Learning concepts through the history of mathematics. The case of symbolic algebra.

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Introduction

The adolescent's notion of rationality often encompasses the epistemological view of mathematics as knowledge which offers absolute certainty. He probably has heard of a geometry in which the parallel postulate does not hold, but most likely believes that Euclidian geometry is the "real one". We can assume that he is not familiar with Gödel's theorems and undecidability. It is further unlikely that he has been taught about the existence of inconsistent arithmetic that performs finite calculations as correct as traditional arithmetic. These findings provide strong arguments against the view that mathematics offers absolute truth. The static and unalterable mode of presentation of concepts in the mathematics curriculum, rather than lack of knowledge, contributes to this misconception. Mathematical concepts, even the most elementary ones, have changed completely and repeatedly over time. Major contributions to the development of mathematics have been possible only because of significant revisions and expansions of the scope and contents of the objects of mathematics. Yet, we do not find this reflected in class room teaching. While the room for integrating philosophy in mathematics education is very limited, an emphasis on the understanding of mathematical concepts is a necessary condition for a philosophical discourse about mathematics. The conceptual history of mathematics provides ample material for such focus and leads to a better understanding of mathematics and our knowledge of mathematics. I will argue for the integration of the history of mathematics within the mathematics curriculum, as a way to teach students about the evolution and context-dependency of human knowledge. Such a view agrees with the contextual approach to rationality as proposed by Batens (2004). As a prime example, I will treat the development of the concept of a symbolic equation before the seventeenth century. In line with Lakatos (1976) and Kitcher (1984) my example is motivated by the epistemological relevance of the history of mathematics.

Living with inconsistencies

When asked for an example of an absolute truth, a student might likely answer "one plus one equals two". This is a grateful example to expand on. One plus one equals two in a current axiomatization of arithmetic, and is therefore true *with respect to that theory*. However, it is rather easy to tailor the axiomatization in order to undermine the truth value of the given statement. Adapting the Peano axioms[1] leading to one being the successor of one, would yield the example false in the new theory. Given that 'one plus one equals two' is true in one theory and not in another, refutes the

[1] The first axiomatization of arithmetic was given by Giuseppe Peano in a Latin publication of 1889, *Arithmetices principia, nova methodo exposita*. For an annotated English translation see van Heijenoort 1967, 83 – 97.

example as an absolute truth. The student might object that changing the rules of arithmetic would lead to complete anarchy in society. The more intelligent student might notice that changing the Peano axioms in the given way would lead to an inconsistent theory and that anything can be derived from inconsistencies. Let us look at these objections. The point that changing the truth value of the given example makes no sense might be true, for now. However, there can be reasons for changing the axioms of arithmetic. Van Bendegem (1994) did develop an inconsistent arithmetic by changing the Peano axioms so that there exist one number which is the successor of itself. His reason for doing so is to demonstrate the feasibility of a strict finite arithmetic. The fifth Peano axiom states that if equality applies for $x = y$ then, x and y are the same number. This is the axiom that is tweaked by Van Bendegem so that starting from some number n, all its successors will be equal to n. If we take n to be one, then in this newly defined arithmetic $1 + 1 = 1$. However, that would be a trivial arithmetic which is not the intention of this entreprise. Rather than using one, the number n can be any number you like. Given a sufficiently large n, all operations of arithmetic behave the same way, as long as this number n is not reached during calculations. Now, a problem arrives when we reach n. The statement $n = n + 1$ is thus both true and false at the same time. This makes the new arithmetic inconsistent. In classical logic you have the rule *ex falso quodlibet* (EFQ) which states that $p \wedge \neg p \rightarrow q$, or from an inconsistency you can derive anything. This would render the arithmetic trivial within classical logic (CL). However, several paraconsistent logics now exist that do not have this problem, as well as inconsistency-adaptive logics, developed at the Center of Logic and Philosophy of Science (Batens, 2001). Van Bendegem used the three-valued paraconsistent logic PL from Priest (1987) in which EFQ does not hold. With this underlying logic he proved that if A is a valid statement in classical elementary number theory, then A is also valid in an elementary numbers theory based on a finite model. Gödel proved that every consistent formal theory, which is rich enough to model arithmetic, will contain true statements which cannot be proved within that theory. In other words, every consistent formal theory is incomplete. Giving up consistency, this new arithmetic, based on a finite model, has the advantage of being complete.

