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Solvability of the halting and reachability problem for binary 2-tag systems

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Abstract. In this paper a detailed proof will be given of the solvability of the halting and reachability problem for binary 2-tag systems.

Keywords: v-Tag systems, halting problem, reachability problem, non-universality

1. Introduction

1.1. Post's frustrating problem of "tag"

Tag systems were invented and studied by Emil Leon post [15, 16] during his Procter fellowship in mathematics at Princeton during the academic year 1920-21. They played an important role in his work on normal systems, which he also developed during that time, and led to the reversal of his program to prove the recursive *solvability* of the Entscheidungsproblem for first-order predicate calculus. Indeed, after 9 months of intensive research on tag systems, Post first came to the conclusion that proving the decidability of the Entscheidungsproblem might be impossible. He never proved that this decision problem is recursively unsolvable. This was done by Church and Turing in their seminal papers published in 1936 [1, 19]. However, already in 1921 he did prove that there are unsolvable decision problems for normal

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systems, on the basis of a thesis similar to Church's and Turing's, called Post's thesis [6, 7]. Unfortunately, he never made any attempt to publish these results at that time. Later, in the forties, he provided a detailed description of his results from the period 1920–1921 in his *Absolutely unsolvable problems and relatively undecidable propositions - Account of an anticipation* [16], a posthumously published manuscript edited by Martin Davis. More detailed information on these more historical matters can be found in [6, 7, 8, 18].

Definition 1. ***(v-tag system)*** A tag system T consists of a finite alphabet Σ of μ symbols, a deletion number $v \in \mathbb{N}$ and a finite set of μ words $w_0, w_1, \ldots, w_{\mu-1} \in \Sigma^*$ called the appendants, where any appendant w_i corresponds to $a_i \in \Sigma$. A v-tag system has a deletion number v.

In a computation step of a tag system T on a word $A \in \Sigma^*$, T appends the appendant associated with the leftmost letter of A at the end of A, and deletes the first v symbols of A.[1] This computational process is iterated until the tag system produces the empty word ϵ. Note that tag systems are monogenic and thus deterministic. Following the notation of [21], $A_i \vdash A_{i+1}$ means that A_{i+1} is produced from A_i after one computation step, $A_i \vdash^n A_{i+n}$ that A_{i+n} is produced after n computation steps from A_i.
To give an example, let us consider the one tag system mentioned by Post with $v = 3, 0 \to 00, 1 \to 1101$ [15, 16]. If the initial word $A_0 = 110111010000$ we get the following productions:

~~110~~111010000
$\vdash$ ~~111~~010000**1101**
$\vdash$ ~~010~~0001101**1101**
$\vdash$ ~~000~~11011101**00**
$\vdash$ ~~110~~1110100**00**

The word A_0 is reproduced after 4 computation steps and is thus an example of a periodic word. Post called the behavior of this one tag system "intractable". Up to now, it is still not known whether this particular example is recursively solvable, despite its formal simplicity. Post also mentioned that he studied the class of tag systems with $v = 2, \mu = 3$ and described this class as being of "*bewildering complexity*" and as "*[...] leading to an overwhelming confusion of classes of cases, with the solution of the corresponding problem depending more and more on problems of ordinary number theory.*" [16].
Post identified three classes of ultimate behavior in tag systems that will be used throughout this paper.

Definition 2. ***(halt)*** A tag system T is said to *halt* on an initial word A_0 when there is an $n \in \mathbb{N}$ such that T produces the empty word ϵ after n computation steps on A_0, i.e., $A_0 \vdash^n \epsilon$ in T.

Definition 3. ***(periodicity)*** A tag system T is said to become *periodic* on an initial word A_0 if there are $n, p \in \mathbb{N}$ such that $A_0 \vdash^n A_n$ and $A_n \vdash^p A_{n+p} = A_n$ in T. A_n is said to be a periodic word in T with period p.

[1]It should be noted that we follow Post's original definition of tag systems, instead of the one that is now commonly used. I.e. instead of first deleting the first v letters and then tagging the appendant, an appendant is first tagged and then the first v symbols are deleted. As a consequence, a tag system will not necessarily halt on a given word when its length has become smaller than v. The proof of the main Theorem only needs some minor changes to be applicable for this slightly different definition.

Definition 4. ***(unbounded growth)*** A tag system T is said to have *unbounded growth* on an initial word A_0, if for every number $n \in \mathbb{N}$ there is an $i \in \mathbb{N}$ such that for every number $j > i$, any word A_j, $A_0 \vdash^j A_j$, $l_{A_j} > n$.

Post considered two decision problems for tag systems, which we will call the *halting problem* and the *reachability problem* for tag systems.

Definition 5. ***(halting problem)*** The halting problem for tag systems is the problem to determine for a given tag system T and any initial word A_0 whether or not T will halt on A_0.

Definition 6. ***(reachability problem)*** The reachability problem for tag systems is the problem to determine for a given tag system T, a fixed initial word A_0 and an arbitrary word $A \in \Sigma^*$, whether or not there is an n such that $A_0 \vdash^n A$ in T.

Note that the halting problem is a special case of the reachability problem.
Post never proved that tag systems are recursively unsolvable. It was Minsky who proved the result in 1961 [13], after the problem was suggested to him by Martin Davis, who was a student of Post. He showed that any Turing machine can be reduced to a tag system with $v = 6$. The result was improved by Cocke and Minsky [2, 3, 14]. They proved that any Turing machine can be reduced to a tag system with $v = 2$. Maslov generalized this result and proved that for any $v > 1$ there exists at least one tag system with an unsolvable decision problem and furthermore proved that any tag system for which $v = 1$ has a solvable reachability problem [12]. This last result was also proven independently by Wang [20].
Both μ and v can be regarded as *decidability criteria* [11] for tag systems, since their solvability depends on the size of these parameters. Another such criterion is the length of the appendants. Wang proved that any tag system for which $l_{\mathbf{min}} \geq v$ or $l_{\mathbf{max}} \leq v$ has a solvable halting and reachability problem [20].
Post mentions that he proved the solvability of the reachability problem (and thus also of the halting problem) for the class of 2-tag systems with $\mu = 2$, but, regretfully, never published the proof. However, he does mention that he used the three classes of ultimate behavior in tag systems he had found. Given any tag system T with $\mu = v = 2$ and any initial word A_0, Post was able to prove that one can decide in a finite number of steps whether or not T will have unbounded growth on A_0. Clearly, if one can determine for any such T and any initial word A_0 it operates on what its ultimate behavior will be one immediately gets that the reachability problem for this class of tag systems is decidable. A proof of this result has recently been found. An outline of this proof can be found in [9]. In this paper we will present the details of this proof.

1.2. Preliminaries

In the remainder of the paper we will use the notations and definitions given in this paragraph.
Let T be a v-tag system in the class of tag systems TS(v, μ) with a deletion number v, μ symbols and appendants $w_0, w_1, ..., w_{\mu-1}$. Then:
a. l_A denotes the *length* of the word A.
b. a^n means that a is repeated n times.
c. $l_{\mathbf{max}}$ denotes the length of the lengthiest appendant w_i, $l_{\mathbf{min}}$ the length of the shortest appendant w_j, $0 \leq i, j < \mu$.

d. $\#a_i$ denotes the total number of occurrences of the symbol a_i in the appendants $w_0, ...w_{\mu-1}$.
e. An odd number is denoted as $\dot{x}$, an even number as $\ddot{x}$. If a number x can be either even or odd, it is denoted as x.
f. $\lfloor x/y \rfloor$ is the largest integer $\leq x/y$, $\lceil x/y \rceil$ is the smallest integer $\geq x/y$, $[x/y]$ denotes either $\lfloor x/y \rfloor$ or $\lceil x/y \rceil$
g. Given some word $W \in \Sigma$ then W^- is W minus its leftmost letter.
h. Given a word $A = a_0 a_1 ... a_{l_A - 1}$, we will say that A is entered with shift x by T, when T erases $a_0..a_{x-1}$ and the first symbol read in A by T is a_x.
i. A round of T on a word A is a number of $\lceil l_A/v \rceil$ computation steps of T on A. Note that one round of T on A is exactly the smallest number of computation steps that result in all the letters of A being deleted by T [20]. For any initial word A_0 the word produced after j rounds on A_0 will be written as Q_j.
j. An s-round of T on a word $A = a_0 a_1 ... a_{l_A - 1}$ produces the word:

$$A' = w_{a_s} w_{a_{v+s}} w_{a_{2v+s}} \dots w_{a_{v(\lceil l_A - s/v \rceil) - 3 + s}}$$

The word A' is thus the result of one round of T on $a_s a_{s+1} a_{s+2} ... a_{l_A - 1}$ (A entered with shift s) without its first $\overline{(l_A - s \bmod v)}$ letters being deleted. I.e., the result of concatenating the appendants associated with every letter read in A when entered with shift s. To explain this with an example, let T be the example provided by Post (Sec. 1.1) and $A = \mathbf{1}1\underline{0}\mathbf{1}1\underline{1}\mathbf{0}1$, then the result of one 0-round on A is the word $A' = 1101110100$, the result of one 2-round on A the word $A' = 001101$. Note that an s-round on A gives the same result as a round on A if $s = 0$ and $l_A \equiv 0 \bmod v$.
k. The additive complement $\overline{(x \bmod y)}$ of a given number x relative to a modulus y is defined as follows:

$$\overline{(x \bmod y)} = \begin{cases} y - (x \bmod y) & \text{if } x \neq 0 \bmod y \\ 0 & \text{if } x \equiv 0 \bmod y \end{cases}$$

2. Solvability of the Halting and Reachability Problem of the Class TS(2,2)

In [16] Post remarks that his proof of the solvability of the reachability problem of the class TS(2, 2) involved "*considerable labor*". This is also true for the proof we have been able to establish, involving the analysis of a large number of cases. One of the major difficulties is that, contrary to classes of Turing machines TM(m, n), one not only has to cope with an infinite number of initial words for each tag system in TS(2,2), but one also has to reduce an infinite number of tag systems to a finite number of cases.
In the current proof Post's approach for proving the decidability of the class TS(2,2) is applied by making use of the three classes of ultimate behavior for tag systems he identified (See Sec. 1.1). I.e., it is proven that given an arbitrary tag system $T \in$ TS(2, 2) and any initial word $A_0 \in \{0, 1\}^*$, one can decide in a finite number of steps whether or not T will halt, become periodic or have unbounded growth and we can thus prove the following Theorem:

Theorem 1. For any given tag system T, if $\mu = v = 2$ then the reachability problem for T is solvable.

Since the halting problem is a special case of the reachability problem, we get the following immediate corollary from Theorem 1:

Corollary 1. For any given tag system T, if $\mu = v = 2$ then the halting problem for T is solvable.

In the remainder of this section we will first explain the main method of the proof (Sec. 2.1) and provide an overview of the general structure of the proof and an explanation of how each of the subcases are determined. This provides an understanding of some of the fundamental differences between the subcases that need to be addressed in order to prove Theorem 1 (Sec. 2.2). We will then turn to the details of the actual proof (Sec. 2.3).

2.1. The table method

The basic technique of the proof is called the *table method* [9]. Intuitively speaking, given a v-tag system T with alphabet Σ and the appendants $w_0, ..., w_{\mu-1}$ this method is used to study all the possible words A that can be contained in any word Q_j produced afer j rounds on some initial word A_0. I.e., if A is produced by the table method, it is possible for T, when started with the proper initial word, to produce a word of the form XAY, with $X, Y \in \Sigma^*$.
The table method is an iterative method. During the n-th iteration step of the table method, first v different words $S_{n,jv+s}$, $s \in \{0, ..., v-1\}$ are produced from each word $S_{n-1,j}$ $0 \leq j < p_{n-1}$ produced in the previous iteration step $n-1$ of the table method. If $n-1 = 0$ then the words $S_{0,0}, S_{0,1}, ..., S_{0,p_0-1}$ are some fixed set of initial words $\in \Sigma^*$, usually the appendants.
Every one of the v words $S_{n,jv+s}$ produced from $S_{n-1,j}$ is the word that results after one s-round of T on $S_{n-1,j}$. For each of the vp_{n-1} words thus produced, if $S_{n,jv+s}$ is equal to ϵ or to one of the $S_{x,y}, y < jv+s, x \leq n$ it is marked. If all the $S_{n,jv+s}$ are marked the method halts. If not, then all $S_{n,jv+s}$ that have been marked are removed, the p_n remaining words are renumbered as $S_{n,0,}, S_{n,1}, S_{n,2}, ..., S_{n,p_n-1}$ and the next iteration can be started.
The method is called the table method because the results from the method can often best be represented through tables. To explain the table method and its representation we will apply it to the example of the 3-tag system mentioned in Sec. 1.1 with $w_0 = 00, w_1 = 1101$, setting $S_{0,0} = w_0, S_{0,1} = w_1$. The following Table shows the first 3 steps of the table method:

	w_0	w_1	w_1w_1	$w_1w_1w_0$	$w_1w_1w_1$	w_0w_1	...
S_0	w_0✓	w_1w_1	$w_1w_1w_0$	$w_1w_1w_0w_0$	$w_1w_1w_0w_1$	w_0w_1✓	...
S_1	w_0✓	w_1✓	$w_1w_1w_1$	$w_1w_1w_1$✓	$w_1w_1w_1w_0$	w_0w_0	...
S_2	ϵ✓	w_0✓	w_0w_1	$w_0w_1w_0$	$w_0w_1w_1w_1$	w_1w_1✓	...

The table is read as follows. Each pair of iterative steps of the table method is separated in the table by a double vertical line. The row headed with S_x gives the word produced at iteration step n after one x-round on the word $S_{n-1,y}$ at the top cell of a column. I.e., the word produced at step n from a given word $S_{n-1,y}$ where the first symbol read in $S_{n-1,y}$ is the $x+1^{th}$ symbol from the left (i.e. the leftmost x symbols in $S_{n-1,y}$ are deleted without being read).
Columns 2 and 3 give the result for step 1. Since there is only one word left unmarked at step 1, i.e., w_1w_1, we need only one column, column 4, for step 2. For step 2 all $vp_1 = v$ words produced are left unmarked. As a result we need 3 columns for step 3. Now, out of the $vp_2 = 9$ words produced, six are left unmarked. This table allows us to study this example in more detail and it seems that, for this

specific example, there will always be words left unmarked.
So why is this method such a useful tool? If we apply the table method to the set of appendants $w_0, w_1, ..., w_{\mu-1}$ of some tag system T (setting each of the $S_{0,i} = w_i$) this method implies that, if it halts at a given iteration step n, then T has a decidable reachability problem. This follows from Lemma 1:

Lemma 1. For any tag system T with deletion number v, alphabet Σ and appendants $w_0, w_1, ..., w_{\mu-1}$, if the table method halts after a finite number of steps n when applied to T with each $S_{0,i} = w_i, 0 \leq i < \mu$ then T will always halt or become periodic on any initial word $A_0 \in \Sigma^*$.

Proof:
Let T be a v-tag system with alphabet Σ and corresponding appendants $w_0, w_1, ..., w_{\mu-1}$ for which the table method halts after a finite number of steps n when applied to T, with each $S_{0,i} = w_i, 0 \leq i < \mu$ and let the union of the appendants and the set of all the different words $S_{i,j}$ that have been produced by the table method after m steps with $0 \leq i \leq m, 0 \leq j < p_i$, including ϵ, be denoted as $\mathbb{S}_m$. It immediately follows that if the table method halts after n iteration steps, $\mathbb{S}_n$ is the finite set of all the possible words that can be produced from the appendants of T by the table method.
Given an initial word $A_0 \in \Sigma^*$, then, after one round of T on A_0 T produces the word:

$$Q_1 = X_0 S_{0,i_1} S_{0,i_2} ... S_{0,i_{\lceil l_{A_0}/v \rceil -1}}$$

with X_0 one of the appendants w_i without its first $\overline{(l_{A_0} \bmod v)}$ letters, each $S_{0,i_m} \in \mathbb{S}_0 = \{w_0, w_1, ..., w_{\mu-1}\}$.
After one more round of T on Q_1 T produces the word:

$$Q_2 = X_1 S_{1,i_1} S_{1,i_2} ... S_{1,i_{\lceil l_{A_0}/v \rceil -1}}$$

with each $S_{1,i_m} \in \mathbb{S}_1$ and X_1 is one of the $S_{i,j} \in \mathbb{S}_1$ minus its first $\overline{(l_{Q_1} \bmod v)}$ letters. Note that the total number of words S_{1,i_m} in Q_2 is equal to $\lceil l_{A_0}/v \rceil - 1$. The reason for this is that each S_{0,i_m} in Q_1 produces exactly one word S_{1,i_m} in Q_2, i.e., the word that is produced after one s-round on S_{0,i_m} where s is determined by the additive complement of the length of the subword preceding S_{0,i_m} in Q_1.
Generally speaking, it easily follows from the table method that after $p+1$ rounds of T on A_0 T produces the word:

$$Q_{p+1} = X_p S_{p,i_1} S_{p,i_2} ... S_{p,i_{\lceil l_{A_0}/v \rceil -1}}$$

with each $S_{p,i_m} \in \mathbb{S}_p$ and X_p is one of the words $S_{i,j} \in \mathbb{S}_p$ minus its first $\overline{(l_{Q_p} \bmod v)}$ letters. Note that the total number of words S_{p,i_m} in Q_{p+1} has remained constant.
Now let us assume that $p = n$. Since we assumed that the table method halts after n steps it follows for any word:

$$Q_{p+k+1} = X_{p+k} S_{p+k,i_1} S_{p+k,i_2} ... S_{p+k,i_{\lceil l_{A_0}/v \rceil -1}}$$

produced after $p+k+1$ rounds of T on A_0, with $k \geq 0$ that each S_{p+k,i_m} in Q_{p+k+1} must be one of the words $S_{i,j} \in \mathbb{S}_p$ since $\mathbb{S}_p = \mathbb{S}_{p+k}$. Similarly, X_{p+k} in Q_{p+k+1}, must be one of the $S_{i,j} \in \mathbb{S}_p$ minus its first $\overline{(l_{Q_{p+k}} \bmod v)}$ letters. It now easily follows that T must either halt or become periodic on A_0. This is the case because the length of any one of the subwords S_{p+k,i_m} and the subword X_{p+k} in any word Q_{p+k+1} produced after $p+k+1$ rounds of T on $A_0, k \in \mathbb{N}$ is bounded by the length of the lengthiest word in $\mathbb{S}_p$. Since the length of the possible productions from A_0 is thus bounded, T must either halt or become periodic on A_0. $\square$

The proof of Lemma 1 reveals a clear connection between the actual productions of a tag system T and the productions of the table method. We also have the following immediate corollary from this proof:

Corollary 1. Given v-tag system T with μ symbols, appendants $w_0, ..., w_{\mu-1}$, $A_0 \in \Sigma^*$ and $\mathbb{S}_i, i \in \mathbb{N}$ the union of the appendants and all the different words that have been produced after i iteration steps of the table method applied to the appendants $w_0, ..., w_{\mu-1}$, then for any word $Q_j, j \geq i+1$ produced after j rounds on A_0:

$$Q_j = X_j V_j$$

where X_j is either one of the words in $\mathbb{S}_i$ minus its first $\overline{(l_{Q_{j-1}} \bmod v)}$ letters or ϵ and $V_j \in \{\mathbb{S}_i\}^*$

The table method is not only useful if it halts when applied to a given tag system. It can also be used to prove that a tag system will either halt or have unbounded growth, resulting in a non-terminating table. This kind of proof is possible because the method also reveals the "structure" of the possible productions of a given tag system. In general, it should be noted that, although this method is very simple, it is an important instrument to study tag systems.
On the basis of the table method we can now introduce the following definition:

Definition 7. ***(s-round)*** We will say that a given tag system T produces a word $\vec{A}_n$ after n s-rounds of T on W, if $\vec{A}_n$ is one of the words produced at step n of the table method, with $p_0 = 1$, $S_{0,0} = W$.

