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NEGATIONS IN THE ADJUNCTIVE DISCURSIVE LOGIC

Abstract

In the logical literature, *Discursive* (or *Discussive*) Logic introduced by Stanisław Jaśkowski is seen as one of the earliest examples of the so-called paraconsistent logic. There is some confusion over what is actually discursive logic nevertheless. One of the possible sources of the confusion is easy to discern; it comes from the fact that Jaśkowski published his two papers in Polish and their English translations appeared many years later.[2] Up till 1999, no one but a Polish reader was able to read Jaśkowski's paper on the discursive conjunction and consequently some authors took discursive logic to be a foremost example of a *non-adjunctive* logic.[3]

The situation became even more complicated when da Costa, Dubikajtis and Kotas presented an axiomatization with discursive connectives as primitive symbols. It turned out that a connective of the discursive conjunction they considered did not coincide with any connective presented by Jaśkowski. Moreover, their axiomatization contained some axiom schemata that were not generally valid in Jaśkowski's logic.[4]

The purpose of this paper is to clarify the confusion surrounding the discursive logic. We will present a new (direct) semantics and axiomatization of Jaśkowski's adjunctive discursive logic and show how to define and axiomatize two additional connectives of negation.

Keywords: discursive (discussive) logic, D_2, paraconsistent logic.

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[2]For details, see *References* ([16] and [17]).

[3]See, for example, [11], [21] and [24].

[4]See, [1], [9] and [12].

1 Introduction

In 1949, Jaśkowski published his second paper on the discursive logic. It was the first time when the discursive conjunction appeared and replaced the conjunction introduced in [16]. The language of the resulting calculus is as follows.

Definition 1. Let *var* denote a non-empty denumerable set of all propositional variables $\{p_1, p_2, ...\}$. For_{D2} is defined to be the smallest set for which the following holds:

(i) $\alpha \in var \Rightarrow \alpha \in For_{D2}$
(ii) $\alpha \in For_{D2} \Rightarrow \sim \alpha \in For_{D2}$
(iii) $\alpha \in For_{D2}$ and $\beta \in For_{D2} \Rightarrow \alpha \bullet \beta \in For_{D2}$, where $\bullet \in \{\vee, \wedge_d, \rightarrow_d\}$.

The symbols: $\sim$, $\vee$, $\wedge_d$, $\rightarrow_d$ denote negation, disjunction, discursive conjunction and discursive implication, respectively. The discursive equivalence, $\alpha \leftrightarrow_d \beta$, is defined by $(\alpha \rightarrow_d \beta) \wedge_d (\beta \rightarrow_d \alpha)$.

Now we determine a translation function of the language of the new calculus, D_2 for short, into the language of S_5 of Lewis, $f : For_{D2} \Rightarrow For_{S5}$, i.e.:

(i) $f(p_i) = p_i$ if $p_i \in var$ and $i \in N$
(ii) $f(\sim \alpha) = \sim f(\alpha)$
(iii) $f(\alpha \vee \beta) = f(\alpha) \vee f(\beta)$
(iv) $f(\alpha \wedge_d \beta) = f(\alpha) \wedge \Diamond f(\beta)$
(v) $f(\alpha \rightarrow_d \beta) = \Diamond f(\alpha) \rightarrow f(\beta)$

and additionally

(vi) $\forall \alpha \in For_{D2} : \alpha \in D_2 \Leftrightarrow \Diamond f(\alpha) \in S_5$.[5]

It is easy to observe that Duns Scotus' thesis $p \rightarrow_d (\sim p \rightarrow_d q)$ and many other classically valid formulas are not valid in Jaśkowski's calculus, for instance,

(1) $p \rightarrow_d (\sim p \rightarrow_d \sim q)$
(2) $p \rightarrow_d (\sim p \rightarrow_d (\sim\sim p \rightarrow_d q))$
(3) $(p \rightarrow_d q) \rightarrow_d ((p \rightarrow_d \sim q) \rightarrow_d \sim p)$
(4) $(p \wedge_d \sim p) \rightarrow_d q$
(5) $(p \rightarrow_d q) \rightarrow_d (\sim q \rightarrow_d \sim p)$
(6) $(\sim p \rightarrow_d \sim q) \rightarrow_d (q \rightarrow_d p)$.[6]

What is worth mentioning is that the calculus is closed under the rule:

(AdR) $\alpha,\ \beta\ /\ \alpha \wedge_d \beta$

since S_5 is closed under $\Diamond\alpha,\ \Diamond\beta\ /\ \Diamond(\alpha \wedge \Diamond\beta)$. Therefore it is an example of an *adjunctive* logic.

[5]See [16] p. 44, [17] p. 57 (we use here the English translations of the Jaśkowski papers that appeared in Logic and Logical Philosophy), [7] pp. 7-9, [8] pp. 285-288.

[6]See [16] pp. 50-53, [17] p. 58 and [8] pp. 289-290.

2 Discursive Logic's New Clothes

The guiding idea behind the semantics we present is to eliminate the translation rules.