There remains the objection of anarchy. What would happen if some people decided to change the rules of arithmetic? Would our accounting and wage calculation programs become unreliable when working with inconsistent arithmetic? In some sense we already use this finite and inconsistent arithmetic in computer programs. An unsigned integer in a programming language such as C is represented by a 32 or 64-bit data structure, depending on the underlying hardware. Our inconsistent number n here becomes $2^{32} - 1$ or $2^{64} - 1$, while its successor is 0. Usually compilers warn for overflow situations such as these. When manipulating the binary structure with bit shift operations, the programmer has to reason within an inconsistent arithmetic and take care of the borderline situations himself. Apparently, many are more worried about giving up absolute certainty in mathematics than they are about their own life by relying on computers in daily situations. We do not have the slightest proof that the current commercial computers and compilers we use to create programs, function the way we think they do. Such programs activate the anti-braking system in our car, guide traffic lights and are used to calculate the structure of bridges and building. If they fail to work, human life may be at risk. There are attempts to prove the correctness of hardware design and computer programs but these are not for practical or commercial use. In fact, we have the proof of the contrary. Commercial computers have been known to be inconsistent in their arithmetic, as was shown with the famous

Intel Pentium bug[2]. The fact therefore is that we live with inconsistencies every day of our life. Why is it so hard to accept this on a philosophical level?

Absolute certainty in mathematics?

> "Gentleman, that $e^{i} + 1 = 0$ is surely true, but it is absolutely paradoxical; we cannot understand it, and we don't know what it means, but we have proved it, and therefore we know it must be the truth".

This well-known quote by Benjamin Peirce, after proving Euler's identity in a lecture, reflects the predominant view of mathematicians before 1930, when mathematical truth equalled provability[3]. When Gödel proved that there are true statements in any consistent formal system that cannot be proved within that system, truth became peremptory decoupled of provability.
However, Peirce seems to imply something stronger: proving things in mathematics leads us to *the truth*. This goes beyond an epistemological view point and is a metaphysical statement about existence of mathematical objects and their truth, independent of human knowledge. The great mathematician Hardy formulates it more strongly, (Hardy 1929):

> "It seems to me that no philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or another, the immutable and unconditional validity of mathematical truth. Mathematical theorems are true or false; their truth or falsity is absolute and independent of our knowledge of them. In some sense, mathematical truth is part of objective reality".

Such statements are more than innocent metaphysical reflections open for discussion. They hide implicit values about the way mathematics develops and have important consequences for the education and research of mathematics. An objective reality implies the fixed and timeless nature of mathematical concepts. The history of mathematics provides evidence of the contrary. Mathematical concepts, even the most elementary ones, like the concept of number, continuously change over time. The objects signified by the ancient Greek concept of *arithmos* differ from that of 'number' by Renaissance mathematicians, which in turn differs from our current view. One could object that not mathematics but our understanding of mathematical reality changes. However, Jacob Klein's landmark study (1934-6) precisely focuses on the *ontological* shift in the number concept. In Greek arithmetic 'one' was not a number, later is was. After that, the root of two was accepted as a number and by the end of the sixteenth century the root of minus fifteen became a number.
Another implicit value hidden in the predominant view is the superiority of modern ideas over past ones, and possibly of Western concepts over non-Western ones. Again, the history of mathematics shows that mathematics always adapted to the needs of society. Mathematics was born in the fertile crescent, extending to the belt from North Africa to Asia, where wild seeds were large enough and mammals capable of

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[2] Given the calculation $x - \left(\frac{x}{y}\right) y = z$, the first Pentium chip produced the solution $z = 256$ for $x = 4195835$ and $y = 3145727$, instead of the correct z = 0. For more see, Coe, e.a. 1995.
[3] Quoted in Kasner and Newman, 1940.

employable domestication[4]. Modern algebra fertilized in the mercantile context of merchants and craftsman of Renaissance Italy. Several important figures in the development of symbolic algebra wrote also on book keeping, as well as on algebra often in one and the same volume[5]. If we accept that double-entry book keeping emerged in the fifteenth century as a result of the expanding commercial structures of sedentary merchant in Renaissance Italy, why not considering symbolic algebra within the same context? Ideas should be interpreted within the historical context in which they emerged and perhaps their superiority is dependent on the degree in which they were adapted to the needs of society.
Finally, the idea of an objective reality of mathematical concepts evades the reality of conceptual problems in mathematics. Time and again there have been serious crisis in the conceptual foundations of mathematics[6]. There have been inconsistent theories, such as the early use of analysis, and set theory which have existed for several decades. It is precisely in times of crisis and conceptual difficulties that new ideas emerge and breakthroughs are made.