2.2. General structure of the proof

In order to prove Theorem 1 only those tag systems with $l_{\mathbf{min}} < 2$ and $l_{\mathbf{max}} > 2$ need to be taken into account. This follows from Wang's decidability criterion which proves that any tag system T with $l_{\mathbf{min}} \geq v$ or $l_{\mathbf{max}} \leq v$ has a decidable reachability problem. In the remainder, we assume that $l_{\mathbf{max}} = l_{w_1}, l_{\mathbf{min}} = l_{w_0}$. The symmetrical case is equivalent to this case.
It now follows from Wang's decidability criterion and the fact that $\mu = 2$, that we only need to prove Theorem 1 for the following words w_0:

I $w_0 = \epsilon$

II $w_0 = 1$

III $w_0 = 0$

These values for w_0 determine the three global cases of the proof. Each of these three cases will be split into several subcases. These are determined by four different parameters: $\#1$, the parity[2] of l_{w_1}, l_{w_1} and the parities of the number of 0 symbols separating consecutive 1 symbols in w_1.

2.2.1. Parameter 1: $\#1$

The total number $\#1$ of 1 symbols in the appendants of a given tag system T is the parameter used for determining the main subcases for each of the three global cases. For each of these three global cases **I**, **II** and **III** treshold values $n_{\mathbf{I}}, n_{\mathbf{II}}$ and $n_{\mathbf{III}}$ for $\#1$ are determined. It is these treshold values that allow

[2]The parity of a number x is the property of it being even or odd.

us to reduce the infinite number of subcases for each of the main cases to a finite number. It will be proven that the infinite number of tag systems covered by the three main cases for which $\#1 \geq n_{\mathbf{I}}$ (for Case **I**), $\#1 \geq n_{\mathbf{II}}$ (for Case **II**) and $\#1 \geq n_{\mathbf{III}}$ (for Case **III**) always have unbounded growth (except, possibly, for a determined set of initial words), while the finite number of tag systems determined by the three gobal cases for which $\#1 < n_{\mathbf{I}}$ (for Case **I**), $\#1 < n_{\mathbf{II}}$ (for Case **II**) and $\#1 < n_{\mathbf{III}}$ (for Case **III**) always halt or become periodic (except, possibly, for a determined class of initial words).
The parameter $\#1$ is varied going from 0 up to its treshold value. Capital letters A, B, C,... will be used to enumerate these subcases.
One of the main reasons for the significance of $\#1$ is that for any tag system T the symbol 1 corresponds with w_1, the longest appendant and the only possibility for increasing the length of a given word. Furthermore, $\#1$ also has an impact on the length of w_1.

2.2.2. Parameter 2: The parity of l_{w_1}

The parity of l_{w_1} plays a major role in Case **I** and is used as a parameter to further split-up each of the main subcases of Case **I** determined by $\#1$. The parameter is, however, not used in the further factorization of the subcases for cases **II** and **III**.
The reason that l_{w_1} only plays a significant role for Case **I** is the fact that $w_0 = \epsilon$. This means that any word Q_j produced after j rounds on some initial word is always either equal to w_1^n or $w_1^- w_1^n$ with $n \in \mathbb{N}$. I.e., every word Q_j is a word consisting entirely of words w_1.

2.2.3. Parameter 3: l_{w_1}

The next parameter l_{w_1} is only significant with respect to Case **II** where it is used as a parameter to further split-up the subcases determined by $\#1$. For tag systems T with $w_0 = 1$ (Case **II**), the "second-order effect" of reading a 0 can be an increase in the length of a word. Indeed, since for any such T a 0 symbol produces a 1 symbol and a 1 symbol produces w_1 the effect of reading a 0 symbol can indirectly result in an increase of the length of a given word. As a consequence, we not only need to determine a treshold value with respect to $\#1$ but also with respect to l_{w_1} for Case **II**. This is not necessary for cases **I** and **III**. For Case **I**, the effect of reading a 0 is the production of ϵ so it always results in a decrease of the length of a given word. For Case **III**, the effect of reading a 0 in a given word Q_j is also a decrease of the length of Q_j since two letters are deleted and only one, i.e., 0, is appended.

2.2.4. Parameter 4: The parity of the number of 0 symbols separating consecutive 1 symbols in w_1

The parity of the number of 0 symbols separating consecutive 1 symbols in w_1 plays a significant role in each of the three global cases **I**, **II** and **III**. It is the parameter used to further split-up the subcases already determined by the other parameters that are relevant for a given case. (For Case **I** this is the first and the second parameter, for Case **II** the first and the third, and for Case **III** only the first).
The significance of the parity of the number of 0 symbols separating consecutive 1 symbols in w_1 has to do with the fact that, since we are dealing with 2-tag systems, an even number of 0 symbols separating two 1 symbols implies that always one of the two will be read by the tag system, while an odd number

implies that either both 1 symbols or none of the two 1 symbols will be read in w_1. Clearly, this parameter can only start to play a role if the number of 1 symbols in $w_1 \geq 2$.
We will now turn to the details of the proof of Theorem 1.

2.3. Proof of Theorem 1

Case I. $\mathbf{w_0} = \epsilon$.
As explained in Sec. 2.2, the parameters used to prove this case are $\#1$, the parity of l_{w_1} and the different parities of the number of 0 symbols between consecutive 1 symbols in w_1. The parity of l_{w_1} only starts to play a role when $\#1 > 0$ (starting from Case I.B). The parities of the number of 0 symbols between consecutive 1 symbols in w_1 is only relevant once $\#1 \geq 2$ (starting from Case I.C).

Case I.A. $\#1 = 0$ $(w_0 = \epsilon)$**.**
If $\#1 = 0$ and $w_0 = \epsilon$ then $w_1 = 0^{l_{w_1}}$. This immediately implies that any tag system T from this class must always halt, irrespective of the length of w_1. The reason for this is that whatever the initial word A_0 the word Q_1 produced after one round of T on A_0 consists entirely of 0 symbols. We then immediately have that the word Q_2 produced after one more round on Q_1 is equal to ϵ since $w_0 = \epsilon$.

Case I.B. $\#1 = 1$ $(w_0 = \epsilon)$**.**
We split this case into two subcases determined by the parity of l_{w_1}.

Case I.B.1. $l_{w_1} \equiv 0 \bmod 2$ $(\#1 = 1, w_0 = \epsilon)$
Since $\#1 = 1$ and $l_{w_1} \equiv 0 \bmod 2$, either $w_1 = 0^{\dot{r}_1}10^{\ddot{t}_1}$ or $w_1 = 0^{\ddot{r}_1}10^{\dot{t}_1}$. Table 2 proves that the table method halts for any tag system T from this class with $w_1 = 0^{\dot{r}_1}10^{\ddot{t}_1}$.

Table 2: $w_1 = 0^{\dot{r}_1}10^{\ddot{t}_1}$

	w_0	w_1
S_0	ϵ	ϵ
S_1	ϵ	w_1✓

Since the table method halts, T will always halt or become periodic on any initial word A_0. The case with $w_1 = 0^{\ddot{r}_1}10^{\dot{t}_1}$ is symmetrical to this case.

Case I.B.2. $l_{w_1} \equiv 1 \bmod 2$ $(\#1 = 1, w_0 = \epsilon)$**.**
Since $\#1 = 1$ and $l_{w_1} \equiv 1 \bmod 2$, either $w_1 = 0^{\dot{r}_1}10^{\dot{t}_1}$ or $w_1 = 0^{\ddot{r}_1}10^{\ddot{t}_1}$. Table 3 proves the case for $w_1 = 0^{\dot{r}_1}10^{\dot{t}_1}$.

Table 3: $w_1 = 0^{\dot{r}_1}10^{\dot{t}_1}$

	w_0	w_1

S_0	ϵ	$\epsilon\checkmark$
S_1	ϵ	$w_1\checkmark$

The table method halts at step 1, all words produced from w_0 and w_1 being marked. It thus follows that any tag system T from this class with $w_1 = 0^{\dot{r}_1}10^{\dot{t}_1}$ will always either halt or become periodic on some initial word A_0.
The case with $w_1 = 0^{\ddot{r}_1}10^{\ddot{t}_1}$ is symmetrical to this case.

Case I.C. $\#1 = 2$ $(w_0 = \epsilon)$**.**
We split the case into two subcases determined by the parity of l_{w_1}. Note that from now on, $\#1 > 1$. Since $w_0 = \epsilon$, this means that parameter 4 from Sec. 2.2, the parity of the number of 0 symbols separating consecutive 1 symbols in w_1, starts to play its role.

Case I.C.1. $l_{w_1} \equiv 0 \bmod 2$ $(\#1 = 2, w_0 = \epsilon)$**.**
We split the case into two subcases determined by the parity of the number of 0 symbols between the two 1 symbols in w_1. I.e., a case with $w_1 = 0^{\ddot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$ (or, equivalently, $w_1 = 0^{\dot{r}_1}10^{\ddot{x}_1}10^{\dot{t}_1}$) and one with $w_1 = 0^{\dot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$ (or, equivalently, $w_1 = 0^{\ddot{r}_1}10^{\dot{x}_1}10^{\dot{t}_1}$).

Case I.C.1.a. $w_1 = 0^{\ddot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$ **or** $w_1 = 0^{\dot{r}_1}10^{\ddot{x}_1}10^{\dot{t}_1}$ $(\#1 = 2, l_{w_1} \equiv 0 \bmod 2, w_0 = \epsilon)$**.**
Let $w_1 = 0^{\ddot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$. The case is proven by Table 4.

Table 4: $w_1 = 0^{\ddot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$

	w_0	w_1
S_0	ϵ	$w_1\checkmark$
S_1	ϵ	$w_1\checkmark$

Since the table method always halts for this case, it follows that any tag system T from this class will always either halt or become periodic. The case with $w_1 = 0^{\dot{r}_1}10^{\ddot{x}_1}10^{\dot{t}_1}$ reduces to this case, since the table method results in the same productions.

Case I.C.1.b. $w_1 = 0^{\dot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$ **or** $w_1 = 0^{\ddot{r}_1}10^{\dot{x}_1}10^{\dot{t}_1}$ $(\#1 = 2, l_{w_1} \equiv 0 \bmod 2, w_0 = \epsilon)$**.**
Let $w_1 = 0^{\dot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$. Table 5 proves this case.

Table 5: $w_1 = 0^{\dot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$

	w_0	w_1	w_1w_1	...	$(w_1w_1)^n$

S_0	ϵ	ϵ	ϵ	...	ϵ
S_1	ϵ	w_1w_1	$w_1w_1w_1w_1$	....	$(w_1w_1)^{2n}$

It easily follows from Table 5 that any tag system T from this case will either halt or have unbounded growth. So, given some initial word A_0, which of the two kinds of behavior will the tag systems have? Let $\vec{B}_0 = \epsilon, \vec{B}_1 = w_1^2$. Clearly, if Q_j, the word produced after j rounds on A_0, is a word of the form w_1^n, then $Q_{j+1} = \vec{B}_0^n = \epsilon$ because l_{w_1} is even. Similarly, if Q_j is a word of the form $w_1^- w_1^n$ then $Q_{j+1} = \vec{B}_1^- \vec{B}_1^n = w_1^- w_1^{2n}$. From this it immediately follows that if l_{A_0} is even that T will halt after at most two rounds on A_0. If l_{A_0} is odd then $Q_1 = w_1^- w_1^n$ and thus it easily follows that T has unbounded growth on A_0.
The case with $w_1 = 0^{\ddot{r}_1}10^{\ddot{x}_1}10^{\dot{t}_1}$ is symmetrical to this case.

Case I.C.2. $l_{w_1} \equiv 1 \bmod 2$ $(\#1 = 2, w_0 = \epsilon)$**.**
We need to split the case into two subcases determined by the parity of the number of 0 symbols separating the two consecutive 1 symbols in w_1. I.e., a case with $w_1 = 0^{\ddot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$ (or, equivalently, $w_1 = 0^{\dot{r}_1}10^{\dot{x}_1}10^{\dot{t}_1}$) and a case with $w_1 = 0^{\dot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$ (or equivalently $w_1 = 0^{\ddot{r}_1}10^{\ddot{x}_1}10^{\dot{t}_1}$).

Case I.C.2.a. $w_1 = 0^{\dot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$ **or** $w_1 = 0^{\ddot{r}_1}10^{\ddot{x}_1}10^{\dot{t}_1}$ $(\#1 = 2, l_{w_1} \equiv 1 \bmod 2, w_0 = \epsilon)$**.**
Let $w_1 = 0^{\dot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$. Table 6 proves that the table method always halts if $w_1 = 0^{\dot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$ and thus that for this case T will always halt or become perodic.

Table 6: $w_1 = 0^{\dot{r}_1}10^{\ddot{x}_1}10^{\ddot{t}_1}$

	w_0	w_1
S_0	ϵ	w_1✓
S_1	ϵ	w_1✓

The case with $w_1 = 0^{\ddot{r}_1}10^{\ddot{x}_1}10^{\dot{t}_1}$ reduces to this case since the tables resulting from the table method are, for both cases, identical.

Case I.C.2.b. $w_1 = 0^{\ddot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$ **or** $w_1 = 0^{\dot{r}_1}10^{\dot{x}_1}10^{\dot{t}_1}$ $(\#1 = 2, l_{w_1} \equiv 1 \bmod 2, w_0 = \epsilon)$**.**
Let $w_1 = 0^{\ddot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$. Table 7 proves that the table method always halts if $w_1 = 0^{\ddot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$ and thus that for this case T will always halt or become perodic.

Table 7: $w_1 = 0^{\ddot{r}_1}10^{\dot{x}_1}10^{\ddot{t}_1}$

	w_0	w_1	w_1w_1

S_0	ϵ	w_1w_1	w_1w_1✓
S_1	ϵ	ϵ✓	w_1w_1✓

The case with $w_1 = 0^{\dot{r_1}}10^{\dot{x_1}}10^{\dot{t_1}}$ is symmetrical to this case. The only difference is that the results from Table 7 for shifts S_0 and S_1 need to be switched.

Case I.D $\#1 \geq 3$ $(w_0 = \epsilon)$.
This case needs to be split in two subcases determined by the parity of l_{w_1}.

Case I.D.1 $l_{w_1} \equiv 0 \bmod 2$ $(\#1 \geq 3, w_0 = \epsilon)$.
We split the case into two subcases determined by the parity of the number of 0 symbols between consecutive 1 symbols in w_1, i.e., a case with $w_1 = 0^{r_1}10^{\dot{x_1}}10^{\dot{x_2}}10^{\dot{x_3}}...0^{\dot{x}_{\#1-1}}10^{t_1}$, with any $\dot{x}_i$, $0 < i < \#1$ odd and a case with $w_1 = 0^{r_1}10^{x_1}10^{x_2}10^{x_3}...0^{x_{\#1-1}}10^{t_1}$, with at least one x_i, $0 < i < \#1$ that is even.

Case I.D.1.a $w_1 = 0^{r_1}10^{x_1}10^{x_2}10^{x_3}...0^{x_{\#1-1}}10^{t_1}$, at least one x_i even $(w_0 = \epsilon, \#1 \geq 3, l_{w_1} \equiv 0 \bmod 2)$.
Note that for any tag system T from this class either at least two 1 symbols or at least one 1 symbol is read after one s-round on w_1 since w_1 contains at least two consecutive 1 symbols that are separated by an even number of 0 symbols.
The ultimate behaviour of any tag system T from this class can be easily determined. Given w_1 one first needs to determine how many 1 symbols will be read by T in w_1 when entered with shift S_0 and S_1 respectively. Let b_0 be the number of 1 symbols read when w_1 is entered with shift S_0 and b_1 the number of 1 symbols read when w_1 is entered with shift S_1 and let B_0 be the word produced after one 0-round on w_1 and B_1 the word produced after one 1-round on w_1. Then, since $l_{w_1} \equiv 0 \bmod 2$, and $w_0 = \epsilon$, for any word Q_j produced after $j > 1$ rounds on some initial word A_0 we have that $Q_j = B_0^m = (w_1)^{mb_0}$ and $Q_{j+1} = B_0^{mb_0}$ (if l_{A_0} is even) or $B_1^- B_1^m = (w_1^{b_1})^- w_1^{mb_1}$ and $Q_{j+1} = (B_1^{b_1})^- B_1^{mb_1}$ (if l_{A_0} is odd). This is the reason why the ultimate behavior of some of the tag system T covered by this case depends on the parity of l_{A_0}. There are two possibilities:

1. $b_0 \geq 2,\ b_1 = 1$ (or vice versa). If at least one 1 is read in the initial word, T will either have unbounded growth or become periodic depending on the parity of the initial word. Else T will halt.

