \subseteq_{\gamma} \Gamma'$ it holds $\Diamond A \in \Gamma'$ and $\Box\Gamma, \Diamond A$ is consistent, since if $\Box\Gamma \vdash_{cKT_{\Box,\Diamond}} \Diamond A$ is admissible, by closure $\Diamond A \in \Gamma$ becomes valid, contradicting the hypothesis. Therefore, there must be a maximal Γ such that $\Box\Gamma, \Diamond A \models \top$: by definition, Γ, A and for some γ' such that Γ, γ' it holds $\Gamma \not\models^{\gamma'} A$.

$\Box$

Now the proof of Completeness reduces to the following argument: Suppose $\Box\Gamma, \Diamond\Gamma' \nvdash_{cKT_{\Box,\Diamond}} \Diamond A$; by Theorem 8 it follows $\models^{\emptyset} \Diamond(\Gamma \cup \Gamma')$ and $\not\models^{\Diamond(\Gamma\cup\Gamma')} \Diamond A$.
$\Box$

Though it is possible to prove model existence also with respect to the extension induced by $\mathcal{L}^{ctx}$, the frames for its models remain intransitive, which induces a basic restriction on the finite model property.

Theorem 9 (Finite Model)**.** *$\models A$ iff $M \models A$ for all finite M^{glob} models.*

Proof. To show that the finite model property can be obtained only by reduction to the $\mathcal{L}^{glob}$ part of the language, it is enough to show the following:

- for $KT_{\Box}$ there is a standard finite model property;
- by admitting the intransitive relation over models defined by axiom $T_{\Diamond}$, a basic condition is lost for a finite filtration model.

Finiteness can be given therefore only for a maximal extension of the model, which reduces to a model of $\mathcal{L}^{glob}$. We refer in the following to the filtration technique presented in [9, §3.4], to transform a Kripke structure M in which a formula A is true, to a finite counterpart of the structure uniformly with respect to A. The same technique is applied to prove the finite model property and decidability for CK in [16], which we exploit here to our purposes.

Let us consider some $M^{ctx} = \{\mathbb{K}, \leq^{\gamma}, v\}$ and the set $T(K_i)$ of subformulas of A valid at some K_i and the set $F(K_i)$ of subformulas of A refutable in some successor state of K_i. The two finite sets $T(K_i)$ and $F(K_i)$ characterize the state K_i in M. By definition of our accessibility relation, it holds $\Box T(K_i) \supseteq T(K_j)$, where $\Box T(K_i)$ refers to the necessitated version of all formulas in $T(K_i)$. Let us notice that it is not the case that $F(K_i) \cap T(K_j) = \emptyset$: i.e., the set of subformulas of A refuted at K_i might intersect the set of truth at K_j accessible from the former. This can happen in particular if an instance function $\leq^{\gamma}$ makes an extension Γ such that $K_i \not\models A$ and $K_j \models^{\Gamma} A$.

Let us now define a filtration of M with respect to the finite set of all pairs $T(K_i), F(K_i)$ of A in M, call it $\Psi(A)$: $M_{|A} = \{\Psi(A), \leq^{\gamma}_{|A}, v_{|A}\}$. We can now show that the filtration model $M_{|A}$ does not satisfy the following condition: $K_i \vDash \phi$ if and only if $T(K_i), F(K_i) \vDash_{|A} \phi$ when $\phi \subseteq A$ and ϕ is of the form $\Diamond B$. This is evident by the mentioned fact that for $\Psi(A)$ it holds $F(K_i) \cap T(K_j) \neq \emptyset$ for some $\phi \subseteq A$ of the form $\Diamond B$, which implies that the filtration model does not need to provide the same set of formulas neither with respect to the ordering $\leq^{\gamma}_{|A}$ nor for the relation $\leq^{\gamma}_{|A}$ of $\mathcal{L}^{ctx}$. Hence, it is not the case that for all formulas $\Diamond B \subseteq A$ in $T(K_i) \in \mathcal{L}^{ctx}$, it holds $K_i \vDash A$ iff $T(K_i), F(K_i) \vDash_{|A} \Diamond B$.