Looking behind the barrier of symbolic thinking

Dealing with the development of symbolic algebra we must define some terms more explicitly. Let us call algebra *an analytical problem-solving method for arithmetical problems in which an unknown quantity is represented by an abstract entity*. There are two crucial conditions in this definition: *analytical*, meaning that the problem is solved by considering some unknown magnitudes as hypothetical and deductively deriving statements so that these unknowns can be expressed as a value, and secondly, an *abstract entity* is used to represent the unknowns. This entity can be a symbol, a figure or even a color as we shall see below. More strictly, symbolic algebra is *an analytical problem-solving method for arithmetical and geometrical problems consisting of systematic manipulation of a symbolic representation of the problem*. Symbolic algebra thus starts from a symbolic representation of a problem, meaning something more than a short-hand notation. There is no room here to expand on this important difference[7]. Instead we will focus on one important misunderstanding: "as arithmetical problems are solved algebraically for over 3000 years, an algebraic equation is a very old concept". This is not the case, as we shall argue. The symbolic equation is an invention of the sixteenth century.

We are all educated in the symbolic mode of thinking which is so predominant that it becomes very difficult to grasp how non-symbolic algebra really works. In fact, in the history of mathematics there are many cases in which one completely ignored the difference. Let us take one example of Babylonian algebra. That Babylonians had an advanced knowledge of algebra is a fact that became known rather late, around 1930. Many thousands of clay cuneiform tablets were found that contained either tables with numbers or the solutions to numerical problems. One such tablet is YBC 6967 from

[4] For an eye-opening study on the relation between these coincidental factors and the development of culture and thus mathematics see the excellent work of Jared Diamond , 1996.
[5] Between 1494 and 1586: Luca Pacioli, Grammateus, Girolamo Cardano, Valentin Mennher, Elcius Mellema, Nicolas Petri and Simon Stevin.
[6] An important case study on crisis in mathematics is Carl Boyer (1959), *The History of the Calculus and Its Conceptual Development*. As the title suggests Boyer concentrates on the conceptual difficulties in developing the modern ideas of the calculus.
[7] Mahony (1980) is one of the few to clarify the distinction. See also my forthcoming "Sixteenth century algebra as a shift in predominant models".

Yale University, written in the Akkadian dialect around 1500 BC. The most prominent scholar having studied and edited these mathematical tablets is Otto Neugebauer (1935-7, 1945). For the problem on YBC 6967 Neugebauer writes the following[8]:

> The problem treated here belongs to a well known class of quadratic equations characterized by the terms igi and igi-bi (in Akkadian *igūm* and *igibūm* respectively) (..) We must here assume the product
>
> $$xy = 60 \qquad (1.1)$$
>
> as the first condition to which the unknowns x and y are subject. The second condition is explicitly given as
>
> $$x - y = 7 \qquad (1.2)$$
>
> From these two equations it follows that x and y can be found from
>
> $$x, y = \sqrt{\left(\frac{7}{2}\right)^2 + 60} \pm \frac{7}{2}$$
>
> a formula which is followed exactly by the text, leading to $x = 12$ and $y = 5$.

Important here is that Neugebauer claims that equations are "explicitly given" and that the problem is "found from a formula which is followed exactly by the text". There are not so many people around who can go back to the cuneiform text and are able to check this claim. Fortunately, Neugebauer added an English translation which allows us to perform the task.

For the "explicitly given" equation we read "The *igibūm* exceeds the *igūm* by 7". This indeed corresponds with the equation (1.2). For the formula we read "As for you – halve 7, by which the *igibūm* exceeded the *igūm*, and the result is 3.5. Multiply together 3.5 with 3.5 and the result is 12.25. To 12.25, which resulted you, add 60, the product and the result is 72.25. What is the square root of 72.25: 8.5. Lay down 8.5, its equal and then subtract 3.5, the *takīlum*, from the one, add it to the other. One is 12, the other 5. 12 is the *igibūm*, 5 the *igūm*". Again the text seems to correspond with the formula. There are two minor details here: the 'lay down' part sounds a little strange in this context, and Neugebauer adds "we have refrained from translating *takīlum*", because no sense could be given to it.