2. $b_0 \geq 2,\ b_1 \geq 2$ (or vice versa). T will always have unbounded growth, whatever the parity of the initial word if at least one 1 is read in the initial word. Else T will halt.

Case I.D.1.b $w_1 = 0^{r_1}10^{\dot{x_1}}10^{\dot{x_2}}10^{\dot{x_3}}...0^{\dot{x}_{\#1-1}}10^{t_1}$, with any $\dot{x}_i$, $0 < i < \#1$, odd $(\#1 \geq 3, l_{w_1} \equiv 0 \bmod 2, w_0 = \epsilon)$
Note that since all 1 symbols in w_1 are separated by an odd number of 0 symbols it immediately follows that after one s-round on w_1 T produces either ϵ or $w_1^{\#1}$. From this it immediately follows that T either halts or has unbounded growth on any word A_0 depending on the parity of A_0 (see Case I.C.1.b. for more details).

Case I.D.2. $\#1 \geq 3, l_{w_1} \equiv 1 \bmod 2$ $(w_0 = \epsilon)$.
The case is split into two subcases determined by the parity of the number of 0 symbols between consecutive 1 symbols in w_1, i.e., a case with $w_1 = 0^{r_1}10^{\dot{x}_1}10^{\dot{x}_2}10^{\dot{x}_3}...0^{\dot{x}_{\#1-1}}10^{t_1}$, with every $\dot{x}_i$, $0 < i < \#1$ odd and a case with $w_1 = 0^{r_1}10^{x_1}10^{x_2}10^{x_3}...0^{x_{\#1-1}}10^{t_1}$ for which there is at least one x_i, $0 < i < \#1$, that is even.

Case I.D.2.a $w_1 = 0^{r_1}10^{x_1}10^{x_2}10^{x_3}...0^{x_{\#1-1}}10^{t_1}$, at least one x_i even $(w_0 = \epsilon,\ \#1 \geq 3, l_{w_1} \equiv 1 \bmod 2)$
Note that any tag system T from this class will either read *at least* two 1 symbols or one 1 symbol in w_1, depending on the shift with which w_1 is entered. This immediately implies that T can never halt if at least one 1 is read in the initial word because the number of w_1 words either increases or stays the same following each round. Now, since l_{w_1} is odd, it follows that after one s-round on w_1^2, T produces $w_1^n, n \geq 3$. The result of unbounded growth thus immediately follows if the word Q_1 produced after one round on A_0 is either $w_1^n, n \geq 2$ or $w_1^- w_1^n, n \geq 1$. This is always the case if at least two 1 symbols are read in A_0. Otherwise, if $Q_1 = w_1$ or $Q_1 = w_1^-$ (only one 1 symbol is read in A_0) then it can be easily checked that $Q_3 = w_1^m, m \geq 2$ or $Q_3 = w_1^- w_1^m, m \geq 1$ and thus we again have unbounded growth. This means that T will always have unbounded growth on any word A_0 if T reads at least one 1 symbol in A_0.

Case I.D.2.b $w_1 = 0^{r_1}10^{\dot{x}_1}10^{\dot{x}_2}10^{\dot{x}_3}...0^{\dot{x}_{\#1-1}}10^{t_1}$ $(\#1 \geq 3, l_{w_1} \equiv 1 \bmod 2, w_0 = \epsilon)$**.**
Note that since all 1 symbols in w_1 are separated by an odd number of 0 symbols we always have that T produces either $w_1^{\#1}$ or ϵ after one s-round on w_1. Since l_{w_1} is odd every second word w_1 is entered with a different shift, and this implies that every pair of words w_1 results in the production of $w_1^{\#1}$. Since $\#1 \geq 3$ it follows easily that any tag system T from this class will always have unbounded growth on any initial word A_0 if the word $Q_1 = w_1^m, m \geq 2$ or $Q_1 = w_1^- w_1^m, m \geq 1$. It is also easily checked that T will have unbounded growth on A_0 if $Q_1 = w_1$ or $Q_1 = w_1^-$ and $Q_2 = w_1^{\#1}$ or $Q_2 = w_1^- w_1^{\#1-1}$. In all other cases, T halts on A_0.

Case II. $w_0 = 1$**.**
As explained in Sec. 2.2, unlike cases **I** and **III**, l_{w_1} is a parameter that needs to be reckoned with for this case. Also $\#1$ and the different parities of the number of 0 symbols between consecutive 1 symbols in w_1 are used in the proof of this case.
Each subcase defined by $\#1$, will be factorized according to increasing values for l_{w_1} up until a certain treshold value. Since $w_0 = 1$, the parities of the number of 0 symbols between consecutive 1 symbols in w_1 only start to play a role once $\#1 > 2$ (starting from Case II.C).
Note that the smallest value for l_{w_1} is always equal to 3 for each subcase defined by $\#1$. The reason for this is that we only need to take into account those cases with $l_{w_1} \geq 3$ because of Wang's decidability criterion which states that any tag systems with $l_{\mathbf{max}} \leq v$ has a decidable reachability problem.
It is trivial to prove that any tag system with $w_0 = 1$ (Case **II**) always halts on $A_0 = 0$. In what follows we will thus only consider initial words $A_0 \neq 0$.
The following Lemma is an important tool for the proofs of cases II.A.3 and II.B.2-6. The reason for this is that for each of these cases it can be easily proven that any word Q_j produced after j rounds on the initial word A_0 is composed of subwords from a given set such that for any such subword T always

produces a word from that same set in Q_{j+1}.

Lemma 2. Given a v-tag system T with alphabet Σ and corresponding appendants $w_0, w_1, ..., w_{\mu-1}$, some initial word $A_0 \in \Sigma^*$ and $\mathbb{W} = \{W_1, W_1..., W_m\}$ some set of words $\in \Sigma^*$. If one can prove that there is an $n \in \mathbb{N}$ and a set $\mathbb{W}' \subseteq \mathbb{W}$ such that for any word:

$$Q_j = X_j V_j \quad X_j \in \{\epsilon, W_1^-, W_2^-, ..., W_m^-\}, V_j \in \mathbb{W}^*$$

that contains at least p words from the set $\mathbb{W}'$ the following holds (a) there is always at least one subword in Q_j from the set $\mathbb{W}$ from which T produces $W_i W_{i'}$ in Q_{j+n} with $W_i, W_{i'} \in \mathbb{W}$ (b) for any word $W_k \in \mathbb{W}, W_i W_{i'} \neq W_k$, (c) for every other subword in Q_j from the set $\mathbb{W}$ T produces at least one word from that same set in Q_{j+n}, and (d) there are at least p words from the set $\mathbb{W}'$ in Q_{j+n}, then T has unbounded growth on any word A_0 from which T produces a word Q_j after j rounds.

Proof:
The proof is trivial □

Case II.A. $\#1 = 1$ $(w_0 = 1)$**.**
The case is split into 4 cases, according to the value of l_{w_1}. Remember that the smallest value for $l_{w_1} = 3$ because of Wang's decidability criterion.

Case II.A.1. $l_{w_1} = 3$ $(w_0 = 1,\ \#1 = 1)$**.**
Note that since $l_{w_1} = 3, \#1 = 1$ and $w_0 = 1$, $w_1 = 000$.
The result is proven through Table 8:

Table 8: Case $w_1 = 000$

	w_0	w_1	w_0w_0
S_0	w_1✓	w_0w_0	w_1✓
S_1	ϵ✓	w_0✓	w_1✓

Since the table method halts, this tag system will always either halt or become periodic on any initial word A_0.

Case II.A.2. $l_{w_1} = 4$ $(w_0 = 1,\ \#1 = 1)$**.**
Since $l_{w_1} = 4, \#1 = 1$ and $w_0 = 1$, $w_1 = 0000$. Table 9 proves that the table method always halts for T and thus that T will always either halt or become periodic on any initial word A_0.

Table 9: $l_{w_1} = 0000$

	w_0	w_1	w_0w_0
S_0	w_1	w_0w_0	w_1✓
S_1	ϵ✓	w_0w_0	w_1✓

Case II.A.3. $l_{w_1} = 5$ $(w_0 = 1,\ \#1 = 1)$**.**
If $l_{w_1} = 5, \#1 = 1, w_1 = 00000$. This tag system always has unbounded growth except for a finite set of initial words on which T either halts or becomes periodic. This can be proven by applying Lemma 2. First of all, note that after one s-round on w_1 T either produces w_0^3 or w_0^2. From w_0^2 T again produces w_1, from w_0^3 T produces either w_1^2 or again w_1. It is the possibility of producing w_1^2 from w_1 after two s-rounds on w_1 that allows for unbounded growth in this tag system.
It follows from Corollary 1 that for any word $Q_j, j \geq 2$ we have that:

$$Q_j = X_jV_j \quad X_j \in \{\epsilon, w_0, w_1^-\}, V_j \in \{w_0^2, w_0^3, w_1\}^*$$

Note that V_j cannot contain a word w_0 that is neither preceded nor followed by a word w_0. The reason for this is that w_0 can only be produced from w_1, but then w_0 must be either preceded or followed by at least one word w_0.
It now easily follows that after one round of T on Q_j every subword in Q_j from the set $\{w_0, w_1^-, w_1, w_0^2, w_0^3\}$ will again result in at least one subword from that same set in Q_{j+1}. Furthermore, if Q_j contains either w_1^2 or $w_1^-w_1$ then we can prove that also the other conditions of Lemma 2 are met. I.e., for any such word Q_j it can be proven that there is at least one subword in Q_j from the set $\{w_1^-, w_1, w_0^2, w_0^3\}$ that results in the production of a new word that consists of at least two words from that same set in $Q_{j+i}, i \in \{2, 4, 6\}$ and differs from any word in that set (conditions a and b), any other subword in Q_j from the set $\{w_0, w_1^-, w_1, w_0^2, w_0^3\}$ again results in at least one word from that same set in Q_{j+i} (condition c), Q_{j+i} again contains either w_1^2 or $w_1^-w_1$ (condition d).
It can be easily checked that if T produces the word Q_1 after one round on A_0 with Q_1 one of the following words $w_0w_1W, w_1w_0W, w_1w_1W$, or $w_0w_0w_0W, W \in \{w_0, w_1\}^*$ then it takes at most four more rounds to produce a word that contains w_1^2 or $w_1^-w_1$. The only initial words that do not result in such words Q_1 are: (**0**), **1**, **0**0, **0**1, **1**0, **1**1, **0**0**0**, **0**1**0**, **0**0**1**, **0**1**1**, **0**0**0**0, **0**1**0**0, **0**0**0**1, **0**1**0**1, **0**0**0**0**0**, **0**1**0**1**0**, **0**0**0**1**0** or **0**1**0**0**0**. T always becomes periodic on any of these words.
We will now prove that Lemma 2 is indeed applicable for words Q_j that contain w_1^2 or $w_1^-w_1$ at least once. Let Q_j be such a word. Clearly, from w_1^2 or $w_1^-w_1$ in Q_j T produces w_0^5 or $w_0^{5^-}$ in Q_{j+1} and thus either again w_1^2 or w_1^3 in Q_{j+2}. If w_1^3 is produced in Q_{j+2} from w_1^2 in Q_j then all conditions of lemma 2 are met. Indeed, from w_1^2 or $w_1^-w_1$ in Q_j a new word is produced in Q_{j+2} that is a combination of two words from the set $\{w_1^-, w_1, w_0^2, w_0^3\}$ and different from any of the words from that set (conditions a and b), every other subword in Q_j from the set $\{w_1^-, w_1, w_0^2, w_0^3\}$ must result in at least one subword from that same set (condition c) and Q_{j+2} again contains w_1^2 (condition d).
If w_1^2 reproduces itself in Q_{j+2} then this means that w_0^5 in Q_{j+1} produced from w_1^2 in Q_j is entered with shift 1. We then have that $Q_{j+1} = w_0^4W_{j+1}$ or $Q_{j+1} = X_{j+1}W_{j+1,1}w_0^5W_{j+1,2}, W_{j+1}, W_{j+1,1}, W_{j+1,2} \in \{w_0^2, w_0^3, w_1\}^*$. Note that in this last case $l_{X_{j+1}W_{j+1,1}}$ is odd since else w_0^5 in Q_{j+1} would be entered with shift 0, thus resulting in the production of w_1^3 instead of w_1^2 in Q_{j+2}.

The following list gives all the possible words Q_j that contain w_1^2 or $w_1^- w_1$ at least once and from which T produces the word $Q_{j+1} = w_0^4 W_{j+1}$. It proves that all conditions of Lemma 2 are met. I.e. there is an i ($i \in \{2,4,6\}$) such that a subword is produced in Q_{j+i} that is the concatenation of at least two words from the set $\{w_0, w_1^-, w_1, w_0^2, w_0^3\}$ and is different from any word from that set (conditions a and b) and Q_{j+i} again contains w_1^2 or $w_1^- w_1$ at least once (condition d). Note that in order to produce a word $Q_{j+1} = w_0^4 W_{j+1}$, l_{Q_j} must be odd.

(1)	$Q_j = w_1^2\mathbf{w_1}W_j$	$Q_{j+1} = w_0^4 w_0^3 W_{j+1}$	$Q_{j+2} = X_{j+2}w_1\mathbf{w_1^2}W_{j+2},\ X_{j+2} \in \{w_1^-, w_1\}$
(2)	$Q_j = w_1^2\mathbf{w_0^3}W_j$	$Q_{j+1} = w_0^4 w_1^2 W_{j+1}$	$Q_{j+2} = X_{j+2}w_1\mathbf{w_0^5}W_{j+2},\ X_{j+2} \in \{w_1^-, w_1\}$
(3)	$Q_j = w_1^2 w_0^2 W_j$	$Q_{j+1} = w_0^4 w_1 W_j$	$Q_{j+2} = (2), (9), (11), (12)$ or (13)
(4)	$Q_j = w_1^-\mathbf{w_1}$	$Q_{j+4} = w_1^-\mathbf{w_1^2}$	
(5)	$Q_j = w_1^- w_1\mathbf{w_1}$	$l_{Q_j} \equiv 0 \bmod v$	$Q_{j+2} = w_1^- w_1\mathbf{w_1^2}$
(6)	$Q_j = w_1^- w_1\mathbf{w_1^2}W_j$	$Q_{j+1} = w_0^4 w_0^5 W_{j+1}$	$Q_{j+2} = X_{j+2}w_1\mathbf{w_1^3}W_{j+2},\ X_{j+2} \in \{w_1^-, w_1\}$
(7)	$Q_j = w_1^- w_1 w_1\mathbf{w_0^3}W_j$	$Q_{j+1} = w_0^4 w_0^2 w_1^2 W_{j+1}$	$Q_{j+2} = X_{j+2}w_1 w_1\mathbf{w_0^5}W_{j+2},\ X_{j+2} \in \{w_1^-, w_1\}$
(8)	$Q_j = w_1^- w_1 w_1 w_0^2 W_j$	$Q_{j+1} = w_0^4 w_0^2 w_1 W_{j+1}$	$Q_{j+2} = (1)$ or (7)
(9)	$Q_j = w_1^- w_1 w_0^3$	$l_{Q_j} \equiv 0 \bmod v$	$Q_{j+2} = w_1^3 w_0^2$
(10)	$Q_j = w_1^- w_1 w_0^2$	$Q_{j+1} = w_0^4 w_1$	$Q_{j+2} = (9)$
(11)	$Q_j = w_1^- w_1 w_0^5$	$Q_{j+1} = w_0^4 w_1^2$	$Q_{j+2} = w_1^2 w_0^5 = (2)$
(12)	$Q_j = w_1^- w_1\mathbf{w_0^6}W_j$	$Q_{j+1} = w_0^4 w_1^3 W_{j+1}$	$Q_{j+2} = X_{j+2}w_1\mathbf{w_0^8}W_{j+2},\ X_{j+2} \in \{w_1^-, w_1\}$
(13)	$Q_j = w_1^- w_1 w_0^5\mathbf{w_1}W_j$	$Q_{j+1} = w_0^4 w_1^2 w_0^3 W_{j+1}$	$Q_{j+2} = X_{j+2}w_1 w_0^5\mathbf{w_1^2}W_{j+2},\ X_{j+2} \in \{w_1^-, w_1\}$

Note that in each equation above W_{j+1} and W_{j+2} respectively contain at least the same number of words from the set $\{w_0, w_1^-, w_1, w_0^2, w_0^3\}$ as W_j and W_{j+1}.
The same result can be proven for words $Q_{j+1} = X_{j+1}W_{j+1,1}w_0^5 W_{j+1,2}$ by using the same method. The proof is left to the reader.

Case II.A.4. $l_{w_1} > 5$ $(w_0 = 1,\ \#1 = 1)$**.**
Note that with $l_{w_1} > 5$, $\#1 = 1$, $w_1 = 0^6 0^{\#0-6}$. Table 10 proves the case for $l_{w_1} = 6$:

Table 10: $w_1 = 0^6$

	w_0	w_1	w_0^3	w_1^2	w_0^6	w_1^3	...	w_1^n	w_0^{3n}
S_0	w_1	w_0^3	w_1^2	w_0^6	w_1^3	w_0^9	...	w_0^{3n}	$w_1^{\lceil 3n/2 \rceil}$
S_1	ϵ✓	w_0^3	w_1✓	w_0^6	w_1^3	w_0^9	...	w_0^{3n}	$w_1^{\lfloor 3n/2 \rfloor}$

It immediately follows from Table 10 that T will always have unbounded growth on any initial word $A_0 \neq 0$ that results in the production of a word that contains w_1^2 (or $w_1^- w_1$) at least once. It can be easily checked that this is the case for any word $A_0 \neq 0$ (Note that for every word Q_1 produced after one round on $A_0 \neq 0$, $Q_1 = 1W, Q_1 = w_1 W$ or $Q_1 = w_1^- W, W \in \{w_0, w_1\}^*$).