A frame (D_2-frame) is a pair $< W, R >$, where W is a non-empty set of points (or possible worlds) and R is the equivalence relation on W. A model (D_2-model) is a triple $< W, R, v >$, where v is a mapping from propositional variables to sets of worlds, $v : var \Rightarrow 2^W$. The satisfaction relation $\models_m$ is defined as follows:

(var) $x \models_m p_i \;\Leftrightarrow\; x \in v(p_i)$ and $i \in N$
$(\sim)$ $x \models_m \sim \alpha \;\Leftrightarrow\; x \not\models_m \alpha$
$(\vee)$ $x \models_m \alpha \vee \beta \;\Leftrightarrow\; x \models_m \alpha$ or $x \models_m \beta$
$(\wedge_d)$ $x \models_m \alpha \wedge_d \beta \;\Leftrightarrow\; x \models_m \alpha$ and $\exists_{y \in W}(xRy$ and $y \models_m \beta)$
$(\rightarrow_d)$ $x \models_m \alpha \rightarrow_d \beta \;\Leftrightarrow\; \forall_{y \in W}(xRy \Rightarrow y \not\models_m \alpha)$ or $x \models_m \beta$.

A formula α is valid in D_2, $\models \alpha$ for short, iff for any model $< W, R, v >$, for every $x \in W$, there exists $y \in W$ such that: xRy and $y \models_m \alpha$.

The conditions (i) – (v) of the translation function were, respectively, replaced with (var) – $(\rightarrow_d)$; (vi) found its expression in the definition of $\models$.

The accessibility relation defined on D_2-frames is reflexive, symmetric and transitive. Any point is accessible from any other. This fact enables us to simplify the notion of the D_2-model.

A model (D_2-model) is a pair $< W, v >$, where W is a non-empty set (of points) and a function, $v : For_{D2} \times W \Rightarrow \{1, 0\}$, is inductively defined:

$(\sim)$ $v(\sim \alpha,\ x) = 1 \;\Leftrightarrow\; v(\alpha,\ x) = 0$
$(\vee)$ $v(\alpha \vee \beta,\ x) = 1 \;\Leftrightarrow\; v(\alpha,\ x) = 1$ or $v(\beta,\ x) = 1$
$(\wedge_d)$ $v(\alpha \wedge_d \beta,\ x) = 1 \;\Leftrightarrow\; v(\alpha,\ x) = 1$ and $\exists_{y \in W}(v(\beta,\ y) = 1)$
$(\rightarrow_d)$ $v(\alpha \rightarrow_d \beta,\ x) = 1 \;\Leftrightarrow\; \forall_{y \in W}(v(\alpha,\ y) = 0)$ or $v(\beta,\ x) = 1$.

$\models \alpha$ iff for any model $< W, R, v >$, there exists $y \in W$ such that $v(\alpha,\ y) = 1$.

Proposition 1. $\forall \alpha \in For_{D2}$: $\models \alpha \;\Leftrightarrow \alpha \in D_2$ $\;(\Leftrightarrow \Diamond f(\alpha) \in S_5)$.

Proof. By induction.

The consequence of Proposition 1 is that we established the equivalence between the translation procedure and our semantics. The translation became redundant.

Now let us focus on the syntactic analysis of D_2.

(A_1) $\alpha \rightarrow_d (\beta \rightarrow_d \alpha)$
(A_2) $(\alpha \rightarrow_d (\beta \rightarrow_d \gamma)) \rightarrow_d ((\alpha \rightarrow_d \beta) \rightarrow_d (\alpha \rightarrow_d \gamma))$
(A_3) $(\alpha \wedge_d \beta) \rightarrow_d \alpha$
(A_4) $(\alpha \wedge_d \beta) \rightarrow_d \beta$
(A_5) $(\alpha \rightarrow_d \beta) \rightarrow_d ((\alpha \rightarrow_d \gamma) \rightarrow_d (\alpha \rightarrow_d (\beta \wedge_d \gamma)))$
(A_6) $\alpha \rightarrow_d (\alpha \vee \beta)$
(A_7) $\beta \rightarrow_d (\alpha \vee \beta)$
(A_8) $(\alpha \rightarrow_d \gamma) \rightarrow_d ((\beta \rightarrow_d \gamma) \rightarrow_d ((\alpha \vee \beta) \rightarrow_d \gamma))$
(A_9) $\alpha \vee (\alpha \rightarrow_d \beta)$

$(A_{10})\ \sim (\sim \alpha \wedge_d \sim\sim \alpha \wedge_d \sim (\alpha \vee \sim \alpha))$
$(A_{11})\ \sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim (\alpha \vee \beta)) \rightarrow_d \sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim \gamma \wedge_d \sim (\alpha \vee \beta \vee \gamma))$
$(A_{12})\ \sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim \gamma \wedge_d \sim (\alpha \vee \beta \vee \gamma)) \rightarrow_d$
$\quad \sim (\sim \alpha \wedge_d \sim \gamma \wedge_d \sim \beta \wedge_d \sim (\alpha \vee \gamma \vee \beta))$
$(A_{13})\ \sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim \gamma \wedge_d \sim (\alpha \vee \beta \vee \gamma)) \rightarrow_d ((\alpha \vee \beta \vee \sim \gamma) \rightarrow_d (\alpha \vee \beta))$
$(A_{14})\ \sim (\sim \alpha \wedge_d \sim \beta) \rightarrow_d (\alpha \vee \beta)$
$(A_{15})\ (\alpha \vee (\beta \vee \sim \beta)) \rightarrow_d \sim (\sim \alpha \wedge_d \sim (\beta \vee \sim \beta))$.

The sole rule of inference is *Detachment Rule*
$(MP)^*\ \alpha,\ \alpha \rightarrow_d \beta\ /\ \beta$.

The set of axiom schemata and $(MP)^*$ define $\vdash_{D2}$ (the consequence relation).

The axiomatization we presented is in fact the first axiomatization of D_2 (with discursive connectives as primitive symbols and positive and negation fragments to be separated).[7]

From now on, let D_2^+ denote the set $\{(A_1), \ldots (A_9)\}$.[8]

THEOREM 1. $\vdash_{D2} \alpha \Leftrightarrow \models \alpha$.