Recently, Jens Høyrup (2002) published a book which completely overthrows the standard interpretation of Babylonian mathematics and adds a new one. For Høyrup, Babylonian algebra works with geometric figures. This went by completely unnoticed because no figures appear on the tablets. But Høyrup's study is very convincing and its importance for the history of mathematics cannot be overestimated. In this problem, the unknowns, *igibūm* and *igūm*, are represented by the sides of a rectangle (Høyrup 2002, 55-6). The term 'product' used by Neugebauer should be read as 'surface', 'square root' as 'equal side' or the side of a square surface and adding means appending in length. According to Høyrup the term *takīlum* which should be read as 'make-hold', or making the sides of a rectangle hold each other. Only within a geometrical interpretation, it makes sense to lay down something. Using a rectangle

[8] Neugebauer and Sachs, 1945, 129-30. The Babylonians used the sexadecimal number system, which unit is represented by Neugebauer as 1,0. I have changed this to decimal numbers and added the reconstructed text fragments for easier reading, which leaves the problem text otherwise intact.

with sides *igibūm* and *igūm*, now everything fits together. The *igibūm* is 7 longer than the *igūm*. Cutting that part in half leads us to figure 1:

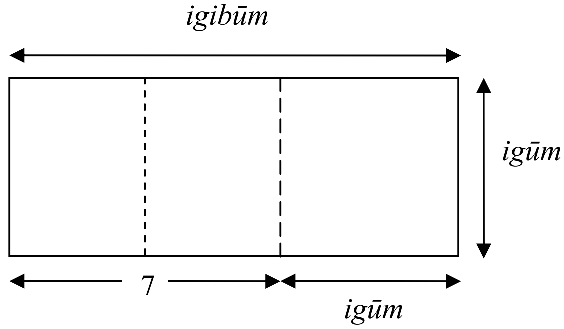

Figure 1: an example of the geometric algebra from the Babylonians.

If we paste one of the halves below the rectangle at the length of the *igūm* we get a figure with the same surface, equal to 60.

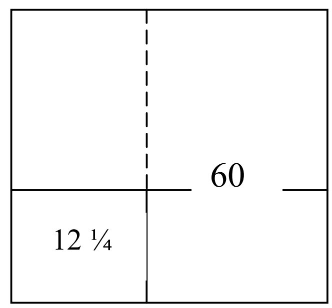

Figure 2: Cut and paste method for solving quadratic problems.

The part in the lower left corner must be a square, as its sides are both 3 ½. We can thus determine its surface as 12 ¼. The complete figure must also be a square with sides equal to *igūm* plus 3 ½. We know that the total surface is 72 ¼, the 'equal side' of that square therefore is 8 ½. That leads us to a value of the *igūm* being 5. Pasting the cut-out half back to its original place gives a length of the *igibūm* of 12.

We are presented here with an interpretation completely different from that of Neugebauer. Høyrup accounts for anomalies in the standard interpretation and gives strong arguments for the reading of terms and actions in the geometrical sense. In this new interpretation it makes no sense to speak about equations. Babylonian algebra does not solve equations, as the concept of an equation was absent. But it fits in with our definition of algebra: the method is unquestionably analytical, it uses the unknowns *igūm* and *igibūm* and they are represented as abstract entities, namely the sides of a rectangle. We cannot blame Neugebauer for his symbolic reading of Babylonian algebra in 1945. Looking behind the barrier of symbolic thinking proves to be a difficult task. His book was a major contribution to the early history of mathematics. But the history of mathematics has changed in the past decades and conceptual analyses such as Høyrup's have become the new methodological standard.

Diophantus: algebra or theory of numbers?