It immediately follows from the generalization of Table 10 that any tag system T with $l_{w_1} > 6$ will always have unbounded growth on any initial word $A_0 \neq 0$.

Case II.B. $\#1 = 2$ $(w_0 = 1)$**.**
We split the case into 6 main cases according to the value of l_{w_1}, i.e., $l_{w_1} = 3, l_{w_1} = 4, l_{w_1} = 5, l_{w_1} = 6, l_{w_1} = 7, l_{w_1} > 7$. We do not consider cases with $l_{w_1} \leq 3$ since it follows from Wang's decidability criterion that for any tag systems T, if $l_{\mathbf{max}} \leq v$ then T has a decidable reachability problem.
It follows from Corollary 1 that for each of the tag systems covered by this case, any word $Q_j, j \geq 2$ produced after j rounds on A_0:

$$(A)\ Q_j = X_j W_j, X_j \in \{\epsilon, \vec{B}_1^-, w_1^-, \vec{A}_1^-\}, W_j \in \{\vec{B}_1, w_1, \vec{A}_1\}^*$$

Case II.B.1. $l_{w_1} = 3$ $(w_0 = 1,\ \#1 = 2)$
It can be determined for any tag system from this class that it will either halt or become periodic. There are three different tag systems covered by this case, i.e., either $w_1 = 100, w_1 = 010$ or $w_1 = 001$. In the following tables it is shown that the table method halts for all three tag systems, and it thus follows that all three tag systems will always halt or become periodic on any initial word.

Table 11: Case $w_0 = 1, w_1 = 100$

	w_0	w_1	w_1w_0	w_0w_1
S_0	w_1	w_1w_0	w_1w_0✓	w_1w_0✓
S_1	ϵ✓	w_0✓	w_0w_1	w_1w_0✓

Table 12: Case $w_0 = 1, w_1 = 010$

	w_0	w_1	w_0w_0
S_0	w_1	w_0w_0	w_1✓
S_1	ϵ✓	w_1✓	w_1✓

Table 13: Case $w_0 = 1, w_1 = 001$

	w_0	w_1	w_0w_1	w_1w_0
S_0	w_1	w_0w_1	w_1w_0	w_0w_1✓
S_1	ϵ✓	w_0✓	w_0w_1✓	w_0w_1✓

Case II.B.2. $l_{w_1} = 4$ $(w_0 = 1,\ \#1 = 2)$**.**

There are exactly 4 tag systems T_1–T_4 covered by this case, i.e., T_1 with $w_{1,1} = 1000$, T_2 with $w_{1,2} = 0100$, T_3 with $w_{1,3} = 0010$ and T_4 with $w_{1,4} = 0001$. For each of these T_i, T_i either produces $\vec{B}_1 = 11$ or $\vec{A}_{1,i} = 1^{\lfloor r_1/2 \rfloor} w_1 1^{\lfloor t_1/2 \rfloor}$ after one s-round on $w_{1,i}$.
Let A_0 be some initial word. It can be proven that if T_i does not result in the production of a word Q_j that contains at least one word $\vec{A}_{1,i}$ or $\vec{A}_{1,i}^-$ after at most 4 rounds on A_0 then T is periodic on A_0. After two rounds on A_0, T produces (Corollary 1):

$$Q_2 = X_2 V_2 \quad X_2 \in \{\epsilon, \vec{B}_1^-, w_{1,i}^-, \vec{A}_{1,i}^-\}, V_2 \in \{\vec{B}_1, w_{1,i}, \vec{A}_{1,i}\}^*$$

If Q_2 does not contain any word $\vec{A}_{1,i}$ or $\vec{A}_{1,i}^-$ then the parity of l_{Q_2} is determined by X_2 since $l_{w_{1,i}}$ and $l_{\vec{B}_1}$ are even. I.e., l_{Q_2} is even if $X_2 = \epsilon$, or else, l_{Q_2} is odd. After one more round on Q_2, T produces:

$$Q_3 = X_3 V_3 \quad X_3 \in \{\epsilon, \vec{B}_1^-, w_{1,i}^-, \vec{A}_{1,i}^-\}, V_2 \in \{\vec{B}_1, w_{1,i}, \vec{A}_{1,i}\}^*$$

If Q_3 does not contain the words $\vec{A}_{1,i}$ or $\vec{A}_{1,i}^-$ then the parity of l_{Q_3} is determined by X_2 in Q_2 since l_{V_2} and l_{V_3} are even and $X_3 = \epsilon$ when l_{Q_2} is even and X_3 is odd when l_{Q_2} is odd. It now follows that if Q_2 and Q_3 do not contain any word $\vec{A}_{1,i}$ or $\vec{A}_{1,i}^-$ then $Q_4 = Q_2$. In order to see this note that every word $w_{1,i}$ and $\vec{B}_1$ in Q_2 is entered with the same shift s (0, when $X_2 = \epsilon$, 1 otherwise) resulting in $\vec{B}_1$ and $w_{1,i}$ in Q_3 respectively. We also have that each word $\vec{B}_1$ and $w_{1,i}$ in Q_3 will be entered with the same shift s resulting in $w_{1,i}$ and $\vec{B}_1$ respectively in Q_4 and $X_4 = X_2$. We thus have that $Q_2 = Q_4$.
We will now prove that T_i always has unbounded growth on any word Q_j that contains $\vec{A}_{1,i}$ or $\vec{A}_{1,i}^-$ at least once by applying Lemma 2. Note first of all that condition c is met. I.e., for any such word Q_j every subword in Q_j from the set $\{\vec{B}_1, w_{1,i}, \vec{A}_{1,i}, \vec{B}_1^-, w_{1,i}^-, \vec{A}_{1,i}^-\}$ will result in the production of at least one word from that same set in Q_{j+1}.
Now, for each T_i we have the following words $\vec{A}_{1,i}$: $\vec{A}_{1,1} = \vec{A}_{1,2} = w_1 1, \vec{A}_{1,3} = \vec{A}_{1,4} = 1 w_1$. After one s-round on $\vec{A}_{1,1}$ and $\vec{A}_{1,4}$ T produces either $\vec{A}_{2,1} = \vec{A}_{1,1} w_{1,1}$ and $\vec{A}_{2,4} = w_{1,4} \vec{A}_{1,4}$, respectively, or $\vec{B}_{2,1} = \vec{B}_{2,4} = \vec{B}_1$. After one s-round on $\vec{A}_{1,2}$ and $\vec{A}_{1,3}$ T produces either $\vec{A}_{2,2} = \vec{B}_1 w_{1,2}$ and $\vec{A}_{2,3} = w_{1,3} \vec{B}_1$ or $\vec{B}_{2,2} = \vec{A}_{1,2}$ and $\vec{B}_{2,3} = \vec{A}_{1,3}$. Since each $l_{\vec{A}_{1,i}}$ is odd, it now easily follows that for any word Q_j that contains at least two words $\vec{A}_{1,i}$ (or one word $\vec{A}_{1,i}^-$ and at least one word $\vec{A}_{1,i}$) there is always at least one word $\vec{A}_{1,i}$ in Q_j that results in two words from the set $\{\vec{B}_1, w_{1,i}, \vec{A}_{1,i}, \vec{B}_1^-, w_{1,i}^-, \vec{A}_{1,i}^-\}$ in Q_{j+1} and Q_{j+1} will again contain at least one word $\vec{A}_{1,i}$. The reason for this is that the only words between two consecutive words $\vec{A}_{1,i}$ are words $\vec{B}_1$ and words $w_{1,i}$. This means that every two words $\vec{A}_{1,i}$ in Q_j are separated by a subword of even length and thus, when the first $\vec{A}_{1,i}$ is entered with shift 0, the second will be entered with shift 1 and vice versa. It thus follows that at least one word $\vec{A}_{1,i}$ in Q_j will result in $\vec{A}_{2,i}$ in Q_{j+1} (Note that $\vec{A}_{2,i}$ is different from any words in the set $\{\vec{B}_1, w_{1,i}, \vec{A}_{1,i}, \vec{B}_1^-, w_{1,i}^-, \vec{A}_{1,i}^-\}$). Thus conditions a, b and c of Lemma 2 are met for words Q_j that contain $\vec{A}_{1,i}$ at least twice (or one word $\vec{A}_{1,i}^-$ and at least one word $\vec{A}_{1,i}$). Furthermore, Q_{j+1} contains at least one word $\vec{A}_{1,i}$ (or $\vec{A}_{1,i}^-$).
The same can be proven if Q_j contains only one word $\vec{A}_{1,i}$ or $\vec{A}_{1,i}$. We can furthermore prove that for these words Q_j property d is met. We will only prove the result for T_1. The proofs for T_2–T_4 follow the same method.
Note that T_1 produces $\vec{A}_{1,1}$ after one 0-round and $\vec{B}_1$ after one 1-round on $w_{1,1}$. Now, given a word Q_j

that contains only one word $\vec{A}_{1,1}$ (or one word $\vec{A}^{-}_{1,1}$), then:

$$Q_j = X_j V_j \vec{A}_{1,1} W_j \text{ (or } Q_j = \vec{A}^{-}_{1,1} W_j) \quad X_j \in \{\epsilon, \vec{B}^{-}_1, w^{-}_{1,1}\}, V_j, W_j \in \{\vec{B}_1, w_{1,1}\}^*$$

If $X_j = \epsilon$ and $Q_j \neq \vec{A}^{-}_{1,1} W_j$ then T produces the word $\vec{A}_{2,1} = \vec{A}_{1,1} w_{1,1}$ in Q_{j+1} and thus conditions a, b,c and d of Lemma 2 are met. I.e., there is at least one subword in Q_j from the set $\{\vec{A}_{1,1}, \vec{B}_1, w_{1,1}, \vec{A}^{-}_{1,1}, \vec{B}^{-}_1, w^{-}_{1,1}\}$ from which T produces a new word that is the concatenation of two words from that same set in Q_{j+1}, this word is different from every word from that set and Q_{j+1} again contains at least one word $\vec{A}_{1,1}$.
If $\vec{A}_{1,1}$ in Q_j is entered with shift 1 then l_{Q_j} is even. This means that after one round on Q_j T produces:

$$Q_{j+1} = V_{j+1} \vec{B}_1 W_{j+1} \quad V_{j+1} \in \{\vec{B}_1, w_{1,1}\}^*, W_{j+1} \in \{w_{1,1}, \vec{A}_{1,1}\}$$

If W_j contains at least one word $w_{1,1}$ then W_{j+1} must contain at least one word $\vec{A}_{1,1}$ (Note that if $\vec{A}_{1,1}$ in Q_j is entered with shift 1 then the leftmost $w_{1,1}$ in W_j is entered with shift 0). It then easily follows that T_1 will produce at least one word $\vec{A}_{2,1}$ in Q_{j+2} since the leftmost word $\vec{A}_{1,1}$ in Q_{j+1} will be entered with shift 0 and thus conditions a–d of Lemma 2 are met.
If W_{j+1} contains no word $\vec{A}_{1,1}$ then $l_{Q_{j+1}}$ is even. In that case, if Q_{j+1} contains at least one word $w_{1,1}$ then it easily follows that it takes two more rounds on Q_{j+1} to produce at least one word $\vec{A}_{2,1}$ in Q_{j+3} (Note that since $l_{Q_{j+1}}$ is even, the first word $\vec{A}_{1,1}$ produced in Q_{j+2} will be entered with shift 0). Finally, if Q_{j+1} contains no word $w_{1,1}$ then, $Q_{j+1} = \vec{B}^{n}_1, n \geq 1$ and it easily follows that it takes three more rounds on Q_{j+1} to produce at least one word $\vec{A}_{2,1}$ in Q_{j+4}. Clearly, also in these last cases conditions a–d of Lemma 2 are met.
The result of unbounded growth now easily follows for words Q_j that contain $\vec{A}_{1,1}$ or $\vec{A}^{-}_{1,1}$ at least once since any such word always satisfies conditions a–d of Lemma 2.

Case II.B.3. $l_{w_1} = 5$ $(w_0 = 1,\ \#1 = 2)$**.**
There are exactly 5 tag systems T_5–T_9 covered by this case. For any T_i, $w_{1,i} = 0^{r_1}10^{t_1}$ and after one s-round on $w_{1,i}$ T_i either produces $\vec{B}_1 = 1^{\lceil r_1 \rceil + \lceil t_1 \rceil}$ or $\vec{A}_{1,i} = 1^{\lfloor r_1/2 \rfloor} w_1 1^{\lfloor t_1/2 \rfloor}$. We split the case according to the value of $\lceil r_1/2 \rceil + \lceil t_1/2 \rceil$, i.e., either $\lceil r_1/2 \rceil + \lceil t_1/2 \rceil = 2$ or $\lceil r_1/2 \rceil + \lceil t_1/2 \rceil = 3$.
Note that for each of these T_i, any word $Q_j, j \geq 2$ is of the form (A).

Case II.B.3.a. $\vec{B}_1 = 11$ $(w_0 = 1,\ \#1 = 2, l_{w_1} = 5)$**.**
There are exactly 3 tag systems covered by this case, i.e., T_5 with $w_{1,5} = 10000$, T_6 with $w_{1,6} = 00100$ and T_7 with $w_{1,7} = 00001$. It is easily proven for each of these tag systems that for any word $A_0 \neq 0$ it takes at most 6 rounds on A_0 to produce a word Q_j that contains at least one word $\vec{A}_{1,i}$ or $\vec{A}^{-}_{1,i}$. The reason for this is that l_{w_1} is odd (note that if Q_2 contains at least two words $\vec{B}_1$ or $w_{1,i}$ then it easily follows that it takes at most two rounds on Q_2 to produce a word that contains at least one word $\vec{A}_{1,i}$ or $\vec{A}^{-}_{1,i}$).
We prove the case by showing that all conditions of Lemma 2 are met for any word $Q_j, j \geq 2$ of form (A) that contains at least one word $\vec{A}_{1,i}$ or $\vec{A}^{-}_{1,i}$. First of all, note that it easily follows that condition (c) is satisified, i.e., every word from the set $\{\vec{B}^{-}_1, w^{-}_1, \vec{A}^{-}_1, \vec{B}_1, w_1, \vec{A}_1\}$ will again result in at least one word from that same set in Q_{j+1}.

The values for the words $\vec{A}_{1,i}$ are $\vec{A}_{1,5} = w_{1,5}\vec{B}_1, \vec{A}_{1,6} = 1w_{1,6}1, \vec{A}_{1,7} = \vec{B}_1 w_{1,7}$. Now, for any word $Q_j = \vec{A}_{1,i}$ or $Q_j = \vec{A}^-_{1,i}$ it can be easily computed that it takes at most 2 rounds of T on Q_j to produce a word that again contains $\vec{A}_{1,i}$ and at least one more word from the set $\{\vec{B}^-_1, w^-_1, \vec{A}^-_1, \vec{B}_1, w_1, \vec{A}_1\}$. This means that we only need to consider words Q_j that contain either one of the following words as a subword:

$$\begin{array}{ll} (a) = & \vec{B}_1\vec{A}_{1,i} \text{ (or } \vec{B}^-_1\vec{A}_{1,i}) \\ (b) = & \vec{A}_{1,i}\vec{B}_1 \text{ (or } \vec{A}^-_{1,i}\vec{B}_1 \\ (c) = & w_{1,i}\vec{A}_{1,i} \text{ (or } w^-_{1,i}\vec{A}_{1,i}) \\ (d) = & \vec{A}_{1,i}w_{1,i} \text{ (or } \vec{A}^-_{1,i}w_{1,i} \\ (e) = & \vec{A}^2_{1,i} \text{ (or } \vec{A}^-_{1,i}\vec{A}_{1,i}) \end{array}$$

For any tag system T_5–T_7 it can be easily checked that if Q_j contains at least one of the subwords (a)–(e) then it takes at most three rounds of T on Q_j to produce from each of these subwords again one of the words (a)–(e) plus at least one word from the set $\{\vec{B}_1, w_{1,i}, \vec{A}_{1,i}, \vec{B}^-_1, w^-_{1,i}, \vec{A}^-_{1,i}\}$ in the word $Q_{j+i}, 0 < i \leq 3$. It thus easily follows that conditions a–d of Lemma 2 are satisfied and the result of unbounded growth immediately follows.