PROOF. See Section 6.

3 Negation as a *Possible − not* Connective

In this section we introduce a new connective of negation. This move allows some of the weaker form of Duns Scotus' thesis to be present in the modified calculus. The definition is the following:

Definition 2. $\sim_d \alpha = (p_1 \vee \sim p_1) \wedge_d \sim \alpha$

Observe that we can apply the translation function to transform the connective into its modal counterpart:

(ii)' $f(\sim_d \alpha) = \Diamond \sim f(\alpha)$.

and extend our semantics by the additional condition:

$(\sim_d)\ x \models_m \sim_d \alpha \ \Leftrightarrow\ \exists_{y \in W}(xRy \text{ and } y \not\models_m \alpha)$

We will henceforth regard $\sim_d$ as a primitive symbol that has replaced the connective of $\sim$. This exchange results in obtaining a quite new calculus, called ND_2^+.

A model (ND_2^+-model) is a pair $< W, v >$, where W is a non-empty set (of points) and a function, $v : For_{ND2+} \times W \Rightarrow \{1, 0\}$, is inductively defined:

$(\sim_d)\ v(\sim_d \alpha,\ x) = 1 \ \Leftrightarrow\ \exists_{y \in W}(v(\alpha,\ y) = 0)$

[7] Most of the authors who dealt with this subject was interested in an alternative strategy. They treated Jaśkowski's calculus as a starting point for pure modal analysis. See [3], [4], [5], [6], [13], [14], [15], [18], [19] and [20].

[8] As before, the discursive equivalence is a definable connective.

($\vee$) $v(\alpha \vee \beta,\ x) = 1 \ \Leftrightarrow\ v(\alpha,\ x) = 1$ or $v(\beta,\ x) = 1$
($\wedge_d$) $v(\alpha \wedge_d \beta,\ x) = 1 \ \Leftrightarrow\ v(\alpha,\ x) = 1$ and $\exists_{y \in W}(v(\beta,\ y) = 1)$
($\rightarrow_d$) $v(\alpha \rightarrow_d \beta,\ x) = 1 \ \Leftrightarrow\ \forall_{y \in W}(v(\alpha,\ y) = 0)$ or $v(\beta,\ x) = 1$.

$\models \alpha$ iff for any model $< W, R, v >$, there exists $y \in W$ such that $v(\alpha,\ y) = 1$.

The idea to treat negation as "possibly-not" is not quite new and was examined by many authors,[9] but any of them hardly studied it in relation to D_2 and even so, they neither axiomatized it nor gave a direct semantics for the resulting system.

Notice that some of the ND_2^+-valid formulas does not correspond to their D_2-counterparts (e.i. after replacing $\sim_d$ with $\sim$), for example,

(1) $\sim_d p \rightarrow_d (\sim_d\sim_d p \rightarrow_d q)$
(2) $\sim_d p \rightarrow_d (\sim_d\sim_d p \rightarrow_d (\sim_d\sim_d\sim_d p \rightarrow_d q))$
(3) $(\sim_d p \wedge_d \sim_d\sim_d p) \rightarrow_d q$
(4) $(\sim_d p \rightarrow_d \sim_d q) \rightarrow_d ((\sim_d p \rightarrow_d \sim_d\sim_d q) \rightarrow_d p)$
(5) $(p \vee \sim_d q) \rightarrow_d ((p \vee \sim_d\sim_d q) \rightarrow_d p)$.[10]

On the other hand, there are many D_2-valid formulas that are not valid in ND_2^+ (after replacing $\sim$ with $\sim_d$), for example,

(6) $(p \rightarrow_d q) \rightarrow_d \sim\sim (p \rightarrow_d q)$
(7) $p \rightarrow_d \sim\sim p$
(8) $\sim (\sim p \wedge_d p)$
(9) $p \rightarrow_d \sim (\sim p \wedge_d \sim q)$
(10) $(p \vee q) \rightarrow_d (p \vee \sim\sim q)$.

Proposition 2. ND_2^+ (with $\sim_d$ as primitive) is not a conservative extension of D_2.

ND_2^+ is axiomatizable by the rule of $(MP)^*$ plus the set of axiom schemata:

(A_1) α, if $\alpha \in D_2^+$
(A_2) $\sim_d (\alpha \wedge_d \sim_d \beta) \rightarrow_d \sim_d\sim_d (\sim_d \alpha \vee \beta)$
(A_3) $\sim_d (\alpha \wedge_d \sim_d \alpha)$
(A_4) $(\alpha \vee \sim_d \beta) \rightarrow_d ((\alpha \vee \sim_d\sim_d \beta) \rightarrow_d \alpha)$
(A_5) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d (\alpha \vee \sim_d\sim_d \beta)$
(A_6) $\sim_d\sim_d \alpha \rightarrow_d \alpha$
(A_7) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d \sim_d\sim_d (\alpha \vee \beta \vee \gamma)$
(A_8) $\sim_d\sim_d (\alpha \vee \beta) \rightarrow_d \sim_d\sim_d (\beta \vee \alpha)$

The consequence relation $\vdash_{ND2+}$ is defined by the set and $(MP)^*$.

Theorem 2. $\vdash_{ND2+} \alpha \Leftrightarrow \models \alpha$.

Proof. See [10] for details.

[9] See, for instance, [2], [13] and [23].
[10] See [16] pp. 46-50, [17] p. 58.

4 Collapse into the Classical Logic

Like in the previous section, we start by characterizing a new connective of negation:

Definition 3. $\neg_d\alpha = \alpha \rightarrow_d \sim (p_1 \vee \sim p_1)$.