The *Arithmetica* of Diophantus is often considered the first work exclusively devoted to algebra[9]. This interpretation is questionable. The discovery of Diophantus in the fifteenth century had an important influence on the development of symbolic algebra. However, its influence is not as decisive as some want us to believe. The prime source for the myth that algebra was invented by Diophantus is Regiomontanus in his Padua lecture of 1464. Just having discovered the manuscript, Regiomontanus describes the *Arithmetica* enthusiastically as a book "in which the flower of the whole of arithmetic is hidden, namely the art of the thing and the *census*, which today is called algebra by an Arabic name. Here and there, the Latins have come in contact with this beautiful art"[10]. Later, humanist mathematicians of the sixteenth century, such as Petrus Ramus, will be more explicit in the idea that algebra originated with Diophantus and the Arabs learned the art from him[11]. Paradoxically, sixteenth-century humanists continued the program of reassessing mathematics from ancient sources, initiated by the Arabs, and in doing so precisely denied the contribution of the Arabs. Høyrup (1998) traces this evolution over several authors in Renaissance Europe. After Ramus, also Bombelli and Viète were well-acquainted with the *Arithmetica* and carefully avoided reference to Arab influences. On the other hand, the Arab roots of algebra have mostly been acknowledged by the Italian abacus tradition from Fibonacci (1202, Boncompagni 1857) through the fifteenth century up to Cardano (1545, Witmer 1968) and the German cossist tradition, with Stifel (1544) as most important author. It is probably thanks to them that we still use the name *algebra* today.

In order to assess the *Arithmetica* it is important to draw a distinction between the context of the original text and its adaptations since its discovery by Regiomontanus. The treatment of problems from the *Arithmetica* by Bombelli (1572) and Simon Stevin (1585) are without doubt algebraic. Several editions of the *Arithmetica* have given an algebraic formulation to problems, as has been done with Euclid's *Elements*. Such reformulation has been historically important for diophantine analysis but was not necessarily a correct interpretation of the original work. Let us look at problem 16 from the first book as an illustration. This is a rather simple problem looking for three numbers given their sum two by two (table 1):

Tannery 1893, p. 39	Ver Eecke, 1926, p. 21
Invenire tres numeros tales ut bini simul additi faciant propositos numeros. Oportet propositorum trium dimidiam summam maiorem esse unoquoque horum.	Trouver trois nombres qui, pris deux à deux, forment des nombres proposés. Il faut toutefois que la moitié de la somme des nombres proposés soit plus grande que chacun de ces nombres.
Proponatur iam $X_1 + X_2 = 20$, $X_2 + X_3 = 30$, $X_3 + X_1 = 40$	Proposons donc que le premier nombre, augmenté du second, forme 20 unités; que le second, augmenté du troisième, forme 30 unités, et que le troisième, augmenté du premier, forme 40 unités.
Ponatur $X_1 + X_2 + X_3 = x$	Posons que la somme des trois nombres est 1 arithme.
Quoniam $X_1 + X_2 = 20$, si a x aufero 20, habebo $X_3 = x - 20$	Dès lors, puisque le premier nombre plus le second forment 20 unités, si nous retranchons

[9] E.g. Varadarajan 1991, Bashmakova 1997.
[10] Regiomontanus, 1972, p. 47, cited and translated by Høyrup 1998, p. 30.
[11] Ramus gives a short history of mathematics in his *Scholae mathematicae* (1569).

	20 unités de 1 arithme, nous aurons comme troisième nombre 1 arithme moins 20 unités.
Eadem ratione erit $X_1 = x - 30$, $X_2 = x - 40$	Pour la même raison, le premier nombre sera 1 arithme moins 30 unités, et le second nombre sera 1 arithme moins 40 unités.
Linquitur summam trium aequari x, sed est haec summa $3x - 90$; ista aequentur x; fit $x = 45$.	Il faut encore que la somme des trois nombres devienne égale à 1 arithme. Mais, la somme des trois nombres forme 3 arithmes moins 90 unités. Egalons-les à 1 arithme, et l'arithme devient 45 unités.
Ad positiones. Erit $X_1 = 15$, $X_2 = 5$, $X_3 = 25$. Probatio evidens est.	Revenons à ce que nous avons posé : le premier nombre sera 15 unités, le second sera 5 unités, le troisième sera, 25 unités, et la preuve est claire.

Table 1: Two interpretations of problem 16 from Book I of the *Arithmetica* by Diophantus

Paul Tannery's respected critical edition of 1893 gives the original Greek text, reconstructed from several manuscripts, together with a Latin translation. As shown, the Latin translation presents the problem as one of three linear equations with three unknowns X_1, X_2 and X_3 and the use of an auxiliary x. However, the idea of linear equations with several unknowns did not emerge before the mid-sixteenth century. Ver Eecke (1926) performed his French translation from the same Greek text as Tannery but gives a more cautious interpretation. He does not use any symbols and draws a distinction between number and *arithmos*. The unknowns X_1, X_2 and X_3 of Tannery are numbers in the French translation. Instead, the *arithmos* designates the unknown. After stating the problem, Diophantus reformulates the problem expressing the numbers in terms of a chosen unknown.