Case II.B.3.b. $\vec{B}_1 = 111$ $(w_0 = 1,\ \#1 = 2, l_{w_1} = 5)$**.**
There are 2 tag systems covered by this case, i.e., T_8 with $w_{1,8} = 01000$, T_9 with $w_{1,9} = 00010$. Now, given some initial word $A_0 \neq 0$ it is easily checked that either T_i is periodic on A_0 or it takes at most 4 rounds on A_0 to produce a word that contains at least one word $\vec{A}_{1,i}$ or $\vec{A}^-_{1,i}$. Both tag systems are periodic on $w_{1,i}$ and 11.
We will now prove that T_i always has unbounded growth on any word Q_j that contains $\vec{A}_{1,i}$ or $\vec{A}^-_{1,i}$ at least once by applying Lemma 2. The main difficulty with these two tag systems is that after one s-round on $\vec{A}_{1,i}$ $(\vec{A}_{1,8} = w_{1,8}1, \vec{A}_{1,9} = 1w_{1,9})$ T_i either produces a word $\vec{A}_{2,i}$ that contains w_1 and $\vec{A}_{1,i}$ *or* $\vec{B}_1 = 111$. We will only consider T_8 since $w_{1,9}$ is the mirror image of $w_{1,8}$ and the proof of T_9 thus easily reduces to the proof of T_8.
Now, it is easily computed that for any word $Q_j = \vec{A}_{1,8}$ or $Q_j = \vec{A}^-_{1,8}$ it takes at most 3 rounds of T_8 on Q_j to produce a word that contains $\vec{A}^-_{1,8}$ and either $\vec{B}_1$ or $w_{1,8}$. We thus only need to consider words Q_j that contains one of the words (a)–(e) with $i = 8$ as a subword.
It can be easily verified that if Q_j contains at least one of the subwords (b)–(e) then it takes at most three rounds on Q_j to produce from any of these subwords (b)-(e) again one of he words (a)–(e) and at least one more word from the set $\{\vec{B}_1, w_{1,8}, \vec{A}_{1,8}, \vec{B}^-_1, w^-_{1,8}, \vec{A}^-_{1,8}\}$ in the word $Q_{j+i}, 0 < i \leq 3$. The problematic cases are words that contain only one word $\vec{A}_{1,i}$ and $\vec{A}_{1,i}$ is preceded by $\vec{B}_1$. There is no problem when (a) is entered with shift 0: after one 0-round on (a) T_8 produces $w^2_{1,8}\vec{A}_{1,8}w_{1,8}$. However, when (a) is entered with shift 1, then T_8 produces $w_{1,8}\vec{B}_1$, after one 0-round on $w_{1,8}\vec{B}_1$ T_8 produces $\vec{B}_1 w_{1,8}$ and after one 1-round on $\vec{B}_1 w_{1,8}$ T_8 again produces $w_{1,8}\vec{B}_1$. This means that there is not necessarily an n such that the subword (a) results again in one of the words (a)–(e) and at least one more subword from the set $\{\vec{B}_1, w_{1,8}, \vec{A}_{1,8}, \vec{B}^-_1, w^-_{1,8}, \vec{A}^-_{1,8}\}$ in Q_{j+n} so we need to consider this special case separately.
Now, if $Q_j = (a)^-$ then it can be easily checked that $Q_{j+3} = w^2_{1,8}\vec{A}_{1,8}$. If $Q_j \neq (a)^-$ but contains (a) and (a) is followed by any word $\vec{B}_1, w_{1,8}$ or $\vec{A}_{1,8}$ or (a) is preceded by $\vec{A}_{1,8}$ then Q_j contains at

least one of the words (b)–(e). If (a) is preceded by $\vec{B}_1$ then it is easily computed that it takes at most two rounds on Q_j to produce from $\vec{B}_1^2\vec{A}_{1,8}$ one word (a)–(e) and at least one word from the set $\{\vec{B}_1, w_{1,8}, \vec{A}_{1,8}, \vec{B}_1^-, w_{1,8}^-, \vec{A}_{1,8}^-\}$. A similar result is easily proven when Q_j contains (a) and $\vec{B}_1^2$ (note that after one s-round of T_8 on $\vec{B}_1^2$ T_8 produces $w_{1,8}^3$). The only remaining possibilities are that Q_j is one of the following words:

$$\begin{array}{ll} Q_j = X_j w_1^{n_1}(\vec{B}_1 w_{1,8})^{n_1}...w_1^{n_i}(\vec{B}_1 w_{1,8})^{n_i}\vec{B}_1\vec{A}_{1,8} & X_j \in \{\epsilon, \vec{B}_1^- w_{1,8}, w_{1,8}^-\}, n_j \in \mathbb{N} \\ Q_j = X_j w_1^{n_1}(w_{1,8}\vec{B}_1)^{n_1}...w_1^{n_i}(w_{1,8}\vec{B}_1)^{n_i}\vec{A}_{1,8} & X_j \in \{\epsilon, w_{1,8}^-\vec{B}_1, w_{1,8}^-\}, n_j \in \mathbb{N} \end{array}$$

It can be easily checked for each of these possibilities that it takes at most 3 rounds to produce a word $Q_{j+i}, 0 < i \leq 3$ such that Q_{j+i} contains at least one of the words (b)–(e). It now easily follows from Lemma 2 that T_8 always has unbounded growth on any word $A_0 \neq 0$ for which T_8 produces a word that contains at least one word $\vec{A}_{1,i}$ or $\vec{A}_{1,i}^-$.

Case II.B.4. $l_{w_1} = 6$ $(w_0 = 1,\ \#1 = 2)$**.**
There are 6 tag systems T_{10}–T_{15} covered by this case, i.e., T_{10} with $w_{1,10} = 100000$ and T_{11} with $w_{1,11} = 010000$, T_{12} with $w_{1,12} = 001000$, T_{13} with $w_{1,13} = 000100$, T_{14} with $w_{1,14} = 000010$,T_{15} with $w_{1,15} = 000001$. For each of these T_i, T_i either produces $\vec{B}_1 = 111$ or $\vec{A}_{1,i} = 1^{\lfloor r_1/2\rfloor}w_1 1^{\lfloor t_1/2\rfloor}$ after one s-round on $w_{1,i}$. For each of these T_i, T_i either produces $\vec{B}_1 = 111$ or $\vec{A}_{1,i} = 1^{\lfloor r_1/2\rfloor}w_1 1^{\lfloor t_1/2\rfloor}$ after one s-round on $w_{1,i}$. Also, every word Q_j is a word of the form (A). From this, it easily follows that condition c of Lemma 2 is satisfied. I.e., for any of the subwords in Q_j from the set $\{\vec{B}_1^-, w_1^-, \vec{A}_1^-, \vec{B}_1, w_1, \vec{A}_1\}$ T_i again produces a word from that same set in Q_{j+1}. We can also prove that all the other conditions of Lemma 2are met.
It is easily proven that T_{10}–T_{15} always have unbounded growth on any word Q_j that contains one of the subwords (a)-(e), $10 \leq i \leq 15$ or:

$$\begin{array}{ll} (f) & w_{1,i}^2 \text{ (or } w_{1,i}^- w_{1,i}) \\ (g) & w_{1,i}\vec{B}_1 \text{ (or } w_{1,i}^-\vec{B}_1) \\ (h) & \vec{B}_1 w_{1,i} \text{ (or } \vec{B}_1^- w_{1,i}) \\ (i) & \vec{B}_1^2 \text{ (or } \vec{B}_1^-\vec{B}_1) \end{array}$$

Indeed, it can be verified that if one of these subwords (a)-(i) is in Q_j then it takes at most two rounds on Q_j to produce from one of these subwords (a)-(i) again one of these subwords and at least one more word from the set $\{\vec{B}_1, w_{1,i}, \vec{A}_{1,i}, \vec{B}_1^-, w_{1,i}^-, \vec{A}_{1,i}^-\}$ in the word $Q_{j+i}, 0 < i \leq 2$.
It is furthermore easily proven that for any word $A_0 \neq 0$ it takes but a finite number of rounds of T_i on A_0 to produce a word that contains at least one of the subwords (a)-(i). Hence it easily follows that each T_i always has unbounded growth on any initial word A_0.

Case II.B.5. $l_{w_1} = 7$ $(w_0 = 1,\ \#1 = 2)$**.**
There are 7 tag systems covered by this case. For any T_i, $w_{1,i} = 0^{r_1}10^{t_1}$ and after one s-round on $w_{1,i}$ T_i either produces $\vec{B}_1 = 1^{\lceil r_1\rceil + \lceil t_1\rceil}$ or $\vec{A}_{1,i} = 1^{\lfloor r_1/2\rfloor}w_1 1^{\lfloor t_1/2\rfloor}$. We split the case according to the value of $\lceil r_1/2\rceil + \lceil t_1/2\rceil$, i.e., either $\lceil r_1/2\rceil + \lceil t_1/2\rceil = 3$ or $\lceil r_1/2\rceil + \lceil t_1/2\rceil = 4$

Case II.B.5.a. $\vec{B}_1 = 111$ $(w_0 = 1,\ \#1 = 2, l_{w_1} = 7)$**.**
There are 4 tag systems covered by this case, i.e., T_{16}, $w_{1,16} = 1000000$, T_{17} with $w_{1,17} = 0010000$, T_{18} with $w_{1,18} = 0000100$ and T_{19} with $w_{1,19} = 0000001$. These tag systems always have unbounded growth on any initial word $A_0 \neq 0$. The proof is similar to the proof of unbounded growth for T_{10}–T_{15} and thus we leave it to the reader.

Case II.B.5.b. $\vec{B}_1 = 1111$ $(w_0 = 1,\ \#1 = 2, l_{w_1} = 7)$**.**
There are exactly 3 tag systems $T_{20} - T_{22}$ for which $\vec{B}_1 = 1111$, i.e., T_{20} with $w_{1,20} = 0100000$, T_{21} with $w_{1,21} = 0001000$ and $T_{22} = 0000010$. After one s-round on $w_{1,i}$ each of these T_i either produces $\vec{B}_1 = 1111$ or $\vec{A}_{1,i} = 1^{\lfloor r_1/2 \rfloor} w_1 1^{\lfloor t_1/2 \rfloor}$. It is easily proven that every T_i has unbounded growth on any word $A_0 \neq 0$. The reason for this is that, on the one hand, it easily follows that $l_{\vec{A}_{1,i}} > l_{w_{1,i}}$ and, on the other hand, every word $\vec{B}_1$ produced from one word $w_{1,i}$ always results in the production of two words $w_{1,i}$, whatever shift $\vec{B}_1$ is entered with. Thus, once w_1 is produced from the initial word A_0 then T must have unbounded growth on A_0. It can be easily checked that it takes at most 2 rounds on any initial word $A_0 \neq 0$ to produce a word that contains w_1 at least once, hence the result of unbounded growth.

Case II.B.6. $_{w_1} > 7$ $(w_0 = 1,\ \#1 = 2)$**.**
Let T be a tag system covered by Case II.B.6. After one s-round on w_1 T either produces $\vec{B}_1 = 1^n$ or $\vec{A}_{1,i} = 1^{\lfloor r_1/2 \rfloor} w_1 1^{\lfloor t_1/2 \rfloor}$. However, since $l_{w_1} > 7$ and w_1 contains only one 1 symbol, it easily follows that $n \geq 4$ in $\vec{B}_1 = 1^n$. The result of unbounded growth thus easily follows (See Case II.B.5.b).

Case II.C. $\#1 > 2$ $(w_0 = 1)$**.**
We split the case into two main cases according to the value of l_{w_1}, i.e.,$l_{w_1} = 3$ and $l_{w_1} > 3$. We do not take into account the case with $l_{w_1} < 3$ due to Wang's decidability criterion.

Case II.C.1. $l_{w_1} = 3$ $(\#1 > 2, w_0 = 1)$**.**
There are exactly four tag systems in this class depending on the value of w_1, i.e., $w_1 = 111, w_1 = 101, w_1 = 110$ and $w_1 = 011$. We need to study each of these tag systems separately.

Case II.C.1.a. $w_1 = 111$ $(l_{w_1} = 3, \#1 > 2, w_0 = 1)$**.**
This tag system T will always have unbounded growth on any word $A_0 \neq 0$. The reason for this is that for this case any word Q_j produced after j rounds on A_0 consists entirely of 1 symbols and with every computation step, two 1 symbols are appended and only 1 deleted.

Case II.C.1.b. $w_1 = 101$ $(l_{w_1} = 3, \#1 > 2, w_0 = 1)$**.**
We will prove that this tag system always has unbounded growth on any initial word $A_0 \neq 0$. Note that T either reads two 1 symbols or one 0 during one s-round on w_1. In order to prove the case we apply the table method for 2 iterations on w_1:

Table 14: $w_1 = 101$

	w_1	$\vec{A}_1$	$\vec{A}_2$	$\vec{B}_2$

S_0	$\vec{A}_1 = w_1^2$	$\vec{A}_2 = \vec{A}_1\vec{B}_1$	$\vec{A}_3 = \vec{A}_2 w_1$	$\vec{B}_3 = w_1\vec{B}_2$
S_1	$\vec{B}_1 = w_0$✓	$\vec{B}_2 = \vec{B}_1\vec{A}_1$	$\vec{B}_2$✓	$\vec{A}_2$✓

It is easily checked that T is periodic on the words w_1^- and $\vec{B}_1$. It can also be checked that it takes at most 3 rounds on any initial word A_0 that differs from these two periodic words and $A_0 \neq 0$ before T produces a word that contains $\vec{A}_1$ or $\vec{A}_1^-$ at least once. Since it follows from Table 14 that after one s-round on $\vec{A}_1$ T produces either $\vec{A}_2$ or $\vec{B}_2$ and after one s-round of T on $\vec{A}_2$ or $\vec{B}_2$ T always produces a word that contains either $\vec{A}_2$ or $\vec{B}_2$ it follows that any word $Q_j, j > 3$ always contains $\vec{A}_2$,$\vec{A}_2^-$,$\vec{B}_2$ or $\vec{B}_2^-$ at least once. Furthermore, for any such word $Q_j, j > 3$ it is impossible that Q_j contains w_0^2 as a subword. I.e., there is no combination of words w_1 and w_0 that results in w_0^2 (Note that the only way to produce w_0 is when w_1 is entered with shift 1). We can now define Q_j as follows:

$$Q_j = X_j V_j Y_j \quad X_j \in \{w_1^- w_0, w_1^- w_1, \epsilon\}, Y_j \in \{w_0, w_1, \epsilon\}$$
$$V_j \in \{w_0 w_1, w_1 w_0, w_1^2, \vec{A}_2, \vec{B}_2\}^*$$

with the following restrictions on X_j, V_j and Y_j: (1) if $X_j = w_1^- w_0$ then the left end of V_j cannot be $w_0 w_1$ or $\vec{B}_2$ (2) no subword $\vec{B}_2$ or $w_0 w_1$ in V_j can be preceded by $\vec{A}_2$ or $w_1 w_0$ and, finally (3) if Y_j is w_0 then the right end of V_j cannot be $\vec{A}_2$ or $w_1 w_0$.

We will now prove that T always has unbounded growth on any such word Q_j. In order to prove this, first of all note that after one round on Q_j every one of the subwords in $X_j V_j$, i.e., words from the set $\{w_1^- w_0, w_1^- w_1, w_0 w_1, w_1 w_0, w_1^2, \vec{A}_2, \vec{B}_2\}$, results in at least one subword from that same set in Q_{j+1} which is at least as long as the original word in Q_j. Furthermore, Q_{j+1} is of the same form as Q_j again containing at least one word $\vec{A}_2$,$\vec{A}_2^-$,$\vec{B}_2$ or $\vec{B}_2^-$. This follows from the productions of table 14. It also follows immediately that the length of Y_j, which is either equal to w_0, w_1 or ϵ, is bounded.

Now, if Q_j is one of the words $\vec{A}_2$, $\vec{A}_2^-$,$\vec{B}_2$ or $\vec{B}_2^-$ then it is easily checked that it takes at most 2 rounds of T on Q_j to produce a word that again contains $\vec{A}_2$, $\vec{A}_2^-$ or $\vec{B}_2$ or $\vec{B}_2^-$ and at least one word w_0 or w_1. This means that we only need to study words Q_j that contain at least one of the following words as a subword:

$$\begin{array}{lllll}
(1) & \vec{A}_2 w_0 = w_1^2 w_0^2 & w_0\vec{A}_2 = w_0 w_1^2 w_0 & \vec{A}_2^- w_0 = w_1^- w_1 w_0^2 & \\
(2) & \vec{A}_2 w_1 = w_1^2 w_0 w_1 & w_1\vec{A}_2 = w_1^3 w_0 & \vec{A}_2^- w_1 = w_1^- w_1 w_0 w_1 & w_1^-\vec{A}_2 = w_1^- w_1^2 w_0 \\
(3) & \vec{B}_2 w_0 = w_0 w_1^2 w_0 & w_0\vec{B}_2 = w_0^2 w_1^2 & \vec{B}_2^- w_0 = w_1^2 w_0 & \\
(4) & \vec{B}_2\vec{w}_1 = w_0 w_1^3 & \vec{w}_1\vec{B}_2 = w_1 w_0 w_1^2 & \vec{B}_2^- w_1 = w_1^3 & w_1^-\vec{B}_2 = w_1^- w_0 w_1^2
\end{array}$$

It is easily checked that if Q_j contains any of the subwords (1)–(4) then it takes at most 3 rounds of T on Q_j to produce from any of these subwords again one of the words (1)–(4) and at least one more subword from the set $\{w_1, \vec{B}_1 w_1, w_1\vec{B}_1\}$ (Note that the words $\vec{A}_2 w_0$, $\vec{A}_2^- w_0$ and $w_0\vec{B}_2$ contain w_0^2 as a subword, so we do not need to take them into account. If Q_j contains $\vec{A}_2^- w_1$, $w_1^-\vec{A}_2$, $\vec{B}_2^- w_0$, $\vec{B}_2^- w_1$ or $w_1^-\vec{B}_2$ these words are at the left end of Q_j and are thus entered with shift 0).

It now easily follows that T always has unbounded growth on any word Q_j that contains one of the words

(1)-(4) at least once.

Case II.C.1.c. $w_1 = 110$ ($l_{w_1} = 3, \#1 > 2, w_0 = 1$).
It can be proven that this tag system is periodic on any word $A_0 \neq 0$ from which T produces $(w_1 1)^n$ after one round and has unbounded growth on any other initial word $A_0 \neq 0$. Note that T always reads one symbol 1 during one s-round on w_1. This immediately implies that T cannot halt on any word A_0 that results in the production of a word that contains at least one w_1 or one w_1^-. It can be easily checked that it takes at most two rounds of T on $A_0 \neq 0$ to produce a word that contains w_1, thus T can never halt on any word $A_0 \neq 0$. In order to prove the case, we need to apply the table method to w_1.