The formula $\sim (p_1 \vee \sim p_1)$ has the same meaning as falsum and Definition 3 looks like the one for intuitionistic logic. It is, however, a bit deceptive since the formula $\alpha \vee \neg_d\alpha$ is generally valid in D_2; in view of that, the translation rules may be of much help, especially the rule

(ii)" $f(\neg_d\alpha) = \sim \Diamond f(\alpha)$.

is of particular interest.

In what follows, we will use $\neg_d$ as a primitive symbol that has replaced $\sim$ and SD_2^+ to denote the resulting calculus.

Here is a direct semantics for SD_2^+.

A model (SD_2^+-model), as before, is a pair $< W, v >$, where W is a non-empty set (of points) and a valuation function, $v : For_{SD2+} \times W \Rightarrow \{1, 0\}$, is defined:

($\neg_d$) $v(\neg_d\alpha,\ x) = 1 \ \Leftrightarrow\ \forall_{y\in W}(v(\alpha,\ y) = 0)$
($\vee$) $v(\alpha \vee \beta,\ x) = 1 \ \Leftrightarrow\ v(\alpha,\ x) = 1$ or $v(\beta,\ x) = 1$
($\wedge_d$) $v(\alpha \wedge_d \beta,\ x) = 1 \ \Leftrightarrow\ v(\alpha,\ x) = 1$ and $\exists_{y\in W}(v(\beta,\ y) = 1)$
($\rightarrow_d$) $v(\alpha \rightarrow_d \beta,\ x) = 1 \ \Leftrightarrow\ \forall_{y\in W}(v(\alpha,\ y) = 0)$ or $v(\beta,\ x) = 1$.

$\models \alpha$ iff for any model $< W, R, v >$, there exists $y \in W$ such that $v(\alpha,\ y) = 1$.

It is remarkable that among the theses of SD_2^+ there are all the laws of the classical propositional calculus (including Duns Scotus' thesis) and the semantics we introduced can be viewed as a new semantics for the classical propositional logic.

A deductive structure of SD_2^+ is given by the set of axiom schemata:

(A_1) α, if $\alpha \in D_2^+$
(A_2) $\neg_d\neg_d\alpha \rightarrow_d \alpha$
(A_3) $\alpha \rightarrow_d \neg_d\neg_d\alpha$
(A_4) $(\alpha \rightarrow_d \beta) \rightarrow_d (\neg_d\beta \rightarrow_d \neg_d\alpha)$

and the rule:

$(MP)^*$ $\alpha,\ \alpha \rightarrow_d \beta\ /\ \beta$

The consequence relation $\vdash_{SD2+}$ is determined by the set and $(MP)^*$.

The axiomatization coincides with the well-known axiom system originated by Hilbert and Bernays, but one might just as well *adopt* a different set of the axiom schemata of the classical propositional calculus.

Proposition 3. Each thesis α of SD_2^+ becomes a thesis α' of the classical propositional logic after replacing in α the connectives $\neg_d$, $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$ by $\neg$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$, respectively.

PROPOSITION 4. Each thesis α' of the classical propositional logic becomes a thesis α of SD_2^+ after replacing in α' the connectives $\neg$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$ by $\neg_d$, $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$, respectively.

PROOF. Apply the method described in [16], pp. 45–46.

THEOREM 3. $\vdash_{SD2+} \alpha \Leftrightarrow \models \alpha$.

PROOF. See Section 6.

5 Da Costa, Dubikajtis and Kotas' system of the Discursive Logic

In the late seventies, da Costa, Dubikajtis and Kotas published a few papers concerned with an axiomatization of the discursive logic. Their axiomatization contains, among others, the formulas

$\sim((p \wedge_d q) \vee r) \rightarrow_d (p \rightarrow_d \sim (q \vee r))$
$\sim(\sim(p \wedge_d q) \vee r) \rightarrow_d (p \wedge_d \sim(\sim q \vee r))$

as axioms. Notice, however, that they are not valid in Jaśkowski's calculus.

We easily fix the problem by shifting the diamond from the right to the left side of the conjunction, i.e.

(iv)' $f(\alpha \wedge_d \beta) = \Diamond f(\alpha) \wedge f(\beta)$

and then replacing the item $(\wedge_d)$ with

$(\wedge_d)$' $v(\alpha \wedge_d \beta,\ x) = 1 \ \Leftrightarrow\ \exists_{y \in W}(v(\alpha,\ y) = 1)$ and $v(\beta,\ x) = 1$.

At cursory glance, the change seems just cosmetic but dig a little deeper into formulas to realize that it is not the point. For example, the formulas

$\sim((q \wedge_d p) \vee r) \rightarrow_d (p \rightarrow_d \sim (q \vee r))$
$\sim(\sim(q \wedge_d p) \vee r) \rightarrow_d (p \wedge_d \sim(\sim q \vee r))$

are valid in Jaśkowski's calculus, whereas it is not so in the case of da Costa, Dubikajtis and Kotas' system of the discursive logic.[11]

Since there are, nevertheless, striking similarities between the two approaches let us just list a few of them, without trying to be complete.

PROPOSITION 5. (i) Each of the axiom schemata of D_2^+ is valid in D_2^* and $(MP)^*$ preserves validity.

(ii) Assume that α includes, besides variables, at most the connectives $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$ and $\vee$. If α is valid in D_2^* (or D_2), then α_{cpc} is valid in CPC, where α_{cpc} is obtained from α by replacing $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$ with $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$, respectively.

[11] For the sake of brevity, we denote the system by D_2^*.