The interpretation of the *Arithmetica* as symbolic algebra is highly problematic. Even its designation as algebra cannot go without careful qualification. Nesselmann (1842) called it *syncopated algebra* as an intermediate stage between rhetoric and symbolic algebra. This would consist of short-hand notations which have not yet developed to full symbolism. The Greek text uses the letters Δ^γ and K^γ which have been interpreted by many as the powers of an unknown, x^2 and x^3. Ver Eecke simply translates this as 'square' and 'cube' respectively. And this is without doubt closer to the original context than Tannery's Latin translation. Diophantus is primarily interested in the properties of numbers. A typical problem sounds like "Find two numbers with their sum and the difference of their squares given" (Book I, Problem 29; Tannery 1893, 65). The aim is to find numbers which satisfy the given property rather than solving the equations $x + y = 20, x^2 - y^2 = 80$. All problems of the *Arithmetica* are stated in the general way. A reading of the *Arithmetica* as a general theory of numbers is further emphasized by the character of diophantine problems having an infinity of numbers satisfying a given property. Diophantus's *Arithmetica* can be equally or better understood as a study on the properties of natural numbers than as early algebra. To read the text as an early form of symbolic algebra cannot be reconciled with the definition we have given above.

The colorful algebra of the Hindus

Hindu tradition has passed down to us several important works on arithmetic and algebra, the importance of which to the development of algebra is still underestimated. The major handicap in drawing a line of influence of Indian sources on the development of Renaissance arithmetic and algebra is its indirect character and lack of written evidence. We can trace some important paths of transmission for arithmetic and the Hindu-Arabic number system we currently use today. Some Arab texts that were translated into Latin clearly refer to Hindu sources[12]. Early arithmetic books are structurally very similar, for example, to Bhramagupta's *Brāhmasphuṭasiddhānta* (*BSS*) of 628 AD (Colebrook 1817). However, there is no textual evidence known which shows a direct influence of Hindu algebra in the West. Comparing many problems treated in Hindu sources as well as in Renaissance algebra, we cannot avoid the particular similarity of both the formulation of the problems and most of the solution methods. Many linear problems solved algebraically in the abacus tradition have their counterpart in Hindu sources, while they are only rarely treated in Arab texts. We can discern an important influence of the oral tradition of recreational problems. Practical and recreational problems have functioned as vehicles for problem prototypes with typical solution patterns[13]. The solution method for typical problems are given as rules in Hindu texts. These rules are mostly formulated in Sanskrit verse, as stanzas or *sūtras*. Given the scarcity and cost of writing aids, memorizing aids in the form of verse has been very important in mathematical texts before the age of printing. As an example, consider the following rule for solving linear problems given both in the *BSS* and the *Bīja-Ganita* (*BG*) of Bhāskarācārya of c. 1150:

> Colebrook, 1817, 227:
> Subtract the first color [or letter] from the other side of the *equation*; and the rest of the colors [or letters] as well as the known quantities, from the first side: the other side being then divided by the [coefficient of the] first, a value of the first color will be obtained. If there be several values of one color, making in such case equations of them and dropping the denominator, the values of the rest of the colors are to be found from them.

> Dvivedi, 1902:
> Removing the other unknowns from [the side of] the first *unknown* and dividing the *coefficient* of the first *unknown*, the value of the first *unknown* [is obtained]. In case of more [values of the first unknown], two and two [of them] should be considered after reducing them to common denominators.

In the English rendition of the Sanskrit verses, Colebrook uses the term 'equation' but he is not followed by Dvivedi. Instead Dvivedi uses the terms 'unknown' and 'coefficient', which in turn are not used by Colebrook. We can therefore cast some doubt about the use of these modern terms. Furthermore, Datta and Singh (1962, II, 9) claim that "in Hindu algebra there is no systematic use of any special term for the coefficient".

Prthūdakasvāmī (860), Srīpati (1039) and later Bhāskara (1150) solved linear problems by the use of several colors, representing the unknowns. In other cases also flavors such as sweet (*madhura*), or flowers were used for the same purpose. Solutions were mostly based on rules for prototypical cases such as the rule of concurrence (*sankramana*) $\{x+y=a, x-y=b\}$ or the pulverizer (*Kuttaka*) $ax-by=c$. In several texts, starting with the *BSS*, we find reference to *samīcarana*, *samīcarā*, or *samīcriyā*, often translated as 'equation'. The rationale for this is that

[12] *Dixit Algorizmi* c. 825. For a French translation see Allard, 1992, 1 – 22.