Table 15: $w_1 = 110$

	w_1	$\vec{A}_1$
S_0	$\vec{A}_1 = w_1 1$	$\vec{A}_1 \checkmark$
S_1	$\vec{B}_1 = w_1 \checkmark$	$\vec{B}_2 = w_1^2$

Given an initial word $A_0 \neq 0$, then it follows from Table 15 and Corollary 1 that any word $Q_j, j > 1$:

$$Q_j = X_j V_j \quad X_j \in \{\epsilon, w_1^-, \vec{B}_2^-, \vec{A}_1^-\}, V_j \in \{w_1, \vec{B}_2, \vec{A}_1\}^*$$

If $Q_j = \vec{A}_1^n, n \in \mathbb{N}$ then Q_j is periodic. The reason for this is that $\vec{A}_1$ reproduces itself when entered with shift 0 and $l_{\vec{A}_1}$ is even.
We can now prove that if $Q_j \neq \vec{A}_1^n, n \in \mathbb{N}, j > 1$ then T has unbounded growth on Q_j. We then have:

(1) $Q_j = w_1^-$
(2) $Q_j = w_1^- \vec{A}_1^n, n \in \mathbb{N}$
(3) $Q_j = \vec{A}_1^- \vec{A}_1^n, n \in \mathbb{N}$
(4) $Q_j = X_j V_j w_1 Y_j W_j \quad X_j \in \{\epsilon, w_1^-, \vec{B}_2^-, \vec{A}_1^-\}, V_j \in \{\vec{A}_1\}^*, Y_j \in \{\epsilon, w_1, \vec{B}_2, \vec{A}_1\}, W_j \in \{w_1, \vec{B}_2, \vec{A}_1\}^*$

If Q_j is one of the words of the forms given in (1), (2) or (3) then it can be easily checked that the word Q_{j+1} produced after one round on Q_j is a word of the form given in (4) and $l_{Q_{j+1}} > l_{Q_j}$. We thus only need to consider words Q_j that are words of form (4).
It easily follows that T must have unbounded growth on any word of form (4). Note first of all that none of the subwords in X_j, V_j and W_j can become shorter. This follows from Table 15.
Now, if $Y_j = w_1$ or $Y_j = \vec{B}_2$ we immediately have that $l_{Q_{j+1}} > l_{Q_j}$ since whatever shift $w_1 Y_j$ is then entered with, it results in the production of a subword in Q_{j+1} that is lengthier than $w_1 Y_j$ (see Table 15). Furthermore Q_{j+1} is again a word of the form (4).
If $Y_j = \vec{A}_1$ and $w_1 \vec{A}_1$ is entered with shift 0 T produces $\vec{A}_1 w_1^2$ in Q_{j+1} from $w_1 \vec{A}_1$ in Q_j. We thus again have that $l_{Q_{j+1}} > l_{Q_j}$ and Q_{j+1} is again a word of form (4). If, on the other hand, $w_1 \vec{A}_1$ is entered with shift 1 then T reproduces $w_1 \vec{A}_1$ in Q_{j+1}. However, if $w_1 \vec{A}_1$ in Q_j is entered with shift 1 then it must be

the case that either $X_j = \vec{B}_2^-$ or $X_j = \vec{A}_1^-$ since $l_{w_1^-}$ and $l_{\vec{A}_1}$ are even and we have assumed that $w_1\vec{A}_1$ is entered with shift 1. But then it must be the case that $l_{Q_{j+1}} > l_{Q_j}$ since after one round on Q_j, $\vec{B}_2^-$ in Q_j results in $w_1\vec{A}_1$ (or $w_1^-\vec{A}_1$) and $\vec{A}_1^-$ results in w_1^2 (or $w_1^- w_1$). Since $l_{w_1^-\vec{A}_1} > l_{\vec{B}_2^-}$ and $l_{w_1^- w_1} > l_{\vec{A}_1^-}$, we again have that $l_{Q_{j+1}} > l_{Q_j}$ and Q_{j+1} is again a word of form (4).
Finally, if $Y_j = \epsilon$ and w_1Y_j is entered with shift 0 then T produces $\vec{A}_1$ from w_1 in Q_j, when entered with shift 1, T produces w_1 in Q_{j+1} from w_1 in Q_j. If w_1Y_j is entered with shift 0 this either means that $Q_j = w_1$, $Q_j = \vec{A}_1^n w_1$ or $Q_j = w_1^-\vec{A}_1^n w_1$. In the first case it follows that $Q_{j+1} = \vec{A}_1^-$ and thus $Q_{j+2} = w_1^- w_1$. We then have that $l_{Q_{j+2}} > l_{Q_j}$ and Q_{j+2} is again a word of form (4). In the second case, if $Q_j = \vec{A}_1^n w_1$ then it easily follows that $Q_{j+1} = \vec{A}_1^-\vec{A}_1^n$ and thus $Q_{j+2} = \vec{B}_2^-\vec{B}_2^n$ and, since $l_{\vec{B}_2} > l_{\vec{A}_1}$ that $l_{Q_{j+2}} > l_{Q_j}$ and Q_{j+2} is again a word of form (4). If $Q_j = w_1^-\vec{A}_1^n w_1$ then $Q_{j+1} = w_1^-\vec{A}_1^{n+1}$ but then $Q_{j+2} = w_1\vec{A}^{n+1}$ and thus $Q_{j+3} = \vec{A}_1^-\vec{B}_2^{n+1}$. Clearly, $l_{Q_{j+3}} > l_{Q_j}$ and Q_{j+3} is again a word of form (4).
If $Y_j = \epsilon$ and w_1Y_j in Q_j is entered with shift 1, then T reproduces w_1Y_j in Q_{j+1}. However, then we again have that either $X_j = \vec{B}_2^-$ or $X_j = \vec{A}_1^-$ and thus that we still have that $l_{Q_{j+1}} > l_{Q_j}$ and Q_{j+1} is again a word of form (4).
It now follows that T always has unbounded growth on any word $A_0 \neq 0$ for which the word Q_2 produced after two rounds on A_0 is not a word $\vec{A}_1^n, n \in \mathbb{N}$.

Case II.C.1.d. $w_1 = 011$ $(l_{w_1} = 3, \#1 > 2, w_0 = 1)$**.**
The proof of this case is very similar to the proof of Case II.C.1.c. The reason for this is that $w_1 = 011$ is the mirror image of $w_1 = 110$. As a consequence the words that can be produced through the table method from w_1 are symmetrical to the words produced from w_1 of Case II.C.1.c. through the table method.

Case II.C.2. $l_{w_1} > 3$ $(w_0 = 1,\ \#1 > 2)$**.**
We split the case into two cases, determined by the parity of the number of 0 symbols between consecutive 1 symbols, i.e., $w_1 = 0^{r_1}10^{\dot{x_1}}10^{\dot{x_2}}10^{\dot{x_3}}...0^{\dot{x}_{\#1-1}}10^{t_1}$, with any $\dot{x}_i$, $0 < i < \#1$, odd and $w_1 = 0^{r_1}10^{x_1}10^{x_2}...0^{x_{\#1-1}}10^{t_1}$, with at least one x_i, $0 < i < \#1$ even.

Case II.C.2.a. $w_1 = 0^{r_1}10^{\dot{x_1}}10^{\dot{x_2}}10^{\dot{x_3}}...0^{\dot{x}_{\#1-1}}10^{t_1}$ $(w_0 = 1,\ \#1 > 2,\ l_{w_1} > 3)$.
Note that since all 1 symbols in w_1 are separated by an odd number of 0 symbols, T will either read all 1 symbols in w_1 or no 1 in w_1 during one round on w_1. We split the case into two cases: $l_{w_1} = 4$ and $l_{w_1} > 4$.

Case II.C.2.a.1. $l_{w_1} = 4$ $(w_0 = 1,\ \#1 > 2)$.
There are two possible tag systems in this class: either $w_1 = 1010$, or $w_1 = 0101$. It is easily proven for both tag systems that they either become periodic or have unbounded growth on any word $A_0 \neq 0$. We will only consider the case with $w_1 = 0101$ as the other case is symmetrical to this case.
First, note that after one 0-round on w_1 T produces $\vec{B}_1 = 11$, after one 1-round, T produces w_1^2. It easily follows that for any word $Q_j, j > 1$ produced after j rounds on A_0 we have:

$$Q_j = X_jV_j \quad X_j \in \{\epsilon, \vec{B}_1^-, w_1^-\}, V_j \in \{\vec{B}_1, w_1\}^*$$

Now, since l_{w_1} and $\vec{B}_1$ are even it easily follows that if $X_j = \epsilon$ then T is periodic on Q_j. It also follows that if $X_j \neq \epsilon$ then T has unbounded growth on Q_j. Indeed, if $X_j \in \{\vec{B}_1^-, w_1^-\}$ then every word w_1 in Q_j produces w_1^2 in Q_{j+1} and every word $\vec{B}_1$ produces w_1 in Q_{j+1} and thus we have that $l_{Q_{j+1}} > l_{Q_j}$. Furthermore, since l_{Q_j} is odd, we again have that $X_{j+1} \in \{\vec{B}_1^-, w_1^-\}$ and thus $l_{Q_{j+1}}$ is odd.

Case II.C.2.a.2. $l_{w_1} > 4$ $(w_0 = 1,\ \#1 > 2.)$ It is easily proven that any tag system T in this class will always have unbounded growth on any initial word $A_0 \neq 0$. This follows from Cases II.B.i, $i > 2$ where it was proven that any tag system with $l_{w_1} > 4, \#1 = 2$ always has unbounded growth on any initial word $A_0 \neq 0$ except for the tag systems with $w_1 = 01000$ and $w_1 = 00010$ for which there are two different periodic words. However, the only reason why these two tag systems can be periodic is that the word $\vec{A}_1$ contains only one word w_1. Clearly, this is impossible for the tag systems covered by Case II.C.2.a.2. Indeed, if $\vec{B}_1 = 111$ for this case then $\vec{A}_1$ must contain at least two words w_1.

Case II.C.2.b. $w_1 = 0^{r_1}10^{x_1}10^{x_2}10^{x_3}...0^{x_{\#1-1}}10^{t_1}$, with at least one x_i even $(w_0 = 1,\ \#1 > 2,\ l_{w_1} > 3)$.
It is easily proven that any tag system from this class always has unbounded growth on any word $A_0 \neq 0$. First of all, it is trivial to prove that it takes at most two rounds of T on A_0 to produce a word Q_j that contains at least one word w_1. Now, depending on the shift w_1 is entered with, either at least two 1 symbols are read (thus resulting in the production of two words w_1) or at least one symbol 1 is read. If only one 1 is read this must result in the production of one word w_1 and at least one word w_0 (since $l_{w_1} > 3$). In other words, whatever shift w_1 is entered with it always results in the production of a word that is lengthier than l_{w_1} and that word again contains w_1. Hence the result of unbounded growth.

Case III. $w_0 = 0$**.**
As explained in Sec. 2.2, the only parameters used to determine the several subcases of Case III are parameter 1, the total number of 1 symbols $\#1$ in the two appendants, and parameter 4, the parity of the number of 0 symbols separating consecutive 1 symbols in w_1. An important feature of tag systems T of this case is the changes in the parity of the number of 0 symbols separating consecutive 1 symbols in w_1 with every new s-round on w_1. To see this, take for example the tag system with $w_1 = 101010100$. After one 0-round on w_1 T produces the word $\vec{A}_1 = w_1w_1w_1w_10$, after one 1-round T produces 000. Now, since the number of 0 symbols between the rightmost 1 in the first, second and third w_1 in $\vec{A}_1$ and the leftmost 1 in the second, third and last w_1 in $\vec{A}_1$, respectively, is even, it immediately follows that whatever shift $\vec{A}_1$ is entered with it results in the production of a word that contains $\vec{A}_1$ twice. This implies that this tag system always has unbounded growth on any initial word A_0 that results in the production of a word that contains $\vec{A}_1$ at least once. If, on the other hand, $w_1 = 10101010$ then this tag system will either halt or have unbounded growth. The reason for this is that the distance between consecutive 1 symbols in the word produced after one 0-round on w_1 will always remain odd.
In the remaining sections any sequence of 0 symbols 0^i will be represented as i to avoid complicated notations. For any tag system T with $\#1 > 0$ the word w_1 will be represented as $w_1 = t_11x_11x_21...x_{\#1-1}1r_1$ where r_1 and t_1 denote the number of 0 symbols to the left-hand side of the leftmost 1 in w_1 and to the right-hand side of the rightmost 1 in w_1, respectively; x_i stands for the sequence of 0 symbols separating consecutive 1 symbols in w_1. Indexed variables k_i represent some sequence of 0 symbols of length k_i.

Let T be a tag system with $\#1 > 0$, $\vec{A}_0 = w_1$. Since $\#1 > 0$ it must be the case that after $n \in \mathbb{N}$ s-rounds on w_1 at least one word $\vec{A}_n$ is produced that again contains at least one word w_1. In what follows any such word $\vec{A}_n$ will be represented as:

$$\vec{A}_n = [X_1/2^n][X_2/2^{n-1}]...[X_{n-1}/4][X_n/2]w_1k_{n_1}w_1k_{n_2}...k_{n_j}w_1[Y_n/2][Y_{n-1}/4]...[Y_2/2^{n-1}][Y_1/2^n]$$

Now, let:

$$\vec{A}_i = [X_1/2^i][X_2/2^{i-1}]...[X_{i-1}/4][X_i/2]w_1k_{i_1}w_1k_{i_2}...k_{i_j}w_1[Y_n/2][Y_{i-1}/4]...[Y_2/2^{i-1}][Y_1/2^i]$$

be some word from which $\vec{A}_n$ is produced after n s-rounds with $\vec{A}_0 = w_1$. Then each X_i in $\vec{A}_n$ represents the total number of 0 and 1 symbols erased in $w_1k_{i_1}w_1k_{i_2}...k_{i_j}w_1$ to the left of the leftmost 1 read in $\vec{A}_{i-1}$, and each Y_i in $\vec{A}_n$ represents the total number of 0 and 1 symbols erased to the right of the rightmost 1 that is read in $\vec{A}_{i-1}$. For example, if $n = 1$ then $\vec{A}_1 = [X_1/2]w_1k_{1_1}w_1k_{1_2}...k_{1_j}w_1[Y_1/2]$ and X_1 is the number of 0 symbols produced from the sequence of 0 symbols preceding the first 1 read in w_1 from which T produces the leftmost w_1 in $\vec{A}_1$ and Y_1 is the number of 0 symbols produced from the sequence of 0 symbols that follows the last 1 read in w_1 from which T produces the leftmost w_1 in $\vec{A}_1$.

Before we begin to give the subcases for Case III we will prove that there is an n such that for any number $i \in \mathbb{N}$ the sequences $[X_1/2^{n+i}][X_2/2^{n+i-1}]...[X_{n+i}/2]$ and $[Y_{n+i}/2]...[Y_2/2^{n+i-1}][Y_1/2^{n+i}]$ in $\vec{A}_{n+i}$ are bounded. It is this property that largely determines the ultimate behavior for any tag system T with $\mu = v = 2, w_0 = 0$ for some initial word A_0. This property is proven through Lemma's 3 and 4. Lemma 3 proves that one can determine values $X_{max}, Y_{max}, X_{min}, Y_{min}$ such that for every X_i and Y_i, $X_i = X_{max}$ or $X_i = X_{min}$ and $Y_i = Y_{max}$ or $Y_i = Y_{min}$. This Lemma is used to prove Lemma 4. This Lemma proves the above boundedness property.

Lemma 3. Given a 2-tag system T with $\mu = v = 2$, $w_0 = 0$ and $\#1 > 0$, $w_1 = t_11x_11x_21...1r_1$, then one can determine values $X_{max}, Y_{max}, X_{min}, Y_{min}$ for any word $\vec{A}_n$:

$$\vec{A}_n = [X_1/2^n][X_2/2^{n-1}]...[X_{n-1}/4][X_n/2]w_1k_{n_1}w_1k_{n_2}...k_{n_j}w_1[Y_n/2][Y_{n-1}/4]...[Y_2/2^{n-1}][Y_1/2^n]$$

produced after n s-rounds on w_1 that contains at least one word w_1, for every $X_i, Y_i \in \vec{A}_n$, $X_i = X_{max}$ or $X_i = X_{min}$ and $Y_i = Y_{max}$ or $Y_i = Y_{min}$.

Proof:
In order to prove the Lemma we split Case III into three global subcases, determined by the parities of the sequences of 0 symbols between consecutive 1 symbols in w_1, i.e., $w_1 = r_11t_1$ $(\#1 = 1)$, $w_1 = r_11x_11x_21x_3...x_{\#1-1}1t_1$, $\#1 \geq 2$, with at least one x_i even, and $w_1 = r_11\dot{x}_11\dot{x}_21\dot{x}_3...\dot{x}_{\#1-1}1t_1, \#1 \geq 2$, with every $\dot{x}_i$ odd.

Case a. $w_1 = r_11t_1, \#1 = 1$.
After one s-round on w_1 T either produces the word $\vec{A}_1 = [X_1/2]w_1[Y_1/2]$, with $X_1 = r_1, Y_1 = t_1$ or a sequence of 0 symbols. After one s-round of T on $\vec{A}_1$ T again produces either a sequence of 0 symbols or the word $\vec{A}_2 = [X_1/4][X_2/2]w_1[Y_2/2][Y_1/4], X_2 = r_1, Y_2 = t_1$. It easily follows from this that after n s-rounds of T on w_1 T produces either a sequence of 0 symbols or a word

$\vec{A_n} = [X_1/2^n][X_2/2^{n-1}]...[X_{n-1}/4][X_n/2]w_1[Y_n/2][Y_{n-1}/4]...[Y_2/2^{n-1}][Y_1/2^n]$ since any word $\vec{A_n}$ contains only one word w_1. It follows that for any such word $\vec{A_n}$ that contains at least one word w_1 it must be the case that each $X_i = r_1, Y_i = t_1$ and thus we have that each $X_i = X_{max} = X_{min} = r_1$ and $Y_i = Y_{max} = Y_{min} = t_1$.