(iii) Suppose that α contains, besides variables, at most the connectives $\wedge$, $\rightarrow$, $\leftrightarrow$ and $\vee$. If α is valid in CPC, then α_d is valid in D_2^* (and D_2), where α_d is obtained from α by replacing $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$ with $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$, respectively.

So long as we deal with negation-free formulas, there is no difference between D_2 and D_2^*; it does not matter which definition of the discursive conjunction we use.

(iv) Let α contain, besides variables, at most the connectives $\vee$ and $\sim$. If α is valid in CPC, then both α and $\alpha \rightarrow_d q$ is valid in D_2^* (and D_2).

In practice, as stated in Proposition 5, we may read off the validity of some formulas of For_{D2^*} (and For_{D2}) directly from a classical true-value analysis.

Now we focus on a new axiomatization of D_2^*.

(A_{1-9}) α, if $\alpha \in D_2^+$
(A_{10}) $\sim (\sim (\alpha \vee \sim \alpha) \wedge_d \sim\sim \alpha \wedge_d \sim \alpha)$
(A_{11}) $\sim (\sim (\alpha \vee \beta) \wedge_d \sim \beta \wedge_d \sim \alpha) \rightarrow_d \sim (\sim (\alpha \vee \beta \vee \gamma) \wedge_d \sim \gamma \wedge_d \sim \beta \wedge_d \sim \alpha)$
(A_{12}) $\sim (\sim (\alpha \vee \beta \vee \gamma) \wedge_d \sim \gamma \wedge_d \sim \beta \wedge_d \sim \alpha) \rightarrow_d$
$\quad \sim (\sim (\alpha \vee \gamma \vee \beta) \wedge_d \sim \beta \wedge_d \sim \gamma \wedge_d \sim \alpha)$
(A_{13}) $\sim (\sim (\alpha \vee \beta \vee \gamma) \wedge_d \sim \gamma \wedge_d \sim \beta \wedge_d \sim \alpha) \rightarrow_d ((\alpha \vee \beta \vee \sim \gamma) \rightarrow_d (\alpha \vee \beta))$
(A_{14}) $\sim (\sim \alpha \wedge_d \sim \beta) \rightarrow_d (\alpha \vee \beta)$
(A_{15}) $(\alpha \vee (\beta \vee \sim \beta)) \rightarrow_d \sim (\sim (\beta \vee \sim \beta) \wedge_d \sim \alpha)$.

plus $(MP)^*$ α, $\alpha \rightarrow_d \beta$ / β as the sole rule of inference.

The axiom schemata and $(MP)^*$ define $\vdash_{D2^*}$ (the consequence relation).
The differences with respect to the axiomatization of D_2 appear in (A_{10}) - (A_{13}) and (A_{15}) where nothing but the variation of the components of the discursive conjunction does change. Metaphorically speaking, the discursive conjunction changes its flow.

THEOREM 4. $\vdash_{D2^*} \alpha \Leftrightarrow \models \alpha$.

PROOF. See [9].

The discursive conjunction has also changed its flow direction in the following:

Definition 2^*. $\sim_d \alpha = \sim \alpha \wedge_d (p_1 \vee \sim p_1)$

and consequently in $(\wedge_d)$ and (A_2) of ND_2^+.
On the other hand, there is no difference which definition of the discursive conjunctive is preferable to use after having introduced $\neg_d$ (Definition 3); the collapse into the classical logic is inevitable.

6 Metalogic of the Discursive Systems

In this section we concentrate on the metalogical properties of the discursive systems.
Notice that the formulas:

$\alpha \rightarrow_d (\beta \rightarrow_d \alpha)$

$(\alpha \to_d (\beta \to_d \gamma)) \to_d ((\alpha \to_d \beta) \to_d (\alpha \to_d \gamma))$

constitute the implicational fragment of D_2^+ and each of the systems is closed under $(MP)^*$ which is the sole rule of inference of all of the systems we mentioned. Therefore a proof of the deduction theorem is standard.

THEOREM 5. $\Phi \vdash_{D_2^+} \alpha \to_d \beta \Leftrightarrow \Phi \cup \{\alpha\} \vdash_{D_2^+} \beta$, where $\alpha, \beta \in For_{D_2^+}$, $\Phi \subseteq For_{D_2^+}$.

PROPOSITION 6. The formulas listed below are provable in all the discursive adjunctive systems:

(T_1) $(\alpha \vee \alpha) \leftrightarrow_d \alpha$
(T_2) $(\alpha \vee \beta) \leftrightarrow_d (\beta \vee \alpha)$
(T_3) $((\alpha \vee \beta) \vee \gamma) \leftrightarrow_d (\alpha \vee (\beta \vee \gamma))$
(T_4) $(\alpha \vee (\beta \wedge_d \gamma)) \leftrightarrow_d ((\alpha \vee \beta) \wedge_d (\alpha \vee \gamma))$
(T_5) $(\alpha \to_d \beta) \to_d ((\alpha \vee \gamma) \to_d (\beta \vee \gamma))$
(T_6) $(\beta \vee \alpha \vee \beta) \leftrightarrow_d (\alpha \vee \beta)$
(T_7) $(\alpha \wedge_d (\alpha \to_d \beta)) \to_d \beta$

and the set of $\{\alpha : \vdash_{D_2^+} \alpha\}$ is closed under the rules:

(R_1) $\alpha, \beta \ / \ \alpha \wedge_d \beta$
(R_2) $\alpha \wedge_d \beta \ / \ \alpha \ (\beta)$
(R_3) $\alpha \ (\beta) \ / \ \alpha \vee \beta$,

where $\vdash_{D_2^+}$ is the consequence relation defined by D_2^+ and $(MP)^*$.