[13] I have argued this more extensively in "How algebra spoiled Renaissance's practical and recreational problems", forthcoming.

sama means ‘equal’ and *cri* stands for ‘to do’. As with the terms *aequatio* and *aequationis* in early Latin works on algebra we should be careful interpreting these terms in the modern way. They basically mean *the act of making even*, an essential operation in the algebraic solution of problems. They not necessarily mean an equation in the sense of symbolic algebra. The basis of Hindu algebra is to reduce problems to the form of given precepts that provide a proven solution to the problem. The method is algebraic as it uses abstract entities for the unknowns and is analytical in its approach. The Hindu methods for solving linear problems were transmitted to the West by prototypical problems, mostly of the recreational type, which served as vehicles for the corresponding problem-solving recipes. An example is the case $\{x+a=c(y-a), y+b=d(x-b)\}$ which we find in the *Ganitasārasangraha* of Mahāvīra and the *BG* but also in several fifteenth-century arithmetics under the name *regula augmentationis*.

Arab algebra

Arab algebra was introduced in Europe by the translations of the *Algebra* of Mohammed ibn Mūsa al-Kwārizmī by Guglielmo de Lunis, Gerhard von Cremona (1145) and Robert of Chester (1450; Hughes, 1981). Most importantly however was the *Liber Abaci* of Fibonacci (1202). Fibonacci devoted the last part of his book to algebra and used mostly problems and solution methods from al-Kwārizmī and Al-Karkhī. Although Arab algebra developed to a high degree of sophistication during the next centuries, it was mostly the content of these early works that were known in Europe. Recent studies have provided us with a new picture on the continuous development of algebra in the Italian abacus schools between Fibonacci and Luca Pacioli’s *Summa de arithmetica geometria proportioni* (1494) (Franci and Rigatelli, 1985). It took about four centuries before the transition to symbolic algebra was completed.

al-Kwārizmī gives solutions to algebraic problems by applying proven procedures in an algorithmic way. The validity of the solution is further demonstrated by geometrical diagrams. In contrast with Babylonian algebra the method is not geometrical in nature, only the demonstration and interpretation is. As an example let us look at the way al-Kwārizmī solves case 4 of the quadratic problem which can be represented by the well-know equation $x^2+10x=39$ (Rosen, 1831, italics mine):

> For instance, one square, and ten roots of the same, amount to thirty-nine dirhems. That is to say, what must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: *you halve the number of the roots*, which in the present instance yields five. *This you multiply by itself*; the product is twenty-five. *Add this to thirty-nine*; the sum is sixty-four. *Now take the root of this*, which is eight, and *subtract from it half the number of the roots*, which is five; the remainder is three. *This is the root of the square which you sought for*; the square itself is nine.

If we write the case as $x^2+bx=c$, the solution fully depends on the application of the procedure which corresponds to $\sqrt{\left(\frac{b}{2}\right)^2+c}-\frac{b}{2}$. Solving problems in Arab algebra consists of formulating the problem in terms of the unknown and reducing the form to one of the known cases. Methods for solving quadratic problems were given before in Babylonian and Hindu algebra. Again we see no equations in Arab algebra. However, new is the explicit treatment of operations on polynomials. The basic operations addition, subtraction, multiplication and division, which were applied before to numbers, are now extended to an aggregation of algebraic terms. A further expansion of these operations would lead to the concept of a symbolic equation in the sixteenth century.

The emergence of the concept of an equation

So, what is it that constitutes the concept of an equation? I propose to adopt an operational definition of the term in order to reconstruct the historical emergence of the concept. We now consider an equation as a mathematical object on which certain operations are allowed. Let us therefore look at the precise point in time in which an equation is named, consistently used and operated upon as a mathematical object. As said before, the use of the term *aequatio* is not a sufficient condition for the existence of an equation. The observation that two polynomials are numerically identical does not in itself constitute an equation. However, an operation on an equation would. The first historical instance that I could find is in Cardano's *Practica arithmetice* (1539, f. 91[r]).