Case b. $w_1 = r_1 1 x_1 1 x_2 1 x_3 ... x_{p_1} 1 t_1$, $(\#1 \geq 2)$, with at least one x_i even.
Note that for any tag system T for which w_1 that contains at least two 1 symbols separated by an even number of 0 symbols, T always reads at least one 1 in w_1 during one s-round on w_1. Let:

$$\vec{A_n} = [X_1/2^n][X_2/2^{n-1}]...[X_{n-1}/4][X_n/2]w_1 k_{n_1} w_1 k_{n_2} ... k_{n_j} w_1 [Y_n/2][Y_{n-1}/4]...[Y_2/2^{n-1}][Y_1/2^n]$$

be some word produced after n s-rounds on w_1 and let:

$$\vec{A_i} = [X_1/2^i][X_2/2^{i-1}]...[X_{i-1}/4][X_i/2]w_1 k_{i_1} w_1 k_{i_2} ... k_{i_j} w_1 [Y_n/2][Y_{i-1}/4]...[Y_2/2^{i-1}][Y_1/2^i]$$

be any of the words from which $\vec{A_n}$ is produced after $n - i$ s-rounds on $\vec{A_i}$ with $\vec{A_0} = w_1$. It then easily follows that for each X_i in $\vec{A_n}$ either $X_i = t_1$ (when the first 1 read in $\vec{A_i}$ from which T produces the leftmost w_1 in $\vec{A_{i+1}}$ is the leftmost 1 of the leftmost word w_1 in $\vec{A_i}$) or $X_i = t_1 + \dot{x}_1 + \dot{x}_2 + ... + \ddot{x}_j + j$ with $\ddot{x}_j$ the first sequence of 0 symbols in w_1 that has even length (when the first 1 read in $\vec{A_i}$ from which T produces the leftmost w_1 in $\vec{A_{i+1}}$ is not the leftmost 1 in the leftmost word w_1 in $\vec{A_i}$). Similarly, for each Y_i in $\vec{A_n}$, either $Y_i = t_1$ or $Y_i = \ddot{x_{j'}} + \dot{x}_{j'+1} + ... + \dot{x}_{\#1-1} + r_1 + \#1 - j'$ where j' is the index of the last sequence of 0 symbols in w_1 that has even length. We then have that $X_{max} = t_1 + \dot{x}_1 + \dot{x}_2 + ... + \ddot{x}_j + j, X_{min} = r_1$ and $Y_{min} = t_1, Y_{max} = \ddot{x_{j'}} + \dot{x}_{j'+1} + ... + \dot{x}_{\#1-1} + r_1 + \#1 - j'$ and for any X_i, Y_i in $\vec{A_n}$, $X_i = X_{max}$ or $X_i = X_{min}$ and $Y_i = Y_{max}$ or $Y_i = Y_{min}$.

Case c. $w_1 = r_1 1 \dot{x}_1 1 \dot{x}_2 1 \dot{x}_3 ... \dot{x}_{p_1} 1 t_1$, $(\#1 \geq 2)$, with every $\dot{x}_i$ odd.
Since all 1 symbols in w_1 are separated by an odd number of 0 symbols, T will either produce a sequence of 0 symbols or the following word after one s-round of T on w_1:

$$\vec{A_1} = [X_1/2]w_1[\dot{x}_1/2]w_1[\dot{x}_2/2]w_1...w_1[\dot{x}_{\#1-1}/2]w_1[Y_1/2]$$

with $X_1 = r_1, Y_1 = t_1$.
Now, if there is at least one j such that $t_1 + [\dot{x}_j/2] + r_1$ is even, where $t_1 + [\dot{x}_j/2] + r_1$ gives the total number of 0 symbols separating the rightmost 1 symbol in a word w_1 from the leftmost 1 symbol in the next word w_1 in $w_1[\dot{x}_1/2]w_1$, then the case reduces to case b since then there is always at least one w_1 for which the total number of 1 symbols in w_1 will be read by T. It then immediately follows that $X_{min} = r_1, X_{max} = jl_{w_1} + [\dot{x}_1/2] + [\dot{x}_2/2] + ... + [\dot{x}_j/2] + r_1, Y_{min} = t_1, Y_{max} = r_1 + [\dot{x_{j'}}/2] + [\dot{x_{j'+1}}/2] + ... + [\dot{x}_{\#1-1}/2] + (\#1 - j')l_{w_1}$ with $t_1 + [\dot{x_j}/2] + r_1$ and $t_1 + [\dot{x_{j'}}/2] + t_1$ respectively the leftmost and rightmost sequence of 0 symbols in $\vec{A_1}$ that is even. For any word $\vec{A_n}$ produced after n s-rounds on w_1 and any X_i and Y_i in $\vec{A_n}$ produced after $n-1$ s-rounds on $\vec{A_1}$, $X_i = X_{max}$ or $X_i = X_{min}$ and $Y_i = Y_{max}$ or $Y_i = Y_{min}$.
If there is no such j then T produces either a sequence of 0 symbols or the following word:

$$\vec{A_2} = [X_1/4]\vec{A_1}[\dot{x}_1/4]\vec{A_1}[\dot{x}_2/4]\vec{A_1}...\vec{A_1}[\dot{x}_{\#1-1}/4]\vec{A_1}[Y_1/4]$$

after one s-round on $\vec{A_1}$. Again, if there is at least one j such that $t_1 + [Y_1/2] + [\dot{x}_j/4] + [X_1/2] + r_1$ is even the case reduces to case b, and we can again determine $X_{max}, X_{min}, Y_{max}$ and Y_{min}.

If there is no such j then T will again produce a sequence of 0 symbols or a word $\vec{A}_3$ after one s-round on $\vec{A}_2$,...
Generally speaking, for any T from this class, there are two possibilities. The first is that there is an n such that the word:

$$\vec{A}_n = [X_1/2^n]\vec{A}_{n-1}[\dot{x}_1/2^n]\vec{A}_{n-1}[\dot{x}_2/2^n]\vec{A}_{n-1}...\vec{A}_{n-1}[\dot{x}_{\#1-1}/2^n]\vec{A}_{n-1}[Y_1/2^n]$$

contains at least one sequence of 0 symbols $t_1 + [Y_{n-1}/2] + [Y_{n-2}/4] + ...[Y_1/2^n] + [\dot{x}_j/2^n] + [X_1/2^n] + ... + [X_{n-2}/4] + [X_{n-1}/2] + r_1$ that is even. The second possibility is that there exists no such n. If there exists such an n, then the case reduces to case b and $X_{max}, X_{min}, Y_{max}$ and Y_{min} can be determined. If there exists no such n then it immediately follows that for any $\vec{A}_n$, $X_{max} = X_{min} = r_1$, $Y_{max} = Y_{min} = t_1$. □

Lemma 4. Given a 2-tag system with $\mu = v = 2$, $w_0 = 0$ and $\#1 > 0$, $w_1 = t_1 1 x_1 1 x_2 1...1 r_1$, then it is possible to determine values $n, \mathbf{Max_X}, \mathbf{Max_Y} \in \mathbb{N}$ such that for any word $\vec{A}_{n+i}, i \in \mathbb{N}$:

$$\vec{A}_{n+i} = [X_1/2^{n+i}]...[X_{n+i-1}/4][X_{n+i}/2]w_1 k_{n+i_1} w_1 ... k_{n+i_j} w_1 [Y_{n+i}/2][Y_{n+i-1}/4]...[Y_1/2^{n+i}]$$

produced after n s-rounds on w_1 that contains at least one word w_1 we have that

$$[X_1/2^{n+i}]...[X_{n+i-1}/4][X_{n+i}/2] \leq \mathbf{Max_X}$$
$$[Y_{n+i}/2][Y_{n+i-1}/4]...[Y_1/2^{n+i}] \leq \mathbf{Max_Y}$$

Proof:
We will only prove that any sequence of 0 symbols $[X_1/2^{n+i}]...[X_{n+i-1}/4][X_{n+i}/2]$ will become bounded by some constant $\mathbf{Max_X}$ after a certain number n of s-rounds of T on w_1. The proof for the right-hand side is symmetrical to this case.[3]
Let T be a 2-tag system with $\mu = v = 2$, $w_0 = 0$ and $\#1 > 0$, then for any word $\vec{A}_m$ produced after m s-rounds on w_1 that contains at least one word w_1:

$$\vec{A}_m = [X_1/2^m][X_2/2^{m-1}]...[X_m/2]w_1 k_{m_1} w_1 k_{m_2} ... k_{m_j} w_1 [Y_m/2]...[Y_2/2^{m-1}][Y_1/2^m]$$

It follows from Lemma 3 that for each X_i and Y_i in $\vec{A}_m$, $X_i \leq X_{max}, Y_i \leq Y_{max}$. Note also that any $[X_i/2^{m-i+1}]$ is in fact $\lfloor X_i/2^{m-i+1} \rfloor$ or $\lceil X_i/2^{m-i+1} \rceil$. In the first case, $[X_{i+1}/2^{m-i+2}]$ is in fact $\lceil X_{i+1}/2^{m-i+2} \rceil$, in the second, $[X_{i+1}/2^{m-i+2}]$ is in fact $\lfloor X_{i+1}/2^{m-i+2} \rfloor$. The reason for this is that if one sequence of 0 symbols $[X_i/2^{m-i+1}]$ is entered with shift 0 and $[X_i/2^{m-i+1}]$ is odd, then the next sequence of 0 symbols will be entered with shift 1, and vice versa.
We can now determine values $\mathbf{Max_X}$ and $\mathbf{Max_Y}$ and n. Let us assume the worst case such that each X_i and Y_i in $\vec{A}_m$ has the maximum value X_{max} and Y_{max}, respectively. Evidently, there must be an n such that either $\lceil X_{max}/2^n \rceil = 1, \lfloor X_{max}/2^{n-1} \rfloor = 0$ or $\lfloor X_{max}/2^n \rfloor = 0, \lceil X_{max}/2^{n-1} \rceil = 1$. Now, given the word:

$$\vec{A}_n = [X_{max}/2^n][X_{max}/2^{n-1}]...[X_{max}/2]w_1 k_{n_1} w_1 k_{n_2} ... k_{n_j} w_1 [Y_{max}/2]...[Y_{max}/2^n]$$

[3] I am indebted to an anonymous referee for pointing out a serious error in a previous version of this proof.

Then, after one more s-round on $\vec{A}_n$ T produces:

$$\vec{A}_{n+1} = [X_{max}/2^{n+1}][X_{max}/2^n][X_{max}/2^{n-1}]...[X_{max}/2]w_1k_{n+1_1}...k_{n+1_j}w_1[Y_{max}/2]...[Y_{max}/2^{n+1}]$$

Now, if $[X_{max}/2^n] = 0$ in $\vec{A}_n$ then $[X_{max}/2^{n+1}] = 0$ in $\vec{A}_{n+1}$. If $[X_{max}/2^n] = 1, [X_{max}/2^{n-1}] = 0$ in $\vec{A}_n$, then $[X_{max}/2^n] = 0$ in $\vec{A}_{n+1}$ and $[X_{max}/2^{n+1}] = [X_{max}/2^n]$. It thus immediately follows that:

$$\vec{A}_{n+1} = [X_{max}/2^n][X_{max}/2^{n-1}]...[X_{max}/2]w_1k_{n+1_1}...k_{n+1_j}w_1[Y_{max}/2]...[Y_{max}/2^{n+1}]$$

and thus that for any word $\vec{A}_{n+i}$ the sequence of 0 symbols $[X_1/2^{n+i}][X_2/2^{n+i-1}]...[X_{n+i-1}/4][X_{n+i}/2], X_i = X_{min}$ or $X_i = X_{max}$ is bounded either by:

$$\lfloor X_{max}/2^n \rfloor \lceil X_{max}/2^{n-1} \rceil ...[X_{max}/2]$$
$$\text{If } \lfloor X_{max}/2^n \rfloor \lceil X_{max}/2^{n-1} \rceil ...[X_{max}/2] > \lceil X_{max}/2^n \rceil \lfloor X_{max}/2^{n-1} \rfloor ...[X_{max}/2]$$

or by

$$\lceil X_{max}/2^n \rceil \lfloor X_{max}/2^{n-1} \rfloor ...[X_{max}/2]$$
$$\text{If } \lceil X_{max}/2^n \rceil \lfloor X_{max}/2^{n-1} \rfloor ...[X_{max}/2] > \lfloor X_{max}/2^n \rfloor \lceil X_{max}/2^{n-1} \rceil ...[X_{max}/2]$$

□

Case III.A. $\#1 = 0$ $(w_0 = 0)$**.**
It is trivial to prove that any tag system T from this class will always halt on any initial word A_0. First of all, after one round of T on A_0 T produces the word $Q_1 = 0^n, n \in \mathbb{N}$. Secondly, T always halts on any sequence of 0 symbols because at each computation step for every 0 appended 2 are deleted.

Case III.B. $\#1 = 1,\ (w_0 = 0)$**.**
Any tag systems T with $w_1 = r_1 1 t_1$ covered by this case will either halt or become periodic on any initial word A_0 because the table method halts for T. Note that this is case **a** of the proof of Lemma 3 and thus we have that after n s-rounds of T on w_1 T produces either a sequence of 0 symbols or a word:

$$\vec{A}_n = [r_1/2^i][r_1/2^{i-1}]...[r_1/2]w_1[t_1/2]...[t_1/2^{n-1}][t_1/2^n]$$

since w_1 contains only one symbol 1. Now, it follows from Lemma 4 that we can determine an n such that for any word $\vec{A}_{n+i}$ produced after $n+i$ s-rounds on w_1 the sequence of 0 symbols to the left-hand of the leftmost and right-hand side of the rightmost 1 in w_1 in $\vec{A}_{n+i}$ is bounded by come constant. This implies that there is thus but a finite number of different words $\vec{A}_i$ that can be produced with the table method and thus the table method halts for T.

Case III.C. $\#1 = 2,\ (w_0 = 0)$**.**
We split the case into two subcases, determined by the parity of the number of 0 symbols between the two 1 symbols in w_1, i.e., $w_1 = r_1 1 \ddot{x}_1 1 t_1$ or $w_1 = r_1 1 \dot{x}_1 1 t_1$.

Case III.C.1. $w_1 = r_1 1 \ddot{x}_1 1 t_1$ $(w_0 = 0,\ \#1 = 2)$**.**
Note that after one s-round of T on w_1, T always produces one w_1, surrounded by a finite number of 0 symbols and thus this case is similar to Case III.B. I.e., only a finite number of words can be produced

by the table method when applied to w_1 and it thus follows that T always halts or becomes periodic on any initial word A_0.

Case III.C.2. $w_1 = r_1 1 \dot{x_1} 1 t_1$ $(w_0 = 0,\ \#1 = 2)$
For any tag system T covered by this case, either T only reads 0 symbols or two 1 symbols during an s-round on w_1. Thus, after one s-round on w_1 T either produces a sequence of 0 symbols, ultimately leading to the production of ϵ, or the following word:

$$\vec{A_1} = [r_1/2] w_1\, [\dot{x_1}/2]\, w_1 [t_1/2]$$

The parity of the length of $t_1\, [\dot{x_1}/2]\, r_1$ plays a significant role in the ultimate behavior of T. Indeed, if $t_1\, [\dot{x_1}/2]\, r_1$ is odd then both words w_1 in $\vec{A_1}$ will be entered with the same shift. If $t_1\, [\dot{x_1}/2]\, r_1$ is even and the first w_1 in $\vec{A_1}$ is entered with shift 0 then the second w_1 will be entered with shift 1 and vice versa.
Then, if $t_1\, [\dot{x_1}/2]\, r_1$ in $\vec{A_1}$ has even length T produces one of the following words:

$$k_{1,1} \vec{A_1} k_{1,2}$$

or:

$$k_{1,3} \vec{A_1} k_{1,4}$$

after one s-round on $\vec{A_1}$, where $k_{i,j}$ indicates some sequence of 0 symbols. It then easily follows that if $t_1 [\dot{x_1}/2] r_1$ in $\vec{A_1}$ has even length, then T will either become periodic or halt on A_0 due to the boundedness property on the number of 0 symbols the left and right of any word $\vec{A_n}$ produced after $n-1$ s-rounds on $\vec{A_1}$ (Lemma 4).
If $t_1\, [\dot{x_1}/2]\, r_1$ has odd length, T produces:

$$\vec{A_2} = [r_1/4] \vec{A_1}\, [\dot{x_1}/4]\, \vec{A_1} [t_1/4]$$

after one s-round on $\vec{A_1}$ or a sequence of 0 symbols depending on the shift with which $\vec{A_1}$ is entered and the number of 0 symbols preceding the first 1 in $\vec{A_1}$.
The ultimate behavior of T depends on the parity of the length of $t_1 [t_1/2]\, [\dot{x_1}/4]\, [r_1/2] r_1$ in $\vec{A_2}$. If $t_1 [t_1/2]\, [\dot{x_1}/4]\, [r_1/2] r_1$ has even length, T produces either one of the following words after one s-round on $\vec{A_2}$:

$$k_{2,1} \vec{A_2} k_{2,2}$$

or:

$$k_{2,3} \vec{A_2} k_{2,4}$$

It thus follows that if $t_1 [t_1/2]\, [\dot{x_1}/4]\, [r_1/2] r_1$ is even that T will always either halt or become periodic on any initial word A_0.
If $t_1 [t_1/2] [\dot{x_1}/4] [r_1/2] r_1$ has odd length, then after one s-round on $\vec{A_2}$ T again either produces a sequence of 0 symbols or a word $\vec{A_3}$ containing two words $\vec{A_2}$,...
It easily follows that tag systems T from this class will always become periodic or halt on any word A_0 if there is an n such that after n s-rounds of T on w_1 T produces the word:

$$\vec{A_n} = [r_1/2^n] \vec{A}_{n-1}\, [\dot{x_1}/2^n]\, \vec{A}_{n-1} [t_1/2^n]$$

with $t_1[t_1/2]...[t_1/2^n][(\dot{x_1}-1)/2^n][r_1/2^n]...[r_1/2]+r_1]$ of even length.
It easily follows from Lemma 4 that we can determine (in a finite number of steps) for any T covered by this case whether or not there exists such an n. If not, T will either halt or have unbounded growth on A_0 since for any word $\vec{A_i}$ produced after i s-rounds on w_1 T either produces a sequence of 0 symbols or a new word $\vec{A}_{i+1}$ containing two words $\vec{A_i}$. In this last case it can be decided whether or not T will have unbounded growth. The reason for this is that (1) one can determine an m such that for any word $\vec{A}_{m+i}$ the length of the sequences of 0 symbols separating consecutive 1 symbols in $\vec{A}_{m+i}$ is bounded (Lemma 4) (2) for any sequence Q_j produced after j rounds on A_0 the length of the sequences of 0 symbols separating consecutive words $\vec{A}_k$ is bounded (3) given m one can determine the parity of any word $\vec{A}_k$. Then, since any word Q_{m+j} produced after $m+j$ rounds on A_0 contains only words $\vec{A}_{m+i}, i \in \mathbb{N}$ and sequences of 0 symbols, the result easily follows.