PROOF. We prove (T_1) - (T_7) in much the same way as it is in the classical propositional logic. (R_1) - (R_3) are obvious due to (A_6), (A_5), (A_4), (A_7), (A_8) of D_2^+ and $(MP)^*$.

PROPOSITION 7. The formulas:

(T_8) $\alpha \vee \sim \alpha$
(T_9) $\sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim (\alpha \vee \beta)) \to_d (\sim (\sim \alpha \wedge_d \sim\sim \beta \wedge_d \sim (\alpha \vee \sim \beta)) \to_d \alpha)$
(T_{10}) $\sim (\sim \alpha \wedge_d \sim \beta \wedge_d \sim (\alpha \vee \beta)) \to_d ((\alpha \vee \sim \beta) \to_d \alpha)$
(T_{11}) $(\alpha \vee \sim \alpha) \to_d \sim (\sim \alpha \wedge_d \sim\sim \alpha \wedge_d \sim (\alpha \vee \sim \alpha))$.

are (schemata of the) theses of D_2.

PROPOSITION 8. The formulas:

(T_{12}) $\alpha \vee \neg_d \alpha$
(T_{13}) $\neg_d (\alpha \wedge_d \neg_d \alpha)$
(T_{14}) $(\alpha \vee \beta) \to_d (\neg_d \beta \to_d \alpha)$

are (schemata of the) theses of SD_2^+.

THEOREM. (i) $\vdash_{D_2} \alpha \Leftrightarrow$ α is valid in D_2.
(ii) $\vdash_{SD_2^+} \alpha \Leftrightarrow$ α is valid in SD_2^+.

PROOF.($\Rightarrow$) By induction.

The initial idea of the proof we present below traces back to [22]. The crucial point is to construct a canonical valuation that falsifies a non-thesis. However, contrary to the Henkin's method, we do not verify, but falsify the sets of formulas we build.

($\Leftarrow$) Assume that $\nvdash_{D_2\ (SD_2^+)} \alpha$ and α is valid in D_2 (SD_2^+). Define a sequence of all the formulas of D_2 (SD_2^+) as follows:

$\Gamma = \gamma_1,\ \gamma_2,\ \gamma_3,\ ...$

The only restriction is that the first element of Γ is α (i.e. $\alpha = \gamma_1$).

Define a family of (finite) subsequences of Γ:

$\Delta_1 = \delta_1 \quad$ where $\delta_1 = \gamma_1 = \alpha$;
$\Delta_2 = \delta_1,\ \delta_2 \quad$ where $\delta_1 = \alpha$ and $\delta_2 = \gamma_2$ if $\nvdash_{D_2} \delta_1 \vee \gamma_2$, otherwise $\delta_2 \neq \gamma_2$ and $\Delta_1 = \Delta_2$;
$\vdots$
$\Delta_n = \delta_1,\ \delta_2 \quad$ where $\delta_1 = \alpha$, $\delta_2 = \gamma_i$ and $\delta_3 = \gamma_{i+k}$
$\vdots$

Define in addition:

$$\nabla_1 = \underbrace{\delta_1,}_{\Delta_1}\ \underbrace{\delta_1, \delta_2,}_{\Delta_2}\ \underbrace{\delta_1, \delta_2, \delta_3,}_{\Delta_3}\ \dots\ ,\ \underbrace{\delta_1, \delta_2, \delta_3,\ \dots\ , \delta_n,}_{\Delta_n}\ \dots$$
$$\nabla_2 = \underbrace{\delta_1, \delta_2,}_{\Delta_2}\ \underbrace{\delta_1, \delta_2, \delta_3,}_{\Delta_3}\ \dots\ ,\ \underbrace{\delta_1, \delta_2, \delta_3,\ \dots\ \delta_n,}_{\Delta_n}\ \dots$$
$$\nabla_3 = \underbrace{\delta_1, \delta_2, \delta_3,}_{\Delta_3}\ \underbrace{\delta_1, \delta_2, \delta_3, \delta_4,}_{\Delta_4}\ \dots\ ,\ \underbrace{\delta_1, \delta_2, \delta_3,\ \dots\ , \delta_n,}_{\Delta_n}\ \dots$$
$$\vdots$$
$$\nabla_n = \underbrace{\delta_1, \delta_2,\ \dots\ , \delta_n,}_{\Delta_n}\ \dots\ ,\ \underbrace{\delta_1, \delta_2,\ \dots\ , \delta_n,\ \dots\ , \delta_{n+k},}_{\Delta_{n+k}}\ \dots$$
$$\vdots$$

Let ∇_i, where $i \in N$, denote both the i-sequence and the set of formulas which contains all the elements of the i-sequence and $\nabla = \{\nabla_1, \nabla_2, ..., \nabla_i, ...\}$.

Lemma 1. (i) $\nvdash_{D_2\ (SD_2^+)} \delta_1 \vee \ldots \vee \delta_n$, for any $n \in N$
(ii) if $\beta \notin \nabla_i$, then $\vdash_{D_2\ (SD_2^+)} \delta_1 \vee \ldots \vee \delta_k \vee \beta$, for some $k \in N$
(iii) $\beta \in \nabla_i \ \Leftrightarrow\ \exists_{\Delta_k \subset \nabla_i} (\beta \in \Delta_k)$, for any $i, k \in N$.

Proof. Straightforward.