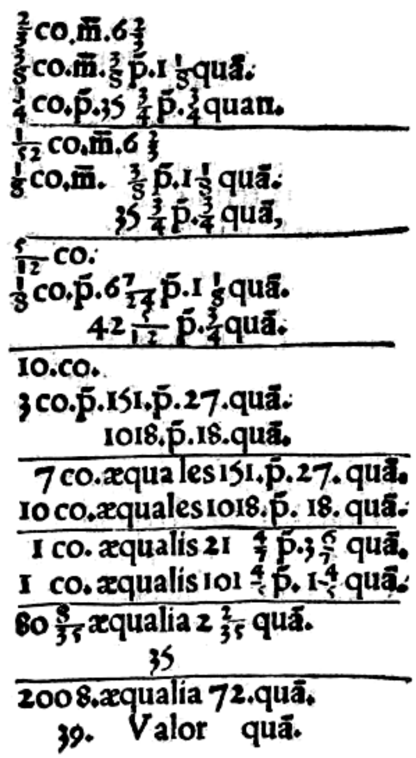
2/3 co. m. 6 2/3
3/5 co. m. 3/5 p. 1 1/5 qua.
3/4 co. p. 35 3/4 p. 3/4 quan.
1/15 co. m. 6 2/3
3/5 co. m. 3/5 p. 1 1/5 qua.
35 3/4 p. 3/4 qua.
5/12 co.
3/5 co. p. 6 7/12 p. 1 1/5 qua.
42 1/12 p. 3/4 qua.
10. co.
3 co. p. 151. p. 27. qua.
1018. p. 18. qua.
7 co. æquales 151. p. 27. qua.
10 co. æquales 1018. p. 18. qua.
1 co. æqualis 21 4/7 p. 3 6/7 qua.
1 co. æqualis 101 4/5 p. 1 4/5 qua.
80 8/35 æqualia 2 2/35 qua.
35
2008. æqualia 72. qua.
39. Valor qua.

Figure 3: The first operation on an equation in Cardano's *Practica arithmetice* of 1539.

This is probably the most important page in the development of symbolic algebra as it combines two important conceptual innovations in a single problem solution: the use of a second unknown and the first operation on an equation. Cardano uses *co*. for the primary unknown and *quan*. for a secondary one. We can justly write this as x and y without misinterpreting the original context. In the example given in figure 3, Cardano manipulates several polynomials, but at some point moves to equations. We find 7 co. aequales 151 p. 27 quan. ($7x = 151 + 27y$) and 10 co. aequales 1018 p 18 quant ($10x = 1018 + 18y$). He divides these equations by 7 and 10 respectively, without explicitly saying so. But then by equating both he arrives at $80\frac{8}{35}=2\frac{2}{35}y$ which he explicitly multiplies by 35 to arrive at 72*y* = 2808 or *y* = 39 (misprinted as 2008 = 72*y*). From this moment onwards algebra changed drastically. Cardano's book was widely read and several authors build further on this milestone. Stifel (1545) introduces the letters 1A, 1B and 1C to differentiate multiple unknowns, which removes most of the ambiguities from earlier notations. But it is Johannes Buteo (1559) who establishes a method for solving simultaneous linear equations by systematically substituting, multiplying and subtracting equations to eliminate unknowns. These developments between 1539 and 1559 constituted the concept of a symbolic equation. The equation became not only a representation of an arithmetical equivalence but also represented the combinatorial operations possible on the symbolic structure. This paved the road for Viète (1591) and Harriot to study the structure of symbolic equations.

Conclusion

We treated the history of 3000 years of algebra in a few paragraphs with the risk of over simplification. However, one important conclusion emerges: at some point in history there was a dramatic change in the way arithmetical problems were solved. By the second half of the sixteenth century algebraic problem solving became the systematic manipulation of symbolic equations. We argued that the concept of an equation, as we understand it today, did not exist before that time. The development of sixteenth-century algebra is one of these occasions in which we see the birth of a new important concept in mathematics. Algebra did exist before, but functioned in a different way. The aim of this paper is to show that the history of mathematics offers ample opportunities to illustrate the plurality of methods and the dynamics of concepts in mathematics. Integrating threads of conceptual development of mathematics in classroom teaching contributes to students' philosophical attentiveness. Such examples will alert students of the relativity of mathematical methods, truth and knowledge and will put mathematics back in the perspective of time, culture and context.

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