Case III.D. $\#1 \geq 3$**,** $(w_0 = 0)$**.**
We split the case into two subcases, determined by the parity of the number of 0 symbols separating consecutive 1 symbols in w_1, i.e., $w_1 = r_1 1\dot{x}_1 1\dot{x}_2 1...1\dot{x}_{\#1-1} 1 t_1$ with every $\dot{x}_i$ odd or $w_1 = r_1 1 x_1 1 x_2 1...x_{\#1-1} 1 t_1$ with at least one x_i even. Note that the fundamental difference between these two cases is the fact that with all 1 symbols in w_1 separated by an odd number of 0 symbols either all 1 symbols or no 1 symbols are read during one s-round on w_1. If at least two 1 symbols are separated by an even number of 0 symbols, either at least one 1 or two 1 symbols will be read by T during one s-round on w_1.

Case III.D.1. $w_1 = r_1 1 x_1 1 x_2 1...x_{\#1-1} 1 t_1$ $(\#1 \geq 3, w_0 = 0)$
Clearly, for any of these tag systems either at least one 1 or at least two 1 symbols are read during one s-round on w_1. This means that none of these tag systems can halt on an initial word A_0 in which T reads at least one 1.
Now, for any tag system T with $\#1 > 3$, if T reads at least two 1 symbols in w_1 whatever shift w_1 is entered with, then T always produces $w_1^n, n \geq 2$ after one s-round on w_1. From this it easily follows that T always has unbounded growth on any initial word A_0 for which T reads at least one 1 during one round on A_0. If no 1 is read in A_0 then T will halt on A_0.
If T always either reads one 1 or at least two 1 symbols during one s-round on w_1, then it follows that there is either one sequence x_i of 0 symbols in w_1 that is even or there are two consecutive sequences x_i, x_{i+1} that are even. If there is only one x_i that is even then either $i = 1$ or $i = \#1-1$. Otherwise, if we have a pair of (x_i, x_{i+1}) that are even then $i \in \{1, ..., \#1-1\}$. For the sake of simplicity we assume that there is only one sequence of 0 symbols x_i that is even and that $i = \#1-1$, thus $w_1 = r_1 1\dot{x}_1 1\dot{x}_2...1\ddot{x}_{\#1-1} 1 r_1$. The other cases easily reduce to this case. After one s-round on w_1 T produces either one of the following words:

$$\vec{A}_1 = [r_1/2]w_1[\dot{x_1}/2]w_1[\dot{x_2}/2]w_1...[\dot{x}_{\#1-2}/2]w_1[\ddot{x}_{\#1-1}/2][t_1/2]$$

$$\vec{B}_1 = k_{1,1} w_1 k_{1,2}$$

Note that there are $\#1-1$ words w_1 in $\vec{A}_1$. The word $\vec{B}_1$ reduces to w_1 since the sequence of 0 symbols $k_{1,1}$ and $k_{1,2}$ will become bounded after a finite number of s-rounds (Lemma 4). This allows for periodicity.
The parity of the length of each of the sequences $t_1[\dot{x}_i/2]r_t$ in $\vec{A}_1$ is a determining feature in the ultimate

behavior of T. Indeed, if there is at least one i such that the sequence of 0 symbols $t_1[\dot{x}_i/2]r_t$ in $\vec{A}_1$ is odd and the word w_1 preceding it is entered with shift 0, the word w_1 following it will be entered with shift 1 and conversely. If there is no such sequence then all words w_1 will be entered with the same shift. Now, if there is an i such that the sequences of 0 symbols $t_1[\dot{x}_i/2]r_t$ in $\vec{A}_1$ has odd length then it follows that T always has unbounded growth on any initial word that produces a word that contains at least one word $\vec{A}_1$. The reason for this is that the two possible words $\vec{C}_2$ and $\vec{D}_2$ that can then be produced after one s-round on $\vec{A}_1$ again contain at least one word $\vec{A}_1$ and one word $\vec{B}_1$. Indeed, the fact that there is at least one sequence of 0 symbols $t_1[\dot{x}_i/2]r_t$ in $\vec{A}_1$ that has odd length implies that if the word w_1 preceding $[\dot{x}_i/2]$ results in $\vec{A}_1$ then the word w_1 following it will result in $\vec{B}_1$ and vice versa. We then have that words $\vec{C}_2$ and $\vec{D}_2$ again result in the production of at least one word $\vec{C}_2$ or $\vec{D}_2$ (from $\vec{A}_1$ in $\vec{C}_2$ and $\vec{D}_2$) and at least one word w_1 (from $\vec{B}_1$ in $\vec{C}_2$ and $\vec{D}_2$). Thus, once either $\vec{C}_2$ or $\vec{D}_2$ are produced the word produced with each successive s-round on $\vec{C}_2$ or $\vec{D}_2$ always contains at least one word $\vec{C}_2$ or $\vec{D}_2$ and at least one additional w_1 as compared to the word from the previous s-round. This gives unbounded growth.

If there is no sequence of 0 symbols $t_1[\dot{x}_i/2]r_t$ in $\vec{A}_1$ that has odd length, then T produces either one of the following words after one s-round on $\vec{A}_1$:

$$\vec{A}_2 = [r_1/4]\vec{A}_1[\dot{x}_1/4]\vec{A}_1[\dot{x}_2/4]\ldots\vec{A}_1[\dot{x}_{\#1-2}/4]\vec{A}_1[\ddot{x}_{\#1-1}/4][t_1/4]$$

$$\vec{B}_2 = [r_1/4]\vec{B}_1[\dot{x}_1/4]\vec{B}_1[\dot{x}_2/4]\ldots\vec{B}_1[\dot{x}_{\#1-2}/4]\vec{B}_1[\ddot{x}_{\#1-1}/4][t_1/4]$$

Again, if $\vec{A}_2$ contains at least one sequence of 0 symbols $t_1[\ddot{x}_{\#1-1}/2][t_1/2][\dot{x}_i/4][r_1/2]r_1, 0 \leq i \leq \#1-1$ that has odd length then it easily follows that T always has unbounded growth on any word A_0 that results in the production of a word that contains $\vec{A}_2$ at least once. The reason is that the two possible words $\vec{C}_3$ and $\vec{D}_3$ that can be produced after one s-round on $\vec{A}_2$ each contain at least one word $\vec{A}_2$ and one word $\vec{B}_2$. Thus, the word produced with each new s-round on one of the words $\vec{C}_3$ or $\vec{D}_3$ must again contain at least one word $\vec{C}_3$ or $\vec{D}_3$ and at least one additional word $\vec{B}_2$. This results in unbounded growth.

Similarly, if there is at least one sequence of 0 symbols $t_1k_{1,2}[\dot{x}_i/4]k_{1,1}r_1$ in $\vec{B}_2$ with odd length, then T always has unbounded growth on any word A_0 that results in the production of a word that contains $\vec{A}_1$ at least once for similar reasons. Note that $\vec{B}_2$ is similar to $\vec{A}_1$. I.e., $\vec{B}_2$ also contains $\#1-1$ words w_1. This allows for periodicity in the same way as the production of $\vec{B}_1$ does. The only difference between $\vec{A}_1$ and $\vec{B}_2$ is the number of 0 symbols separating consecutive words w_1 in $\vec{B}_2$. In this sense the possible productions from this word reduce to that of the word $\vec{A}_1$ and we will thus not consider these productions here.

In general we have two possibilities. In the first possibility there is an n such that one of the words $\vec{A}_n$ or $\vec{B}_n$ given below exists. In the second possibility there exist no such words $\vec{A}_n$ or $\vec{B}_n$. The word $\vec{A}_n$ has the following form:

$$\vec{A}_n = [r_1/2^n]\vec{A}_{n-1}[\dot{x}_1/2^n]\vec{A}_{n-1}[\dot{x}_2/2^n]\vec{A}_{n-1}[\dot{x}_{\#1-2}/2^n]\vec{A}_{n-1}[\ddot{x}_{\#1-1}/2^n][t_1/2^n]$$

where A_n contains the following odd length subword consisting entirely of 0 symbols:

$$t_1[\ddot{x}_{\#1-1}/2][t_1/2]...[\ddot{x}_{\#1-1}/2^{n-1}][t_1/2^{n-1}][\dot{x}_i/2^n][r_1/2^{n-1}]...r_1/2r_1, 0 \leq i \leq \#1-1$$

B_n has the following form:

$$\vec{B}_n = [r_1/2^n]\vec{B}_{n-1}[\dot{x}_1/2^n]\vec{B}_{n-1}[\dot{x}_2/2^n]\vec{B}_{n-1}[\dot{x}_{\#1-2}/2^n]\vec{B}_{n-1}[\ddot{x}_{\#1-1}/2^n][t_1/2^n]$$

where $\vec{B}_n$ contains the following subword consisting entirely of 0 symbols of odd length:

$$t_1k_{1,2}[\ddot{x}_{\#1-1}/4][t_1/4]...[\ddot{x}_{\#1-1}/2^{n-1}][t_1/2^{n-1}][\dot{x}_i/2^n][r_1/2^{n-1}]...[r_1/4]k_{1,1}r_1$$

It easily follows from Lemma 4 that one can decide in a finite number of steps whether or not there exists such an n. Now, if there is such an n then T has unbounded growth on any word A_0 that results in the production of a word that contains $\vec{A}_n$ or $\vec{B}_n$ at least once, else T halts or becomes periodic on A_0. Clearly, it can be decided in a finite number of rounds j whether or not T will produce a word Q_j that contains $\vec{A}_n$ or $\vec{B}_n$. The reason for this is that it follows from Lemma 4 that there is an $m \in \mathbb{N}$ such that for any word $\vec{A}_{m+i}$ and $\vec{B}_{m+i}, i \in \mathbb{N}$ the length of any sequence of 0 symbols between two consecutive 1 symbols is bounded by some constant and it must be the case that $n \leq m$. One thus only has to wait at most $m+1$ rounds on A_0 to see whether or not T will produce these words.
If there is no such n then we again have that T will either halt, become periodic or have unbounded growth on A_0. Also here it can be decided in a finite number of rounds of T on A_0 whether or not T will have unbounded growth on A_0 for similar reasons.

Case III.D.2. $w_1 = r_1 1\dot{x}_1 1\dot{x}_2 1...1\dot{x}_{\#1-1}1t_1$ $(\#1 \geq 3, w_0 = 0)$
For any tag system T covered by this case, all 1 symbols in w_1 are separated by an odd number of 0 symbols. As a consequence, either zero or $\#1$ 1 symbols are read during one s-round on w_1. After one s-round on w_1 T thus produces either a sequence of 0 symbols which ultimately results in the production of ϵ or the word:

$$\vec{A}_1 = [r_1/2]w_1[\dot{x_1}/2]w_1[\dot{x_2}/2]w_1...[\dot{x}_{\#1-1}/2]w_1[t_1/2]$$

Note that the word $\vec{A}_1$ contains $\#1$ words w_1.
Now, if there is at least one sequence of 0 symbols $t_1[\dot{x}_i/2]r_1$ that has even length then the case reduces to the previous Case III.D.1. I.e., for tag systems covered by Case III.D.1., either at least one word w_1 or at least two words w_1 are produced after one s-round on w_1. Similarly, for any tag system T if the word $\vec{A}_1$ produced after one s-round on w_1 contains at least one sequence of 0 symbols $t_1[\dot{x}_i/2]r_1$ that is even, either at least one word $\vec{A}_1$ or at least two words $\vec{A}_1$ are produced after one s-round on $\vec{A}_1$. The ultimate behavior of T then depends on the evolution of the parity of the number of 0 symbols separating consecutive words w_1 in the words produced from $\vec{A}_1$. Note that for any initial word A_0, if after two rounds of T on A_0 T does not produce a word Q_2 that contains $\vec{A}_1$ at least once, then Q_2 is a sequence of 0 symbols and thus T will halt on A_0. Otherwise, T will either become periodic or have unbounded growth on A_0. If there is no sequence of 0 symbols $r_1[\dot{x}_i/2]t_1$ of even length in $\vec{A}_1$ T either produces a sequence of 0 symbols or:

$$\vec{A}_2 = [r_1/4]\vec{A}_1[\dot{x_1}/4]\vec{A}_1[\dot{x_2}/4]\vec{A}_1...[\dot{x}_{\#1-1}/4]\vec{A}_1[t_1/4]$$

Again, if there is at least one sequence of 0 symbols $t_1[t_1/2][\dot{x}_i/4][r_1/2]r_1$ of even length then the case reduces to Case III.D.1. If this occurs and after three rounds of T on A_0 T does not produce a word Q_3 that contains $\vec{A}_2$ at least once, then Q_3 is a sequence of 0 symbols and thus T will halt on A_0. Otherwise, T will either become periodic or have unbounded growth on A_0.
As is clear from these productions either there is or is no n such that after n s-rounds of T on A_0 T produces the word:

$$\vec{A}_n = [r_1/2^n]\vec{A}_{n-1}[\dot{x_1}/2^n]\vec{A}_{n-1}[\dot{x_2}/2^n]\vec{A}_{n-1}...[\dot{x}_{\#1-1}/2^n]\vec{A}_{n-1}[t_1/2^n]$$

and $\vec{A}_n$ contains at least one sequence of 0 symbols $t_1[t_1/2]...[t_1/2^{n-1}][\dot{x}_i/2^n][r_1/2^{n-1}]...[r_1/2]r_1$ that has even length. Note that T either always produces a word $\vec{A}_j$ or a sequence of 0 symbols after j s-rounds on w_1, with $j \leq n$ if there is such an n and $j \in \mathbb{N}$ if there is no such n.
Clearly, if there is such an n then the case reduces to Case III.D.1. It then follows that given some initial word A_0, if after $n+1$ rounds of T on A_0 T does not produce a word Q_{n+1} that contains $\vec{A}_n$ at least once, then Q_{n+1} is a sequence of 0 symbols and thus T will halt on A_0. Otherwise, T will either become periodic or have unbounded growth on A_0. If there is no such n then the behavior of T reduces to those subcases of Case III.C.2 for which there is no n such that the rightmost and leftmost 1 in the words $\vec{A}_{n-1}$ in $\vec{A}_n$ are separated by an even number of 0 symbols. I.e., T will either halt or have unbounded growth on any initial word A_0 and this can be decided in a finite number of steps.

Given the proofs of Cases I–III Theorem 1 follows

□

3. Discussion

It might be very hard, if not impossible, to prove the solvability of those classes of tag systems that are closest to TS(2, 2), i.e., TS(2,3) and TS(3,2). The class TS(2,3) contains the example provided by Post, which is known for its complexity. The class TS(3,2) contains a tag system that is capable to simulate the $3n+1$ problem, a number theoretical problem that is still open [10]. As far as our experience goes, it seems that the methods used in the present proof cannot be used to prove these classes decidable. For example, consider the application of the table method to Post's example from Sec. 1.1. Still, the table method is a very useful and simple tool to study and prove certain properties of tag systems. This method can also be automated and thus used in computer-based research on tag systems. It also allows us to reveal the structure of the kind of words that can be produced by a given tag system T. As is clear from the proof of Theorem 1, the different kinds of structures found for tag systems from the class TS(2,2), with $l_{min} < v, l_{max} > v$ is very simple and predictable. It is exactly this simplicity of the structure of the words that can be produced by tag systems T from this class that allows us to decide the two decision problem discussed here.
In recent years there has been a lot of research on non-standard models of computability. One example comes from the context of Turing machines, where the standard model was generalized by allowing an infinitely repeated word to the left and right of the input (weak Turing machines) and left or right of the input (semi-weak Turing machines). This generalization has made it possible to find smaller universal Turing machines than those known for the standard model (see e.g. [4, 22]). It would be interesting to extend this research to tag systems. One could, for example, consider (universal) tag systems that cannot halt, but always have unbounded growth. Let us call such (universal) tag systems weak (universal) tag systems.[4] However, even if one considers such more general models, the simplicity of the structure of the words that can be produced by any tag system in the class TS(2,2), with $l_{min} < v, l_{max} > v$, seems

[4]Note that it is impossible to *directly* apply weak or semi-weak universality as defined for cellular automata and Turing machines to tag systems, because one always has to tag something at the *end* of a word, on the basis of what happens at the *beginning* of a word. One round on a word A would then take infinite time.

to exclude the possibility of universal encoding for any of these tag systems.[5] In this sense, the proof of Theorem 1 gives strong support for the following conjecture:

Conjecture 1. The class of tag systems TS(2,2), $l_{min} < v, l_{max} > v$ is not (weak) universal.

Given Wang's theorem (Sec. 1.1) we only had to consider tag systems with $l_{min} < v, l_{max} > v$ to prove Theorem 1 and thus we did not study the more general class, including those cases for which $l_{min} \geq v$.[6] A more intensive analysis would be needed to know whether the following statement is true or false:

Statement 1. There exists no unsolvable decision problem for tag systems from the class TS(2,2).

If false, this could imply that there is weak universality in the class TS(2,2).

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