Lemma 2. For every $\beta, \gamma \in For_{D2+}$, for any $\nabla_i, \nabla_k \in \nabla$:
(i) $\beta \vee \gamma \in \nabla_i \ \Leftrightarrow\ \beta \in \nabla_i$ and $\gamma \in \nabla_i$
(ii) $\beta \wedge_d \gamma \in \nabla_i \ \Leftrightarrow\ \beta \in \nabla_i$ or $\forall_{\nabla_k \in \nabla} (\gamma \in \nabla_k)$
(iii) $\beta \rightarrow_d \gamma \in \nabla_i \ \Leftrightarrow\ \exists_{\nabla_k \in \nabla} (\beta \notin \nabla_k)$ and $\gamma \in \nabla_i$.

Proof. See [10].

Lemma 3. For every $\beta, \gamma \in For_{SD2+}$, for any $\nabla_i, \nabla_k \in \nabla$:
(i) $\neg_d \beta \in \nabla_i \ \Leftrightarrow\ \exists_{\nabla_k \in \nabla}\ (\beta \notin \nabla_k)$.

Proof. (i)$\Rightarrow$. Assume that (1) $\neg_d \beta \in \nabla_i$ and (2) $\forall_{\nabla_k \in \nabla}\ (\beta \in \nabla_k)$. In particular, (3) $\beta \in \nabla_i$. Apply Lemma 1(i) to get (4) $\nvdash_{SD2+}\ \beta \vee \neg_d \beta$. A contradiction due to (T_{12}).
(i)$\Leftarrow$. Suppose that (1) $\exists_{\nabla_k \in \nabla}\ (\beta \notin \nabla_k)$ and (2) $\neg_d \beta \notin \nabla_i$. Obviously, $i \geq k$ or $k > i$.

Let $i \geq k$. Since $\nabla_k = \Delta_k,\ \ldots\ ,\ \underbrace{\Delta_i, \Delta_{i+1},\ \ldots}_{\nabla_i}$ then $\nabla_i \subseteq \nabla_k$ and (3) $\beta \notin \nabla_i$.
Now use Lemma 1(ii), to obtain (4) $\vdash_{SD2+} \delta_1 \vee \ldots \vee \delta_m \vee \beta$, for some $m \in N$, and (5) $\vdash_{SD2+} \delta_1 \vee \ldots \vee \delta_n \vee \neg_d \beta$, for some $n \in N$. Observe that $m \geq n$ or $n > m$. Suppose then that $m \geq n$ (the case $n > m$ is similar to $m \geq n$). Apply (R_3), (T_2), (T_3), $(MP)^*$ to (5), to deduce (6) $\vdash_{SD2+} \delta_1 \vee \ldots \vee \delta_m \vee \neg_d \beta$. Use (R_1), to get (7) $\vdash_{SD2+} (\delta_1 \vee \ldots \vee \delta_m \vee \beta) \wedge_d (\delta_1 \vee \ldots \vee \delta_m \vee \neg_d \beta)$ and (T_4) to receive (8) $\vdash_{SD2+} (\delta_1 \vee \ldots \vee \delta_m) \vee (\beta \wedge_d \neg_d \beta)$. Since (T_{13}), (T_{14}) are theses of SD_2^+ and the system is closed under $(MP)^*$, we finally obtain (11) $\vdash_{SD2+} \delta_1 \vee \ldots \vee \delta_m$. Notice, however, that $\delta_1, \ldots, \delta_m \in \nabla_i$. A contradiction due to Lemma 1(i).

Let $k > i$. Then $\nabla_k \subseteq \nabla_i$ and (3) $\neg_d \beta \notin \nabla_k$ since $\nabla_i = \Delta_i,\ \ldots\ ,\ \underbrace{\Delta_k, \Delta_{k+1},\ \ldots}_{\nabla_k}$
Next proceed analogously to $i \geq k$.

Lemma 4. For every $\beta, \gamma \in For_{D2}$, for any $\nabla_i, \nabla_k \in \nabla$:
(i) $\sim \beta \in \nabla_i \ \Leftrightarrow\ \beta \notin \nabla_i$.

Proof. (i)$\Rightarrow$. Analogous to the proof of Lemma 3(i)$\Rightarrow$.
(i)$\Leftarrow$. Assume that (1) $\beta \notin \nabla_i$ and (2) $\sim \beta \notin \nabla_i$. Apply Lemma 1 (ii) to (1) and (2), to obtain (3) $\vdash_{D2} \delta_1 \vee \ldots \vee \delta_m \vee \sim \beta$, for some $m \in N$, (4) $\vdash_{D2} \delta_1 \vee \ldots \vee \delta_p \vee \beta$, for some $p \in N$. Clearly, $m \geq p$ or $m < p$.

Let $m \geq p$ (the case $m < p$ is similar to $m \geq p$). Observe that the formula $(\alpha \vee \beta) \rightarrow_d (\sim \beta \rightarrow_d \alpha)$ is not valid in D_2 (for every $\alpha, \beta \in For_{D2}$) we are not, then, allowed to proceed analogously to Lemma 3(i)$\Leftarrow$. In order to solve the problem, some terminological explanations will be of much help.

Definition 4. We call a formula β *classical* if it does not include constant symbols other than $\vee$ and $\sim$. We call a formula β *discursive* if it contains at least one of the symbols: $\rightarrow_d$, $\wedge_d$, $\leftrightarrow_d$. A formula β is a *discursive thesis* if it is a thesis and discursive.