But for the model existence theorem 8, there must be a maximal Γ such that $\Box\Gamma, \Diamond A \vDash \top$ and for some γ' such that $\Gamma \leq \gamma'$ it holds $\Gamma \nvDash^{\gamma'} A$. So we can pick up the filtration model up to such Γ, instantiate its $\Box\Gamma$ counterpart and apply Nec^{glob}. This will be a model of $KT_{\Box}$ and so a M^{glob} model. $\Box$

Theorem 10 (Decidability)**.** *$\mathcal{L}^{glob}$ is the language of the decidable fragment of the theory $cKT_{\Box,\Diamond}$ whose class of models is reflexive and transitive.*

Proof. Immediate from soundness and completeness properties expressed by Theorem 7 and the restricted finite model property of Theorem 9. $\Box$

3.1 Translation to the two-variable monadic fragment

The decidability result shown above can be reduced to standard decidability results for intuitionistic modal logics. In fact, it is well-known from [8] that given the smallest set that contains all first-order atoms and is closed under boolean connectives, formulas containing at most two free or bound variables and quantifiers, also known as the two-variable guarded fragment, when adding transitive relations, it becomes undecidable, while the translation of (multi-)modal propositional logics $K4, S4, S5$ to the same fragment of first-order logic is decidable. In [1], these results are applied to intuitionistic modal logic: it shows that the translation of the standard interpretation of modalities holds for the same guarded fragment and that such a translation is decidable for a logic that adds definable closure conditions on the accessibility relation of models. The translation proves, based on the result given in [8], decidability for the class of models with reflexive transitive guards.

Theorem 10 establishes the same result for the class of transitive and reflexive models, based on the restricted model-existence for $\mathcal{L}^{glob}$. In order to explain in more detail how the extension to our $\Diamond$ operator, and thus the inclusion of $\mathcal{L}^{ctx}$ models, induces indecidability over the reflexive, transitive and symmetryc models, we start from presenting the translation of our satisfiability clauses to the two-variable guarded fragment of first-order logic without equality. In the standard translation, a first order formula $t_i\phi$ (with i in one of the two translation variables x, y) contains the free variable i which represents the state at which ϕ is evaluated in the model. It can be proven that for any intuitionistic modal formula A and model M in the class of intuitionistic modal models, if a formula is valid in a model at a certain point in the appropriate structure, then the corresponding first-order translation will be valid in the same model. Hence the satisfiability problem for the two-variable guarded fragment and for intuitionistic modal logic is decidable over the same class of models. Decidability for intutionistic modal logics with no conditions follows from decidability of the monotone two-variables guarded fragment with reflexivity and transitivity and the expressibility in the same fragment of the upward persistency for propositional variables occuring in formulas. Decidability of the guarded fragment with transitive guards is proven in [29] and for the two variable case in [12]. Decidability of reflexivity-and-transitivity as one closure condition on relations in a formula obtained by translation of an intuitionistic modal formula into the guarded two-variable fragment of first-order logic is proven in [1].

In our translation, we set co-inductively (we skip the obvious inductive definition on the atom ϕ) the evaluation of formulas *conditionally* on the formula representing our contextual information in Γ. We indicate the contextually valid formula as $[\phi]\psi$ and obtain an appropriate translation for ϕ and ψ with the same variable. (In the following $\sim$ and $\supset$ are classical connectives).

$$t_x\phi := \phi(x)$$

$$t_x\neg A := \forall y(xRy(t_yA \to \bot))$$

$$t_x[\phi] :=\sim t_x\neg\phi$$

$$t_x[\phi]\psi := t_x\phi \supset \psi(x)$$

$$t_x[A']\neg A := t_xA' \supset \forall y(xRy(t_yA \to \bot))$$

$$t_x[A']A \wedge B := t_xA' \supset (t_xA \wedge t_xB)$$

$$t_x[A']A \vee B := t_xA' \supset (t_xA \vee t_xB)$$

$$t_x[A']A \to B := t_xA' \supset (t_x\neg A \vee t_xB)$$

$$t_x[A']\Box A := t_xA' \supset \forall y(xRy \supset t_yA)$$

$$t_x[A']\Diamond A := t_xA' \supset \exists y(xRy \wedge t_yA)$$

We have seen in Section 2.2 how the frames defined over the knowledge set and the accessibility relation among those states are reflexive but intransitive, by which one means that $\neg\forall x, y, z(xRy \wedge yRz \to xRz)$. In the translation, transitivity means that if $t_x\phi \supset t_x\psi$ and $t_x\psi \supset t_x\xi$, then $t_x\phi \supset \xi$; now take ϕ to be of the form $[\phi]\Diamond\psi$; then the first clause in the translation above requires that $\exists y(xRy \wedge t_y\psi)$; given another relation $\exists y(xRz \wedge t_z\neg\psi)$, transitivity is falsified.

4 Conclusions

The system introduced via the semantics $\mathcal{L}^{cc}$ and the axiomatic setting of $cKT_{\Box\Diamond}$ is inspired by applications of logics for modeling knowledge processes in the context of exchange of *unverified* or *uncertain* information. A multi-modal type system inspired by similar principles, with signatures on verification functions for formalizing trusted communications of uncertain information is presented in [23]. A different application is given by programming languages that use contextual verification methods in distributed and staged computation, see [24].

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