Now for each ∇_i, where $i \in N$, define an additional sequence of formulas, $\nabla_i^\star$, in the following way:

$$\nabla_i^\star = \delta_1^\star,\ \ldots\ ,\ \delta_k^\star,\ \ldots$$

where

(a) $\delta_1^\star = \delta_1 = \gamma_1 = \alpha$
(b) $(\delta_k^\star = \delta_n)$ if $\nvdash_{D2} \sim (\sim \delta_1^\star \wedge_d \ldots \sim \delta_k^\star \wedge_d \sim (\delta_1^\star \vee \ldots \vee \delta_k^\star))$,
for any $\delta_n \in \nabla_i$, $n \geq k$ and $i, k, n \in N$; otherwise $\delta_k^\star \neq \delta_n$.

Proposition 9. (i) $\nabla_i^\star \subseteq \nabla_i$, for every $i \in N$

(ii) $\nvdash_{D2} \sim (\sim \delta_1^\star \wedge_d \ldots \wedge_d \sim \delta_n^\star \wedge_d \sim (\delta_1^\star \vee \ldots \vee \delta_n^\star))$, for every $n \in N$

(iii) if β is not a *discursive thesis*, $\beta \notin \nabla_i$,

then $\vdash_{D2} \sim (\sim \delta_1^\star \wedge_d \ldots \wedge_d \sim \delta_k^\star \wedge_d \sim \beta \wedge_d \sim (\delta_1^\star \vee \ldots \vee \delta_k^\star \vee \beta))$, for some $k \in N$.

Proof. (i) - (ii) Straightforward.
(iii) By the construction of ∇_i $(\nabla_i^\star)$, the fact that $\nabla_i^\star \subseteq \nabla_i$ holds for every $i \in N$, (A_{11}), (A_{12}), (A_{15}), (T_{11}) and $(MP)^*$.

Now we return to the proof of the main lemma. If $\beta \notin \nabla_i$, $\sim \beta \notin \nabla_i$ and $\nabla_i^\star \subseteq \nabla_i$, then (5) $\beta \notin \nabla_i^*$, (6) $\sim \beta \notin \nabla_i^*$. There are three subcases to examine.

(a) neither β nor $\sim \beta$ is a *discursive thesis*

(b) β is a *discursive thesis*, but $\sim \beta$ is not a *discursive thesis*

(c) $\sim \beta$ is a *discursive thesis*, but β is not a *discursive thesis*.

The 4th subcase (both β and $\sim \beta$ is a *discursive thesis*) is impossible due to *Soundness*.

Subcase (a). Let m = 1. Then (7) $\vdash_{D2} \sim (\sim \delta_1^\star \wedge_d \sim \beta \wedge_d \sim (\delta_1^\star \vee \beta))$ and (8) $\vdash_{D2} \sim (\sim \delta_1^\star \wedge_d \sim\sim \beta \wedge_d \sim (\delta_1^\star \vee \sim \beta))$ by Proposition 9(iii), (2), (1). Apply (T_9) and $(MP)^*$ to (7) and (8), to get $\vdash_{D2} \delta_1^\star$. But $\delta_1^\star = \delta_1 = \gamma_1 = \alpha$ and $\nvdash_{D2} \alpha$. A contradiction.

Let m > 1. (7)' $\vdash_{D2} \sim (\sim \delta_1^\star \wedge_d \ldots \sim \delta_k^\star \wedge_d \sim \beta \wedge_d \sim (\delta_1^\star \vee \ldots \vee \delta_k^\star \vee \beta))$, for some $k \in N$. Note that $k < m$ or, at most, $k = m$. If $k < m$, then make use of (A_{11}), (A_{12}) and $(MP)^*$ to get (8)' $\vdash_{D2} \sim (\sim \delta_1^\star \wedge_d \ldots \sim \delta_m^\star \wedge_d \sim \beta \wedge_d \sim (\delta_1^\star \vee \ldots \vee \delta_m^\star \vee \beta))$, where $\delta_1^\star = \delta_1$, $\delta_2^\star = \delta_2$, ..., $\delta_m^\star = \delta_m$. Now take (8)', (3) and use (A_{13}) and $(MP)^*$, to obtain (10)' $\vdash_{D2} \delta_1 \vee \ldots \vee \delta_m$. But $\delta_1, \ldots, \delta_m \in \nabla_i$. A contradiction due to Lemma 1(i).

We prove the subcases (b) and (c) in a very similar way, but you are not allowed to apply Proposition 9(iii) to $\beta \notin \nabla_i$ (in Subcase (b)) and to $\sim \beta \notin \nabla_i$ (in Subcase (c)).

Let $< \nabla, v_c >$ be a canonical model for D_2 (SD_2^+). The canonical valuation $v_c : For_{D2(SD2^+)} \times \nabla \Rightarrow \{1, 0\}$ is defined:

$$v_c(\beta, \nabla_i) = \begin{cases} 1, & \text{if } \beta \notin \nabla_i \\ 0, & \text{if } \beta \in \nabla_i \end{cases}$$

Apply Lemma 2 and Lemma 4 (Lemma 3) to show that the conditions $(\sim)$, $(\vee)$, $(\wedge_d)$ and $(\rightarrow_d)$ $((\neg_d)$, $(\vee)$, $(\wedge_d)$ and $(\rightarrow_d))$ hold for v_c.

Now assume that $\nvdash_{D2(SD2^+)} \alpha$ and α is valid in D_2 (SD_2^+). Notice that α is the very first element of each i-sequence we defined (i.e. for every $i \in N$, $\alpha \in \nabla_i$). Then the formula α is not valid in D_2 (SD_2^+) since there exists a model $< \nabla, v_c >$ such that $v_c(\alpha, \nabla_i) = 0$, for every $\nabla_i \in \nabla$. A contradiction.

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